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ORIGINAL ARTICLE

Why FDE might be too strong for Beall

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Abstract
In his "The simple argument for subclassical logic," Jc Beall advances an argument that led him to take **FDE** as the one true logic (the latter point is explicitly made clear in his "**FDE** as the One True Logic"). The aim of this article is to point out that if we follow Beall's line of reasoning for endorsing **FDE**, there are at least two additional reasons to consider that **FDE** is too strong for Beall's purposes. In fact, we claim that Beall should consider another weaker subclassical logic as the logic adequate for his project. To this end, we first briefly present Beall's argument for **FDE**. Then, we discuss two specific topics that seem to motivate us to weaken **FDE**. We then introduce a subsystem that will enjoy all the benefits of Beall's suggestion.

Keywords Subclassical logic · **FDE** · Jc Beall · Double negation · Distribution law

1 Introduction

In his "The simple argument for subclassical logic" Beall (2018), Jc Beall advances an argument to the effect that **FDE** is the one true logic (the same claim is also advanced in Beall (2019)). As Beall himself recognizes, the argument is extremely simple, relying on the typical expectations we have on the role of logic and on its traditional features: logic is *universal, topic neutral, and intransgressible*. In a nutshell, the reasons for adopting **FDE** as the one true logic may be put as follows: "we lose no true theories by accepting the **FDE** account, and we gain live options for true theories" (Beall, 2019, p.119).

Our aim in this paper is very simple: by applying the same kind of reasoning used by Beall, we suggest that one need not stop at **FDE**; there may be even weaker subclassical

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systems that are available and that allow us to lose nothing, with the benefit that we gain still further options for true theories. In this sense, our suggestion is that **FDE** may be too strong for Beall's purposes. We advance our claim based on Beall's own considerations leading him to prefer subclassical logic.

Let us spell some of the details behind Beall's argument because they will be important for our aims in this paper. We begin with the role of logic. According to Beall (Beall, 2018, p.31):

> what logic (qua logical consequence) does is both firmly mark the remotest boundaries of theoretical possibilities and also furnish the weakest skeletal structure of our true (closed) theories. And that's pretty much it.
> …at the very least, the sheer weakness of logic makes it plain that much of our rational inferential behavior is in many (most?) cases phenomenon-/theory-specific, unlike logic, which is universal.

That is, logic furnishes the inferential basis of *every* theory. What distinguishes such theories — in terms of what counts as good or bad inferential moves —, is very much theory-specific, depending on the theory. So, according to Beall, we add some determinations that fill in the details of inferential validity on the top of logic, in order to adjust our inferences to the particularities of the topic being dealt with.

That characterization of the role of logic relies on the usual features of logic: *universality, topic neutrality, and irrefutability*. In a nutshell, being universal means that logic is involved in *every* theory, it sets the basis for each of such attempts at a true description of the world. Being topic-neutral indicates that logic does not discriminate between topics; it applies equally to any subject whatsoever — something cannot be logically valid for one topic, but not for another, i.e., such distinctions are left for each topic to stipulate, and result, by that very fact, non-logical. Now, given that logic is involved in every theory, logic cannot be violated by the specific features of any particular theory, i.e., logic is intransgressible.

Given such features and such a role attributed to logic, how should one evaluate the dispute between classical logic and a subclassical logic such as **FDE**? The answer is simple, according to Beall: **FDE** scores very high on topic neutrality, universality, and intransgressibility, while classical logic introduces *ad hoc* restrictions that limit the scope of logic. More specifically, **FDE** allows for more options by encompassing the treatment of predicates that are gappy and/or glutty as open possibilities. Classical logic, on the other hand, requires predicates to be always exhaustive and exclusive, ruling such possibilities out by *fiat*. As a result, **FDE** allows for more theories than classical logic, it is more general (see (Beall, 2018, p.35)). To put the same point in different terms, given that logic, according to the conception advanced, delimits the most general space of possibilities, we should choose the one allowing for more possible theories; **FDE** does just that, without imposing *ad hoc* restrictions:

> we gain the possibility of true glutty theories and true (and prime) gappy theories. These possibilities mightn't be strikingly important in the face of normal phenomena with which natural science or even mathematics deals; however, they are strikingly important for the strange phenomena at the heart of other theories (e.g., paradoxes, weird metaphysical entities, more). (Beall, 2018, p.48)

Also, adopting **FDE** does not imply that we lose theories available for classical logicians: by applying the well-known Shrieking and Shrugging strategies, we can obtain classical theories without any restriction. The point is that on this account of logic as delimiting the most general space of possibilities, demanding exhaustion and exclusion of every predicate is not the task of logic, but rather a matter left to be decided by the subject matter (i.e., the topic) being dealt with.

However, given that the space of possibilities is enlarged by allowing gappy and glutty theories, why stop there? In this paper, we suggest that by adopting Beall's line of argument, one needs not to rest content with the new theories allowed by **FDE**. Indeed, there seem to be good reasons to think that, just as classical logic introduces strong requirements by enforcing exhaustion and exclusion for predicates, **FDE** may also be seen as adding constraints that one could consider as *ad hoc* too. This will lead us to even weaker systems. Where exactly should such a process stop is a delicate issue we shall comment on later.

The structure of this article is as follows. In §2, we recall the basics of **FDE**, which shall be used in our discussion throughout the other sections of this article. In §3 we advance two kinds of reasons that could lead one to think that **FDE** is too strong. Then, in §4, we present a system incorporating such restrictions, which is followed by some reflections in §5. We close the paper with our conclusions in §6. A proof of one of the propositions is left to the appendix.

2 Basics of FDE

The propositional language $\mathcal{L}$ consists of a set $\{\neg, \wedge, \vee\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. We denote by Form the set of formulas defined as usual in $\mathcal{L}$, denote a formula of $\mathcal{L}$ by A, B, C, etc. and a subset of Form by Γ, Δ, Σ, etc.

As is well-known, there are a few presentations of the semantics of **FDE**. For our purpose, let us first recall the two-valued *relational* presentation, and then the four-valued *functional* presentation. We name the interpretation after Michael Dunn, who introduced the relational semantics in Dunn (1976).

Definition 1 A *Dunn-interpretation* of $\mathcal{L}$ is a relation, r, between propositional variables and the values 1 and 0, namely, $r \subseteq \mathsf{Prop} \times \{1, 0\}$. Given an interpretation, r, this is extended to a relation between all formulas and truth values by the following clauses:

$$\neg A r 1 \text{ iff } A r 0, \quad A \wedge B r 1 \text{ iff } A r 1 \text{ and } B r 1, \quad A \vee B r 1 \text{ iff } A r 1 or B r 1,$$
$$\neg A r 0 \text{ iff } A r 1, \quad A \wedge B r 0 \text{ iff } A r 0 \text{ or } B r 0, \quad A \vee B r 0 \text{ iff } A r 0 \text{ and } B r 0.$$

Definition 2 For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, A is a *two-valued semantic consequence* of Γ ($\Gamma \models_2 A$) iff for all Dunn-interpretations r, if $B r 1$ for all $B \in \Gamma$, then $A r 1$.

Assuming a *classical* metatheory, or something strong enough, we may present the semantics as a four-valued logic as follows.

Definition 3 A *four-valued-interpretation* of $\mathcal{L}$ is a function v_4 : Prop $\longrightarrow \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$. Given a four-valued interpretation v_4, this is extended to a function I_4 that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

	$\neg$
t	**f**
b	**b**
n	**n**
f	**t**

$\wedge$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$\vee$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

Then, the semantic consequence relation based on the four-valued interpretations for **FDE** (notation: $\models_4$) is defined as follows.

Definition 4 For all $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \models_4 A$ iff for all four-valued **FDE**-interpretations v_4, $I_4(A) \in \mathcal{D}$ if $I_4(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$.

Remark 5 For a *mechanical* procedure of turning four-valued truth tables into two-valued relational clauses, see Omori and Sano (2015).

Note that for the purpose of discussing Beall's idea, the two-valued relational presentation is the most suitable.[1] Indeed, Shrieking and Shrugging are captured in a very natural and intuitive manner. However, as we will observe later, the four-valued presentation will be also helpful to illustrate our proposal.

3 Two reasons for going below FDE

Let us now turn to discuss how one could provide reasons to think that some weaker systems should be preferred by Beall. The point, in general, is that it seems that even **FDE** seems to embrace forms of reasoning that are topic-specific, and that may fail in different topics (thus, violating the demand that it obeys the traditional features of logic: universality, topic neutrality and intransgressibility). Let us check two particular cases that seem to point in that direction.

3.1 Double negation

Recall the demand by Beall that our theories should be closed under logical consequence, and that logic should be on the background of every such theory (and it is the same logic everywhere if the argument is to have any kind of bite). Given a set of principles, logic allows us to derive consequences of such principles, and the theory should include such consequences. This should hold for every topic, given the universality of logic.

The problem with the demand of topic neutrality is that it is conceivable that there are some topics that seem to introduce restrictions on Beall's favorite system, **FDE**. Consider a case where one's theory consists of a database including information that

[1] We shall also briefly discuss how our observations can be seen in the light of the star semantics for **FDE** later (see §5).

is being fed by some source. For our purposes, the set of information constituting the database could be seen as a kind of description of a world, with worlds considered as states of information (for a brief discussion, see (Priest, 2006, §5.2)). Such states, of course, may be highly incomplete, and whatever else is incorporated, besides what has been informed, depends on the logic legislating such worlds. In other words, what counts as a legitimate member of the set constituting the world is whatever has been informed or what follows from what has been informed by the source. This is a variation of the well-known "being told" interpretation of **FDE** due to Nuel Belnap (cf. Belnap (1977a, b)). However, with some constraints on the information that is available, it seems that **FDE** adds "information" not warranted to the set constituting the theory; i.e., it allows for expansions of the initial information set that are not in complete agreement with the constraints of information addition. The reason is that one could distinguish between being informed that p and being informed that $\neg\neg p$. By adopting some reasonable constraints on the allowability of information, one can see how such inferences may be resisted.

Let us consider that the system is being informed true that $\neg\neg p$, for some p. That is, the system is informed false that $\neg p$. On the restriction that only informed propositions should be allowed, one can clearly find settings where it is not the case that the source has also informed true that p. If the source informs false that it is not the case that Peter is in Paris, this is still not the same as informing us true that Peter is in Paris (that particular latter information is still missing). In this sense, then, we have some reasons to be diffident of double negation elimination in these scenarios. One could motivate such restriction in this scenario by adopting some kind of restriction on the allowance of a proposition as legitimate information; for instance, that negative information and positive information have different statuses, so that each of them requires distinct kinds of verification, resulting in that they must be separately informed, and introducing thus a distinction between affirming the negation of a proposition and affirming the proposition itself.

One may complain that this is going too far: the behavior of the system in this case is just too specific and seems to be developed precisely to bar double negation. However, our point here is not to use these scenarios as actual cases where such laws do break down. Rather, our major point is methodological, in the sense that if logic is going to be universal, these seem also possibilities that one should take into account. Just as the classical logician is seen as introducing *ad hoc* limitations on the theories that are contemplated by logic, such scenarios that are opened when we consider some wilder possibilities should just as well be contemplated. After all, logic is universal, according to Beall.

3.2 Distribution

In the first half of the twentieth century, the debate on the very nature of logic was boosted by some discoveries in empirical sciences. Some discoveries in quantum mechanics, in particular, led some philosophers to think that logic may be empirical, that is, that its laws and valid inferences could depend on the features of reality as described by our best scientific theories (see the discussion in (Jammer, 1974, chap.8), and also Bueno and Colyvan (2004) for a more recent account). The most prominent of such example concerns the law of distribution, which is valid in classical logic but which was said to be violated by quantum mechanics: from $p \wedge (q \vee r)$ it is said that one cannot validly infer $(p \wedge q) \vee (p \wedge r)$.

Keeping this particular case with a minimum of technical details, we can briefly sketch the reason advanced for that failure: given Heisenberg's indeterminacy principle, a quantum state $|\psi\rangle$ cannot always attribute a determinate value to every observable O that could be *prima facie* measured for the system; such attributions may be just incompatible. So, to put it very lightly, whenever we get to determine that a system definitively has one property as represented by an observable, we lose information as to the value attributed to other incompatible properties. The famous example by Heisenberg is related to position and momentum: getting to know very sharply the position of a particle makes it impossible to determine its momentum, and vice versa. One just cannot have both properties sharply attributed to one single particle.

But how does that impact on the logic behind the treatment of propositions in quantum mechanics? In order to illustrate the case, let us consider the simple example of the property "spin." Just as position and momentum are incompatible properties, it is known that some spatial directions of spin are incompatible; once we know the spin of a particle in one direction, we lose information about spin in other incompatible directions.[2] However, still we have it from quantum mechanics that for any given spatial direction y, quantum systems may have either spin up in y, or spin down in y. So, let us consider orthogonal spatial directions x and y. As a result from the theory, we cannot have both that a particle has spin determined in x and y. Assuming that the spin of a specific particle is known to be up in the x direction, it is still the case that spin is up or down in the y direction, that is, we have

$$\text{spin up in } x \wedge (\text{spin up in } y \vee \text{spin down in } y)$$

At the same time, given the limitations imposed by quantum theory, we cannot have

$$(\text{spin up in } x \wedge \text{spin up in } y) \vee (\text{spin up in } x \wedge \text{spin down in } y)$$

Now, although the claim that quantum theory generates a non-classical logic is a much-discussed topic (see, for instance, Maudlin (2005)), the example presented here is of a concern to Beall: it could be the case that some topic being studied demanded that such a law be violated. In a sense, this is similar to the story Beall tells us about gappy and glutty theories: they are not to be ignored, because there are investigations into such realms and logic should be open for such possibilities. What such quantum mechanical case indicates is that one can at least in principle tell a story against the law of distribution being a logical law, given the failure of universality, topic neutrality (the law is valid in classical mechanics), and intransgressibility. The solution: go even weaker than **FDE** again. Just as exclusion and exhaustion, distribution is a topic-relative kind of inference.

A remark similar to the previous one is in place here. We are not contemplating that quantum logic could actually be the perfect couterexample to distributivity. Rather, what this kind of scenario illustrates is that restrictions to the validity of this law

[2] The same point may be interpreted in ontological terms: once the spin of a particle has a definite value in one direction, it no longer has a definite value in other incompatible directions.

are possible. Considering the methodology of Beall's claims for going subclassical, if logic is to be completely general and topic neutral, — to embrace this one further possibility contemplated by the quantum case — it needs to be even weaker than **FDE**. Basically, **FDE** has been too demanding and ends up cutting some possibilities out. Of course, if we can have more possibilities and still not lose anything, we are better off, right?

4 A subsystem of FDE

Given the above motivations for going below **FDE**, we would now like to propose how one may proceed from a more technical perspective. Since we are inspired by Beall's general approach, we shall refer to the subsystem of **FDE** as **BFDE**, for *Beallean* **FDE**.

Definition 6 A **BFDE**-*relational-interpretation* is a relation, ρ, between formulas and the values 1 and 0 (i.e., $\rho \subseteq \mathsf{Form} \times \{0, 1\}$) such that ρ satisfies the following:

$$\neg A r 1 \text{ iff } A r 0, \qquad A \wedge B r 1 \text{ iff } A r 1 \text{ and } B r 1, \qquad A \vee B r 0 \text{ iff } A r 0 \text{ and } B r 0.$$

Remark 7 What we are proposing here is to drop the falsity condition for negation, conjunction, and the truth condition for disjunction, when **FDE** is seen in the light of two-valued relational semantics. If we consider other ways to weaken **FDE** in a similar manner, there are at least seven other ways. Our choice may then seem arbitrary.[3] However, our major concern here is of a methodological nature; we want to advance possible ways that **FDE** may be too strong, and not really the only one such way. If it is possible to come up with distinct ways to do that in even more convincing ways that are not exactly the ones we advanced here, then, our point is still granted. Let us emphasize this: what we are trying to establish is that there are good reasons to believe that if Beall is going to follow his own argument for going subclassical, then he should not stop at **FDE**.

We can then define the semantic consequence relation as follows.

Definition 8 For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, A is a *two-valued* **BFDE** *semantic consequence* of Γ ($\Gamma \models_{r\mathbf{BFDE}} A$) iff for all **BFDE**-relational-interpretations r, if $B r 1$ for all $B \in \Gamma$, then $A r 1$.

[3] We must confess, though, that there is a strong temptation on our side to believe that these are the conditions that characterize the basic connectives. But, for this belief to become more like a statement, we would need to provide much more argument with full details. A referee correctly pressed us that such a position "runs counter to many theories of meaning for the connectives. It goes against Gentzen, it goes against Dummett, it goes against Rumfitt, etc." Let us make it clear that we are in complete agreement with the referee. However, note also that such discussions do not operate on the basis of contemplating so many distinct possibilities, as we are doing here. More specifically, none of the mentioned authors will consider **FDE** as a logical system to operate with, and therefore, our context is very far from their contexts. Still, if one is to follow Beall in the search for full generality, then, it seems something has to go. That does not mean, however, that one cannot recover the full meaning, positive and negative conditions, when one is dealing with less general domains.

Remark 9 Note that the most important part of the original story told by Beall is kept even by working with **BFDE** instead of **FDE**. That is, once we Shriek and Shrug fully, then we will still be back in classical logic, as Beall emphasizes in various contexts. Indeed, Shrieking and Shrugging fully will imply that the relation is actually functional, and once that is back, the missing half of the truth/falsity condition will be restored by the conditions we assume for **BFDE**.

However, it is not the case that the original story told by Beall will be fully kept. For example, if we only Shriek fully, then the resulting theory will be closed under **K3** in the original proposal made by Beall, but in our case, it will be closed under a subsystem of **K3**.

One may wonder how the weakening will be implemented from the four-valued perspective. The short answer is that it will have some *non-deterministic* flavor. We will give a somewhat more detailed answer in the rest of this section. First, let us recall the general definition of non-deterministic semantics, due to Arnon Avron and Iddo Lev (cf. Avron and Lev (2005)).[4] Although we will state the semantics for the specific language $\mathcal{L}$, specified above, this is not essential and can be taken to be any propositional language.

Definition 10 A *non-deterministic matrix* (*Nmatrix* for short) for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V}$ is a non-empty set of truth values.
(b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$.

We say that M is (in)finite if so is $\mathcal{V}$. A *legal valuation* in an Nmatrix M is a function $v : \mathsf{Form} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $*$ of $\mathcal{L}$ and $A_1, \dots, A_n \in \mathsf{Form}$:

$$v(*(A_1, \dots, A_n)) \in \tilde{*}(v(A_1), \dots, v(A_n)). \qquad \text{(gHom)}$$

Remark 11 The condition above can be interpreted as a generalized homomorphism condition, and therefore we refer to it as (gHom).

Then, the specific Nmatrix for **BFDE** will be as follows.

Definition 12 A *four-valued* **BFDE**-*Nmatrix* for $\mathcal{L}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$,
(b) $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$,
(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

A	$\tilde{\neg}A$
t	**n**, **f**
b	**t**, **b**
n	**n**, **f**
f	**t**, **b**

$A \tilde{\wedge} B$	**t**	**b**	**n**	**f**
t	**t**, **b**	**t**, **b**	**n**, **f**	**n**, **f**
b	**t**, **b**	**t**, **b**	**n**, **f**	**n**, **f**
n	**n**, **f**	**n**, **f**	**n**, **f**	**n**, **f**
f	**n**, **f**	**n**, **f**	**n**, **f**	**n**, **f**

$A \tilde{\vee} B$	**t**	**b**	**n**	**f**
t	**t**, **n**	**t**, **n**	**t**, **n**	**t**, **n**
b	**t**, **n**	**b**, **f**	**t**, **n**	**b**, **f**
n	**t**, **n**	**t**, **n**	**t**, **n**	**t**, **n**
f	**t**, **n**	**b**, **f**	**t**, **n**	**b**, **f**

[4] For a general overview of non-deterministic semantics, see Avron and Zamansky (2011).

A *four-valued* **BFDE**-*valuation* in an **BFDE**-Nmatrix M is a function $v : \mathsf{Form} \to \mathcal{V}$ that satisfies (gHom).

Remark 13 Note that **BFDE** is not a completely isolated system. Indeed, if the truth table for disjunction is replaced by the following table, then we obtain the conditional-free fragment of **CLoN**, and the truth table can be found in (Avron, 2005, Definition 3.6),[5]

$A\tilde{\vee}B$	**t**	**b**	**n**	**f**
t	**t**, **b**	**t**, **b**	**t**, **b**	**t**, **b**
b	**t**, **b**	**t**, **b**	**t**, **b**	**t**, **b**
n	**t**, **b**	**t**, **b**	**n**, **f**	**n**, **f**
f	**t**, **b**	**t**, **b**	**n**, **f**	**n**, **f**

The difference, of course, is that this corresponds to the case in which the truth condition, rather than the falsity condition, for disjunction, is kept from the two-valued relational perspective. Note also that one of the referees observed that some entries of the table for disjunction produce "mixed" sets **t**, **n** and **b**, **f**, having both designated and non-designated values as options. This may look unusual at first blush, but there are some examples in the literature. See, for example, (Avron et al., 2007, p.62) for an example involving disjunction, and for negation, see (Avron and Konikowska, 2005, Example 2.8).

Finally, we can define the semantic consequence relation-building on the Nmatrix for **BFDE**.

Definition 14 For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_4 A$ iff for all four-valued **BFDE**-interpretations v_4, $I_4(A) \in \mathcal{D}$ if $I_4(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$.

Note that, as expected, we obtain the following equivalence result for the above two semantic consequence relations.[6]

Proposition 1 For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{r\mathbf{BFDE}} A$ iff $\Gamma \models_4 A$.

Proof The details are spelled out in the Appendix. □

For those who are interested in the proof systems, we may devise a Gentzen-style sequent calculus, by building on an extremely general and powerful result established by Arnon Avron, Jonathan Ben-Naim, and Beata Konikowska in Avron and Konikowska (2005); Avron et al. (2007).

[5] For the details of **CLoN** see Batens (1980).

[6] For those who are wondering about the intuitive connection between the two presentations, we would like to suggest such readers to have a look at the proof of Proposition 1 in the Appendix.

Definition 15 The system $\mathcal{G}$**BFDE** consists of the following rules.

$$\overline{A \Rightarrow A}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (LW) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ (RW)$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}\ (Cut)$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (L\wedge) \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ (R\wedge)$$

$$\frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta}\ (L\neg\vee) \qquad \frac{\Gamma \Rightarrow \neg A, \Delta \quad \Gamma \Rightarrow \neg B, \Delta}{\Gamma \Rightarrow \neg(A \vee B), \Delta}\ (R\neg\vee)$$

Then, given any set $\Gamma \cup \{A\}$ of formulas, $\Gamma \vdash_{\mathbf{BFDE}} A$ if there is a finite labelled tree whose nodes are labelled by sequents, in such a way that its root is $\Gamma \Rightarrow A$, its leaves are labelled by axioms and each sequent at a node is obtained from sequents at immediate predecessor(s) node(s) according to one of the above rules.

Remark 16 Note that **BFDE** is a subsystem of intuitionistic logic, unlike **FDE**. This may suggest that there is a possibility to follow Beall's strategy with the reference logic being intuitionistic logic rather than classical logic. We leave this topic, however, for interested readers.

Then, we obtain the following soundness and completeness results, as well as the cut admissibility, by removing some of the cases for the proof of (Avron et al., 2007, Theorem 5.1).

Theorem 1 *For all* $\Gamma \cup \{A\} \subseteq$ Form, $\Gamma \vdash_{\mathbf{BFDE}} A$ *iff* $\Gamma \models_4 A$. *Moreover, the cut rule is admissible in the system.*

Remark 17 Note that in (Avron et al., 2007, §5), Avron, Ben-Naim, and Konikowska consider four systems, as information source logics. The closest system to **BFDE**, discussed in some detail in (Avron et al., 2007, §5.1), is obtained, from a semantic perspective, by adding the falsity condition for negation, as well as the right-to-left direction of both the falsity condition for conjunction and the truth condition for disjunction. Note also that the double negation laws hold in their system, but not the law of distribution. In other words, **BFDE** can be strengthened by the above additional conditions.

Before moving further, note further that we lose a number of familiar inferences involving disjunction, including the following.[7] These are implications of keeping the falsity condition for disjunction, but removing the truth condition.

Proposition 2 $p \nvdash_{\mathbf{BFDE}} p \vee q$, $p \vee p \nvdash_{\mathbf{BFDE}} p$, and $p \vee q \nvdash_{\mathbf{BFDE}} q \vee p$.

Proof For the first and the third item, it suffices to assign $\mathbf{t}$ to both p and q. For the second item, it suffices to assign $\mathbf{f}$ to p. □

[7] We would like to thank one of the referees for suggesting us to make this point explicit. Note that in addition to the inferences mentioned in Proposition 2, there are more inferences that we lose, such as the de Morgan law for negated conjunction.

5 Reflections

There are many points that may be raised concerning what we have already developed in the paper. Here, we point to some of them, acknowledging that the discussion of these topics is not exhaustive.

Why BFDE ? The first point to be discussed concerns what is really achieved by such system and its motivations. Notice that by presenting **BFDE** we are merely pushing even further Beall's own strategy for going subclassical, and in doing that, we claim, just as Beall, we lose nothing, and gain even more possibilities concerning the field of allowable theories (see again (Beall, 2019, p.119)). That is, if a system of logic is going to score better in a dispute against classical logic the more general it is, i.e., the more possible theories it allows, then, the option we have advanced seems to have such features as to recommend itself as the appropriate choice. Now, it is important that it be clear that we are *not* actually claiming that **BFDE** is the correct choice of logic; rather, what we are advancing here is that *by following Beall's premises*, one may be tempted to go for even weaker systems. How to stop such descending into always weaker systems is an issue that is certainly important. One way to do that, it seems to us, would involve the introduction of some substantial demands on what are the minimal truth and falsity conditions for the connectives that cannot be dispensed with, and, on the basis of that, stop the descent to weaker systems at some minimal point satisfying such conditions. What are those conditions, however, is a very delicate topic.

Does BFDE contain logical connectives? One may raise the worry that by discarding some of the truth/falsity conditions for some connectives we may be going too far, and not being able to really claim that the result is actually a logical connective (this is directly connected with the previous point just discussed). If we follow the path that has been explored here, then there is a danger of having a connective that is both conjunction and disjunction at the same time.[8] Notice, however, that in the absence of any account of the nature of logical connectives that could introduce some minimal demands for what are legitimate minimal conditions a connective must satisfy, of course, such worries are not going to take us very far. A discussion on the nature of logical connectives is beyond the scope of this paper, however.

We can try to avoid such arbitrary connectives by following Beall's informal account and specify that, just as for classical logicians, the connectives we have are the traditional ones, and something deviating too much from that is just out of the question.

Another way — in a completely distinct direction — to go would consist of following Beall himself concerning the status of negation as a logical operator. In (Beall, 2017, p.25), Beall says of negation in **FDE**:

> the familiar behavior of so-called logical negation is in fact behavior imposed by individual theories — and in particular their respective closure relations. Such theories impose constraints of exhaustion and exclusion, which are warranted,

[8] More specifically, consider the binary connective known as *informational meet* in the context of bilattice. Then, this connective can be seen as having the truth condition for conjunction and the falsity condition for disjunction.

by the theorist's lights, by the phenomena in question. Such constraints are not imposed by logic itself.

In a nutshell, the claim is that **FDE** by itself imposes no constraints on negation. All that negation does, in terms of logic, is related to its connection with other connectives. So, in a sense, there is no "logical" theory of negation. Now, what the view being explored here seems to suggest is that a similar claim could now be extended also to other connectives, in the measure that we allow that truth and falsity conditions for them to become weaker. Of course, we are not claiming (again) that this is the right move to take, but only that Beall, embracing his approach to negation and extending it to other connectives, may end up with a very weak logic, or no logic at all.[9]

Beyond Shrieking and Shrugging? Suppose that one is convinced about the failure of the distribution law. Then, just as the failures of the ex contradictione quodlibet and the law of excluded middle motivate to introduce the notions of Shrieking and Shrugging, respectively, doesn't the failure of distribution law motivate us to introduce another notion that will let us recover the validity of the distribution law?[10] We believe that this seems to be a very natural move, and worth exploring further. Note that from the perspective of non-deterministic semantics, it seems to be interesting to observe the following difference: while Shrieking and Shrugging involve the elimination of one of the values, **b** or **n**, the new notion will not necessarily involve the elimination of values, but only eliminating non-determinacies. This seems to suggest that there can be some further detailed classification for the notions that recover certain laws.

Can we make similar moves in the Australian plan? We mentioned earlier that **FDE** can be also formulated in terms of star semantics, known as the Australian plan for negation.[11] We may then wonder if Beall's story can be told along the Australian plan instead of the American plan. The answer is yes, if we take into account the observation reported in Omori and De (2022).[12] However, when it comes to the case of distribution, assuming that one is convinced about the failure of the law, this might give us a reason to favor the American plan. This is because for the purpose of relativizing the distribution law, one would need to give up the truth condition for conjunction and/or disjunction, and therefore it becomes harder to justify that the binary connectives involved in the distribution law are conjunction and disjunction. Of course, there might be a way to overcome the difficulty, but that does not seem to be obvious, at least to the authors at the time of writing.

Other ways to weaken FDE? There is a further point concerning Beall's favorite system **FDE**. As we have been pointing out in this paper, why go subclassical only

9 The more possibilities one opens up, the weaker the logic must be, of course. The fear is that by allowing every possibility one may end up with no universally valid logic was a serious proposal considered by Newton C. A. da Costa in da Costa (1997); see Arenhart (2022) for a discussion of da Costa's views.

10 We must confess that we could not come up with a good name for such a notion.

11 For some discussions on the Australian plan and the American plan, see Berto (2015); De and Omori (2018); Berto and Restall (2019).

12 In brief, one needs to introduce a distinguished state to separate Shrieking and Shrugging.

by privileging only failure of exclusion and exhaustion? One may go for subsystems of **FDE** by relaxing other clauses for the truth and falsity conditions for connectives, as we have seen. But that is not the only way to go. One could certainly also consider infectious weakening of FDE.[13] That would certainly still be very much in the spirit of admitting more theories in the space of possibilities. This is one possible way that we shall not explore here, but that is certainly within the scope of Beall's consideration regarding logic as the responsible for the space of possibilities.[14] If one follows this reading, then the infectious value may be considered as representing off-topicness, and given the emphasis on the topic neutrality of logic by Beall, this might not be a suitable way to weaken **FDE.**

6 Conclusion

In this paper, we have presented Beall's argument for subclassical logic, and have briefly seen how such argument is used by him to propose **FDE** as the one true logic. Now, departing from the same premises that Beall adopts, we have suggested that it can be equally employed to motivate the adoption of even weaker systems. Given that **FDE** is considered to be better than classical logic on the grounds that it allows for more possible theories, and that we lose nothing by adopting it, we have suggested that even weaker logics allow for even more theories, and we lose nothing by adopting them.

As we have mentioned throughout the discussions, we are not here endorsing the claim that some weaker subclassical logic is the one true logic, but rather that for those considering logic in the same terms as Beall, it seems that such logics have something to recommend themselves, unless one has additional reasons to stop the descent into weaker systems. Besides, it is important to mention that there are other possible ways to weaken **FDE**, but we believe that our suggestion is possibly one of the most well-motivated, though we do not have arguments against other options, and that will be left for the readers, including Beall.

Appendix. Proof of proposition 1

We prove both directions by contraposition. For the left-to-right direction, assume that $\Gamma \not\models_4 A$. Then, there is a four-valued **BFDE**-valuation v_0 such that $v_0(B) \in \mathcal{D}$ for all

13 We would like to thank Michael De for the suggestion. One of the referees pressed us on why not weaken **FDE** along the systems discussed in Angell (1977); Ferguson (2016); Fine (2016); Ciuni et al. (2018). Our reason in this context is that all the systems discussed in these references satisfy the double negation laws, as well as the law of distribution, and therefore, these systems are not suitable for our purposes, given the cases we considered in §3.

14 One may also borrow another theme from Beall, namely on his interpretation of Weak Kleene logic, presented in Beall (2016).

$B \in \Gamma$, and $v_0(A) \notin \mathcal{D}$. Then, define a relation, ρ_0, between formulas and the values 1 and 0 (i.e., $\rho_0 \subseteq \mathsf{Form} \times \{0, 1\}$), as follows:

- $A\rho_0 1$ iff $v_0(A) \in \{\mathbf{t}, \mathbf{b}\}$;
- $A\rho_0 0$ iff $v_0(A) \in \{\mathbf{b}, \mathbf{f}\}$.

Then, we can check that ρ_0 is a **BFDE**-relational interpretation. Indeed, we have:

- $\neg A\rho_0 1$ iff $v_0(\neg A) \in \{\mathbf{t}, \mathbf{b}\}$ (by the def. of ρ_0) iff $v_0(A) \in \{\mathbf{b}, \mathbf{f}\}$ (by the Nmatrix for $\neg$) iff $A\rho_0 0$ (by the def. of ρ_0);
- $A \wedge B\rho_0 1$ iff $v_0(A \wedge B) \in \{\mathbf{t}, \mathbf{b}\}$ (by the def. of ρ_0) iff $v_0(A) \in \{\mathbf{t}, \mathbf{b}\}$ and $v_0(B) \in \{\mathbf{t}, \mathbf{b}\}$ (by the Nmatrix for $\wedge$) iff $A\rho_0 1$ and $B\rho_0 1$ (by the def. of ρ_0);
- $A \vee B\rho_0 0$ iff $v_0(A \vee B) \in \{\mathbf{b}, \mathbf{f}\}$ (by the def. of ρ_0) iff $v_0(A) \in \{\mathbf{b}, \mathbf{f}\}$ and $v_0(B) \in \{\mathbf{b}, \mathbf{f}\}$ (by the Nmatrix for $\vee$) iff $A\rho_0 0$ and $B\rho_0 0$ (by the def. of ρ_0);

Therefore, we obtain that there is a **BFDE**-relational interpretation ρ_0 such that $B\rho_0 1$ for all $B \in \Gamma$, and not $A\rho_0 1$, that is $\Gamma \not\models_{r\mathbf{BFDE}} A$, as desired.

For the right-to-left direction, assume that $\Gamma \not\models_{r\mathbf{BFDE}} A$. Then, there is a **BFDE**-relational interpretation ρ_0 such that $B\rho_0 1$ for all $B \in \Gamma$, and not $A\rho_0 1$. Then, define a function v_0 from Form to $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ as follows.

$$v_0(A) := \begin{cases} \mathbf{t} \text{ iff } A\rho_0 1 \text{ and not } A\rho_0 0; \\ \mathbf{b} \text{ iff } A\rho_0 1 \text{ and } A\rho_0 0; \\ \mathbf{n} \text{ iff not } A\rho_0 1 \text{ and not } A\rho_0 0; \\ \mathbf{f} \text{ iff not } A\rho_0 1 \text{ and } A\rho_0 0. \end{cases}$$

Then, we can show that v_0 is a four-valued **BFDE**-valuation. Then the desired result is proved by induction on the number n of connectives. For the base case, for atomic formulas, it is obvious by the definition. For the induction step, we split the cases based on the connectives.

Case 1: If $A = \neg B$, then we have the following two cases.

Cases	$v_0(B)$	condition for B	$v_0(A)$	condition for A, i.e., $\neg B$
(i)	$\mathbf{t}, \mathbf{n}$	not $B\rho_0 0$	$\mathbf{n}, \mathbf{f}$	not $\neg B\rho_0 1$
(ii)	$\mathbf{b}, \mathbf{f}$	$B\rho_0 0$	$\mathbf{t}, \mathbf{b}$	$\neg B\rho_0 1$

By induction hypothesis, we have the conditions for B, and in view of the condition for ρ_0, we obtain the conditions for A, i.e., $\neg B$.

Case 2: If $A = B \wedge C$, then we have the following three cases.

Cases	$v_\Sigma(B)$	condition for B	$v_\Sigma(C)$	condition for C	$v_\Sigma(A)$	condition for A
(i)	**n**, **f**	not $B\rho_0 1$	any	—	**n**, **f**	not $B \wedge C\rho_0 1$
(ii)	any	—	**n**, **f**	not $C\rho_0 1$	**n**, **f**	not $B \wedge C\rho_0 1$
(iii)	**t**, **b**	$B\rho_0 1$	**t**, **b**	$C\rho_0 1$	**t**, **b**	$B \wedge C\rho_0 1$

By induction hypothesis, we have the conditions for B and C, and we can see that the conditions for A, i.e., $B \wedge C$, are easily provable by making use of the conditions for ρ_0.

Case 3: If $A = B \vee C$, then we have the following three cases.

Cases	$v_\Sigma(B)$	condition for B	$v_\Sigma(C)$	condition for C	$v_\Sigma(A)$	condition for A
(i)	**t**, **n**	not $B\rho_0 0$	any	—	**t**, **n**	not $B \vee C\rho_0 0$
(ii)	any	—	**t**, **n**	not $C\rho_0 0$	**t**, **n**	not $B \vee C\rho_0 0$
(iii)	**b**, **f**	$B\rho_0 0$	**b**.**f**	$C\rho_0 0$	**b**, **f**	$B \vee C\rho_0 0$

By induction hypothesis, we have the conditions for B and C, and we can see that the conditions for A, i.e., $B \vee C$, are easily provable by making use of the conditions for ρ_0.

Therefore, we obtain that there is a four-valued **BFDE**-valuation v_0 such that $v_0(B) \in \mathcal{D}$ for all $B \in \Gamma$, and $v_0(A) \notin \mathcal{D}$, that is, $\Gamma \not\models_4 A$.

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Declarations

Conflict of interest The authors declare no competing interests.

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