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RESEARCH ARTICLE

Lie Symmetry Analysis, Power Series Solutions and Conservation Laws of (2+1)-Dimensional Time Fractional Modified Bogoyavlenskii–Schiff Equations

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Abstract
In this paper, Lie symmetry analysis method is applied to the (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations, which is an important model in physics. The one-dimensional optimal system composed by the obtained Lie symmetries is utilized to reduce the system of (2+1)-dimensional fractional partial differential equations with Riemann–Liouville fractional derivative to the system of (1+1)-dimensional fractional partial differential equations with Erdélyi–Kober fractional derivative. Then the power series method is applied to derive explicit power series solutions for the reduced system. In addition, the new conservation theorem and the generalization of Noether operators are developed to construct the conservation laws for the equations studied.

Keywords Lie symmetry analysis · Fractional modified Bogoyavlenskii–Schiff equations · Riemann–Liouville fractional derivative · Erdélyi–Kober fractional derivative · Conservation laws.

Mathematics Subject Classification 76M60 · 35G50 · 37C79 · 34K37

1 Introduction

The (2+1)-dimensional modified Bogoyavlenskii–Schiff equations are the important mathematical physical equations to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the

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x-axis. The equations are firstly derived by Bogoyavlenskii [1] and Schiff [2], which are given by

$$\begin{cases} u_t + u_{xxy} - 4u^2u_y - 2u_xv = 0, \\ v_x = 2uu_y. \end{cases} \tag{1}$$

Since then, Eq. (1) have been studied by many scholars with different methods in [3–8]. Among them, in [8], Kumar and Manju studied the integral order (2+1)-dimensional modified Bogoyavlenskii–Schiff equations by using Lie symmetry analysis method. They obtained the one-dimensional optimal subalgebras, optimal reductions and analytical solutions for Eq. (1).

As a generalization of the classical calculus, fractional calculus can be traced back to the letter written by L'Hôspital to Leibniz in 1695. Since then, it has gradually gained the attention of mathematicians. Especially in recent decades, it has developed rapidly and been successfully applied in many fields of science and technology [9–12]. Therefore, it is very important to find the solution of fractional differential equation. So far, there have been some numerical and analytical methods, such as Adomian decomposition method [13], finite difference method [14], homotopy perturbation method [15], the sub-equation method [16], the variational iteration method [17], Lie symmetry analysis method [18], invariant subspace method [19] and so on. Among them, Lie symmetry analysis method has received an increasing attention.

Lie symmetry analysis method was founded by Norwegian mathematician Sophus Lie at the end of the nineteenth century and then further developed by some other mathematicians, such as Ovsiannikov [20], Olver [21], Ibragimov [22–24] and so on. As a modern method among many analytic techniques, Lie symmetry analysis has been extended to fractional differential equations (FDEs) by Gazizov et al. [18] in 2007. It was then effectively applied to various models of the FDEs occurring in different areas of applied science (see [25–41]).

In this paper, Lie symmetry analysis method is extended to the following (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations:

$$\begin{cases} D_t^\alpha u + u_{xxy} - 4u^2u_y - 2u_xv = 0, \\ v_x = 2uu_y, \end{cases} \tag{2}$$

with $0 < \alpha < 1$. When $y = x$, the above equations become an (1+1)-dimensional time fractional mKdV equation, i.e.,

$$D_t^\alpha u - 6u^2u_x + u_{xxx} = 0, \tag{3}$$

which is also researched by several scholars. As we all know, there are many types of definitions for fractional derivative, such as Riemann–Liouville type, Caputo type, Weyl type and so on. This paper adopts Riemann–Liouville fractional derivative defined by

$$
{}_aD_t^\alpha f(t,x) = D_t^n\, {}_aI_t^{n-\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1<\alpha<n, n\in\mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N} \end{cases}
$$

for $t > a$. We denote the operator ${}_0D_t^\alpha$ as D_t^α throughout this paper.

The aim of this paper is to find all Lie symmetries for Eq. (2) by using Lie symmetry analysis method and reduce Eq. (2) to (1+1)-dimensional time fractional partial differential equations or time fractional ordinary differential equations. Moreover, we will derive explicit power series solutions by the power series method for the reduced equations and construct the conservation laws by the new conservation theorem and the generalization of Noether operators.

This paper is organized as follows. In Sect. 2, Lie symmetry analysis of Eqs. (2) is presented. In Sect. 3, the reduced equations, power series solutions and some trivial solutions for Eq. (2) are obtained. The conserved vectors for the symmetries admitted by Eq. (2) are constructed in Sects. 4, and the conclusion is given in the last section.

2 Lie Symmetry Analysis of Eq. (2)

2.1 Lie Symmetries

Consider the (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations (2), which are assumed to be invariant under the one-parameter (ϵ) Lie group of continuous point transformations, i.e.

$$
\begin{aligned}
t^* &= t + \epsilon\tau(t,x,y,u,v) + o(\epsilon), \quad x^* = x + \epsilon\xi(t,x,y,u,v) + o(\epsilon),\\
y^* &= y + \epsilon\theta(t,x,y,u,v) + o(\epsilon), \quad u^* = u + \epsilon\eta(t,x,y,u,v) + o(\epsilon),\\
v^* &= v + \epsilon\zeta(t,x,y,u,v) + o(\epsilon), \quad D_{t^*}^\alpha u^* = D_t^\alpha u + \epsilon\eta^{\alpha,t} + o(\epsilon),\\
D_{x^*}u^* &= D_x u + \epsilon\eta^x + o(\epsilon), \quad D_{x^*}v^* = D_x v + \epsilon\zeta^x + o(\epsilon),\\
D_{y^*}u^* &= D_y u + \epsilon\eta^y + o(\epsilon), \quad D_{y^*}v^* = D_y v + \epsilon\zeta^y + o(\epsilon),\\
D_{x^*}^2 u^* &= D_x^2 u + \epsilon\eta^{xx} + o(\epsilon), \quad D_{x^*x^*y^*}u^* = D_{xxy}u + \epsilon\eta^{xxy} + o(\epsilon),
\end{aligned} \tag{4}
$$

where τ, ξ, θ, η and ζ are infinitesimals and $\eta^{\alpha,t}$, η^x, ζ^x, η^y, ζ^y, η^{xx}, η^{xxy} are the corresponding prolongations of orders α, 1, 2 and 3, respectively.

The corresponding group generator is defined by

$$
X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \theta\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial v}. \tag{5}
$$

So the prolongation of the above group generator X has the form

$$
prX = X + \eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \eta^x\frac{\partial}{\partial u_x} + \zeta^x\frac{\partial}{\partial v_x} + \eta^y\frac{\partial}{\partial u_y} + \eta^{xxy}\frac{\partial}{\partial u_{xxy}}, \tag{6}
$$

where

$$\begin{aligned}\eta^{\alpha,t} &= D_t^{\alpha}(\eta) - D_t^{\alpha+1}(\tau u) + D_t^{\alpha}[D_t(\tau)u] + \tau D_t^{\alpha+1}(u) + \xi D_t^{\alpha}(u_x) - D_t^{\alpha}(\xi u_x) \\ &\quad + \theta D_t^{\alpha}(u_y) - D_t^{\alpha}(\theta u_y) \\ &= \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + (\eta_u - \alpha D_t(\tau))\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} + (\eta_v\frac{\partial^{\alpha}v}{\partial t^{\alpha}} - v\frac{\partial^{\alpha}\eta_v}{\partial t^{\alpha}}) + \mu_1 + \mu_2 \\ &\quad + \sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(u) + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\eta_v}{\partial t^n}D_t^{\alpha-n}(v) \\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\theta)D_t^{\alpha-n}(u_y),\end{aligned} \tag{7}$$

with

$$\begin{aligned}\mu_1 &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}, \\ \mu_2 &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-v)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m v^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v^k},\end{aligned}$$

and

$$\eta^x = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) - u_yD_x(\theta), \tag{8}$$

$$\eta^y = D_y(\eta) - u_tD_y(\tau) - u_xD_y(\xi) - u_yD_y(\theta), \tag{9}$$

$$\zeta^x = D_x(\zeta) - v_tD_x(\tau) - v_xD_x(\xi) - v_yD_x(\theta), \tag{10}$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\theta), \tag{11}$$

$$\eta^{xxy} = D_y(\eta^{xx}) - u_{xxt}D_y(\tau) - u_{xxx}D_y(\xi) - u_{xxy}D_y(\theta), \tag{12}$$

where D_t, D_x and D_y are the total derivative with respect to t, x and y, respectively.

Remark 1 The infinitesimal transformations (4) should conserve the structure of the Riemann–Liouville fractional derivative operator, of which the lower limit in the integral is fixed. Therefore, the manifold $t = 0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t, x, y, u, v)|_{t=0} = 0. \tag{13}$$

Remark 2 From the expressions of μ_1 and μ_2, if the infinitesimals η and ζ be linear with respect to the variables u and v, then $\mu_1 = \mu_2 = 0$, that is,

$$\frac{\partial^2\eta}{\partial u^2} = \frac{\partial^2\eta}{\partial v^2} = 0. \tag{14}$$

The one-parameter Lie symmetry transformations (4) are admitted by the system (2), if the following invariance criterion holds:

$$\begin{cases} prX\left(D_t^\alpha u + u_{xxy} - 4u^2u_y - 2u_xv\right)|_{(1.2)} = 0, \\ prX(v_x - 2uu_y)|_{(1.2)} = 0, \end{cases} \tag{15}$$

which can be rewritten as

$$\begin{cases} \left(\eta^{\alpha,t} + \eta^{xxy} - 4u^2\eta^y - 2v\eta^x - 8uu_y\eta - 2u_x\zeta\right)|_{(1.2)} = 0, \\ \left(\zeta^x - 2u\eta^y - 2u_y\eta\right)|_{(1.2)} = 0. \end{cases} \tag{16}$$

Putting $\eta^{\alpha,t}$, η^x, ζ^x, η^y and η^{xxy} into (16) and letting coefficients of various derivatives of u and v to be zero, we can obtain the over-determined system of differential equations as follows:

$$\tau_x = \tau_y = \tau_u = \tau_v = 0, \quad \theta_t = \theta_x = \theta_u = \theta_v = 0, \tag{17}$$

$$\xi_t = \xi_u = \xi_v = \eta_v = \zeta_u = 0, \quad \eta_{uu} = \zeta_{uu} = \zeta_{uv} = \zeta_{vv} = 0, \tag{18}$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \tag{19}$$

$$2\xi_x + \theta_y - \alpha\tau_t = 0, \tag{20}$$

$$v(\xi_x - \alpha\tau_t) - \zeta = 0, \tag{21}$$

$$u(\theta_y - \alpha\tau_t) - 2\eta = 0, \tag{22}$$

$$u(\zeta_v + \theta_y - \eta_u - \xi_x) - \eta = 0, \tag{23}$$

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha} + \eta_{xxy} - 4u^2\eta_y - 2v\eta_x = 0. \tag{24}$$

Solving these equations altogether, with the conditions (13) and (14), we can obtain infinitesimals as follows:

$$\tau = c_1 t, \quad \xi = c_2 x + c_3, \quad \theta = (\alpha c_1 - 2c_2)y + c_4, \quad \eta = -c_2 u, \quad \zeta = (c_2 - \alpha c_1)v, \tag{25}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

So the system (2) admitted the four-dimension Lie algebra spanned by

$$X_1 = t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} - \alpha v\frac{\partial}{\partial v}, \quad X_2 = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial y}. \tag{26}$$

Table 1 The commutation table of Lie algebra

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$-\alpha X_4$
X_2	0	0	$-X_3$	$2X_4$
X_3	0	X_3	0	0
X_4	αX_4	$-2X_4$	0	0

Table 2 The adjoint representation of Lie algebra

$Ad(exp(\varepsilon X_i))X_j$	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$e^{\alpha\varepsilon} X_4$
X_2	X_1	X_2	$e^{\varepsilon} X_3$	$e^{-2\varepsilon} X_4$
X_3	X_1	$e^{-\varepsilon} X_2$	X_3	X_4
X_4	$X_1 - \alpha\varepsilon X_4$	$X_2 + 2\varepsilon X_4$	X_3	X_4

2.2 One-Dimensional Optimal System

It is easy to check that the group generators in (26) are closed under the Lie bracket defined by

$$[X_i, X_j] = X_i X_j - X_j X_i, \quad (i, j = 1, 2, 3, 4). \tag{27}$$

The commutation relationships of these group generators can be seen in Table 1.

Then we consider the action of the adjoint operator which is given by the Lie series

$$Ad(exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \cdots, \tag{28}$$

where ε is an arbitrary parameter. According to (28), we calculate the adjoint action of the group generators in (26) which is listed in Table 2.

From Tables 1 and 2, by Olver's method [21], the optimal system for the one-dimensional subalgebras of Eq. (2) can be obtained as the following forms:

$$X_1, \; X_2, \; X_3, \; X_4, \; X_1 + \beta X_2, \; X_3 + \gamma X_4. \tag{29}$$

3 Similarity reductions and invariant solutions of Eq. (2)

In this section, the aimed equations (2) can be reduced to (1+1)-dimensional time fractional partial differential equations with the left-hand Erdélyi–Kober fractional derivative. Then the reduced equations can be solved by the power series method and Laplace transform method to construct the invariant solutions for Eq. (2). In what follows, we consider the following cases.

Case 1: $X_1 + \beta X_2$

For the convenience of calculation, assuming $\beta = \frac{\alpha}{2}$, we can obtain the following group generator:

$$X_1 + \frac{\alpha}{2}X_2 = t\frac{\partial}{\partial t} + \frac{\alpha}{2}x\frac{\partial}{\partial x} - \frac{\alpha}{2}u\frac{\partial}{\partial u} - \frac{\alpha}{2}v\frac{\partial}{\partial v},$$

which can also be obtained by making $c_2 = \frac{\alpha}{2}c_1$ in (25). The characteristic equation corresponding to the group generator $X_1 + \frac{\alpha}{2}X_2$ is

$$\frac{\mathrm{d}t}{t} = \frac{2\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}y}{0} = \frac{2\mathrm{d}u}{-\alpha u} = \frac{2\mathrm{d}v}{-\alpha v}, \tag{30}$$

from which, we obtain the similarity variables $xt^{-\frac{\alpha}{2}}$, y, $ut^{\frac{\alpha}{2}}$ and $vt^{\frac{\alpha}{2}}$. So we get the invariant solutions of the system (2) as follows:

$$u(t,x,y) = t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2), \;\; v(t,x,y) = t^{-\frac{\alpha}{2}}g(\omega_1,\omega_2), \tag{31}$$

with $\omega_1 = xt^{-\frac{\alpha}{2}}$, $\omega_2 = y$.

Theorem 2.1 *The similarity transformations* $u(t,x,y) = t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2)$, $v(t,x,y) = t^{-\frac{\alpha}{2}}g(\omega_1,\omega_2)$ *with the similarity variables* $\omega_1 = xt^{-\frac{\alpha}{2}}$, $\omega_2 = y$ *reduce the system* (2) *to the system of* (1+1)*-dimensional fractional differential equations given by*

$$\begin{cases} (\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\infty}f)(\omega_1,\omega_2) + f_{\omega_1\omega_1\omega_2} - 4f^2f_{\omega_2} - 2f_{\omega_1}g = 0, \\ g_{\omega_1} - 2ff_{\omega_2} = 0, \end{cases} \tag{32}$$

where $(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2})$ *is the left-hand Erdélyi–Kober fractional differential operator defined by*

$$(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2}\psi)(\omega_1,\omega_2) := \prod_{j=0}^{m-1}\left(\iota + j - \frac{1}{\delta_1}\omega_1\frac{\partial}{\partial\omega_1} - \frac{1}{\delta_2}\omega_2\frac{\partial}{\partial\omega_2})(\mathcal{K}^{\iota+\kappa,m-\kappa}_{\delta_1,\delta_2}\psi\right)(\omega_1,\omega_2), \; \kappa > 0,$$
$$m = \begin{cases} [\kappa]+1, & if\ \kappa \notin \mathbb{N}, \\ \kappa, & if\ \kappa \in \mathbb{N}, \end{cases} \tag{33}$$

where

$$(\mathcal{K}^{\iota,\kappa}_{\delta_1,\delta_2}\psi)(\omega_1,\omega_2) := \begin{cases} \frac{1}{\Gamma(\kappa)}\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega_1 s^{\frac{1}{\delta_1}}, \omega_2 s^{\frac{1}{\delta_2}})\mathrm{d}s, & \kappa > 0, \\ \psi(\omega_1,\omega_2), & \kappa = 0, \end{cases} \tag{34}$$

is the left-hand Erdélyi–Kober fractional integral operator.

Proof For $0 < \alpha < 1$, the Riemann–Liouville time fractional derivative of $u(t, x)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2)) = \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\frac{-\alpha}{2}}f(xs^{-\frac{\alpha}{2}},y)\mathrm{d}s\right].$$

Assuming $r = \frac{t}{s}$, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t}\left[\frac{t^{1-\frac{3\alpha}{2}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha}r^{\frac{3\alpha}{2}-2}f(\omega_1 r^{\frac{\alpha}{2}},\omega_2)\mathrm{d}r\right] = \frac{\partial}{\partial t}\left[t^{1-\frac{3\alpha}{2}}(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\infty}f)(\omega_1,\omega_2)\right].$$

Because of $\omega_1 = xt^{-\frac{\alpha}{2}}$ and $\omega_2 = y$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega_1,\omega_2) = tx\left(-\frac{\alpha}{2}\right)t^{-\frac{\alpha}{2}-1}\psi_{\omega_1} = -\frac{\alpha}{2}\omega_1\frac{\partial}{\partial\omega_1}\psi(\omega_1,\omega_2).$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial}{\partial t}\left[t^{1-\frac{3\alpha}{2}}\left(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\infty}f\right)(\omega_1,\omega_2)\right] =& t^{-\frac{3\alpha}{2}}\left[\left(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega_1\frac{\partial}{\partial\omega_1}\right)\left(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\infty}f\right)(\omega_1,\omega_2)\right]\\ =& t^{-\frac{3\alpha}{2}}\left(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\infty}f\right)(\omega_1,\omega_2).\end{aligned}$$

Meanwhile,

$$\begin{aligned}u_{xxy} - 4u^2u_y - 2u_xv &= t^{-\frac{3\alpha}{2}}(f_{\omega_1\omega_1\omega_2} - 4f^2f_{\omega_2} - 2f_{\omega_1}g),\\ v_x - 2uu_y &= t^{-\alpha}(g_{\omega_1} - 2ff_{\omega_2}).\end{aligned}$$

This completes the proof. □

Next we use the power series method introduced in [42, 43] to derive the power series solutions of the reduced equations (32). Let us introduce the variable $\omega = \omega_1 + a\omega_2$ and assume

$$f(\omega_1,\omega_2) = f(\omega) = \sum_{k=0}^{\infty} a_k(\omega_1 + a\omega_2)^k = \sum_{k=0}^{\infty} a_k\omega^k, \tag{35}$$

$$g(\omega_1,\omega_2) = g(\omega) = \sum_{k=0}^{\infty} b_k(\omega_1 + a\omega_2)^k = \sum_{k=0}^{\infty} b_k\omega^k, \tag{36}$$

where a_k and b_k are constants to be determined later. Substituting (35) and (36) into the reduced equations (32) and using the derivations in [43] yields that

$$\sum_{k=0}^{\infty}\frac{\Gamma\left(1-\frac{(k+1)\alpha}{2}\right)}{\Gamma\left(1-\frac{(k+3)\alpha}{2}\right)}a_k\omega^k+a\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)a_{k+3}\omega^k$$
$$-4a\sum_{k=0}^{\infty}\sum_{l+m+n=k}(n+1)a_la_ma_{n+1}\omega^k-2\sum_{k=0}^{\infty}\sum_{m+n=k}(n+1)b_ma_{n+1}\omega^k=0, \tag{37}$$

$$\sum_{k=0}^{\infty}(k+1)b_{k+1}\omega^k-2a\sum_{k=0}^{\infty}\sum_{m+n=k}(n+1)a_ma_{n+1}\omega^k=0. \tag{38}$$

Equating the coefficients of different powers of ω arrives at the following system:

$$\begin{cases}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k-4a\sum_{l+m+n=k}(n+1)a_la_ma_{n+1}-2\sum_{m+n=k}(n+1)b_ma_{n+1}\\+a(k+3)(k+2)(k+1)a_{k+3}=0,\\(k+1)b_{k+1}-2a\sum_{m+n=k}(n+1)a_ma_{n+1}=0,\end{cases} \tag{39}$$

from which, we can obtain the explicit expressions of a_k and b_k. For $k=0$, we have

$$\begin{cases}b_1=2aa_0a_1,\\a_3=-\frac{1}{6a}\left[\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{3\alpha}{2})}a_0-4aa_0^2a_1-2b_0a_1\right].\end{cases} \tag{40}$$

For $k=1$, we have

$$\begin{cases}b_2=a(a_1^2+2a_0a_2),\\a_4=-\frac{1}{24a}\left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}a_1-8a(a_0a_1^2+a_0^2a_2)-2(a_1b_1+2b_0a_2)\right].\end{cases} \tag{41}$$

For $k=2$, we have

$$\begin{cases}b_3=2a(a_1a_2+a_0a_3),\\a_5=-\frac{1}{60a}\left[\frac{\Gamma(1-\frac{3\alpha}{2})}{\Gamma(1-\frac{5\alpha}{2})}a_2-4a(a_1^3+3a_0^2a_3+6a_0a_1a_2)-2(a_1b_2+2b_1a_2+3b_0a_3)\right].\end{cases} \tag{42}$$

For $k>2$, we have

$$\begin{cases}b_{k+1}=\frac{2a}{(k+1)}\sum_{m+n=k}(n+1)a_ma_{n+1},\\a_{k+3}=\frac{-1}{a(k+3)(k+2)(k+1)}\left[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k-4a\sum_{l+m+n=k}(n+1)a_la_ma_{n+1}\right.\\\qquad\left.-2\sum_{m+n=k}(n+1)b_ma_{n+1}\right].\end{cases} \tag{43}$$

In what follows, applying the method in [44, 45], the convergence analysis of the power series solutions for the system (32) will be presented. From (40)–(43), we get

$$\begin{cases} |b_{k+1}| \le 2a\sum_{m+n=k}|a_m||a_{n+1}|, \\ |a_{k+3}| \le M\Big[|a_k| + \sum_{l+m+n=k}|a_l||a_m||a_{n+1}| + \sum_{m+n=k}|b_m||a_{n+1}|\Big], \end{cases} \tag{44}$$

with $M = \max\left(\frac{\Gamma\left(1-\frac{(k+1)\alpha}{2}\right)}{6a\Gamma\left(1-\frac{(k+3)\alpha}{2}\right)}, \frac{2}{3}, \frac{1}{3a}\right)$. Assuming

$$C(\theta) = \sum_{k=0}^{\infty} c_k\theta^k, \quad D(\theta) = \sum_{k=0}^{\infty} d_k\theta^k, \tag{45}$$

with $c_0 = |a_0|$, $d_0 = |b_0|$, $c_1 = |a_1|$, $c_2 = |a_2|$ and

$$\begin{cases} d_{k+1} = 2a\sum_{m+n=k} c_m c_{n+1}, \ k = 0, 1, 2\dots, \\ c_{k+3} = M(c_k + \sum_{l+m+n=k} c_l c_m c_{n+1} + \sum_{m+n=k} d_m c_{n+1}), \ k = 0, 1, 2\dots, \end{cases} \tag{46}$$

that is, $|a_k| \le c_k, |b_k| \le d_k, k = 0, 1, 2\dots$, we can get

$$\begin{aligned} C(\theta) &= c_0 + c_1\theta + c_2\theta^2 + \sum_{k=0}^{\infty} c_{k+3}\theta^{k+3} \\ &= c_0 + c_1\theta + c_2\theta^2 + M\sum_{k=0}^{\infty}\left(c_k + \sum_{l+m+n=k} c_l c_m c_{n+1} + \sum_{m+n=k} d_m c_{n+1}\right)\theta^{k+3} \\ &= c_0 + c_1\theta + c_2\theta^2 + MC(\theta)\theta^3 + M\sum_{k=0}^{\infty}\left[\sum_{l+m+n=k} c_l c_m c_{n+1} + \sum_{m+n=k} d_m c_{n+1}\right]\theta^{k+3}, \end{aligned} \tag{47}$$

$$D(\theta) = d_0 + \sum_{k=0}^{\infty} d_{k+1}\theta^{k+1} = d_0 + 2a\sum_{k=0}^{\infty}\sum_{m+n=k} c_m c_{n+1}\theta^{k+1}. \tag{48}$$

Consider the implicit functional system with respect to the independent variable θ,

$$\begin{aligned} H(\theta, C, D) = C - c_0 - c_1\theta - c_2\theta^2 - MC\theta^3 - M\sum_{k=0}^{\infty}\Big[\sum_{l+m+n=k} c_l c_m c_{n+1} \\ + \sum_{m+n=k} d_m c_{n+1}\Big]\theta^{k+3}, \end{aligned} \tag{49}$$

$$I(\theta, C, D) = D - d_0 - 2a\sum_{k=0}^{\infty}\sum_{m+n=k} c_m c_{n+1}\theta^{k+1}, \tag{50}$$

from which, H and I are analytic in the neighborhood of point $(0, c_0, d_0)$ and

$$H(0, c_0, d_0) = 0, \quad I(0, c_0, d_0) = 0. \tag{51}$$

The Jacobian determinant is

$$J = \frac{\partial(H, I)}{\partial(C, D)} \neq 0. \tag{52}$$

Then the two series $C = C(\theta)$ and $D = D(\theta)$ are analytic in the neighborhood of $(0, c_0, d_0)$ with positive radius by implicit function theorem, that is, the series $f(\omega)$ and $g(\omega)$ are convergent in the neighborhood of $(0, c_0, d_0)$.

Therefore, the power series solutions of (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations (2) have the form

$$\begin{aligned} u(t,x,y) &= t^{-\frac{\alpha}{2}} f(\omega) = t^{-\frac{\alpha}{2}} \sum_{k=0}^{\infty} a_k (xt^{-\frac{\alpha}{2}} + ay)^k = a_0 t^{-\frac{\alpha}{2}} + a_1 (xt^{-\frac{\alpha}{2}} + ay) t^{-\frac{\alpha}{2}} \\ &+ a_2 (xt^{-\frac{\alpha}{2}} + ay)^2 t^{-\frac{\alpha}{2}} + \sum_{k=0}^{\infty} \frac{-t^{-\frac{\alpha}{2}}}{a(k+3)(k+2)(k+1)} \Bigg[\frac{\Gamma\Big(1 - \frac{(k+1)\alpha}{2}\Big)}{\Gamma\Big(1 - \frac{(k+3)\alpha}{2}\Big)} a_k \\ &- 4a \sum_{l+m+n=k} (n+1) a_l a_m a_{n+1} - 2 \sum_{m+n=k} (n+1) b_m a_{n+1} \Bigg] (xt^{-\frac{\alpha}{2}} + ay)^{k+3}, \end{aligned} \tag{53a}$$

$$\begin{aligned} v(t,x,y) &= t^{-\frac{\alpha}{2}} g(\omega) = t^{-\frac{\alpha}{2}} \sum_{k=0}^{\infty} b_k (xt^{-\frac{\alpha}{2}} + ay)^k \\ &= b_0 t^{-\frac{\alpha}{2}} + \sum_{k=0}^{\infty} \frac{2at^{-\frac{\alpha}{2}}}{(k+1)} \sum_{m+n=k} (n+1) a_m a_{n+1} (xt^{-\frac{\alpha}{2}} + ay)^{k+1}, \end{aligned} \tag{53b}$$

where a_k and b_k are defined by (40)–(43) with arbitrary initial conditions $a_0 = f(0,0)$, $b_0 = g(0,0)$, $a_1 = f'(0,0)$ and $a_2 = \frac{1}{2} f''(0,0)$.

Case 2: X_1

The characteristic equation corresponding to the group generator X_1 is

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{\alpha y} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{-\alpha v}, \tag{54}$$

from which, we obtain the similarity variables x, $yt^{-\alpha}$, u and vt^{α}. So we get the invariant solutions of the system (2) as follows:

$$u(t,x,y) = f(\omega_1, \omega_2), \quad v(t,x,y) = t^{-\alpha} g(\omega_1, \omega_2), \tag{55}$$

with $\omega_1 = x$, $\omega_2 = yt^{-\alpha}$.

Theorem 2.2 *The similarity transformations* $u(t,x,y) = f(\omega_1, \omega_2)$, $v(t,x,y) = t^{-\alpha} g(\omega_1, \omega_2)$ *with the similarity variables* $\omega_1 = x$, $\omega_2 = yt^{-\alpha}$ *reduce the system* (2) *to the system of* (1+1)-*dimensional fractional differential equations given by*

$$\begin{cases} (\mathcal{P}^{1-\alpha,\alpha}_{\infty,\frac{1}{\alpha}}f)(\omega_1,\omega_2) + f_{\omega_1\omega_1\omega_2} - 4f^2 f_{\omega_2} - 2f_{\omega_1} g = 0, \\ g_{\omega_1} - 2ff_{\omega_2} = 0. \end{cases} \tag{56}$$

The proof of Theorem 2.2 is similar to that of Theorem 2.1. Meanwhile, assuming $\omega = \omega_2 + a\omega_1$ *and using the procedure in Case* 1, *we can obtain the power series solution of Eqs.* (2) *as follows*:

$$\begin{aligned} u(t,x,y) = f(\omega) = &\sum_{k=0}^{\infty} a_k (yt^{-\alpha} + ax)^k = a_0 + a_1(yt^{-\alpha} + ax) \\ &+ a_2(yt^{-\alpha} + ax)^2 + \sum_{k=0}^{\infty} \frac{-1}{(k+3)(k+2)(k+1)a^2} \Big[\frac{\Gamma(1-k\alpha)}{\Gamma(1-(k+1)\alpha)} a_k \\ &- 4 \sum_{l+m+n=k} (n+1) a_l a_m a_{n+1} - 2a \sum_{m+n=k} (n+1) b_m a_{n+1} \Big] (yt^{-\alpha} + ax)^{k+3}, \end{aligned} \tag{57a}$$

$$\begin{aligned} v(t,x,y) = t^{-\alpha} g(\omega) = &\, t^{-\alpha} \sum_{k=0}^{\infty} b_k (yt^{-\alpha} + ax)^k \\ &= b_0 t^{-\alpha} + \sum_{k=0}^{\infty} \frac{2t^{-\alpha}}{(k+1)a} \sum_{m+n=k} (n+1) a_m a_{n+1} (yt^{-\alpha} + ax)^{k+1}, \end{aligned} \tag{57b}$$

with arbitrary initial conditions $a_0 = f(0,0)$, $b_0 = g(0,0)$, $a_1 = f'(0,0)$ *and* $a_2 = \frac{1}{2} f''(0,0)$.

Case 3: X_2

The characteristic equation corresponding to the group generator X_2 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{-2y} = \frac{\mathrm{d}u}{-u} = \frac{\mathrm{d}v}{v}, \tag{58}$$

from which, we obtain the similarity variables t, yx^2, ux and vx^{-1}. So we get the invariant solutions of the system (2)

$$u = x^{-1} f(t,X), \quad v = xg(t,X), \quad X = yx^2, \tag{59}$$

and the reduced equations

$$\begin{cases} D_t^\alpha f + 4X^2 f_{XXX} + 6X f_{XX} - 4f^2 f_X - 4Xg f_X + 2fg = 0, \\ 2ff_X - 2Xg_X - g = 0, \end{cases} \tag{60}$$

which is (1+1)-dimensional time fractional partial differential equations. For Eqs. (60), we can once again use the Lie symmetry analysis method to further reduce them to fractional ordinary differential equations, and obtain the following generators:

$$\Lambda_1 = \frac{\partial}{\partial X}, \quad \Lambda_2 = t\frac{\partial}{\partial t} + \alpha X\frac{\partial}{\partial X} - \alpha g\frac{\partial}{\partial g}. \tag{61}$$

For $\Lambda_1 = \frac{\partial}{\partial X}$, Eqs. (60) have the following invarant solutions:

$$f(t,X) = F(t), \quad g(t,X) = G(t). \tag{62}$$

Substituting (62) into Eqs. (60), we get the following solutions:

$$F(t) = \frac{C_0}{\Gamma(\alpha)}t^{\alpha-1}, \quad G(t) = 0, \tag{63}$$

That is, Eqs. (2) have one trivial solution, i.e.,

$$u = \frac{C_0}{\Gamma(\alpha)}x^{-1}t^{\alpha-1}, \quad v = 0. \tag{64}$$

For $\Lambda_2 = t\frac{\partial}{\partial t} + \alpha X\frac{\partial}{\partial X} - \alpha g\frac{\partial}{\partial g}$, Eqs. (60) have the following invariant solutions:

$$f(t,X) = F(\sigma), \quad g(t,X) = t^{-\alpha}G(\sigma), \quad \sigma = Xt^{-\alpha}. \tag{65}$$

Substituting (65) into Eqs. (60), we can obtain the following fractional ordinary differential equations:

$$\begin{cases} (\mathcal{P}^{1-\alpha,\alpha}_{\frac{1}{\alpha}}F)(\sigma) + 4\sigma^2 F_{\sigma\sigma\sigma} + 6\sigma F_{\sigma\sigma} - 4F^2F_\sigma - 4\sigma GF_\sigma - 2FG = 0, \\ 2FF_\sigma - 2\sigma G_\sigma - G = 0. \end{cases} \tag{66}$$

The power series solution of (66) can be obtained as

$$F(\sigma) = \sum_{k=0}^{\infty} a_k\sigma^k, \quad G(\sigma) = \sum_{k=0}^{\infty} b_k\sigma^k, \tag{67}$$

where a_k and b_k are defined by $a_0 = f(0)$, $a_1 = \frac{1}{4a_0\Gamma(1-\alpha)} - \frac{1}{2}$, $b_0 = 2a_0a_1$, $a_2 = \left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}a_1 - 8a_0a_1^2 + 4b_0a_1 - 4a_0a_1\right]/(8a_0^2 - 12)$ and

$$\begin{cases} b_{k+1} = \frac{2}{2k+3}\sum_{m+n=k+1}(n+1)a_m a_{n+1}, \\ a_{k+3} = \frac{\frac{\Gamma(1-(k+2)\alpha)}{\Gamma(1-(k+3)\alpha)}a_{k+2} - 4\sum_{l+m+n=k+2}^{n\neq k+2}(n+1)a_l a_m a_{n+1} - 4\sum_{m+n=k+1}(n+1)b_m a_{n+1} - 2\sum_{m+n=k+2}a_m a_n}{4(k+3)a_0^2 - 4(k+3)(k+2)(k+1) - 6(k+3)(k+2)}. \end{cases} \tag{68}$$

That is, Eqs. (2) have the following power series solution:

$$
\begin{aligned}
u(t,x,y) =&x^{-1}f(t,X) = x^{-1}F(\sigma) = a_0x^{-1} + \left(\frac{1}{4a_0\Gamma(1-\alpha)} - \frac{1}{2}\right)xyt^{-\alpha} \\
&+ \left[\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}a_1 - 8a_0a_1^2 + 4b_0a_1 - 4a_0a_1\right)/(8a_0^2-12)\right]x^3y^2t^{-2\alpha} \\
&+ \sum_{k=0}^{\infty}\left[\frac{\frac{\Gamma(1-(k+2)\alpha)}{\Gamma(1-(k+3)\alpha)}a_{k+2} - 4\sum_{l+m+n=k+2}^{n\neq k+2}(n+1)a_la_ma_{n+1}}{4(k+3)a_0^2 - 4(k+3)(k+2)(k+1) - 6(k+3)(k+2)}\right. \\
&\left. - \frac{4\sum_{m+n=k+1}(n+1)b_ma_{n+1} + 2\sum_{m+n=k+2}a_ma_n}{4(k+3)a_0^2 - 4(k+3)(k+2)(k+1) - 6(k+3)(k+2)}\right]x^{2k+5}y^{k+3}t^{-(k+3)\alpha},
\end{aligned} \tag{69a}
$$

$$
\begin{aligned}
v(t,x,y) =&xg(t,X) = xt^{-\alpha}G(\sigma) = 2a_0a_1xt^{-\alpha} \\
&+ \sum_{k=0}^{\infty}\frac{2}{2k+3}\sum_{m+n=k+1}(n+1)a_ma_{n+1}x^{2k+3}y^{k+1}t^{-(k+2)\alpha}.
\end{aligned} \tag{69b}
$$

Case 4: X_3

The characteristic equation corresponding to the group generator X_3 is

$$
\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0}, \tag{70}
$$

from which, we obtain the similarity variables t, y, u and v. So we get the invariant solutions of the system (2) as follows:

$$
u = f(t,y), \quad v = g(t,y). \tag{71}
$$

Substituting (71) into Eqs. (2), we have the following reduced equations:

$$
\begin{cases} D_t^{\alpha}f - 4f^2f_y = 0, \\ 2ff_y = 0, \end{cases} \tag{72}
$$

which is rewritten as

$$
D_t^{\alpha}f(t) = 0. \tag{73}
$$

We can easily use the Laplace transform of Riemann–Liouville fractional derivative, i.e.,

$$
\mathcal{L}\{D_t^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^kf^{(\alpha-k-1)}(0), \quad n-1 < \alpha \leq n, n \in N \tag{74}
$$

to obtain the exact solutions of (73) as

$$
f(t) = \frac{k_1}{\Gamma(\alpha)}t^{\alpha-1}, \tag{75}
$$

where k_1 is a constant determined by initial condition, i.e., $k_1 = f^{(\alpha-1)}(0)$.

In this case, we obtain the following trivial solutions:

$$u = \frac{k_1}{\Gamma(\alpha)} t^{\alpha-1}, \quad v = g(t, y), \tag{76}$$

where $g(t, y)$ is an arbitrary function.

Case 5: X_4

The characteristic equation corresponding to the group generator X_4 is

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0} = \frac{dv}{0}, \tag{77}$$

from which, we obtain the similarity variables t, x, u and v. So we get the invariant solutions of the system (2)

$$u = f(t, x), \quad v = g(t, x). \tag{78}$$

and the reduced equations

$$\begin{cases} D_t^\alpha f - 2f_x g = 0, \\ g_x = 0. \end{cases} \tag{79}$$

Now we use invariant subspace method for (79) and find that they admit the following invariant subspace:

$$W_2^1 \times W_2^2 = \{1, x\} \times \{1, x\},$$

from which, the solutions of (79) have the form $f = f_1(t) + f_2(t)x$, $g = g(t)$. So (79) can be rewritten as

$$D_t^\alpha f_1 + x D_t^\alpha f_2 - 2f_2 g = 0, \tag{80}$$

from which, we can get $f_2(t) = \frac{k_2}{\Gamma(\alpha)} t^{\alpha-1}$ with $k_2 = f_2^{(\alpha-1)}(0)$ and

$$D_t^\alpha f_1 - \frac{2k_2}{\Gamma(\alpha)} t^{\alpha-1} g = 0. \tag{81}$$

Then solving the equation (81), we obtain

$$f_1(t) = k_1 t^\beta, \quad g(t) = \frac{k_1 \Gamma(\alpha)\Gamma(\beta+1)}{2k_2 \Gamma(\beta-\alpha+1)} t^{\beta-2\alpha+1}, \tag{82}$$

where k_1 and β are arbitrary constants.

In this case, we obtain the following trivial solutions:

$$u = k_1 t^\beta + \frac{k_2}{\Gamma(\alpha)} t^{\alpha-1} x, \quad v = \frac{k_1 \Gamma(\alpha)\Gamma(\beta+1)}{2k_2 \Gamma(\beta-\alpha+1)} t^{\beta-2\alpha+1}. \tag{83}$$

Case 6: $X_3 + \gamma X_4$

The characteristic equation corresponding to the group generator $X_3 + \gamma X_4$ is

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{\gamma} = \frac{du}{0} = \frac{dv}{0}, \tag{84}$$

from which, we obtain the similarity variables t, $y - \gamma x$, u and v. So we get the invariant solutions of the system (2)

$$u = f(t, X), \quad v = g(t, X), \quad X = y - \gamma x, \tag{85}$$

and the reduced equations

$$\begin{cases} D_t^\alpha f + \gamma^2 f_{XXX} - 4f^2 f_X + 2\gamma f_X g = 0, \\ 2ff_X + \gamma g_X = 0, \end{cases} \tag{86}$$

which can be further reduced to fractional ordinary differential equations by using Lie symmetry analysis method once again. The following generators are admitted by (86):

$$\Lambda_1 = \frac{\partial}{\partial X}, \quad \Lambda_2 = t\frac{\partial}{\partial t} + \frac{\alpha}{3}X\frac{\partial}{\partial X} - \frac{\alpha}{3}f\frac{\partial}{\partial f} - \frac{2\alpha}{3}g\frac{\partial}{\partial g}. \tag{87}$$

For $\Lambda_1 = \frac{\partial}{\partial X}$, similarly to Case 3, we can get another one trivial solution, i.e.,

$$u = \frac{C_0}{\Gamma(\alpha)}t^{\alpha-1}, \quad v = 0. \tag{88}$$

For $\Lambda_2 = t\frac{\partial}{\partial t} + \frac{\alpha}{3}X\frac{\partial}{\partial X} - \frac{\alpha}{3}f\frac{\partial}{\partial f} - \frac{2\alpha}{3}g\frac{\partial}{\partial g}$, Eqs. (86) have the following invarant solutions:

$$f(t, X) = t^{-\frac{\alpha}{3}}F(\sigma), \quad g(t, X) = t^{-\frac{2\alpha}{3}}G(\sigma), \quad \sigma = Xt^{-\frac{\alpha}{3}}. \tag{89}$$

Substituting (89) into Eqs. (86), we can obtain the following fractional ordinary differential equations:

$$\begin{cases} (\mathcal{P}_{\frac{3}{\alpha}}^{1-\frac{4\alpha}{3},\alpha}F)(\sigma) + \gamma^2 F_{\sigma\sigma\sigma} - 4F^2 F_\sigma + 2\gamma G F_\sigma = 0, \\ 2FF_\sigma + \gamma G_\sigma = 0. \end{cases} \tag{90}$$

The power series solution of (3.61) can be obtained as

$$F(\sigma) = \sum_{k=0}^{\infty} a_k \sigma^k, \quad G(\sigma) = \sum_{k=0}^{\infty} b_k \sigma^k, \tag{91}$$

where a_k and b_k are defined by $a_0 = f(0)$, $b_0 = g(0)$, $a_1 = f'(0)$, $a_2 = \frac{1}{2}f''(0)$ and

$$\begin{cases} b_{k+1} = -\frac{2}{(k+1)\gamma}\sum_{m+n=k}(n+1)a_m a_{n+1}, \\ a_{k+3} = \frac{-1}{(k+3)(k+2)(k+1)\gamma^2}\left[\frac{\Gamma(1-\frac{(k+1)\alpha}{3})}{\Gamma(1-\frac{(k+4)\alpha}{3})}a_k - 4\sum_{l+m+n=k}(n+1)a_l a_m a_{n+1} \right. \\ \left. \quad +2\gamma\sum_{m+n=k}(n+1)b_m a_{n+1}\right]. \end{cases} \tag{92}$$

That is, Eqs. (2) have the following power series solution:

$$\begin{aligned} u(t,x,y) =& f(t,X) = t^{-\frac{\alpha}{3}}F(\sigma) = a_0 t^{-\frac{\alpha}{3}} + a_1(y-\gamma x)t^{-\frac{2\alpha}{3}} + a_2(y-\gamma x)^2 t^{-\alpha} \\ &+ \sum_{k=0}^{\infty}\frac{-1}{(k+3)(k+2)(k+1)\gamma^2}\left[\frac{\Gamma(1-\frac{(k+1)\alpha}{3})}{\Gamma(1-\frac{(k+4)\alpha}{3})}a_k - 4\sum_{l+m+n=k}(n+1)\right. \\ &\left. a_l a_m a_{n+1} + 2\gamma\sum_{m+n=k}(n+1)b_m a_{n+1}\right](y-\gamma x)^{k+3}t^{-\frac{(k+4)\alpha}{3}}, \end{aligned} \tag{93a}$$

$$\begin{aligned} v(t,x,y) =& g(t,X) = t^{-\frac{2\alpha}{3}}G(\sigma) \\ =& b_0 t^{-\frac{2\alpha}{3}} + \sum_{k=0}^{\infty}\frac{-2}{(k+1)\gamma}\sum_{m+n=k}(n+1)a_m a_{n+1}(y-\gamma x)^{k+1}t^{-\frac{(k+3)\alpha}{3}}. \end{aligned} \tag{93b}$$

4 Conservation Laws of Eq. (2)

In this section, we will construct conservation laws of Eq. (2) by using the generalization of the Noether operators and the new conservation theorem [46, 47].

The system (2) is denoted as

$$\begin{cases} F_1 = D_t^{\alpha}u + u_{xxy} - 4u^2u_y - 2u_x v = 0, \\ F_2 = v_x - 2uu_y = 0, \end{cases} \tag{94}$$

of which the formal Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= p(t,x,y)F_1 + q(t,x,y)F_2 \\ &= p(t,x,y)\left(D_t^{\alpha}u + u_{xxy} - 4u^2u_y - 2u_x v\right) + q(t,x,y)\left(v_x - 2uu_y\right), \end{aligned} \tag{95}$$

where $p(t, x, y)$ and $q(t, x, y)$ are new dependent variables. The Euler–Lagrange operators are

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial(D_t^{\alpha}u)} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} - D_x^2 D_y\frac{\partial}{\partial u_{xxy}}, \tag{96}$$

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_x\frac{\partial}{\partial v_x}, \tag{97}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α. It is defined by the right-sided of Caputo fractional derivative, i.e.

$$(D_t^\alpha)^* f(t,x) \equiv {}_t^cD_T^\alpha f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^n}{\partial s^n} f(s,x)\mathrm{d}s, & n-1<\alpha<n, n\in\mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}. \end{cases}$$

The system of adjoint equations to (94) is given by

$$\begin{cases} F_1^* = \frac{\delta\mathcal{L}}{\delta u} = (D_t^\alpha)^* p + 2vp_x + 2v_x p + 4u^2 p_y + 2uq_y - p_{xxy} = 0, \\ F_2^* = \frac{\delta\mathcal{L}}{\delta v} = -2u_x p - q_x = 0. \end{cases} \tag{98}$$

Next we will use the above adjoint equations and the new conservation theorem to construct conservation laws of Eq. (2). From the classical definition of the conservation laws, a vector $C = (C^t, C^x, C^y)$ is called the conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x + D_y C^y]_{F_1,F_2=0} = 0$. We can obtain the components of the conserved vector by using the generalization of the Noether operators.

Firstly, from the fundamental operator identity, i.e.

$$prX + D_t\tau\cdot\mathcal{I} + D_x\xi\cdot\mathcal{I} + D_y\theta\cdot\mathcal{I} = W^u\cdot\frac{\delta}{\delta u} + W^v\cdot\frac{\delta}{\delta v} + D_t\mathcal{N}^t + D_x\mathcal{N}^x + D_y\mathcal{N}^y, \tag{99}$$

where prX is mentioned in (6), $\mathcal{I}$ is the identity operator and $W^u = \eta - \tau u_t - \xi u_x - \theta u_y$, $W^v = \zeta - \tau v_t - \xi v_x - \theta v_y$ are the characteristics for group generator X, we can get the Noether operators as follows:

$$\mathcal{N}^t = \tau\mathcal{I} + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W^u) D_t^k \frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J\left(W^u, D_t^n\frac{\partial}{\partial(D_t^\alpha u)}\right), \tag{100}$$

$$\mathcal{N}^x = \xi\mathcal{I} + W^u\left(\frac{\partial}{\partial u_x} + D_x D_y\frac{\partial}{\partial u_{xxy}}\right) + W^v\frac{\partial}{\partial v_x} - D_x(W^u)D_y\frac{\partial}{\partial u_{xxy}} + D_x D_y(W^u)\frac{\partial}{\partial u_{xxy}}, \tag{101}$$

$$\mathcal{N}^y = \theta\mathcal{I} + W^u\left(\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{yxx}}\right) - D_x(W^u)D_x\frac{\partial}{\partial u_{yxx}} + D_x^2(W^u)\frac{\partial}{\partial u_{yxx}}, \tag{102}$$

where $n = [\alpha] + 1$ and J is given by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x,y)g(\theta,x,y)}{(\theta-\tau)^{\alpha+1-n}}\mathrm{d}\theta\mathrm{d}\tau. \tag{103}$$

The components of conserved vector are defined by

$$C^t = \mathcal{N}^t\mathcal{L}, \quad C^x = \mathcal{N}^x\mathcal{L}, \quad C^y = \mathcal{N}^y\mathcal{L}. \tag{104}$$

Case 1: $X_1 = t\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial y} - \alpha v\frac{\partial}{\partial v}$

The characteristics of X_1 are

$$W^u = -tu_t - \alpha y u_y, \quad W^v = -\alpha v - tv_t - \alpha y v_y. \tag{105}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned} C^t &= pD_t^{\alpha-1}(W^u) + J(W^u, p_t) \\ &= -pD_t^{\alpha-1}(tu_t + \alpha y u_y) - J(tu_t + \alpha y u_y, p_t), \end{aligned} \tag{106}$$

$$\begin{aligned} C^x &= (p_{xy} - 2vp)W^u + qW^v - p_y D_x(W^u) + pD_x D_y(W^u) \\ &= (2vp - p_{xy})(tu_t + \alpha y u_y) - q(\alpha v + tv_t + \alpha y v_y) \\ &\quad + p_y(tu_{xt} + \alpha y u_{xy}) - p(tu_{xyt} + \alpha y u_{xyy}), \end{aligned} \tag{107}$$

$$\begin{aligned} C^y &= (p_{xx} - 4u^2 p - 2uq)W^u - p_x D_x(W^u) + pD_x^2(W^u) \\ &= (4u^2 p + 2uq - p_{xy})(tu_t + \alpha y u_y) + p_x(tu_{xt} + \alpha y u_{xy}) \\ &\quad - p(tu_{xxt} + \alpha y u_{xxy}). \end{aligned} \tag{108}$$

Case 2: $X_2 = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$

The characteristics of X_2 are

$$W^u = -u - xu_x + 2yu_y, \quad W^v = v - xv_x + 2yv_y. \tag{109}$$

Therefore, for $0 < \alpha < 1$,

$$\begin{aligned} C^t &= pD_t^{\alpha-1}(W^u) + J(W^u, p_t) \\ &= -pD_t^{\alpha-1}(u + xu_x - 2yu_y) - J(u + xu_x - 2yu_y, p_t), \end{aligned} \tag{110}$$

$$\begin{aligned} C^x &= (p_{xy} - 2vp)W^u + qW^v - p_y D_x(W^u) + pD_x D_y(W^u) \\ &= (2vp - p_{xy})(u + xu_x - 2yu_y) + q(v - xv_x + 2yv_y) \\ &\quad + p_y(2u_x + xu_{xx} - 2yu_{xy}) - p(xu_{xxy} - 2yu_{xyy}), \end{aligned} \tag{111}$$

$$\begin{aligned} C^y &= (p_{xx} - 4u^2 p - 2uq)W^u - p_x D_x(W^u) + pD_x^2(W^u) \\ &= (4u^2 p + 2uq - p_{xy})(u + xu_x - 2yu_y) + p_x(2u_x \\ &\quad + xu_{xx} - 2yu_{xy}) - p(3u_{xx} + xu_{xxx} - 2yu_{xxy}). \end{aligned} \tag{112}$$

Case 3: $X_3 = \frac{\partial}{\partial x}$

The characteristics of X_3 are

$$W^u = -u_x, \quad W^v = -v_x. \tag{113}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = pD_t^{\alpha-1}(W^u) + J(W^u, p_t) = -pD_t^{\alpha-1}u_x - J(u_x, p_t), \tag{114}$$

$$\begin{aligned} C^x &= (p_{xy} - 2vp)W^u + qW^v - p_y D_x(W^u) + pD_x D_y(W^u) \\ &= (2vp - p_{xy})u_x - qv_x + p_y u_{xx} - pu_{xxy}, \end{aligned} \tag{115}$$

$$\begin{aligned} C^y &= (p_{xx} - 4u^2 p - 2uq)W^u - p_x D_x(W^u) + pD_x^2(W^u) \\ &= (4u^2 p + 2uq - p_{xy})u_x + p_x u_{xx} - pu_{xxx}. \end{aligned} \tag{116}$$

Case 4: $X_4 = \frac{\partial}{\partial y}$

The characteristics of X_4 are

$$W^u = -u_y, \quad W^v = -v_y. \tag{117}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = pD_t^{\alpha-1}(W^u) + J(W^u, p_t) = -pD_t^{\alpha-1}u_y - J(u_y, p_t), \tag{118}$$

$$\begin{aligned} C^x &= (p_{xy} - 2vp)W^u + qW^v - p_y D_x(W^u) + pD_x D_y(W^u) \\ &= (2vp - p_{xy})u_y - qv_y + p_y u_{xy} - pu_{xyy}, \end{aligned} \tag{119}$$

$$\begin{aligned} C^y &= (p_{xx} - 4u^2 p - 2uq)W^u - p_x D_x(W^u) + pD_x^2(W^u) \\ &= (4u^2 p + 2uq - p_{xy})u_y + p_x u_{xy} - pu_{xxy}. \end{aligned} \tag{120}$$

5 Conclusion

This paper repeatedly uses the Lie symmetry analysis method to reduce the (2+1)-dimensional time fractional modified Bogoyavlenskii–Schiff equations to some (1+1)- and (0+1)- dimensional fractional differential equations. For the reduced fractional differential equations with Riemann–Liouville fractional derivative, we use the Laplace transform method and the invariant subspace method to obtain some analytical solutions. For the reduced fractional differential equations with Erdélyi–Kober fractional derivative, we use the power series method to obtain some power series solutions. The main results of this study are summarized in Table 3. This paper demonstrates that the Lie symmetry analysis method can effectively reduce the dimensionality of high-dimensional fractional differential equations until they are reduced to solvable equations. In addition, the nonlocal symmetries have gradually been applied to treat integer order differential equations [48–50]. Inspired by this, our next step is to study nonlocal symmetries for fractional differential equations.

Table 3 The reduced equations and exact solutions of Eq. (2)

Case	Invariant solution	Reduced equation	Exact solution
X_1	(3.26)	(3.27)	(3.28)
X_2	(3.30)	(3.31)	(3.35)(3.40)
X_3	(3.42)	(3.43)	(3.47)
X_4	(3.49)	(3.50)	(3.54)
$X_1 + \beta X_2$	(3.2)	(3.3)	(3.24)
$X_3 + \gamma X_4$	(3.56)	(3.57)	(3.59)(3.64)

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