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RESEARCH ARTICLE

The Ristić–Balakrishnan–Topp–Leone–Gompertz-G Family of Distributions with Applications

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Abstract

In this paper, we introduce the newly generated Ristić–Balakrishnan–Topp–Leone–Gompertz-G family of distributions. Statistical and mathematical properties of this new family including moments, moment generating function, incomplete moments, conditional moments, probability weighted moments, distribution of the order statistics, stochastic ordering, and Rényi entropy are derived. The unknown parameters of the family are inferred using the maximum likelihood estimation technique. A Monte Carlo simulation study is performed to investigate the convergence of the maximum likelihood estimation. Three real-life data sets are used to demonstrate the flexibility and capacity of the new family of distributions.

Keywords Generalized distributions · Gamma generator · Topp–Leone distribution · Statistical properties · Maximum likelihood estimation · Goodness-of-fit tests

Mathematics Subject Classification 62E99 · 60E05

Abbreviations

cdf	Cumulative distribution function
pdf	Probability density function
TL-G	Topp–Leone-G
RB	Ristić and Balakrishnan
Gom-G	Gompertz-G
RB–TL–Gom-G	Ristić–Balakrishnan–Topp–Leone-G–Gompertz-G

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hrf	Hazard rate function
rhrf	Reverse hazard function
Exp-G	Exponentiated-G
PWMs	Probability weighted moments
MLE	Maximum likelihood estimation
ABIAS	Average bias
RMSEs	Root mean squared errors
TL-OBIII-LLoG	Topp–Leone odd Burr III Log-logistic
GWLLoG	Generalized Weibull log-logistic
GELLoG	Gamma exponentiated Lindley-log-logistic
APTLW	Alpha power Topp–Leone Weibull
AIC	Akaike Information Criterion
CAIC	Consistent Akaike Information Criterion
BIC	Bayesian Information Criterion
W^*	Cramér–von Mises statistic
A^*	Anderson–Darling statistic
K-S	Kolmogorov–Smirnov statistic

1 Introduction

Probability distributions are utilized extensively in statistical analysis especially when modeling and predicting real-world phenomena. There is a growing demand for more flexible distributions due to the variety and complexity of big data encountered nowadays. Recent work on the extensions of the existing distributions includes the Gompertz–Topp Leone-G family of distributions [33], the Marshall–Olkin-Odd power generalized Weibull-G family of distributions [10], Topp–Leone odd Fréchet generated family of distributions [4], a new extended Rayleigh distribution [6], the odd Weibull inverse Topp–Leone distribution [5], Type I half logistic Burr X-G family of distributions [2], and the Marshall–Olkin–Weibull-H family of distributions [1].

The gamma generator proposed by [35] has been one of the most explored approaches for creating new distributions. The cumulative distribution function (cdf) of the gamma generator is provided by

$$F_{RB}(x;\delta) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log(G(x))} t^{\delta-1} e^{-t} dt, \quad \delta > 0 \tag{1}$$

with a probability density function (pdf)

$$f_{RB}(x;\delta) = \frac{1}{\Gamma(\delta)} \left[-\log(G(x))\right]^{\delta-1} g(x), \quad x \in \mathbb{R}. \tag{2}$$

Some existing distributions developed via the gamma transformation method include the gamma odd power generalized Weibull-G family of distributions by [14], a new gamma generalized Lindley-log-logistic distribution by [24], the gamma modified Weibull distribution by [11], the gamma-exponentiated Weibull distribution by [31], the gamma odd Burr III-G family of distributions by [30], the gamma Weibull-G family of distributions by [28], the Zografos–Balakrishnan-G family of distributions by [27], Zografos–Balakrishnan Burr XII distribution by [8], the Zografos–Balakrishnan Lindley distribution by [21], the Zografos–Balakrishnan odd log-logistic generalized half-normal distribution by [26], the gamma-generalized inverse Weibull distribution by [29], the Ristić and Balakrishnan Lindley–Poisson distribution by [13], and Ristić–Balakrishnan extended exponential distribution by [16].

In this paper, we develop a novel family of distributions using the gamma generator method in conjunction with the Topp–Leone-G (TL-G) family of distributions [7] and the Gompertz-G (Gom-G) family of distributions [3]. The cdf and pdf of TL-G are given by

$$F_{TL-G}(x;b,\psi) = \left[1 - \overline{G}^2(x;\psi)\right]^b \tag{3}$$

and

$$f_{TL-G}(x;b,\psi) = 2b\left[1 - \overline{G}^2(x;\psi)\right]^{b-1}\overline{G}(x;\psi)g(x;\psi),$$

respectively, for $b > 0$ and parameter vector ψ. Note that $\overline{G}(x;\psi) = 1 - G(x;\psi)$ is the baseline survival function (also called reliability function), and $g(x;\psi) = dG(x;\psi)/dx$ where $G(x;\psi)$ is the cdf of any continuous distribution. The cdf and pdf of the Gompertz-G (Gom-G) family of distributions are

$$F(x;\theta,\gamma,\psi) = 1 - \exp\left\{\frac{\gamma}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}, \tag{4}$$

and

$$f(x;\theta,\gamma,\psi) = \gamma[1 - G(x;\psi)]^{-\theta-1}\exp\left\{\frac{\gamma}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}g(x;\psi),$$

where $\theta, \gamma > 0$, and ψ is the baseline parameter vector.

The objectives of developing this new family of distributions are as follows: to create new statistical distributions with more shape properties than previous distributions; to extend the possible shapes of the density and hazard rate functions of the baseline distribution; to skew any symmetrical distribution; and to modulate the weight of the tails of any parental distribution.

The remaining sections are organized as follows. In Sect. 2, we introduce the new RB–TL–Gom-G distribution family, its subfamilies, the hazard rate function, and the quantile function. Some special cases of the new family of distributions are described in Sect. 3. The expansion of density functions is demonstrated in Sect. 4. Section 5 derives the mathematical and statistical properties of the RB–TL–Gom-G family of distributions. In Sect. 6, the model parameters of the RB–TL–Gom-G family of distributions are estimated using the maximum likelihood approach, and in Sect. 7, convergence properties are validated using a simulation study. The flexibility and applicability of the RB–TL–Gom-G family of distributions are exemplified by three real-world data sets in Sect. 8. Section 9 concludes with some final remarks.

In this study, all simulations, parameter estimates, and model comparisons are conducted in R (an open-source software environment maintained by R Core Team and the R Foundation for Statistical Computing (https://www.r-project.org/).

2 The New Family of Distributions

Based on the TL-G and Gom-G families of distributions, we define a new family of distributions namely the Ristić–Balakrishnan–Topp–Leone–Gompertz-G (RB–TL–Gom-G) distribution.[1] The cdf of the RB–TL–Gom-G family of distributions is denoted by

$$\begin{aligned} F_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) &= 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\left(\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)} t^{\delta-1}\mathrm{e}^{-t}dt \\ &= 1-\frac{\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)}{\Gamma(\delta)}, \end{aligned} \tag{5}$$

where $\gamma(\delta,y)=\int_0^y t^{\delta-1}e^{-t}dt$ is the lower incomplete gamma function, and $\Gamma(\delta)$ is the gamma function. Note that $g(x;\psi)=dG(x;\psi)/dx$ is the pdf of any continuous baseline distribution, and $G(x;\psi)$ is the corresponding cdf. The pdf of the RB–TL–Gom-G family of distributions is

$$\begin{aligned} f_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) &= \frac{2b}{\Gamma(\delta)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)^{\delta-1} \\ &\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b-1} \\ &\times\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\} \\ &\times[1-G(x;\psi)]^{-\theta-1}g(x;\psi), \end{aligned} \tag{6}$$

for $\delta,b,\theta>0$ and baseline parameter vector ψ.

2.1 Sub-families

Several sub-families of the RB–TL–Gom-G family of distributions are presented in this sub-section.

- When $\delta=1$, we obtain the Topp–Leone–Gompertz-G (TL-Gom-G) family of distributions with the cdf

[1] The parameter γ was set to 1 in the Gompertz-G family of distributions (Eqn.(4)) to avoid issues of over-parameterization.

$$F(x;b,\theta,\psi) = \left[1 - \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}\right]^{b},$$

for $b, \theta > 0$ and parameter vector ψ.

- Setting $b = 1$, we find a new family of distributions namely Ristić–Balakrishnan–Gompertz-G (RB-Gom-G) with the cdf

$$F(x;\delta,\theta,\psi) = 1 - \frac{\gamma\left(\delta, -\log\left[1 - \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}\right]\right)}{\Gamma(\delta)},$$

for $\delta, \theta > 0$ and parameter vector ψ.

- When $\theta = 1$, a new family of distributions is generated with the cdf

$$F(x;\delta,b,\psi) = 1 - \frac{\gamma\left(\delta, -\log\left[1 - \exp\left\{2\left(1 - [1 - G(x;\psi)]^{-1}\right)\right\}\right]^{b}\right)}{\Gamma(\delta)}, \quad (7)$$

for $\delta, b > 0$ and parameter vector ψ.

- When $\delta = b = 1$, it becomes the Gompertz-G (Gom-G) family of distributions with the cdf

$$F(x;\theta,\psi) = \left[1 - \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}\right],$$

for $\theta > 0$ and parameter vector ψ.

- When $\delta = \theta = 1$, another new family of distributions is obtained with the cdf

$$F(x;b,\psi) = \left[1 - \exp\left\{2\left(1 - [1 - G(x;\psi)]^{-1}\right)\right\}\right]^{b},$$

for $b > 0$ and parameter vector ψ.

- Let $b = \theta = 1$, then we obtain one more new family of distributions with the cdf

$$F(x;\delta,\psi) = 1 - \frac{\gamma\left(\delta, -\log\left[1 - \exp\left\{2\left(1 - [1 - G(x;\psi)]^{-1}\right)\right\}\right]\right)}{\Gamma(\delta)},$$

for $\delta > 0$ and parameter vector ψ.

- As $\delta = b = \theta = 1$, we obtain a special case of the Gom-G family of distributions with the cdf

$$F(x;\psi) = \left[1 - \exp\left\{2\left(1 - [1 - G(x;\psi)]^{-1}\right)\right\}\right],$$

for parameter vector ψ.

2.2 Hazard Rate and Quantile Functions

For a continuous random variable X with cdf $F(x)$, survival function $\bar{F}(x) = 1 - F(x)$, and pdf $f(x)$, its hazard rate function (hrf), reverse hazard function (rhf), and mean residual life function are defined as $h_F(x) = f(x)/\bar{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \frac{\bar{F}(u)}{\bar{F}(x)} du$ respectively. It has been shown by [36] that the behaviors of $h_F(x)$, $\tau_F(x)$, and $\delta_F(x)$ are equivalent. Here we will only present the hazard rate function, but the reverse hazard and residual life functions can be obtained in a similar way. Given the cdf (Eq. (5)) and pdf (Eq. (6)), the hrf of the RB–TL–Gom-G family of distributions is

$$\begin{aligned} h_{RB-TL-Gom-G}(x) = {} & 2b\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1} \\ & \times\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b-1} \\ & \times\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\} \\ & \times\frac{[1-G(x;\psi)]^{-\theta-1}g(x;\psi)}{\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)}. \end{aligned} \tag{8}$$

The quantile function of the RB–TL–Gom-G family of distributions can be obtained by solving for the inverse cumulative distribution function as

$$\begin{aligned} & F_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) \\ & = 1-\frac{\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)}{\Gamma(\delta)} = p, \end{aligned}$$

for $0 \le p \le 1$, which is equivalent to

$$-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right] = \frac{\gamma^{-1}[\Gamma(\delta)(1-p),\delta]}{b},$$

and

$$\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\} = 1-\mathrm{e}^{-\gamma^{-1}(\Gamma(\delta)(1-p),\delta)/b}.$$

Thus, it is sufficient to solve

$$G(x;\psi) = 1-\left[1-\frac{\theta}{2}\log\left(1-\mathrm{e}^{-\gamma^{-1}(\Gamma(\delta)(1-p),\delta)/b}\right)\right]^{-1/\theta} := q. \tag{9}$$

Therefore, the quantile function of the RB–TL–Gom-G family of distributions reduces to the quantile x_q of the baseline distribution with cdf $G(x;\psi)$ which is given by

$$x_q = G^{-1}(q), \tag{10}$$

where q is defined in Eq. (9).

3 Some Special Cases

Special cases of the RB–TL–Gom-G family of distributions are presented in this section when the baseline cdf $G(x;\psi)$ is specified as the Burr XII, Weibull, or Uniform distributions.

3.1 Ristić–Balakrishnan–Topp–Leone–Gompertz–Burr XII (RB–TL–Gom–BXII) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;c,\lambda) = 1-(1+x^c)^{-\lambda}$ and $g(x;c,\lambda) = \lambda c x^{c-1}(1+x^c)^{-\lambda-1}$ for $c,\lambda > 0$ and $x > 0$. The cdf and pdf of the new RB–TL–Gom–BXII distribution are

$$F_{RB-TL-Gom-BXII}(x;\delta,b,\theta,c,\lambda) = 1 - \frac{\gamma\left(\delta, -\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^b\right)}{\Gamma(\delta)}, \tag{11}$$

and

$$\begin{aligned} f_{RB-TL-Gom-BXII}(x;\delta,b,\theta,c,\lambda) &= \frac{2b}{\Gamma(\delta)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^b\right)^{\delta-1}\}\\ &\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^{b-1}\\ &\times\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\\ &\times\left[(1+x^c)^{-\lambda}\right]^{-\theta-1}\lambda c x^{c-1}(1+x^c)^{-\lambda-1}, \end{aligned}$$

for $\delta, b, \theta, c, \lambda > 0$. The hrf of the RB–TL–Gom–BXII distribution is given by

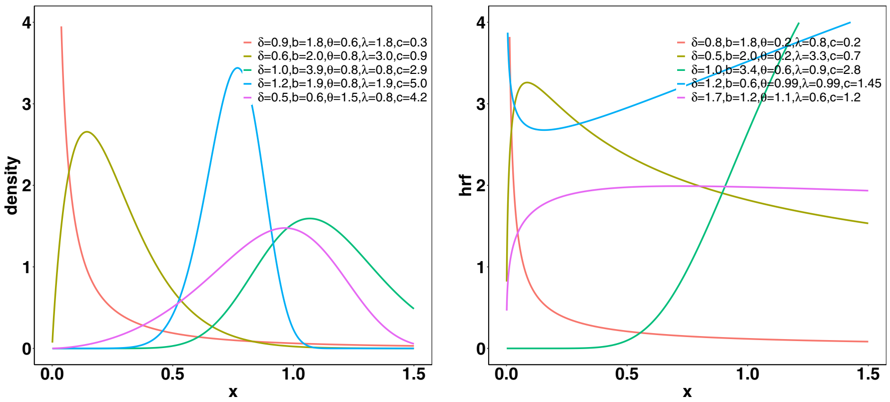

Fig. 1 Density and hazard plots for RB–TL–Gom–BXII distribution

$$
\begin{aligned}
&h_{RB-TL-Gom-BXII}(x;\delta,b,\theta,c,\lambda)\\
&\quad= 2b\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^b\right)^{\delta-1}\\
&\qquad\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^{b-1}\\
&\qquad\times\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\\
&\qquad\times\left[(1+x^c)^{-\lambda}\right]^{-\theta-1}\lambda c x^{c-1}(1+x^c)^{-\lambda-1}\\
&\qquad\times\left(\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[(1+x^c)^{-\lambda}\right]^{-\theta}\right)\right\}\right]^b\right)\right)^{-1},
\end{aligned}
$$

for $\delta, b, \theta, c, \lambda > 0$. If $\lambda = 1$, we obtain the RB–TL–Gom-Log-Logistic (RB–TL–Gom-LLoG) distribution. Also, if $c = 1$, the RB–TL–Gom–BXII distribution reduces to the RB–TL–Gom–Lomax (RB–TL–Gom–Lx) distribution.

Figure 1 shows several typical configurations of the pdf and hrf of the RB–TL–Gom–BXII distribution. The pdf can take various shapes including almost symmetric, reverse-J, left-skewed, and right-skewed. Furthermore, plots of the hrf for the RB–TL–Gom–BXII distribution exhibit monotonically increasing, monotonically decreasing, bathtub, and upside-down bathtub shapes.

3.2 Ristić–Balakrishnan–Topp–Leone-Gompertz–Weibull (RB–TL–Gom–W) Distribution

If we consider the Weibull distribution with cdf and pdf given by $G(x;\lambda) = 1 - \exp(-x^{\lambda})$ and $g(x;\lambda) = \lambda x^{\lambda-1}\exp(-x^{\lambda})$ respectively, for $\lambda > 0$ and $x > 0$, as the baseline distribution, then the RB–TL–Gom–W distribution has cdf and pdf given by

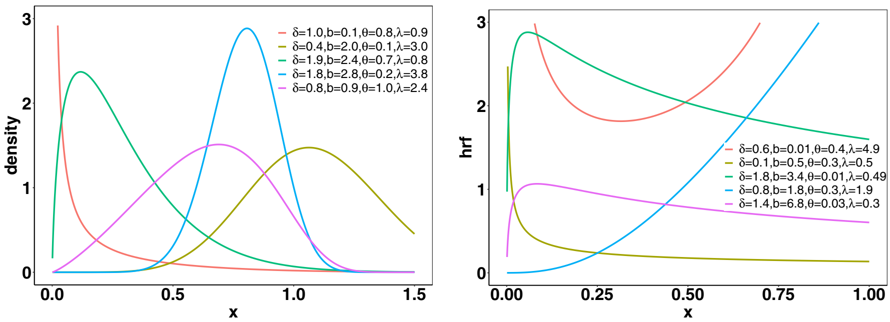

Fig. 2 Density and hazard plots for RB–TL–Gom–W distribution

$$F_{_{RB-TL-Gom-W}}(x;\delta,b,\theta,\lambda)=1-\frac{\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b}\right)}{\Gamma(\delta)},$$

and

$$\begin{aligned}f_{_{RB-TL-Gom-W}}(x;\delta,b,\theta,\lambda)=&\frac{2b}{\Gamma(\delta)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1}\\&\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b-1}\\&\times\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\\&\times\left[\exp(-x^{\lambda})\right]^{-\theta-1}\lambda x^{\lambda-1}\exp(-x^{\lambda}),\end{aligned}$$

for $\delta, b, \theta, \lambda > 0$. The corresponding hrf is given by

$$\begin{aligned}h_{_{RB-TL-Gom-W}}(x;\delta,b,\theta,\lambda)=&2b\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1}\\&\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b-1}\\&\times\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\\&\times\left[\exp(-x^{\lambda})\right]^{-\theta-1}\lambda x^{\lambda-1}\exp(-x^{\lambda})\\&\times\left(\gamma\left(\delta,-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[\exp(-x^{\lambda})\right]^{-\theta}\right)\right\}\right]^{b}\right)\right)^{-1},\end{aligned}$$

for $\delta, b, \theta, \lambda > 0$.

Figure 2 illustrates the adaptability of the RB–TL–Gom-W pdf and hrf distributions. The pdf produces a variety of forms, such as unimodal, reverse-J, left-skewed, and right-skewed. In addition, hrf plots for the RB–TL–Gom–W distribution exhibit growing, decreasing, bathtub, and inverted bathtub forms.

3.3 Ristić–Balakrishnan–Topp–Leone–Gompertz-Uniform (RB–TL–Gom-U) Distribution

If we consider the uniform distribution as the baseline distribution with cdf and pdf $G(x;\lambda) = \frac{x}{\lambda}$ and $g(x;\lambda) = \frac{1}{\lambda}$, for $\lambda > 0$ and $0 < \lambda < x$, then the RB–TL–Gom-U distribution has the cdf and pdf given by

$$F_{RB-TL-Gom-U}(x;\delta,b,\theta,\lambda) = 1 - \frac{\gamma\left(\delta, -\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b}\right)}{\Gamma(\delta)},$$

and a pdf of

$$\begin{aligned} f_{RB-TL-Gom-U}(x;\delta,b,\theta,\lambda) = {} & \frac{2b}{\Gamma(\delta)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1} \\ & \times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b-1} \\ & \times\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\left[1-\frac{x}{\lambda}\right]^{-\theta-1}\frac{1}{\lambda}, \end{aligned}$$

for $\delta, b, \theta, \lambda > 0$. The hrf for the RB–TL–Gom-U distribution is

$$\begin{aligned} h_{RB-TL-Gom-U}(x;\delta,b,\theta,\lambda) = {} & \frac{2b}{\lambda}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1} \\ & \times\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b-1} \\ & \times\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\left[1-\frac{x}{\lambda}\right]^{-\theta-1} \\ & \times\left(\gamma\left(\delta, -\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-\left[1-\frac{x}{\lambda}\right]^{-\theta}\right)\right\}\right]^{b}\right)\right)^{-1}, \end{aligned}$$

for $\delta, b, \theta, \lambda > 0$.

Figure 3 presents the pdf and hrf of the RB–TL–Gom-U distribution. The pdf can take numerous shapes such as almost symmetric, J, reverse-J, left-skewed, and right-skewed. Furthermore, the hrf of the RB–TL–Gom-U distribution captures a variety of possibilities including decreasing, bathtub, and bathtub followed upside-down bathtub shapes.

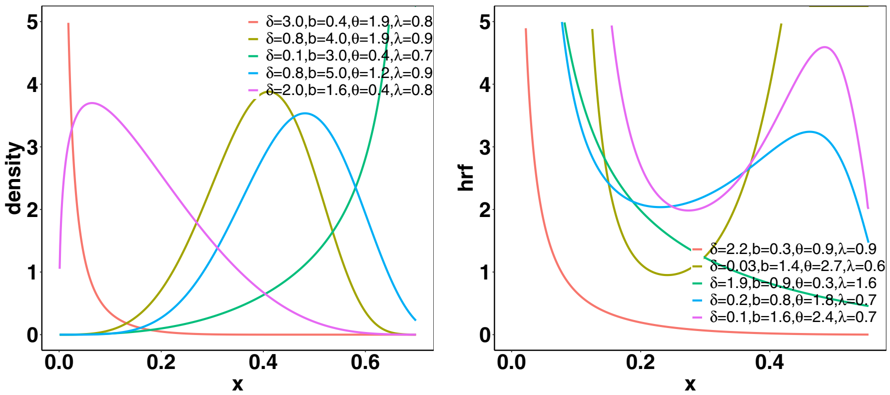

Fig. 3 Density and hazard plots for RB–TL–Gom-U distribution

4 Expansion of the Density Function

In this section, we will present a series expansion of the RB–TL–Gom-G density function. First, let $y = \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}$, and note that $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$. Then

$$\left(-\log\left[1 - \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1} = b^{\delta-1} y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m} y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right]. \tag{12}$$

Moreover, let $a_s = (s+2)^{-1}$, then $\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m = \left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s$, where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$ [17, 32]. Then Eq. (12) can be simplified as

$$\left(-\log\left[1 - \exp\left\{\frac{2}{\theta}\left(1 - [1 - G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)^{\delta-1} = b^{\delta-1} y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m} y^{m}\sum_{s=0}^{\infty} b_{s,m} y^{s}\right] = b^{\delta-1}\left[\sum_{m,s=0}^{\infty}\binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1}\right]. \tag{13}$$

Now, the pdf in Eq. (6) can be written as

$$
\begin{aligned}
f_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) &= \frac{2b^{\delta}}{\Gamma(\delta)}\left[\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}y^{m+s+\delta-1}\right] \\
&\quad\times(1-y)^{b-1}y[1-G(x;\psi)]^{-\theta-1}g(x;\psi) \\
&= \frac{2b^{\delta}}{\Gamma(\delta)}\left[\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}y^{m+s+\delta}\right] \\
&\quad\times\sum_{n=0}^{\infty}\binom{b-1}{n}(-1)^{n}y^{n}[1-G(x;\psi)]^{-\theta-1}g(x;\psi) \\
&= \frac{2b^{\delta}}{\Gamma(\delta)}\left[\sum_{m,s,n=0}^{\infty}\binom{\delta-1}{m}\binom{b-1}{n}(-1)^{n}b_{s,m}y^{m+s+\delta+n}\right] \\
&\quad\times[1-G(x;\psi)]^{-\theta-1}g(x;\psi).
\end{aligned}
\tag{14}
$$

Note that $y=\exp\left(\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right)$ and

$$
\begin{aligned}
y^{m+s+\delta+n} &= \left[\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{m+s+\delta+n} \\
&= \sum_{p=0}^{\infty}\frac{2^{p}(m+s+\delta+n)^{p}}{\theta^{p}p!}\sum_{q=0}^{\infty}\binom{p}{q}(-1)^{q}[1-G(x;\psi)]^{-\theta q} \\
&= \sum_{p,q=0}^{\infty}\binom{p}{q}\frac{2^{p}(m+s+\delta+n)^{p}}{\theta^{p}p!}(-1)^{q}[1-G(x;\psi)]^{-\theta q}.
\end{aligned}
\tag{15}
$$

Substitute Eq. (15) into Eq. (14), and the pdf of RB–TL–Gom-G distribution can be written as

$$
\begin{aligned}
&f_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) \\
&= \frac{2b^{\delta}}{\Gamma(\delta)}\sum_{m,s,n,p,q=0}^{\infty}\binom{\delta-1}{m}\binom{b-1}{n}\binom{p}{q} \\
&\quad\times\frac{2^{p}(m+s+\delta+n)^{p}}{\theta^{p}p!}(-1)^{n+q}b_{s,m}[1-G(x;\psi)]^{-\theta(q+1)-1}g(x;\psi) \\
&= \frac{2b^{\delta}}{\Gamma(\delta)}\sum_{m,s,n,p,q,u=0}^{\infty}\binom{\delta-1}{m}\binom{b-1}{n}\binom{p}{q}\binom{-\theta(q+1)-1}{u} \\
&\quad\times\frac{2^{p}(m+s+\delta+n)^{p}}{\theta^{p}p!}(-1)^{n+q+u}b_{s,m}G^{u}(x;\psi)g(x;\psi).
\end{aligned}
\tag{16}
$$

To simplify the notation, we let

$$w_{u+1} = \sum_{m,s,n,p,q=0}^{\infty} \frac{2b^{\delta}}{\Gamma(\delta)} \binom{\delta-1}{m} \binom{b-1}{n} \binom{p}{q} \binom{-\theta(q+1)-1}{u} \times \frac{2^p (m+s+\delta+n)^p}{(u+1)\theta^p p!} (-1)^{n+q+u} b_{s,m} \tag{17}$$

be the weights. Then the RB–TL–Gom-G density function can be written as

$$f_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi) = \sum_{u=0}^{\infty} w_{u+1}(u+1)G^u(x;\psi)g(x;\psi) \tag{18}$$

$$= \sum_{u=0}^{\infty} w_{u+1} g_{u+1}(x;\psi), \tag{19}$$

where $g_{u+1}(x;\psi)$ is the pdf of the exponentiated-G (Exp-G) distribution [20] with power parameter $u+1$.

Therefore, we can obtain the statistical properties of the RB–TL–Gom-G family of distributions from the well-established properties of the Exp-G distributions.

5 Mathematical and Statistical Properties

The moments, moment generating function, incomplete and conditional moments, Reńyi entropy, order statistics, stochastic orderings, and probability weighted moments of the RB–TL–Gom-G family of distributions are presented in this section. Throughout this section X denotes a random variable with a density function $f_{RB-TL-Gom-G}(x)$ as in Eq. (6), and Y_{u+1} is a random variable with the exponentiated-G distribution with power parameter $u+1$.

5.1 Moments and Generating Functions

The r^{th} moment of the RB–TL–Gom-G family of distributions can be derived as

$$E(X^r) = \int_{-\infty}^{\infty} x^r f_{RB-TL-Gom-G}(x)dx = \sum_{u=0}^{\infty} w_{u+1} E(Y^r_{u+1}), \tag{20}$$

where $E(Y^r_{u+1})$ is the r^{th} moment of the Exp-G distribution with power parameter $u+1$, and w_{u+1} is given in Eq. (17). Furthermore, the moment generating function (mgf) for $s < 1$ is

$$M_X(s) = \sum_{u=0}^{\infty} w_{u+1} M_{u+1}(s), \tag{21}$$

where $M_{u+1}(s)$ is the mgf of Y_{u+1}, and w_{u+1} is defined in Eq. (17).

5.2 Incomplete and Conditional Moments

Incomplete and conditional moments are widely used in lifetime models and measures of inequality such as Bonferroni and Lorenz curves. The rth incomplete moment of X can be obtained as

$$\phi_r(z) = \int_{-\infty}^{z} x^r f_{RB-TL-Gom-G}(x)dx = \sum_{u=0}^{\infty} w_{u+1} \int_{-\infty}^{z} x^r g_{u+1}(x;\psi)dx, \tag{22}$$

where $g_{u+1}(x;\psi)$ is the pdf of Exp-G distribution with power parameter $u+1$. By setting $r=1$ in Eq. (22), we obtain the first incomplete moment of the RB–TL–Gom-G family of distributions. The rth conditional moments of the RB–TL–Gom-G family of distributions is given by

$$\begin{aligned} E(X^r|X \geq a) &= \frac{1}{\bar{F}_{RB-TL-Gom-G}(a;\delta,b,\theta,\psi)} \int_t^{\infty} x^r f_{RB-TL-Gom-G}(x;\delta,b,\theta,\psi)dx \\ &= \frac{1}{\bar{F}_{RB-TL-Gom-G}(a;\delta,b,\theta,\psi)} \sum_{u=0}^{\infty} w_{u+1} E(Y^r_{u+1} I_{Y^r_{u+1}\geq t}), \end{aligned} \tag{23}$$

where $\bar{F}_{RB-TL-Gom-G}(a;\delta,b,\theta,\psi) = 1 - F_{RB-TL-Gom-G}(a;\delta,b,\theta,\psi)$ and

$$\begin{aligned} E(Y^r_{u+1} I_{Y^r_{u+1}\geq t}) &= \int_t^{\infty} y^r g_{u+1}(x;\psi)dy \\ &= (u+1)\int_{G(z;\psi)}^{1} [Q_G(z;\psi)]^r z^{u+1} dz, \end{aligned} \tag{24}$$

for parameter vector ψ. The mean residual life function is given by $E(X-a|X>a) = E(X|X>a) - a = V_F(a) - a$, where $V_F(a)$ is referred to as the vitality function of the distribution function F. The mean deviations, Bonferroni, and Lorenz curves can be readily obtained from the conditional and incomplete moments.

5.3 Reńyi Entropy

In information theory, the Reńyi entropy [34] is a measurement that generalizes various related entropies including Hartley entropy, Shannon entropy, collision entropy, and min-entropy. Reńyi entropy plays an important role as an index of diversity in fields such as ecology and statistics. The Reńyi entropy of the RB–TL–Gom-G family of distributions is given by

$$I_R(v) = \frac{1}{1-v} \log\left(\int_0^{\infty} f^v_{RB-TL-Gom-G}(x)dx\right), v > 0 \text{ and } v \neq 1, \tag{25}$$

where

$$f^{v}_{RB-TL-Gom-G}(x)=\left(\frac{2b}{\Gamma(\delta)}\right)^{v}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)^{v(\delta-1)}$$
$$\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{v(b-1)}$$
$$\times\exp\left\{\frac{2v}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}$$
$$\times[1-G(x;\psi)]^{-v\theta-v}g^{v}(x;\psi). \tag{26}$$

Let $y=\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}$. Consider a similar expansion for the pdf, as in Eq. (13), we can obtain that

$$\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b}\right)^{v(\delta-1)}$$
$$=b^{v\delta-v}\left[\sum_{m,s=0}^{\infty}\binom{v\delta-v}{m}b_{s,m}y^{m+s+v\delta-v}\right]. \tag{27}$$

Therefore, Eq. (26) can be written as

$$f^{v}_{RB-TL-Gom-G}(x)=\frac{2^{v}b^{v\delta}}{\Gamma^{v}(\delta)}\left[\sum_{m,s=0}^{\infty}\binom{v\delta-v}{m}b_{s,m}y^{m+s+v\delta-v}\right]$$
$$\times(1-y)^{v(b-1)}y^{v}[1-G(x;\psi)]^{-v\theta-v}g^{v}(x;\psi)$$
$$=\frac{2^{v}b^{v\delta}}{\Gamma^{v}(\delta)}\left[\sum_{m,s,n=0}^{\infty}\binom{v\delta-v}{m}\binom{vb-v}{n}(-1)^{n}b_{s,m}y^{m+s+v\delta+n}\right]$$
$$\times[1-G(x;\psi)]^{-v\theta-v}g^{v}(x;\psi). \tag{28}$$

Note that with $y=\exp\left(\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right)$, then

$$y^{m+s+v\delta+n}=\exp\left(\frac{2(m+s+v\delta+n)}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right)$$
$$=\sum_{p=0}^{\infty}\frac{2^{p}(m+s+v\delta+n)^{p}}{\theta^{p}}\left(1-[1-G(x;\psi)]^{-\theta}\right)^{p}$$
$$=\sum_{p,q=0}^{\infty}\binom{p}{q}\frac{2^{p}(m+s+v\delta+n)^{p}}{\theta^{p}p!}(-1)^{q}[1-G(x;\psi)]^{-\theta q}. \tag{29}$$

It follows that

$$f^{v}_{RB-TL-Gom-G}(x) = \frac{2^v b^{v\delta}}{\Gamma^v(\delta)} \sum_{m,s,n,p,q=0}^{\infty} \binom{v\delta - v}{m} \binom{vb - v}{n} \binom{p}{q} \frac{2^p(m+s+v\delta+n)^p}{\theta^p p!} \times (-1)^{n+q} b_{s,m} [1 - G(x;\psi)]^{-\theta(v+q)-v} g^v(x;\psi)$$
$$= \frac{2^v b^{v\delta}}{\Gamma^v(\delta)} \sum_{m,s,n,p,q,u=0}^{\infty} \binom{v\delta - v}{m} \binom{vb - v}{n} \binom{p}{q} \binom{-\theta(v+q) - v}{u} \times \frac{2^p(m+s+v\delta+n)^p}{\theta^p p!} (-1)^{n+q+u} b_{s,m} G^u(x;\psi) g^v(x;\psi).$$

Thus, the Reńyi entropy of the RB–TL–Gom-G family of distributions can be written as

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log\left(\frac{2^v b^{v\delta}}{\Gamma^v(\delta)} \sum_{m,s,n,p,q,u=0}^{\infty} \binom{v\delta - v}{m} \binom{vb - v}{n} \binom{p}{q} \binom{-\theta(v+q) - v}{u} \right. \\ &\quad \left. \times \frac{2^p(m+s+v\delta+n)^p}{\theta^p p!} (-1)^{n+q+u} b_{s,m} \int_0^{\infty} G^u(x;\psi) g^v(x;\psi) dx \right) \\ &= \frac{1}{1-v} \log\left(\frac{2^v b^{v\delta}}{\Gamma^v(\delta)} \sum_{m,s,n,p,q=0}^{\infty} \binom{v\delta - v}{m} \binom{vb - v}{n} \binom{p}{q} \frac{2^p(m+s+v\delta+n)^p}{\theta^p p!} \right. \\ &\quad \times \sum_{u=0}^{\infty} \binom{-\theta(v+q) - v}{u} (-1)^{n+q+u} b_{s,m} \frac{1}{(1+u/v)^v} \\ &\quad \left. \times \int_0^{\infty} \left[(1+u/v) G^{u/v}(x;\psi) g(x;\psi) \right]^v dx \right) \\ &= (1-v)^{-1} \log\left(\sum_{u=0}^{\infty} C_u e^{(1-v) I_{REG}} \right), \end{aligned} \tag{30}$$

where $I_{REG} = (1-v)^{-1} \log\left[\int_0^{\infty} \left((1+u/v) G^{u/v}(x;\psi) g(x;\psi) \right)^v dx \right]$ is the Reńyi entropy of exponentiated-G distribution with power parameter $1 + u/v$, and the weights are

$$\begin{aligned} c_u &= \frac{2^v b^{v\delta}}{\Gamma^v(\delta)} \sum_{m,s,n,p,q=0}^{\infty} \binom{v\delta - v}{m} \binom{vb - v}{n} \binom{p}{q} \frac{2^p(m+s+v\delta+n)^p}{\theta^p p!} \\ &\quad \times \binom{-\theta(v+q) - v}{u} (-1)^{n+q+u} b_{s,m} \frac{1}{(1+u/v)^v}. \end{aligned} \tag{31}$$

As a result, we can obtain the Reńyi entropy of the RB–TL–Gom-G family of distributions from that of the exponentiated-G distribution by Eq. (30).

5.4 Order Statistics

Order statistics have a wide range of applications such as in actuarial science, modeling auctions, optimizing production processes, and estimating parameters of distributions. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed RB–TL–Gom-G random variables, $f(x)$ and $F(x)$ be the pdf and cdf of the RB–TL–Gom-G family of distributions, and denote $\bar{F}(x) = 1 - F(x)$. The pdf of the ith order statistic, $f_{i:n}(x)$, can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F(x)]^{i-1+m} \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m\left[\sum_{u=0}^{\infty}w_{u+1}(u+1)G^u(x;\psi)g(x;\psi)\right] \\ &\quad\times\left[\sum_{u=0}^{\infty}w_{u+1}G^{u+1}(x;\psi)\right]^{i+m-1}. \end{aligned} \tag{32}$$

Here we can write

$$\left[\sum_{u=0}^{\infty}w_{u+1}G^{u+1}(x;\psi)\right]^{i+m-1} = \sum_{u=0}^{\infty}d_{u,i+m-1}G^{u+1}(x;\psi), \tag{33}$$

where $d_{0,i+m-1} = (w_1)^{i+m-1}$,

$$d_{u,i+m-1} = [(u+1)w_1]^{-1}\sum_{z=1}^{u}[z(i+m)-u-1]w_{z+1}d_{u-z,i+m-1},$$

for $u = 1, 2, \ldots$, and w_{u+1} is given in Eq. (17). Then

$$f_{k:n}(x) = \sum_{m=0}^{n-i}\sum_{u,r=0}^{\infty}s_{m,u,r}g_{u+r+1}(x;\psi), \tag{34}$$

where

$$s_{m,u,r} = \frac{n!w_{u+1}d_{u,i+m-1}(u+1)}{(i-1)!(n-i)!(u+r+1)}(-1)^m\binom{n-i}{m}, \tag{35}$$

and $g_{u+r+1}(x;\psi)$ is the exponentiated-G distribution with parameter $u+r+1$. Therefore, we can derive properties of the distribution of the order statistics from the RB–TL–Gom-G family of distributions from those of the exponentiated-G distribution.

5.5 Stochastic Orderings

In this subsection, we present the commonly used stochastic orders for the RB–TL–Gom-G family of distributions including stochastic order, hazard rate order and the likelihood ratio order [37].

Let $F_X(t)$ and $F_Y(t)$ be the cdfs of two random variables X and Y, and define $\overline{F}_X(t) = 1 - F_X(t)$ and $\overline{F}_Y(t) = 1 - F_Y(t)$ as the corresponding reliability or survival functions. The random variable X is stochastically smaller than Y if, for all t, $\overline{F}_X(t) \le \overline{F}_Y(t)$ (or $F_X(t) \ge F_Y(t)$). It is represented by $X <_{st} Y$ or $X \preceq Y$. Moreover, if $\overline{F}_X(t) < \overline{F}_Y(t)$ for some t, then X is stochastically strictly less than Y and denoted as $X \prec Y$. In the hazard rate order given by $X \preceq_{hr} Y$ X, $h_X(t) \ge h_Y(t)$ for all t. Similarly, X is said to be smaller than Y in the likelihood ratio order denoted by $X \preceq_{lr} Y$ if $\dfrac{f_X(t)}{f_Y(t)}$ is decreasing in t. It has been shown that $X \preceq_{lr} Y \implies X \preceq_{hr} Y \implies X \preceq Y$ [37].

Now consider two independent random variables X_1 and X_2 following RB–TL–Gom-G family of distributions with $X_1 \sim F_1(x;\delta_1, b, \theta, \psi)$ and $X_2 \sim F_2(x;\delta_2, b, \theta, \psi)$ and their pdfs given by

$$
\begin{aligned}
f_1(x) = \frac{2b}{\Gamma(\delta_1)}&\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)^{\delta_1-1}\\
&\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b-1}\\
&\times\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\\
&\times[1-G(x;\psi)]^{-\theta-1}g(x;\psi),
\end{aligned}
$$

and

$$
\begin{aligned}
f_2(x) = \frac{2b}{\Gamma(\delta_2)}&\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)^{\delta_2-1}\\
&\times\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^{b-1}\\
&\times\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\\
&\times[1-G(x;\psi)]^{-\theta-1}g(x;\psi)
\end{aligned}
$$

respectively. Then

$$
\frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]^b\right)^{\delta_1-\delta_2}. \quad (36)
$$

Differentiating Eq. (36) with respect to x yields

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)}\left(-\log\left[1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}\right]\right)^{\delta_1-\delta_2-1} \times (\delta_1-\delta_2) b^{\delta_1-\delta_2} \frac{\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}}{1-\exp\left\{\frac{2}{\theta}\left(1-[1-G(x;\psi)]^{-\theta}\right)\right\}} \times 2[1-G(x;\psi)]^{\theta-1} g(x;\psi), \tag{37}$$

which is negative if $\delta_1 < \delta_2$. Therefore, the likelihood ratio order $X \preceq_{lr} Y$ exists, and we can conclude that the random variables X_1 and X_2 are stochastically ordered.

5.6 Probability Weighted Moments

Distributions may be characterized by the probability weighted moments (PWMs) [18], defined as

$$M_{i,j,k} = E\left(X^i F^j(X)[1-F(X)]^k\right) = \int_{-\infty}^{\infty} x^i F^j(x)[1-F(x)]^k f(x)\, dx,$$

where the random variable X is distributed as $F \equiv F(x) = P(X \le x)$, i, j and k are real numbers. When j and k are nonnegative integers, then the probability weighted moment of order (i, j, k) is proportional to the ith moment about the origin of the $(j+1)$th order statistic for a sample of size $n = k + j + 1$. If X is a random variable that follows the RB–TL–Gom-G family of distributions, then

$$F^j(x)[1-F(x)]^k f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l [F(x)]^{j+l} f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l \left[\sum_{u=0}^{\infty} w_{u+1} G^{u+1}(x;\psi)\right]^{j+l} \left[\sum_{u=0}^{\infty} w_{u+1}(u+1) G^u(x;\psi) g(x;\psi)\right], \tag{38}$$

and apply a similar expansion in Eq. (33),

$$F^j(x)[1-F(x)]^k = \sum_{l=0}^{k} \binom{k}{l} (-1)^l \left[\sum_{u=0}^{\infty} d_{u,j+l} G^{u+1}(x;\psi)\right] \left[\sum_{u=0}^{\infty} w_{u+1}(u+1) G^u(x;\psi) g(x;\psi)\right] = \sum_{l=0}^{k} \sum_{u,r=0}^{\infty} s_{l,u,r}\, g_{u+r+1}(x;\psi), \tag{39}$$

where

$$s_{m,u,r} = \frac{w_{u+1}(u+1) d_{u,j+l}}{u+r+1} \binom{k}{l} (-1)^l, \tag{40}$$

$d_{0,j+l} = (w_1)^{j+l}$,

$$d_{u,j+l} = [(u+1)w_1]^{-1} \sum_{z=1}^{u} [z(j+l+1) - u - 1] w_{z+1} d_{u-z,j+l},$$

for $u = 1, 2, \ldots$, and w_{u+1} is given in Eq. (17). As a result, the PWMs of the RB–TL–Gom-G family of distributions can be written as

$$M_{i,j,k} = \sum_{l=0}^{k} \sum_{u,r=0}^{\infty} s_{l,u,r} \int_{-\infty}^{\infty} x^i g_{u+r+1}(x;\psi)\, dx, \tag{41}$$

i.e., the PWMs of the RB–TL–Gom-G family of distributions can be obtained from the moments of the exponentiated-G distribution.

6 Maximum Likelihood Estimation

In this section, we estimate the unknown parameters of the RB–TL–Gom-G family of distributions using the maximum likelihood estimation (MLE) method. Assuming that the independent random sample $\left(X_1, X_2, \ldots, X_n\right)$ is observed from the RB–TL–Gom-G family of distributions with the vector of model parameters $\boldsymbol{\Delta} = (\delta, b, \theta, \psi)^T$. Then, the log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ for the parameters from the observed values has the form

$$\begin{aligned} \ell_n(\boldsymbol{\Delta}) = & \; n \ln(2b) + (b-1) \sum_{i=1}^{n} \log \left[1 - \exp\left\{ \frac{2}{\theta} \left(1 - \left[1 - G(x_i;\psi) \right]^{-\theta} \right) \right\} \right] \\ & + (\delta - 1) \sum_{i=1}^{n} \log \left(- \log \left[1 - \exp\left\{ \frac{2}{\theta} \left(1 - \left[1 - G(x_i;\psi) \right]^{-\theta} \right) \right\} \right]^{b} \right) \\ & + \sum_{i=1}^{n} \left\{ \frac{2}{\theta} \left(1 - \left[1 - G(x_i;\psi) \right]^{-\theta} \right) \right\} + (-\theta - 1) \sum_{i=1}^{n} \log \left[1 - G(x_i;\psi) \right] \\ & - n \ln(\Gamma(\delta)) + \sum_{i=1}^{n} \log(g(x_i;\psi)). \end{aligned} \tag{42}$$

To obtain the maximum likelihood estimates (MLEs), we maximize the log-likelihood function $\ell_n(\boldsymbol{\Delta})$ numerically. This can be obtained by setting the nonlinear system of equations $(\frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \psi_k})^T = \mathbf{0}$ and solving them simultaneously. However, since these equations are not in closed form, the MLEs can be found by maximizing $\ell_n(\boldsymbol{\Delta})$ numerically with respect to the parameters using a numerical method such as Newton–Raphson procedure. The partial derivatives of the log-likelihood function with respect to each component of the parameter vector are presented in the Supplementary Information.

7 Monte Carlo Simulations

This section is devoted to assessing the asymptotic convergence property of the estimated parameters of the RB–TL–Gom-LLoG distribution. A Monte Carlo simulation study was performed based on the following: $N = 1000$ samples of size n = 25, 50, 100, 200, 400, 800, 1600 generated from the RB–TL–Gom-LLoG distribution with different parameter values. The assessment was done based on the average bias (ABIAS) and root mean squared errors (RMSEs).

The ABIAS and RMSE for the estimated parameter, $\hat{\theta}$, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

The results of the simulation are summarized in Tables 1 and 2. These Tables report the average estimates (mean), average bias (ABIAS), and root mean squared errors (RMSEs). These results indicate the convergence of the mean estimations to the true parameters as the sample size (n) increases. Moreover, the RMSEs and ABIAS converge to zero as n increases further validating that the estimates are robust and consistent.

8 Applications

In this section we illustrate the flexibility and functionality of the RB–TL–Gom-G family of distributions by fitting its special case, the RB–TL–Gom-LLoG distribution, to several real-life data sets. The computations in the applications are based on the definitions in Eqs. (5) and (6). Parameters are estimated using the maximum likelihood estimation method, with the box-constrained optimization using PORT routines [15]. Moreover, we apply the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm, a quasi-Newton method, to find the best-fit parameter sets and their corresponding goodness-of-fit. The performance of the new distribution is compared to that of other recent models including: Topp-Leone odd Burr III Log-logistic (TL-OBIII-LLoG) by [25], generalized Weibull log-logistic (GWLLoG) distribution by [12], a new gamma exponentiated Lindley-log-logistic (GELLoG) distribution by [24], heavy-tailed log-logistic distribution, named alpha power log-logistic distribution by [38], and alpha power Topp–Leone Weibull (APTLW) distribution by [9]. The pdf's of the models of comparison are given in the Supplementary Information.

To assess the goodness-of-fit of all the fitted distributions, the well-known goodness-of-fit statistics such as −2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), (n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises (W^*) statistic, Anderson-Darling statistic (A^*), Kolmogorov-Smirnov

Table 1 Monte Carlo simulation results

(1.0, 1.0, 2.0, 1.0,)					(0.5, 1.0, 2.0, 0.2,)			(0.2, 0.2, 2.0, 0.2)		
Parameter	Sample size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
δ	25	1.6556	1.2775	0.6556	2.0786	5.6666	1.5786	0.3208	2.1091	0.1208
	50	1.5513	1.1427	0.5513	1.4147	3.5233	0.9147	0.2155	0.0514	0.0155
	100	1.3980	0.9695	0.3980	0.9432	2.4842	0.4432	0.2127	0.0369	0.0127
	200	1.3037	0.8218	0.3037	0.6918	2.5435	0.1918	0.2094	0.0267	0.0094
	400	1.1992	0.6721	0.1992	0.4818	0.0859	−0.0181	0.2040	0.0146	0.0040
	800	1.1037	0.4764	0.1037	0.4822	0.0659	−0.0177	0.2063	0.0122	0.0063
	1600	1.0545	0.3591	0.05451	0.4851	.0526	−0.0148	0.2035	0.0056	0.0035
b	25	1.2437	3.7410	0.24373	1.8242	3.9990	0.8242	0.8374	6.2033	0.6374
	50	0.8057	0.4104	−0.1942	1.5890	5.3110	0.5890	0.3712	2.0735	0.1712
	100	0.8466	0.3309	−0.1533	1.1687	1.0244	0.1687	0.2273	0.1560	0.0273
	200	0.8914	0.2601	−0.1085	1.1328	0.4934	0.1328	0.2093	0.1066	0.0093
	400	0.9342	0.1835	−0.0657	1.1074	0.3799	0.1074	0.2030	0.0670	0.0030
	800	0.9657	0.1119	−0.0342	1.0834	0.3079	0.0834	0.1972	0.0359	−0.0027
	1600	0.9814	0.0740	−0.0185	1.0470	0.2340	0.0470	0.2000	0.0251	0.0000
θ	25	4.1813	3.6372	2.1813	2.4728	2.0382	0.4728	2.7216	1.6107	0.7216
	50	3.7077	2.9793	1.7077	2.6041	1.9232	0.6041	2.5220	1.2649	0.5220
	100	3.1611	2.2921	1.1611	2.3914	1.4966	0.3914	2.4237	0.9127	0.4237
	200	2.7682	1.7478	0.7682	2.3079	1.2439	0.3079	2.2559	0.6228	0.2559
	400	2.4054	1.1293	0.4054	2.1756	0.9152	0.1756	2.1682	0.4386	0.1682
	800	2.1643	0.5624	0.1643	2.0891	0.6987	0.0891	2.1096	0.3045	0.1096
	1600	2.0784	0.3505	0.0784	2.0058	0.5322	0.0058	2.0017	0.0051	0.0017
c	25	0.8342	0.8115	−0.1657	0.3067	0.3407	0.1067	0.3888	0.2599	0.1888
	50	0.8457	0.8003	−0.1542	0.2495	0.2340	0.0495	0.3314	0.1907	0.1314
	100	0.8833	0.7722	−0.1166	0.2479	0.1800	0.0479	0.3083	0.1555	0.1083

Table 1 (continued)

(1.0, 1.0, 2.0, 1.0,)					(0.5, 1.0, 2.0, 0.2,)			(0.2, 0.2, 2.0, 0.2)		
Parameter	Sample size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
	200	0.9483	0.7213	−0.0516	0.2441	0.1605	0.0441	0.2787	0.1141	0.0787
	400	0.9605	0.6245	−0.0395	0.2378	0.1404	0.0378	0.2574	0.0899	0.0574
	800	1.0008	0.5494	0.0008	0.2317	0.1061	0.0317	0.2436	0.0716	0.0436
	1600	1.0077	0.4168	0.0077	0.2205	0.0850	0.0205	0.2106	0.0106	0.0106

Table 2 Monte Carlo simulation results

(0.8, 0.3, 2.2, 0.3)					(0.8, 0.3, 2.2, 1.5)			(1.5, 1.5, 2.2, 0.3)		
Parameter	Sample size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
δ	25	1.4044	6.7963	0.6044	1.1054	3.5976	0.3054	3.8529	13.0147	2.3529
	50	0.7324	0.2507	−0.0675	0.9005	0.5258	0.1005	2.2900	6.6557	0.7900
	100	0.7455	0.1984	−0.0544	0.8603	0.4000	0.0603	1.2510	0.5330	−0.2489
	200	0.7467	0.1826	−0.0532	0.8574	0.3130	0.0574	1.3083	0.4312	−0.1916
	400	0.7675	0.1388	−0.0324	0.8549	0.2407	0.0549	1.3401	0.3735	−0.1598
	800	0.7942	0.0965	−0.0057	0.8529	0.1780	0.0529	1.3839	0.3026	−0.1160
	1600	0.7950	0.0718	−0.0049	0.8379	0.1011	0.03791	1.4107	0.2520	−0.0892
b	25	0.4030	0.4655	0.1030	0.3935	2.2072	0.0935	2.5653	9.0881	1.0653
	50	0.3549	0.2198	0.0549	0.2870	0.1522	−0.0129	2.0476	2.9905	0.5476
	100	0.3433	0.1563	0.0433	0.2889	0.0943	−0.0110	1.6074	0.8901	0.1074
	200	0.3308	0.1201	0.0308	0.2943	0.0648	−0.0056	1.5369	0.6633	0.0369
	400	0.3202	0.0890	0.0202	0.2956	0.0477	−0.0043	1.4746	0.5258	−0.0253
	800	0.3069	0.0661	0.0069	0.2985	0.0355	−0.0014	1.4752	0.4044	−0.0247
	1600	0.3042	0.0529	0.0042	0.2989	0.0255	−0.00109	1.4761	0.3406	−0.0238
θ	25	2.5026	1.6527	0.3026	2.9283	1.6395	0.7283	2.6443	2.8861	0.4443
	50	2.4726	1.2974	0.27264	2.6237	1.1421	0.4237	2.6338	2.3801	0.4338
	100	2.3291	1.0453	0.1291	2.4617	0.7962	0.2617	2.6065	1.9528	0.4065
	200	2.2322	0.8413	0.0322	2.3951	0.5934	0.1951	2.6030	1.5066	0.4030
	400	2.2143	0.6056	0.0143	2.3472	0.4410	0.1472	2.5388	1.1900	0.3388
	800	2.2721	0.5088	0.0721	2.3366	0.3366	0.1366	2.3911	0.8288	0.1911
	1600	2.2582	0.4258	0.0582	2.2752	0.2291	0.0752	2.3337	0.6523	0.1337
c	25	0.5556	.7389	0.2556	0.9998	0.8698	−0.5001	0.5032	0.4517	0.2032
	50	0.4670	0.5930	0.1670	1.0489	0.7725	−0.4510	0.4837	0.3998	0.1837
	100	0.4487	0.4964	0.1487	1.1130	0.6408	−0.3869	0.4358	0.3268	0.1358

Table 2 (continued)

(0.8, 0.3, 2.2, 0.3)					(0.8, 0.3, 2.2, 1.5)			(1.5, 1.5, 2.2, 0.3)		
Parameter	Sample size	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
	200	0.4239	0.4078	0.1239	1.1765	0.5379	−0.3234	0.4207	0.3179	0.1207
	400	0.4125	0.3743	0.1125	1.2802	0.4043	−0.2197	0.3965	0.3098	0.0965
	800	0.3654	0.2569	0.0654	1.4189	0.1607	−0.0810	0.3678	0.2643	0.0678
	1600	0.3193	0.1643	0.0193	1.4339	0.1206	−0.0660	0.3471	0.2328	0.0471

(K-S) statistic and its $p-$value were performed. It is known that, except for the $p-$value of the K-S statistic, the smaller the values of all the goodness-of-fit statistics, the better the model for fitting the data set.

8.1 Survival Times of Guinea Pigs Data

This data set consists of the survival times of guinea pigs injected with tubercle bacilli and was analyzed by [23] and [19]. The observations are as follows:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Estimates of the parameters of the fitted distributions (standard error in parentheses), AIC, BIC, W*, and A* are given in Table 3. Plots of the fitted densities along with the histogram and the observed probability vs predicted probability are presented in Fig. 4. We can conclude that the RB-TL-Gom-LLoG distribution is significantly better than other distributions since the values of all goodness-of-fit statistics are the smallest, and the p-value of the K-S statistic is the largest.

Graphical representations such as Kaplan–Meier (K–M) survival curve, theoretical and empirical cumulative distribution functions (ECDF), and total time on test (TTT) scaled are plotted in Fig. 5. It is clear that the fitted empirical and theoretical plots are close to each other, hence we conclude that our model provides a very good fit for the data. Moreover, the TTT scaled plot clearly demonstrates that the model fits a non-monotonic hazard rate structure.

8.2 Active Repair Times Data

The data set was reported by [22], and it represents active repair times for airborne communication transceivers. The observations are

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Table 4 reports the maximum likelihood estimates (MLEs) of the parameters of all fitted models, as well as the standard errors (in parenthesis) and various goodness-of-fit statistics for the active repair times data. We can see that the RB–TL–Gom-LLoG distribution is the best model for describing the data set since it has the smallest values among all goodness-of-fit statistics and the largest p-value of the K–S statistic. Plots of the fitted densities and the histogram, observed probability against predicted probability are given in Fig. 6.

Moreover, the fitted Kaplan–Meier survival curves, theoretical and ECDF plots, and TTT scaled plot are presented in Fig. 7. From the Kaplan-Meier and ECDF plots, it is clear that the RB–TL–Gom-LLoG distribution provides a perfect description of the active repair times data. The TTT scaled plot demonstrates that the data follow a non-monotonic hazard rate shape.

Table 3 MLEs and goodness-of-fit statistics for guinea pigs data

	Estimates				Statistics							
Model	δ	b	θ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
RB-TL-Gom-LLoG	7.1266	194.6200	0.4701	0.2079	780.4776	788.4776	789.0746	797.5843	0.1415	0.7746	0.0999	0.4684
	(0.0185)	(0.0026)	(0.2559)	(0.0532)								
	α	β	b	λ								
TL-OBIII-LLoG	0.8187	0.0041	2.0677×10^{07}	0.8872	790.321	798.321	798.9181	807.4277	0.2019	1.2075	0.1480	0.0851
	(0.9049)	(8.5367×10^{-04})	(4.1832×10^{-13})	(0.9806)								
	α	β	a	θ								
GWLLoG	6.1149×10^{-05}	29.1020	4.0531×10^{-04}	0.0888	793.0675	801.0675	801.6645	810.1742	0.4124	2.2682	0.1435	0.1029
	(1.9177×10^{-04})	(3.5254×10^{-05})	(9.9565×10^{-04})	(8.7845×10^{-03})								
	λ	c	α	δ								
GELLoG	0.0195	3.4911	5.2062	14.3628	784.4124	792.4124	793.0094	801.5191	0.2253	1.2061	0.1375	0.1313
	(0.0089)	(2.0808)	(4.2466)	(8.7835)								
	α	a	b	c								
APExLLD	0.1087	2.7804	147.9757	0.6664	784.1520	792.1519	792.749	801.2586	0.1973	1.0876	0.1032	0.4260
	(0.1328)	(0.4561)	(44.2313)	(0.1693)								
	θ	α	β	λ								
APTLW	1.8968	0.0607	1.2367	0.0013	783.1013	791.1014	791.6984	800.208	0.2232	1.2154	0.1164	0.2827
	(0.4336)	(0.0706)	(0.1322)	(0.0010)								

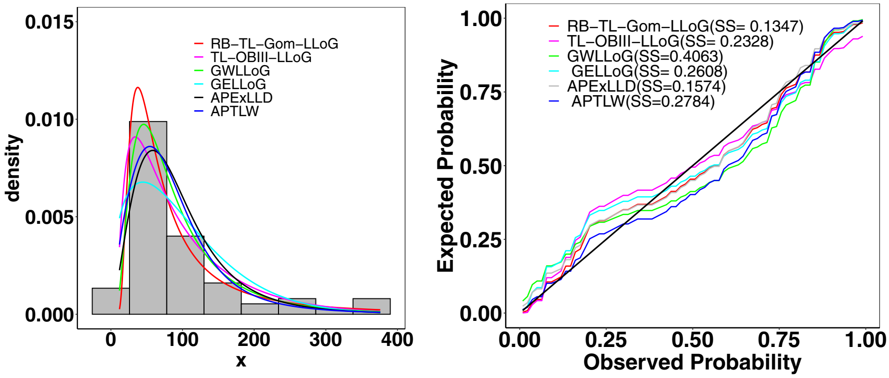

Fig. 4 Fitted density superposed on the histogram (left) and observed probability vs expected probability plots (right) for the Guinea Pigs Data

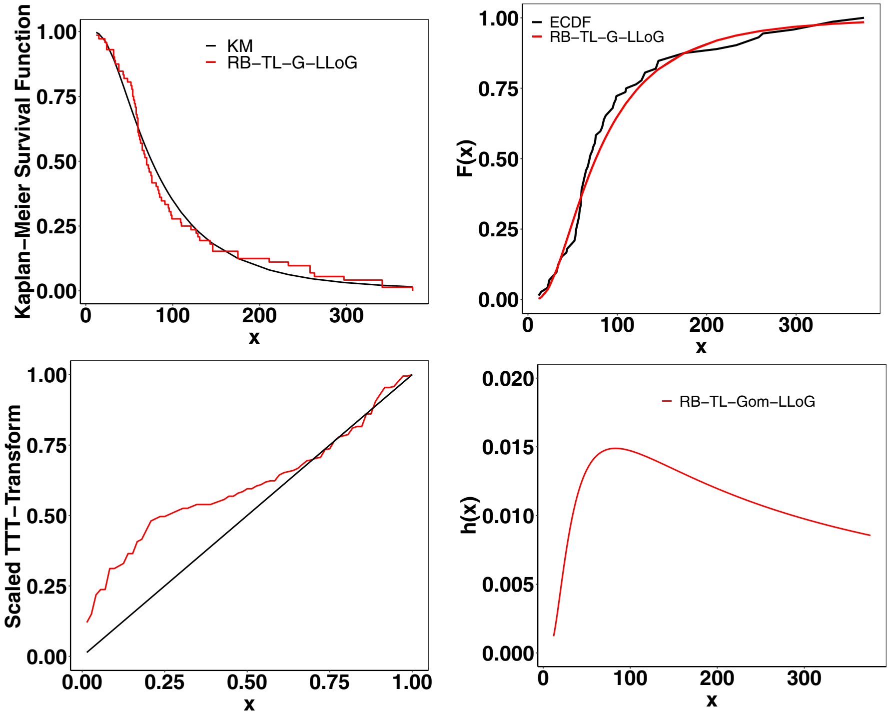

Fig. 5 Fitted Kaplan–Meier survival curve, theoretical and empirical cumulative distribution functions, the total time on test statistics, and the hazard rate function for the Guinea Pigs Data

8.3 Eruptions Data

The last data set was reported by Professor Jim Irish, which can be accessed at http://www.statsci.org/data/oz/kiama.html. Regarding the Kiama Blowhole eruptions, the following data is provided:

Table 4 MLEs and goodness-of-fit statistics for active repair times data

	Estimates				Statistics							
Model	δ	b	θ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
RB-TL-Gom-LLoG	2.0150	10.6950	1.8067×10^{-10}	0.5879	180.6067	188.6067	189.7495	195.3622	0.0647	0.4385	0.1172	0.6412
	(0.2521)	(0.05615)	(0.0023)	(0.0825)								
	α	β	b	λ								
TL-OBIII-LLoG	1.2238	19.5592	0.1277	0.9931	180.9412	188.9412	190.0841	195.6967	0.0653	0.4488	0.1181	0.6315
	(0.9458)	(30.2032)	(0.2093)	(0.7675)								
	α	β	a	θ								
GWLLoG	9.1827×10^{-04}	11.5960	3.9311×10^{-04}	0.1800	188.8792	196.8793	198.0221	203.6348	0.1245	0.9043	0.1203	0.6085
	(3.4919×10^{-03})	(7.9634×10^{-04})	(9.2160×10^{-04})	(4.2675×10^{-02})								
	λ	c	α	δ								
GELLoG	0.1164	0.9998	5.5095	0.460	182.1902	190.1902	191.333	196.9457	0.0661	0.4979	0.1276	0.5323
	(0.0835)	(0.1920)	(6.0111)	(0.5023)								
	α	a	b	c								
APExLLD	0.0026	1.7305	24.2383	0.5278	190.4375	198.4375	199.5803	205.1930	0.1154	0.8056	0.1587	0.2655
	(0.0040)	(0.3693)	(0.0019)	(0.1287)								
	θ	α	β	λ								
APTLW	0.1631	238.4700	1.1866	0.0630	199.5483	207.5487	208.6915	214.3042	0.2206	1.5189	0.1857	0.1266
	(0.0566)	(3.6715×10^{-05})	(0.1655)	(0.0327)								

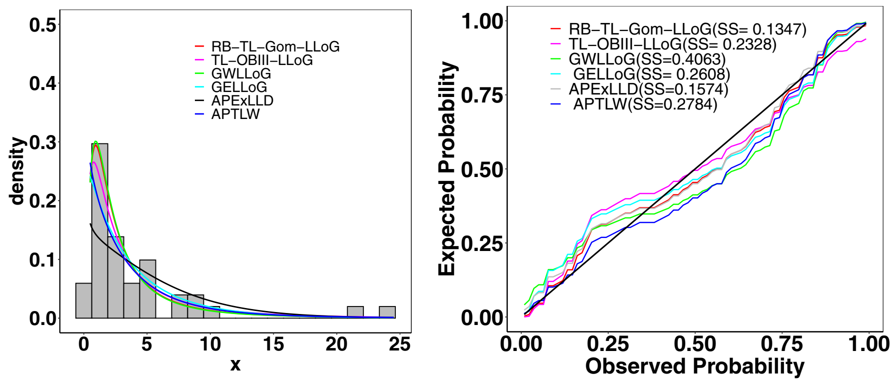

Fig. 6 Fitted density vs histogram (left) and observed probability vs expected probability plots (right) for the Active Repair Times Data

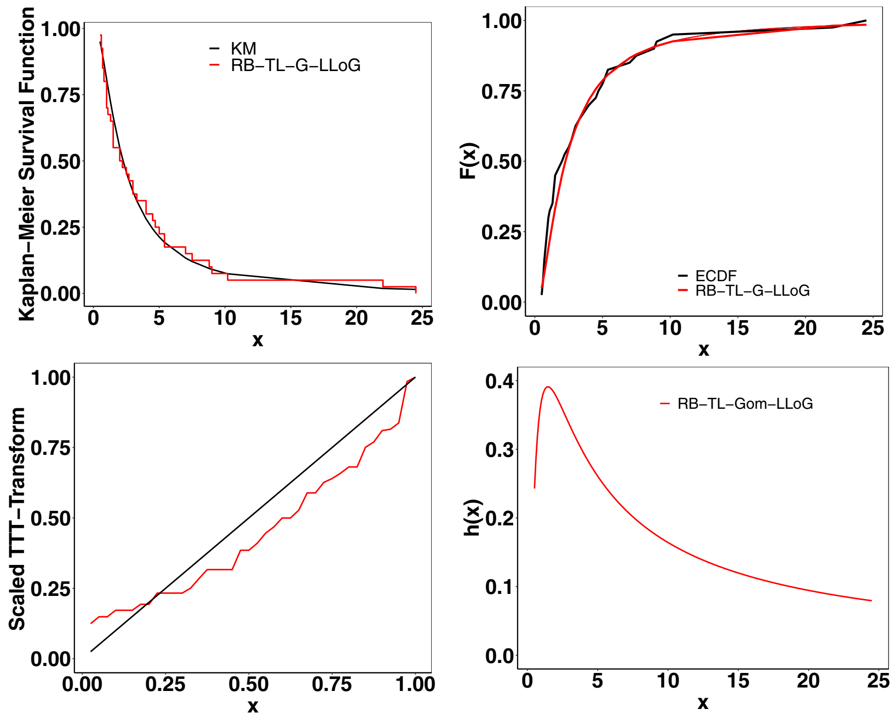

Fig. 7 Fitted Kaplan–Meier survival curve, theoretical and empirical cumulative distribution functions, the total time on test statistics, and the hazard rate function for the Active Repair Times Data

83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 5 MLEs and goodness-of-fit statistics for eruptions data

Model	Estimates				Statistics							
	δ	b	θ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
RB-TL-Gom-LLoG	27.7100	218.7800	0.2498	0.1372	587.7877	595.7877	596.4657	604.4232	0.1097	0.8069	0.0947	0.6134
	(4.5333×10^{-03})	(8.3521×10^{-04})	(0.10268)	(0.0200)								
	α	β	b	λ								
TL-OBIII-LLoG	0.8236	0.0018	2.0484×10^{07}	0.8491	590.745	598.745	599.423	607.3805	0.1735	1.1989	0.0952	0.6065
	(0.0374)	(2.9590×10^{-04})	(6.8063×10^{-14})	(0.0362)								
	α	β	a	θ								
GWLLoG	2.3314×10^{-04}	24.2860	3.9245×10^{-04}	0.0975	592.9843	600.9843	601.6622	609.6198	0.1405	0.9694	0.1116	0.4026
	(6.7889×10^{-04})	(9.3830×10^{-05})	(0.0012)	(0.015)								
	λ	c	α	δ								
GELLoG	0.0946	1.2788	0.003	12.6659	599.9101	607.9101	608.5881	616.5456	0.2086	1.3568	0.1369	0.1809
	(0.0190)	(0.4142)	(0.0032)	(0.0057)								
	α	a	b	c								
APExLLD	2.3071×10^{06}	1.4194	3.3085	0.8865	591.4674	599.4674	600.1453	608.1029	0.1668	1.1671	0.0972	0.5800
	(1.0000×10^{-07})	(0.1453)	(2.5287)	(0.8229)								
	θ	α	β	λ								
APTLW	0.6297	15.9900	1.0971	0.0115	598.4024	606.4024	607.0803	615.0379	0.1905	1.2501	0.1221	0.2957
	(0.2647)	(0.0037)	(0.1979)	(0.0115)								

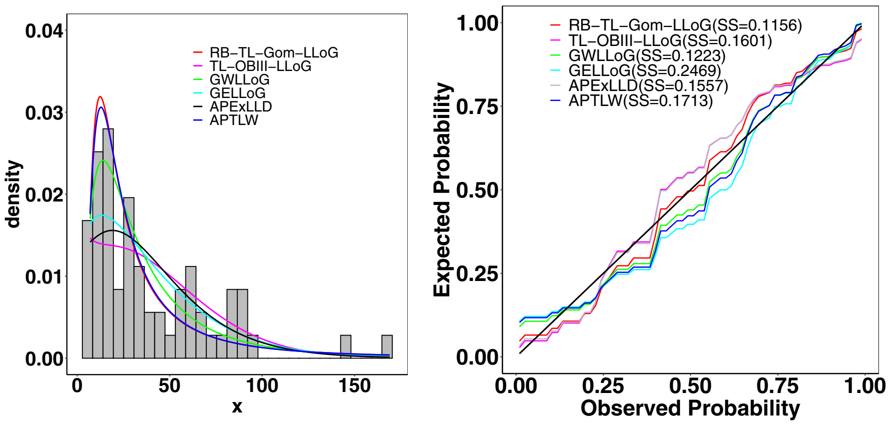

Fig. 8 Fitted density vs histogram (left) and observed probability vs expected probability plots (right) for the Eruptions Data

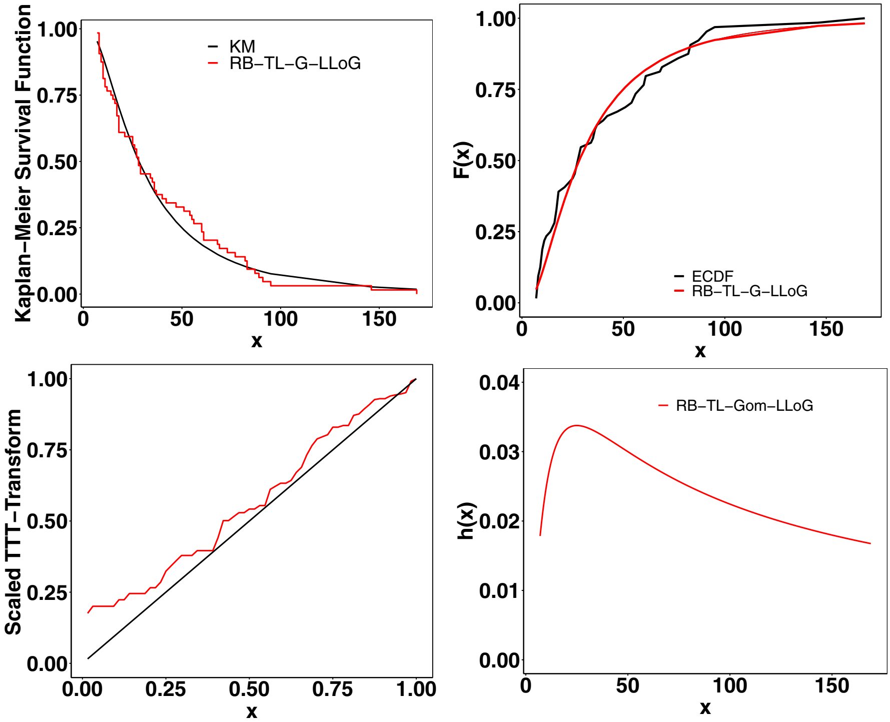

Fig. 9 Fitted Kaplan–Meier survival curve, theoretical and empirical cumulative distribution functions, the total time on test statistics, and the hazard rate function for the Eruptions Data

The parameter estimates (standard errors in parentheses), the goodness-of-fit statistics: AIC, BIC, CAIC, W^*, A^*, K-S statistic, and its p-value are given in Table 5. The numbers in Table 5 illustrate that the RB–TL–Gom-LLoG distribution has the smallest values for the goodness-of-fit statistics and the largest p-value of the K–S statistic. Thus, the RB–TL–Gom-LLoG distribution can fit the eruptions data better than the rest of the distributions. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Fig. 8.

Figure 9 shows the observed and the fitted Kaplan-Meier survival curves, ECDF, and TTT scaled plot. We can see that the new proposed distribution follows the Kaplan-Meier survival and empirical cdf curves very closely. The TTT scaled plot demonstrates a non-monotonic hrf for the eruptions data.

9 Concluding Remarks

The Ristić–Balakrishnan–Topp–Leone–Gompertz-G (RB–TL–Gom-G) family of distributions has been proposed and studied. Major statistical properties of this novel distribution family are derived. Using the maximum likelihood estimation approach, the parameters of the RB–TL–Gom-G distribution family have been estimated. The accuracy and convergence of the maximum likelihood estimates are assessed using Monte Carlo simulations. By fitting three real-world data sets, the flexibility and properties of the RB–TL–Gom-G family of distributions are presented. In conclusion, the RB–TL–Gom-G family of distributions can leverage the performance of existing distributions to generate a range of densities and hazard rate functions with a variety of shapes.

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Declarations

Conflict of interest The authors declare no competing interests.

References

1. Afify, A.Z., Al-Mofleh, H., Aljohani, H.M., Cordeiro, G.M.: The Marshall-Olkin-Weibull-H family: estimation, simulations, and applications to covid-19 data. J. King Saud Univ. Sci. **34**(5), 102115 (2022)
2. Algarni, A., Almarashi, A.M., Elbatal, I., Hassan, A.S., Almetwally, E.M., Daghistani, A.M., Elgarhy, M.: Type I half logistic Burr X-G family: properties, Bayesian, and non-Bayesian estimation under censored samples and applications to covid-19 data. Math. Probl. Eng. (2021)
3. Alizadeh, M., Cordeiro, G.M., Pinho, L.G.B., Ghosh, I.: The Gompertz-G family of distributions. J. Stat. Theory Pract. **11**(1), 179–207 (2017)
4. Al-Marzouki, S., Jamal, F., Chesneau, C., Elgarhy, M.: Topp–Leone odd Fréchet generated family of distributions with applications to covid-19 data sets. Comput. Model. Eng. Sci. **125**(1), 437–458 (2020)
5. Almetwally, E.M.: The odd Weibull inverse Topp–Leone distribution with applications to covid-19 data. Ann. Data Sci. **9**(1), 121–140 (2022)
6. Almongy, H.M., Almetwally, E.M., Aljohani, H.M., Alghamdi, A.S., Hafez, E.: A new extended Rayleigh distribution with applications of covid-19 data. Results Phys. **23**, 104012 (2021)
7. Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S., Shahbaz, M.Q.: Topp–Leone family of distributions: Some properties and application. Pak. J. Stat. Oper. Res. **12**(3), 443–451 (2016)
8. Altun, E., Yousof, H.M., Chakraborty, S., Handique, L.: Zografos–Balakrishnan Burr XII distribution: regression modeling and applications. Int. J. Math. Stat. **19**(3), 46–70 (2018)
9. Benkhelifa, L.: Alpha power Topp–Leone Weibull distribution: properties, characterizations, regression modeling and applications. J. Stat. Manag. Syst., 1–26 (2022)
10. Chipepa, F., Moakofi, T., Oluyede, B.: The Marshall–Olkin-odd power generalized Weibull-G family of distributions with applications of covid-19 data. J. Probab. Stat. Sci. **20**(1), 1–20 (2022)
11. Cordeiro, G.M., Aristizábal, W.D., Suárez, D.M., Lozano, S.: The gamma modified Weibull distribution. Chil. J. Stat. **6**(1), 37–48 (2015)
12. Cordeiro, G.M., Ortega, M.E.M., Ramires, T.G.: A new generalized Weibull family of distributions: mathematical properties and applications. J. Stat. Distrib. Appl. **2**(1), 1–25 (2015)
13. Fagbamigbe, A.F., Melamu, P., Oluyede, B.O., Makubate, B.: The Ristić and Balakrishnan Lindley–Poisson distribution: model, theory and application. Afr. Stat. **13**(4), 1837–1864 (2018)
14. Gabanakgosi, M., Moakofi, T., Oluyede, B., Makubate, B.: The gamma odd power generalized Weibull-G family of distributions with applications. J. Stat. Model. Theory Appl. **2**(2), 79–101 (2021)
15. Gay, D.M.: Usage summary for selected optimization routines. Comput. Sci. Tech. Rep. **153**, 1–21 (1990)
16. Gomes-Silva, F., de Andrade, T.A.N., Bourguignon, M.: Ristić–Balakrishnan extended exponential distribution. Acta Sci. Technol. **40**, e34963 (2018)
17. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic press, New York (2014)
18. Greenwood, J.A., Landwehr, J.M., Matalas, N.C., Wallis, J.R.: Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form. Water Resour. Res. **15**(5), 1049–1054 (1979)
19. Gupta, R.C., Kundu, D.: Weighted inverse Gaussian-a versatile lifetime model. J. Appl. Stat. **38**(12), 2695–2708 (2011)
20. Gupta, R.C., Gupta, P.L., Gupta, R.D.: Modeling failure time data by Lehman alternatives. Commun. Stat. Theory Methods **27**(4), 887–904 (1998)
21. Irshad, M.R., D'cruz, V., Maya, R.: The Zografos–Balakrishnan Lindley distribution: properties and applications. Statistica (Bologna) **81**(1), 45–64 (2021)
22. Jorgensen, B.: Statistical Properties of the Generalized Inverse Gaussian Distribution, vol. 9. Springer, Berlin (2012)
23. Kundu, D., Gupta, R.D.: An extension of the generalized exponential distribution. Stat. Methodol. **8**(6), 485–496 (2011)

24. Moakofi, T., Oluyede, B., Makubate, B.: A new gamma generalized Lindley-log-logistic distribution with applications. Afr. Stat. **15**(4), 2451–2481 (2020)
25. Moakofi, T., Oluyede, B., Gabanakgosi, M.: The Topp–Leone odd Burr III-G family of distributions: model, properties and applications. Stat. Optim. Inf. Comput. **10**(1), 236–262 (2022)
26. Mozafari, M., Afshari, M., Alizadeh, M., Karamikabir, H.: The Zografos–Balakrishnan odd log-logistic generalized half-normal distribution with mathematical properties and simulations. Stat. Optim. Inf. Comput. **7**(1), 211–234 (2019)
27. Nadarajah, S., Cordeiro, G.M., Ortega, E.M.: The Zografos–Balakrishnan-G family of distributions: mathematical properties and applications. Commun. Stat. Theory Methods **44**(1), 186–215 (2015)
28. Oluyede, B., Pu, S., Makubate, B, Qui, Y.: The gamma-Weibull-G family of distributions with applications. Aust. J. Stat. **47**(1), 45–76 (2018)
29. Oluyede, B.O., Makubate, B., Wanduku, D., Elbatal, I., Sherina, V.: The gamma-generalized inverse Weibull distribution with applications to pricing and lifetime data. J. Comput. Model. **7**(2), 1 (2017)
30. Peter, P.O., Oluyede, B., Bindele, H.F., Ndwapi, N., Mabikwa, O.: The gamma odd Burr III-G family of distributions: model, properties and applications. Rev. Colomb. Estad. **44**(2), 331–368 (2021)
31. Pinho, L., Cordeiro, G., Nobre, J.: The gamma-exponentiated Weibull distribution. J. Stat. Theory Appl. **11**(4), 379–395 (2012)
32. Pu, S., Oluyede, B.O., Qiu, Y., Linder, D.: A generalized class of exponentiated modified Weibull distribution with applications. J. Data Sci. **14**(4), 585–613 (2016)
33. Rannona, K., Oluyede, B., Chamunorwa, S.: The Gompertz–Topp–Leone-G family of distributions with applications. J. Probab. Stat. Sci. **20**(1), 108–126 (2022)
34. Rényi, A.: On measures of entropy and information. In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, vol. 1, pp. 547–561 (1961)
35. Ristić, M.M., Balakrishnan, N.: The gamma-exponentiated exponential distribution. J. Stat. Comput. Simul. **82**(8), 1191–1206 (2012)
36. Shaked, M., Shanthikumar, J.G.: Stochastic Orders and Their Applications. Academic press, New York (1994)
37. Szekli, R.: Stochastic Ordering and Dependence in Applied Probability, vol. 97. Springer, Berlin (2012)
38. Teamah, A.E.A., Elbanna, A.A., Gemeay, A.M.: Heavy-tailed log-logistic distribution: properties, risk measures and applications. Stat. Optim. Inf. Comput. **9**(4), 910–941 (2021)