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RESEARCH ARTICLE

Ricci Solitons on Riemannian Hypersurfaces Arising from Closed Conformal Vector Fields in Riemannian and Lorentzian Manifolds

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Abstract

This paper investigates Ricci solitons on Riemannian hypersurfaces in both Riemannian and Lorentzian manifolds. We provide conditions under which a Riemannian hypersurface, exhibiting specific properties related to a closed conformal vector field of the ambiant manifold, forms a Ricci soliton structure. The characterization involves a delicate balance between geometric quantities and the behavior of the conformal vector field, particularly its tangential component. We extend the analysis to ambient manifolds with constant sectional curvature and establish that, under a simple condition, the hypersurface becomes totally umbilical, implying constant mean curvature and sectional curvature. For compact hypersurfaces, we further characterize the nature of the Ricci soliton.

Keywords Ricci soliton · Conformal vector field · Hypersurfaces with constant mean curvature · Maximal hypersurfaces · de Sitter and anti-de Sitter spaces

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1 Introduction

The exploration of Ricci solitons on Riemannian and Lorentzian manifolds, with a specific focus on hypersurfaces, has been an active and dynamic area of study within both differential geometry and mathematical physics. Valuable insights into this field can be found in references such as [2, 11, 13, 15, 18, 20, 25].

Richard S. Hamilton's introduction of Ricci solitons in his seminal work on Ricci flow [19] has been instrumental in enhancing our understanding of the long-term behavior of the Ricci flow and its interplay with geometric structures. These solitons serve as self-similar solutions, providing valuable perspectives on the geometric and topological properties of manifolds.

The well-established connection between Ricci solitons and Ricci flows, as elucidated in [14], is widely recognized. A metric g on a manifold M induces a Ricci soliton on M if and only if there exists a positive function $\sigma(t)$ and a one-parameter family $\psi(t)$ of diffeomorphisms of M. The family of metrics, expressed as $g(t) = \sigma(t)\psi(t)^*g$, satisfies the Ricci flow equation:

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)},$$

with the initial condition $g = g(0)$. In this equation, $\psi(t)^*$ represents the pullback along the diffeomorphism $\psi(t)$, and $Ric_{g(t)}$ denotes the Ricci curvature of $g(t)$.

It is known that compact Ricci solitons and noncompact shrinking Ricci solitons fall under the category of gradient Ricci solitons (see [22, 24]). Trivial solitons, characterized by a constant function f, lead to (M, g) becoming an Einstein manifold.

The study of Ricci solitons on hypersurfaces gained momentum, focusing on investigating conditions under which hypersurfaces in Riemannian and Lorentzian manifolds can admit Ricci soliton structures (cf. [2, 9, 13]). Research has also delved into hypersurfaces with constant curvature and their correlation with Ricci solitons, including classification results and geometric properties (cf. [21]). The non-existence of gradient Ricci solitons are investigated in [4–8, 26, 27].

This paper focuses on Ricci solitons present on Riemannian hypersurfaces within both Riemannian and Lorentzian manifolds. Section 2 introduces preliminaries, covering basic concepts and fundamental formulas in the theories of Ricci solitons and Riemannian (or spacelike) hypersurfaces in Riemannian (or Lorentzian) manifolds. In Section 3, we provide an exhaustive list of totally umbilical Riemannian hypersurfaces in Riemannian and Lorentzian space forms. These hypersurfaces serve as models for Riemannian hypersurfaces that, under certain assumptions, can support Ricci solitons.

Section 4 presents the main results of the paper, focusing on characterizing the conditions under which a Riemannian hypersurface (or a spacelike hypersurface) in a Riemannian (or Lorentzian) manifold possessing a closed conformal vector field, exhibits a Ricci soliton structure. This characterization intricately involves the tangential component of the vector field, with conditions specifying a nuanced equilibrium between various geometric quantities and the behavior of the conformal vector field. The analysis extends to cases where the ambient manifold has constant sectional curvature. In such instances, we prove that the hypersurface is totally umbilical, exhibiting constant mean curvature and constant sectional curvature, aligning it with the models detailed in Section 3. We also consider the case where the hypersurface is compact.

2 Preliminaries

In this paper, we adopt to the convention introduced in [23] for defining the Riemannian tensor on an n-dimensional pseudo-Riemannian manifold (M, g). According to this formulation, the Riemannian tensor is characterized as the $(1, 3)$ tensor field, and it is given by:

$$R(X, Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$, the set of all vector fields on M.

Consider an orthonormal frame $E_1, ..., E_n$ on M, and set $\epsilon_i = g(E_i, E_i)$. The Ricci curvature tensor $Ric(X, Y)$ of M is expressed as:

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i g\left(R(X, E_i)Y, E_i\right), \tag{1}$$

for all $X, Y \in \mathfrak{X}(M)$. The scalar curvature S of M is defined as:

$$S = \sum_{i=1}^{n} \epsilon_i Ric(E_i, E_i). \tag{2}$$

The divergence of $X \in \mathfrak{X}(M)$ is given by:

$$div(X) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{E_i} X, E_i). \tag{3}$$

Additionally, a Ricci soliton is identified by the presence of a vector field X on M and a constant λ (called the soliton constant) that satisfy the equation

$$Ric + \frac{1}{2} L_X g = \lambda g, \tag{4}$$

where $L_X g$ is the Lie derivative of g in the direction of X.

A Ricci soliton (M, g, X, λ) is considered trivial when the vector field X is a Killing vector field. In such instances, the manifold (M, g) transforms into an Einstein manifold. A Ricci soliton (M, g, X, λ) is categorized as steady if $\lambda = 0$, shrinking if $\lambda > 0$, and expanding if $\lambda < 0$. In cases where X is expressible as ∇f for some function f on M, the Ricci soliton (M, g, f, λ) is called a gradient Ricci soliton. In this situation, Eq. 4 transforms into

$$Ric + \text{Hess } f = \lambda g, \tag{5}$$

where Hess f denotes the Hessian of f with respect to the metric g.

If (M, g, X, λ) is a Ricci soliton, then by taking the trace of Eq. 4, we derive

$$S + divX = n\lambda, \tag{6}$$

In the specific case of a gradient Ricci soliton (M, g, f, λ), Eq. 6 transforms into

$$S + \triangle f = n\lambda, \tag{7}$$

where $\triangle f$ is the Laplacian of f.

We conclude this preliminary section by providing pertinent equations related to hypersurfaces in both Riemannian and Lorentzian manifolds, which are essential for the content presented in this paper. To do so, we consider an orientable Riemannian manifold (M^n, g) of dimension $n \geq 3$, which is immersed isometrically as a hypersurface in a Riemannian or Lorentzian manifold $(M^{n+1}, \bar{g})$. When M^{n+1} takes on the Lorentzian metric, the corresponding hypersurface M^n is identified as a spacelike hypersurface.

Let $\mathfrak{X}(M^n)$ be the set of all vector fields tangent to M^n, and let N be a globally defined normal vector field on M^n, which is timelike when M^n is spacelike. Let ∇ and $\overline{\nabla}$ denote the Levi-Civita connections of (M^n, g) and $\left(M^{n+1}, \bar{g}\right)$, respectively, and let A be the shape operator of M^n with respect to N. The Gauss and Weingarten formulae for M^n as a hypersurface of M^{n+1} are expressed as:

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon g\left(A\left(X\right), Y\right) N, \tag{8}$$

$$A\left(X\right) = -\overline{\nabla}_X N, \tag{9}$$

for all $X, Y \in \mathfrak{X}(M^n)$, where $\epsilon = \bar{g}(N, N)$.

We note that the curvature tensor R of M^n can be expressed in terms of the curvature tensor $\bar{R}$ of M^{n+1} and the shape operator using the Gauss equation for the curvature tensor. This equation is given by

$$R(X, Y)Z = \left(\bar{R}(X, Y)Z\right)^{\top} + \epsilon\left(g(A(Y), Z)A(X) - g(A(X), Z)A(Y)\right), \tag{10}$$

for all $X, Y, Z \in \mathfrak{X}(M^n)$, where the symbol $\top$ refers to the tangential part on M^n.

Recalling that the mean curvature of M is defined to be

$$H = \frac{\epsilon}{n} trace(A), \tag{11}$$

it follows from Eq. 10 that the Ricci curvatures Ric and $\overline{Ric}$ of M^n and M^{n+1} are related as follows

$$Ric(X, Y) = \overline{Ric}(X, Y) - \epsilon\overline{g}(\overline{R}(N, X)Y, N) + g(A\left(X\right), nHY - \epsilon A\left(Y\right)), \tag{12}$$

for all $X, Y \in \mathfrak{X}(M^n)$.

Moreover, by tracing (12), we see that the scalar curvatures S and $\overline{S}$ of M^n and M^{n+1} are related as follows

$$S = \overline{S} - 2\epsilon \overline{Ric}(N, N) + \epsilon \left(n^2 H^2 - |A|^2\right). \tag{13}$$

3 Totally Umbilical Riemannian Hypersurfaces in Riemannian and Lorentzian Space Formsficci Solitons on Riemannian hypersurfaces

Let $\mathbf{R}^n_\nu$ denote the pseudo-Euclidean space of dimension n and signature $(\nu, n-\nu)$, that is, the vector space $\mathbf{R}^n$ endowed with the scalar product

$$< x, y >= -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j,$$

for all vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$.

For $\bar{c} > 0$, we define the pseudo-sphere of radius $\frac{1}{\sqrt{\bar{c}}}$ to be the hyperquadric

$$S^n_\nu(\bar{c}) = \left\{x \in \mathbf{R}^{n+1}_\nu :< x, x >= \frac{1}{\bar{c}}\right\}.$$

Similarly, for $\bar{c} < 0$, we define the pseudo-hyerbolic space of radius $\frac{1}{\sqrt{|\bar{c}|}}$ to be the hyperquadric

$$H^n_\nu(\bar{c}) = \left\{x \in \mathbf{R}^{n+1}_{\nu+1} :< x, x >= \frac{1}{\bar{c}}\right\}.$$

Take note that these spaces are totally umbilical hypersurfaces in the pseudo-Euclidean space $\mathbf{R}^{n+1}_\nu$. These are both geodesically complete and of constant sectional curvature $\bar{c}$. It is worth mentioning that spaces denoted as $S^n_1(\bar{c})$ and the universal covering of $H^n_1(\bar{c})$ are referred to as the *de Sitter* and *anti-de Sitter* spaces, respectively.

We now draw upon [1] (See also [3]) to quote the Riemannian hypersurfaces of Riemannian and Lorentzian space forms that are totally umbilical, serving as the models for our discussion.

1. Totally umbilical hypersurfaces of Euclidean space $\mathbf{R}^{n+1}$:

   (a) The totally geodesic hyperplane

   $$\mathbf{R}^n = \left\{x \in \mathbf{R}^{n+1} : x_1 = 0\right\}.$$

   (b) The sphere

   $$\mathbf{S}^n(c) = \left\{x \in \mathbf{R}^{n+1} :< x, x >= \frac{1}{c}\right\},$$

   which has shape operator $A = \pm\sqrt{c} I$ and constant sectional curvature $c > 0$.

2. Totally umbilical hypersurfaces of the sphere $\mathbf{S}^{n+1}(\bar{c}) \subset \mathbf{R}^{n+2}$:
   (a) The totally geodesic great hypersphere

$$\mathbf{S}^n(\bar{c}) = \left\{x \in \mathbf{S}^{n+1}(\bar{c}) : x_{n+2} = 0\right\},$$

   which has constant sectional curvature $\bar{c}$.
   (b) The small hypersphere

$$\mathbf{S}^n(c) = \left\{x \in \mathbf{S}^{n+1}(\bar{c}) : x_{n+2} = \sqrt{\frac{1}{\bar{c}} - \frac{1}{c}}\right\},$$

   which has shape operator $A = \pm\sqrt{c - \bar{c}}I$ and constant sectional curvature c, subject to the constraint $\bar{c} < c$.

3. Totally umbilical spacelike hypersurfaces of the hyperbolic space $\mathbf{H}^{n+1}(\bar{c}) \subset \mathbf{R}_1^{n+2}$:
   (a) The Euclidean space

$$\mathbf{R}^n = \left\{x \in \mathbf{H}^{n+1}(\bar{c}) : x_2 = \frac{1}{\sqrt{|\bar{c}|}} - x_1\right\},$$

   which has shape operator $A = \pm\sqrt{-\bar{c}}I$.
   (b) The sphere

$$\mathbf{S}^n(c) = \left\{x \in \mathbf{H}^{n+1}(\bar{c}) : x_1 = \sqrt{\frac{1}{c} - \frac{1}{\bar{c}}}\right\},$$

   which has shape operator $A = \pm\sqrt{c - \bar{c}}I$ and constant sectional curvature $c > 0$.
   (c) The hyperbolic space

$$\mathbf{H}^n(c) = \left\{x \in \mathbf{H}^{n+1}(\bar{c}) : x_{n+2} = \sqrt{\frac{1}{\bar{c}} - \frac{1}{c}}\right\},$$

   which has shape operator $A = \pm\sqrt{c - \bar{c}}I$ and constant sectional curvature c, subject to the constraint $\bar{c} \le c < 0$.

4. Totally umbilical spacelike hypersurfaces of Minkowski space $\mathbf{R}_1^{n+1}$:
   (a) The totally geodesic hyperplane

$$\mathbf{R}^n = \left\{x \in \mathbf{R}_1^{n+1} : x_1 = 0\right\}.$$

(b) The hyperbolic space

$$\mathbf{H}^n(c) = \left\{ x \in \mathbf{R}_1^{n+1} :< x, x >= \frac{1}{c} \right\},$$

which has shape operator $A = \pm\sqrt{-c}I$ and constant sectional curvature $c < 0$.

5. Totally umbilical spacelike hypersurfaces of the de Sitter space $\mathbf{S}_1^{n+1}(\bar{c}) \subset \mathbf{R}_1^{n+2}$:

(a) The Euclidean space

$$\mathbf{R}^n = \left\{ x \in \mathbf{S}_1^{n+1}(\bar{c}) : x_2 = \frac{1}{\sqrt{\bar{c}}} - x_1 \right\},$$

which has shape operator $A = \pm\sqrt{\bar{c}}I$. For $n \geq 3$, it can be proved that $\mathbf{R}^n$ can be represented through an isometric immersion $f : \mathbf{R}^n \to S_1^{n+1}(\bar{c})$ which is defined by the function

$$f(x) = \left(\frac{\sqrt{\bar{c}}}{2}|x|^2, \frac{1}{\sqrt{\bar{c}}} - \frac{\sqrt{\bar{c}}}{2}|x|^2, x \right),$$

as indicated in [16].

(b) The sphere

$$\mathbf{S}^n(c) = \left\{ x \in \mathbf{S}_1^{n+1}(\bar{c}) : x_1 = \sqrt{\frac{1}{c} - \frac{1}{\bar{c}}} \right\},$$

which has shape operator $A = \pm\sqrt{\bar{c} - c}I$ and constant sectional curvature c, subject to the constraint $0 < c \leq \bar{c}$.

(c) The hyperbolic space

$$\mathbf{H}^n(c) = \left\{ x \in \mathbf{S}_1^{n+1}(\bar{c}) : x_2 = \sqrt{\frac{1}{\bar{c}} - \frac{1}{c}} \right\},$$

which has shape operator $A = \pm\sqrt{\bar{c} - c}I$ and constant sectional curvature $c < 0$.

6. Totally umbilical spacelike hypersurfaces of the anti-de Sitter space $\mathbf{H}_1^{n+1}(\bar{c}) \subset \mathbf{R}_2^{n+2}$:

(a) The only totally umbilical spacelike hypersurface of $\mathbf{H}_1^{n+1}(\bar{c})$ is the hyperbolic space

$$\mathbf{H}^n(c) = \left\{ x \in \mathbf{H}_1^{n+1}(\bar{c}) : x_1 = \sqrt{\frac{1}{c} - \frac{1}{\bar{c}}} \right\},$$

which has shape operator $A = \pm\sqrt{\bar{c} - c}I$ and constant sectional curvature c, subject to the constraint $c \leq \bar{c} < 0$.

4 Ricci Solitons on Riemannian Hypersurfaces

Let $n \geq 3$, and consider an n-dimensional orientable Riemannian manifold (M^n, g) that is isometrically immersed as a hypersurface in either a Riemannian or Lorentzian manifold $\left(M^{n+1}, \bar{g}\right)$. Assume N is a unit normal vector field on M^n, supposed to be timelike when M^{n+1} is Lorentzian. Let $\bar{\xi}$ be a closed conformal vector field on M^{n+1}, meaning

$$\overline{\nabla}_X \bar{\xi} = \psi X, \tag{14}$$

for all $X \in \mathfrak{X}(M^{n+1})$, where ψ is a function on M^{n+1}, called the potential function of $\bar{\xi}$. The vector field $\bar{\xi}$ is timelike when the manifold M^{n+1} is Lorentzian.

If we define the restriction of $\bar{\xi}$ to M^n as ξ, it can be represented as follows:

$$\xi = \xi^T + \epsilon \theta N, \tag{15}$$

where $\theta =< \xi, N >$ is the support function on M^n, ξ^T is the tangential component of ξ, and $\epsilon = \bar{g}\,(N, N)$.

Now, because ξ is a closed conformal vector field, it becomes evident when utilizing (14) and (15) that

$$\nabla_X \xi^T = \psi X + \epsilon \theta A(X), \tag{16}$$

and

$$\nabla \theta = -A(\xi^T). \tag{17}$$

From Eq. 16, it is straightforward to derive

$$\mathrm{div} \xi^T = n(\psi + \theta H). \tag{18}$$

Before we present the theorem that characterizes Ricci solitons on Riemannian hypersurfaces in both Riemannian and Lorentzian manifolds, let us introduce certain notations. Let Q and $\bar{Q}$ be self-adjoint operators on M^n and M^{n+1}, respectively. Their definitions are such that $Ric(X, Y) = g(QX, Y)$ and $\overline{Ric}(X, Y) = \bar{g}(\bar{Q}X, Y)$. As the vector field $\bar{R}(N, X)N$ is tangent to M^n for all $X \in \mathfrak{X}(M^n)$, the *normal Jacobi operator* $R_N : TM^n \to TM^n$ is defined as follows:

$$R_N(X) = \bar{R}(N, X)N.$$

Ricci solitons have been demonstrated to occur on hypersurfaces (and more generally on submanifolds) of pseudo-Riemannian manifolds, as outlined in references [10, 12, 17].

Theorem 1 *Given an orientable Riemannian manifold (M^n, g) immersed isometrically as a hypersurface in either a Riemannian or Lorentzian manifold $(M^{n+1}, \bar{g})$ of dimension $n+1$, where $n \geq 3$, and assuming $\bar{\xi}$ is a closed conformal vector field of M^{n+1} (timelike if M^{n+1} is Lorentzian), with ψ as its potential function, and ξ as its restriction to M^n and ξ^T represent the tangential component of ξ. Let $\epsilon = \pm 1$ be defined as $\epsilon = \bar{g}(N, N)$, where N is a unit normal vector field on M^n. Then, the* *quadruple* (M^n, g, ξ^T, λ) *constitutes a Ricci soliton if and only if the equation below is satisfied:*

$$\epsilon A^2 - (\epsilon\theta + nH)A + (\lambda - \psi)I = \bar{Q} - \epsilon R_N. \tag{19}$$

Here, A, θ, H, λ, ψ, I, $\bar{Q}$, *and* R_N *hold their defined meanings within the context of the mathematical formulation provided.*

Proof Assume that (M^n, g, ξ^T, λ) is a Ricci soliton, satisfying the equation:

$$Ric + \frac{1}{2}L_{\xi^T} g = \lambda g. \tag{20}$$

For any pair of vector fields $X, Y \in \mathfrak{X}(M^n)$, utilizing (16), we derive the expression:

$$L_{\xi^T} g(X, Y) = 2g((\psi I + \epsilon\theta A)X, Y). \tag{21}$$

Additionally, Eq. 12 provides the following representation:

$$QX = \bar{Q}X - \epsilon\bar{R}(N, X)N + nHA(X) - \epsilon A^2(X) \tag{22}$$

By substituting (21) and (22) into (20), we obtain the expression labeled as Eq. 19. The converse is easily deduced. □

When the ambient manifold has constant sectional curvature, Theorem 1 leads to the following conclusion:

Theorem 2 *Let* (M^n, g) *be an orientable Riemannian manifold immersed isometrically as a hypersurface in either a Riemannian or Lorentzian manifold* $(M^{n+1}(\bar{c}), \bar{g})$ *having constant sectional curvature* $\bar{c}$ *and being of dimension* $n+1$, *with* $n \geq 3$. *Under the conditions and notations fulfilled in* Theorem 1, *it follows that if the quadruple* (M^n, g, ξ^T, λ) *constitutes a Ricci soliton, then the following equation is satisfied:*

$$|A|^2 - nH^2 + \epsilon n(\lambda - \psi - \theta H) - n(n-1)(H^2 + \epsilon\bar{c}) = 0, \tag{23}$$

or equivalently

$$|A|^2 - nH^2 + \epsilon S - n(n-1)(H^2 + \epsilon\bar{c}) = 0. \tag{24}$$

Here, A, H, *and* S *refer to the shape operator, mean curvature, and scalar curvature of the manifold* M^n, *respectively, as described earlier.*

Proof Since $M^{n+1}(\bar{c})$ is of constant curvature $\bar{c}$, then $\bar{Q} = n\bar{c}I$ and $R_N = \epsilon\bar{c}I$. Equation 19 becomes

$$\epsilon A^2 - (\epsilon\theta + nH)A + (\lambda - \psi - (n-1)\bar{c})I = 0.$$

By tracing the above equation, we get

$$\epsilon|A|^2 - \epsilon n(\epsilon\theta + nH)H + n\,(\lambda - \psi - (n-1)\bar{c}) = 0,$$

or equivalently

$$|A|^2 - nH^2 + \epsilon n(\lambda - \psi - \theta H) - n(n-1)(H^2 + \epsilon \bar{c}) = 0. \tag{25}$$

Given that ξ^T represents the potential field of (M^n, g, ξ^T, λ), according to Eq. 6, we can express this relationship as:

$$S + div\xi^T = n\lambda, \tag{26}$$

where S represents the scalar curvature of M^n. Substituting Eqs. 18 into 26, we obtain:

$$S = n(\lambda - \psi - \theta H),$$

Consequently, the modified form of Eq. 25 is:

$$|A|^2 - nH^2 + \epsilon S - n(n-1)(H^2 + \epsilon \bar{c}) = 0.$$

The results presented below are consequences of the formulas Eqs. 23 and 24 as specified in Theorem 2. The inequality presented in the following theorem's implication extends the scope of Theorem 4.2 in [2], which was originally established for hypersurfaces with constant mean curvature in Euclidean space, under the assumption that ξ serves as the position vector field.

Theorem 3 *Consider the manifolds* (M^n, g) *and* $(M^{n+1}(\bar{c}), \bar{g})$ *as defined in* Theorem 2, *with the additional assumption that* M^n *is compact. If the quadruple* (M^n, g, ξ^T, λ) *forms a non-trivial Ricci soliton, then it implies* $\epsilon\lambda \leq (n-1)(H^2 + \epsilon\bar{c})$, *with equality occurring if and only if* M^n *is a sphere* $\mathbf{S}^n(c)$, *where* $c = H^2 + \epsilon\bar{c} > 0$. *Notably, the Ricci soliton is shrinking when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Riemannian, while it becomes trivial when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Lorentzian.*

Proof If M^n is a compact manifold, by integrating (23) and considering (26), we get:

$$\int_{M^n} \left(nH^2 - |A|^2\right) dV = n \int_{M^n} \left(\epsilon\lambda - (n-1)(H^2 + \epsilon\bar{c})\right) dV. \tag{27}$$

By utilizing Schwartz's inequality, it follows that $\epsilon\lambda \leq (n-1)(H^2 + \epsilon\bar{c})$. Therefore, if equality is achieved, then M^n is totally umbilical. Referring to Lemma 35 on page 116 in [23], we can deduce that H is constant and M^n has a constant sectional curvature $c = \bar{c} + \epsilon H^2$. As M^n is compact, we conclude that M^n is a sphere with a constant positive curvature $c = H^2 + \epsilon\bar{c}$. This implies that $\epsilon\lambda > 0$. Consequently, when $\epsilon = 1$, the Ricci soliton is shrinking, whereas it is expanding when $\epsilon = -1$. Since it is compact and expanding, it necessarily qualifies as a trivial Ricci soliton (cf. [18]). □

The following corollary arises as a consequence of Theorem 3.

Corollary 1 *Consider the manifolds* (M^n, g) *and* $(M^{n+1}(\bar{c}), \bar{g})$ *as defined in* Theorem 2*, with the additional assumptions that* M^n *is compact and* $H^2+\epsilon\bar{c} \le 0$*. If the quadruple* (M^n, g, ξ^T, λ) *forms a Ricci soliton, then it is shrinking when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Lorentzian, while it becomes trivial when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Riemannian.*

Proof In line with Theorem 3, it follows that $\epsilon\lambda \le (n-1)(H^2+\epsilon\bar{c})$. Therefore, if the condition $H^2+\epsilon\bar{c} \le 0$ is assumed, it now implies $\epsilon\lambda \le 0$. The case $\lambda = 0$ is clearly ruled out. Consequently, with $\epsilon\lambda < 0$, for $\epsilon = -1$, the Ricci soliton displays a shrinking behavior, while for $\epsilon = 1$, it expands and becomes trivial. The shift to triviality is attributed to the compactness of M^n, as mentioned in the proof of Theorem 3. □

Theorem 4 *Consider the manifolds* (M^n, g) *and* $(M^{n+1}(\bar{c}), \bar{g})$ *as defined in* Theorem 2*, and assume in addition that* M^n *is connceted and geodesically complete. If the quadruple* (M^n, g, ξ^T, λ) *constitutes a Ricci soliton and satisfies the condition* $\epsilon S \ge n(n-1)(H^2+\epsilon\bar{c})$*, then the mean curvature* H *of* M^n *is constant, and* M^n *has a constant sectional curvature* $c = \bar{c} + \epsilon H^2$*. More specifically,* M^n *can be identified as either a Euclidean space* $\mathbf{R}^n$*, a sphere* $\mathbf{S}^n(c)$*, or a hyperbolic space* $\mathbf{H}^n(c)$*, with all possibilities detailed in* Section 3.

Proof The condition $\epsilon S \ge n(n-1)(H^2+\epsilon\bar{c})$ implies $|A|^2 - nH^2 = 0$, which results directly from Eq. 24. This leads to the conclusion that M^n is totally umbilical, by application of Schwartz's inequality. Similarly to what is stated in the proof of Theorem 3, we conclude that H is constant and M^n has a constant sectional curvature $c = \bar{c} + \epsilon H^2$. Now, it is clear that M^n is either a Euclidean space $\mathbf{R}^n$, a sphere $\mathbf{S}^n(c)$, or a hyperbolic space $\mathbf{H}^n(c)$, with all possibilities detailed in Section 3. □

The following theorem generalizes Theorem 4.3 in [2], established for hypersurfaces in Euclidean space, under the assumption that ξ serves as the position vector field.

Theorem 5 *Consider* (M^n, g) *and* $(M^{n+1}(\bar{c}), \bar{g})$ *as defined in* Theorem 2*, and assume in addition that* M^n *is compact. If the quadruple* (M^n, g, ξ^T, λ) *forms a Ricci soliton and satisfies the condition* $\epsilon\lambda \ge (n-1)(H^2+\epsilon\bar{c})$*, then it follows that* M^n *is a sphere* $\mathbf{S}^n(c)$*, with* $c = H^2 + \epsilon\bar{c} > 0$*. Notably, the Ricci soliton is shrinking when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Riemannian, while it becomes trivial when* $(M^{n+1}(\bar{c}), \bar{g})$ *is Lorentzian.*

Proof If M is compact, we encounter the same integral formula Eq. 27 as presented in the proof of Theorem 3.

The condition $\epsilon\lambda \ge (n-1)(H^2+\epsilon\bar{c})$ indicates that $|A|^2 - nH^2 = 0$. This also implies $\lambda = (n-1)(\bar{c}+\epsilon H^2)$ and that M^n is totally umbilical, resulting in the same conclusion as observed in the proof of Theorem 3. □

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Declarations

Conflicts of interest We declare that we have no competing interests.

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