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RESEARCH ARTICLE

Weighted Sobolev Type Inequalities in a Smooth Metric Measure Space

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Abstract

In this paper, we obtain weighted Sobolev type inequalities with explicit constants that extend the inequalities obtained by Guo et al. (Math Res Lett 28(5):1419–1439, 2021) in the Riemannian setting. As an application, we prove some new logarithmic Sobolev type inequalities in some smooth metric measure spaces.

Keywords Sobolev inequality · Reilly type formula · Smooth metric measure space · Weighted Laplacian

Mathematics Subject Classification 53C21 · 58J32

1 Introduction

The study of Sobolev inequalities with sharp constants has a long tradition in analysis and geometry. For example, on the unit sphere S^{n-1} endowed with its standard metric, Escobar [7] classified all positive solutions of

$$\begin{cases} \Delta u = 0 & in\ B^n, \\ u_\nu + \frac{n-2}{2} u = u^{\frac{n}{n-2}} & on\ S^{n-1}, \end{cases} \tag{1.1}$$

by an integral method and hence [8] proved that, for all $u \in C^\infty(\overline{B}^n)$,

$$\mathrm{vol}(S^{n-1})^{\frac{1}{n-1}} \left(\int_{S^{n-1}} u^{\frac{2(n-1)}{n-2}} dv \right)^{\frac{n-2}{n-1}} \leq \frac{2}{n-2} \int_{B^n} |\nabla u|^2 dv + \int_{S^{n-1}} u^2 dv, \tag{1.2}$$

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where dv and $\mathrm{vol}(S^{n-1})$ are respectively the Riemannian measure and the Riemannian volume of S^{n-1}. This inequality plays an important role in the study of the Yamabe problem on Riemannian manifolds. Note that, using harmonic analysis, Beckner [1] derived a family of inequalities

$$\mathrm{vol}(S^{n-1})^{\frac{q-1}{q+1}}\left(\int_{S^{n-1}} u^{q+1}dv\right)^{\frac{2}{q+1}} \le (q-1)\int_{B^n} |\nabla u|^2 dv + \int_{S^{n-1}} u^2 dv, \forall u \in C^\infty(\overline{B}^n), \tag{1.3}$$

provided $1 < q < \infty$, if $n = 2$, and $1 < q \le \frac{n}{n-2}$, if $n \ge 3$. The corresponding Euler–Lagrange equation to (1.3) is

$$\begin{cases} \Delta u = 0 & in\ B^n, \\ u_\nu + \frac{1}{q-1}u = u^q & on\ S^{n-1}. \end{cases} \tag{1.4}$$

It is apparent that the case $n \ge 2$ and $q = \frac{n}{n-2}$ of (1.3) and (1.4) are just (1.2) and (1.1) respectively. Also, in the same paper, Beckner [1] confirmed

$$\mathrm{vol}(S^{n-1})^{\frac{q-1}{q+1}}\left(\int_{S^{n-1}} u^{q+1}dv\right)^{\frac{2}{q+1}} \le \frac{q-1}{n-1}\int_{S^{n-1}} |\nabla u|^2 dv + \int_{S^{n-1}} u^2 dv, \forall u \in C^\infty(S^{n-1}), \tag{1.5}$$

provided $1 < q < \infty$, if $n = 2$ or 3, and $1 < q \le \frac{n+1}{n-3}$, if $n \ge 3$. By considering the Euler–Lagrange equation (1.4) and using integral methods, Bidaut-Véron and Véron [2] were able to give another proof of the inequality (1.5). More results about the Sobolev inequalities on the unit sphere can be found in [5, 10, 11].

Guo et al. [9] recently generalized the spherical inequality (1.3) to any smooth compact Riemannian manifolds with nonnegative sectional curvature and strictly convex boundary. They proved the following Sobolev inequality.

Theorem 1.1 [9] *Let (M^n, g) be a smooth compact Riemannian manifold with non-negative sectional curvature and $II \ge 1$ on the boundary ∂M. Assume $2 \le n \le 8$ and $1 < q \le \frac{4n}{5n-9}$. Then for any $u \in C^\infty(M)$, we have*

$$\left(\frac{1}{\mathrm{vol}(\partial M)}\int_{\partial M} |u|^{q+1}\, d\Sigma\right)^{\frac{2}{q+1}} \le \frac{q-1}{\mathrm{vol}(\partial M)}\int_M |\nabla u|^2\, d\Omega + \frac{1}{\mathrm{vol}(\partial M)}\int_{\partial M} u^2\, d\Sigma. \tag{1.6}$$

In the limiting case Theorem 1.1 implies the following logarithmic Sobolev inequality.

Corollary 1.2 [9] *Let (M^n, g) be a smooth compact Riemannian manifold with non-negative sectional curvature and $II \ge 1$ on the boundary ∂M. Assume $2 \le n \le 8$. Then for any $u \in C^\infty(M)$ with $\frac{1}{\mathrm{vol}(\partial M)}\int_{\partial M} u^2\, d\Sigma = 1$, we have*

$$\frac{1}{\mathrm{vol}(\partial M)}\int_{\partial M} |u|^2 \log u^2\, d\Sigma \le \frac{2}{\mathrm{vol}(\partial M)}\int_M |\nabla u|^2\, d\Omega.$$

The purpose of the present paper is to adapt the technique that has been used in [9] to the setting of smooth metric measure spaces with nonnegative sectional curvature and strictly convex boundary. We generalize Theorem 1.1 and Corollary 1.2 to the smooth metric measure space. Let us fix some required notations before stating our results.

Let (M, g) be a smooth compact n-dimensional Riemannian manifold and ϕ be a $C^2(M)$ function. We denote ∇, Δ and ∇^2 the gradient, Laplacian and Hessian operator on M with respect to g, respectively. Ric and R denote Ricci curvature and scalar curvature, respectively. An n-dimensional *smooth metric measure space* $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ is a smooth compact n-dimensional Riemannian manifold (M, g) endowed with a weighted measure $e^{-\phi}\, d\Omega$ and $d\Omega$ is the Riemannian volume element of the metric g. On a smooth metric measure space $(M, g, d\sigma = e^{-\phi}\, d\Omega)$, we let

$$\mathrm{Ric}_\phi = \mathrm{Ric} + \nabla^2\phi, \tag{1.7}$$

stand for the Bakry–Émery Ricci curvature which is also called ∞-Bakry–Émery Ricci curvature, i.e., the $m = \infty$ case of the following m-Bakry–Émery Ricci curvature defined by

$$\mathrm{Ric}_\phi^m = \mathrm{Ric}_\phi - \frac{1}{m-n}\nabla\phi \otimes \nabla\phi \tag{1.8}$$

with some constant $m \geq n$, and $m = n$ if and only if ϕ is a constant. The equation $\mathrm{Ric}_\phi = \lambda g$ for some constant λ is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow. The equation $\mathrm{Ric}_\phi^m = \lambda g$ corresponds to the quasi-Einstein equation [4], which has been studied by many mathematicians. In recent years, the smooth metric measure space received much attention from many mathematicians, see [6, 12–16, 19–22, 24–26] and the references therein.

Let $\boldsymbol{\nu}$ be the unit outward normal of ∂M. Define the second fundamental form of ∂M by $II(X, Y) = \langle \nabla_X \boldsymbol{\nu}, Y\rangle$ for any two tangent vector fields X and Y on M, and the mean curvature by $H = tr(II)$. The f-mean curvature (see [22, p. 398]) at a point $x \in M$ with respect to $\boldsymbol{\nu}$ is given by $H_\phi(x) = H(x) - \langle \nabla\phi(x), \boldsymbol{\nu}(x)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric g.

On $(M, g, d\sigma = e^{-\phi}\, d\Omega)$, we consider the weighted Laplacian as follows:

$$\mathbb{L}_\phi \cdot := e^\phi div(e^{-\phi}\nabla \cdot) = \Delta - g(\nabla\phi, \nabla \cdot), \tag{1.9}$$

where ∇ denotes the Levi-Civita connection, $div = tr(\nabla \cdot)$ denotes the Riemannian divergence operator, and $\Delta = div\nabla$ is the Laplace–Beltrami operator. Notice that the Green formula (the integration by parts formula)

$$\begin{aligned}\int_M h\mathbb{L}_\phi u \, d\sigma &= \int_{\partial M} hu_{\boldsymbol{\nu}}\, dv - \int_M \langle \nabla u, \nabla h\rangle \, d\sigma \\ &= \int_{\partial M} (hu_{\boldsymbol{\nu}} - uh_{\boldsymbol{\nu}})\, dv + \int_M u\mathbb{L}_\phi h \, d\sigma\end{aligned}$$

holds provided u or h belongs to $C^2(M)$, where $u_{\boldsymbol{\nu}} = \langle \boldsymbol{\nu}, \nabla u\rangle$, and $dv = e^{-\phi}\, d\Sigma$ and $d\Sigma$ is the volume form on ∂M.

The following is one of our main results.

Theorem 1.3 *Let* $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ *be a smooth compact metric measure space with nonnegative sectional curvature and* $II \geq c$ *for a positive constant* c *on the boundary* ∂M. *Let* ϕ *be a potential function such that* $\nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi \geq 0$ *on* M. *Assume* $2 \leq m \leq 8$ *and* $1 < q \leq \frac{4m}{5m-9}$. *Then for any* $u \in C^\infty(M)$, *we have*

$$\left(\frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} |u|^{q+1}\, dv\right)^{\frac{2}{q+1}} \leq \frac{q-1}{c\big(\mathrm{vol}_\phi(\partial M)\big)} \int_M |\nabla u|^2\, d\sigma + \frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} u^2\, dv, \tag{1.10}$$

where $\mathrm{vol}_\phi(\partial M)$ *is the weighted area of* ∂M.

Note that Theorem 1.3 recovers Theorem 1.1 obtained by Guo–Hang–Wang. Moreover, Theorem 1.3 implies the logarithmic type Sobolev inequality.

Corollary 1.4 *Let* $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ *be a smooth compact metric measure space with nonnegative sectional curvature and* $II \geq c$ *for a positive constant* c *on the boundary* ∂M. *Let* ϕ *be a potential function such that* $\nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi \geq 0$ *on* M. *Assume* $2 \leq m \leq 8$. *Then for any* $u \in C^\infty(M)$ *with* $\frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} u^2\, dv = 1$, *we have*

$$\frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} |u|^2 \log u^2\, dv \leq \frac{2}{c\big(\mathrm{vol}_\phi(\partial M)\big)} \int_M |\nabla u|^2\, d\sigma.$$

The proof of Theorem 1.3 is based on the uniqueness results, which we state in this setting as follows.

Theorem 1.5 *Let* $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ *be a smooth compact metric measure space with boundary* ∂M. *Assume that the sectional curvature is nonnegative on* M, *and the second fundamental form* $II \geq c$ *for a positive constant* c *on* ∂M. *Let* ϕ *be a potential function such that* $\nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi \geq 0$ *on* M. *Let* u *be a positive solution of the following system*:

$$\begin{cases} \mathbb{L}_\phi u = 0 & \text{in} \quad M, \\ u_\nu + \lambda u = u^q & \text{on} \quad \partial M. \end{cases} \tag{1.11}$$

Then the only positive solution to the Eq. (1.11) *is constant if* $\lambda \leq \frac{c}{q-1}$, *provided* $2 \leq m \leq 8$ *and* $1 < q \leq \frac{4m}{5m-9}$.

If we take $\phi = constant$, then Theorem 1.5 becomes Theorem 2 proved by Guo et al. [9].

The rest of this paper is organized as follows. In Sect. 2, we establish some elementary lemmas (Lemmas 2.1, 2.2, 2.3). The uniqueness results (Theorem 1.5) are discussed in Sect. 3. Finally, Sect. 4 is dedicated to the proof of Theorem 1.3 and Corollary 1.4, respectively.

2 Preliminaries

In this section, we drive some useful lemmas that will be used later.

Lemma 2.1 (Weighted Reilly formula) *Let $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ be a smooth compact metric measure space with boundary ∂M and $V : M \to R$ be a twice differential function. Given a smooth function f on M, we have*

$$\begin{aligned}
&\int_M \Big(-J + \mathrm{Ric}^m_\phi(\nabla f, \nabla f)\Big) V\, d\sigma + \int_M \Big(1 - \frac{1}{m}\Big)(\mathbb{L}_\phi f)^2 V\, d\sigma \\
&\quad = \int_M \big(\nabla^2 V(\nabla f, \nabla f) - |\nabla f|^2 \mathbb{L}_\phi V + \mathrm{Ric}^m_\phi(\nabla f, \nabla f) V\big)\, d\sigma \\
&\qquad + \int_{\partial M} V\Big(2 f_\nu \overline{\mathbb{L}_\phi} f + H_\phi f_\nu^2 + II(\overline{\nabla} f, \overline{\nabla} f)\Big)\, dv + \int_{\partial M} V_\nu |\overline{\nabla} f|^2\, dv,
\end{aligned} \tag{2.1}$$

where $J := |\nabla^2 f|^2 - \frac{1}{m}\big(\mathbb{L}_\phi f\big)^2 + \mathrm{Ric}_\phi(\nabla f, \nabla f)$, $\overline{\mathbb{L}_\phi}\cdot = \overline{\Delta} - g(\overline{\nabla}\phi, \overline{\nabla}\cdot)$, $\overline{\nabla}$ *and* $\overline{\Delta}$ *are respectively the gradient operator and the Laplace operator on* ∂M.

Proof Now in the calculations that follow (at a point $x \in \partial M$), we will use an orthonormallocal frame $\{e_1, \dots, e_n\}$ such that $e_1, \dots, e_{n-1}$ are tangent to the boundary ∂M and $e_n = \boldsymbol{\nu}$ is the outward unit normal to ∂M.

We use the integration by parts and the Ricci identity to derive that

$$\begin{aligned}
\int_M V|\nabla^2 f|^2\, d\sigma &= \int_M V \sum_{i,j=1}^n f_{ij} f_{ij} e^{-\phi}\, d\Omega \\
&= \int_{\partial M} V \sum_{i=1}^n f_{i\nu} f_i e^{-\phi}\, d\Sigma - \int_M \sum_{i,j=1}^n V_j f_{ij} f_i e^{-\phi}\, d\Omega - \int_M V \sum_{i,j=1}^n f_{ijj} f_i e^{-\phi}\, d\Omega \\
&\quad + \int_M V \sum_{i,j=1}^n f_{ij} f_i \phi_j e^{-\phi}\, d\Omega \\
&= \int_{\partial M} V \sum_{i=1}^n f_{i\nu} f_i e^{-\phi}\, d\Sigma - \int_M \sum_{j=1}^n V_j \Big(\frac{1}{2}|\nabla f|^2\Big)_j e^{-\phi}\, d\Omega \\
&\quad - \int_M V \sum_{i=1}^n \Bigg((\Delta f)_i + \sum_{j=1}^n R_{ij} f_j\Bigg) f_i e^{-\phi}\, d\Omega + \int_M V \sum_{i,j=1}^n f_{ij} f_i \phi_j e^{-\phi}\, d\Omega.
\end{aligned} \tag{2.2}$$

Using the integration by parts again, we obtain

$$
\begin{aligned}
&-\int_M \sum_{j=1}^n V_j\left(\frac{1}{2}|\nabla f|^2\right)_j e^{-\phi}\,d\Omega \\
&= -\frac{1}{2}\int_{\partial M} V_\nu |\nabla f|^2 e^{-\phi}\,d\Sigma + \frac{1}{2}\int_M \sum_{j=1}^n V_{jj}|\nabla f|^2 e^{-\phi}\,d\Omega - \frac{1}{2}\int_M \sum_{j=1}^n V_j\phi_j |\nabla f|^2 e^{-\phi}\,d\Omega \\
&= -\frac{1}{2}\int_{\partial M} V_\nu |\nabla f|^2 e^{-\phi}\,d\Sigma + \frac{1}{2}\int_M \left(\mathbb{L}_\phi V\right)|\nabla f|^2 e^{-\phi}\,d\Omega
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
&-\int_M V \sum_{i=1}^n \left((\Delta f)_i + \sum_{j=1}^n R_{ij} f_j\right) f_i e^{-\phi}\,d\Omega \\
&= -\int_M V \sum_{i=1}^n \left(\left(\mathbb{L}_\phi f + \sum_{j=1}^n \phi_j f_j\right)_i + \sum_{j=1}^n R_{ij} f_j\right) f_i e^{-\phi}\,d\Omega \\
&= -\int_M V \sum_{i=1}^n \left(\left(\mathbb{L}_\phi f\right)_i + \sum_{j=1}^n \left(f_{ji}\phi_j + \phi_{ji} f_j\right) + \sum_{j=1}^n R_{ij} f_j\right) f_i e^{-\phi}\,d\Omega.
\end{aligned}
\tag{2.4}
$$

Inserting (2.3) and (2.4) into (2.2), we attain

$$
\begin{aligned}
&\int_M V|\nabla^2 f|^2\,d\sigma \\
&= \int_{\partial M} V \sum_{i=1}^n f_{i\nu} f_i e^{-\phi}\,d\Sigma - \frac{1}{2}\int_{\partial M} V_\nu |\nabla f|^2 e^{-\phi}\,d\Sigma \\
&\quad + \frac{1}{2}\int_M \left(\mathbb{L}_\phi V\right)|\nabla f|^2 e^{-\phi}\,d\Omega + \int_M V \sum_{i,j=1}^n f_{ij} f_i \phi_j e^{-\phi}\,d\Omega \\
&\quad - \int_M V \sum_{i=1}^n \left(\left(\mathbb{L}_\phi f\right)_i + \sum_{j=1}^n \left(f_{ji}\phi_j + \phi_{ji} f_j\right) + \sum_{j=1}^n R_{ij} f_j\right) f_i e^{-\phi}\,d\Omega.
\end{aligned}
\tag{2.5}
$$

By the integration by parts, we have

$$
\begin{aligned}
&-\int_M V\sum_{i=1}^n \left(\mathbb{L}_\phi f\right)_i f_i e^{-\phi}\, d\Omega \\
&= -\int_{\partial M} V(\mathbb{L}_\phi f) f_\nu e^{-\phi}\, d\Sigma + \int_M (\mathbb{L}_\phi f)\sum_{i=1}^n V_i f_i e^{-\phi}\, d\Omega + \int_M V(\mathbb{L}_\phi f)\sum_{i=1}^n f_{ii} e^{-\phi}\, d\Omega \\
&\quad - \int_M V(\mathbb{L}_\phi f)\sum_{i=1}^n f_i\phi_i e^{-\phi}\, d\Omega.
\end{aligned}
\tag{2.6}
$$

Substituting (2.6) into (2.5), we can get

$$
\begin{aligned}
&\int_M V|\nabla^2 f|^2\, d\sigma \\
&= \int_{\partial M} V\sum_{i=1}^n f_{i\nu} f_i e^{-\phi}\, d\Sigma - \frac{1}{2}\int_{\partial M} V_\nu |\nabla f|^2 e^{-\phi}\, d\Sigma + \frac{1}{2}\int_M (\mathbb{L}_\phi V)|\nabla f|^2 e^{-\phi}\, d\Omega \\
&\quad - \int_{\partial M} V(\mathbb{L}_\phi f) f_\nu e^{-\phi}\, d\Sigma + \int_M (\mathbb{L}_\phi f)\sum_{i=1}^n V_i f_i e^{-\phi}\, d\Omega + \int_M V(\mathbb{L}_\phi f)\sum_{i=1}^n f_{ii} e^{-\phi}\, d\Omega \\
&\quad - \int_M V(\mathbb{L}_\phi f)\sum_{i=1}^n f_i\phi_i e^{-\phi}\, d\Omega - \int_M V\sum_{i,j=1}^n (R_{ij}+\phi_{ij}) f_i f_j e^{-\phi}\, d\Omega \\
&= \int_{\partial M}\left(V\sum_{i=1}^n f_{i\nu} f_i - \frac{1}{2}V_\nu|\nabla f|^2 - V f_\nu \mathbb{L}_\phi f\right)e^{-\phi}\, d\Sigma + \frac{1}{2}\int_M (\mathbb{L}_\phi V)|\nabla f|^2 e^{-\phi}\, d\Omega \\
&\quad + \int_M \mathbb{L}_\phi f\sum_{i=1}^n V_i f_i e^{-\phi}\, d\Omega + \int_M V(\mathbb{L}_\phi f)^2 e^{-\phi}\, d\Omega - \int_M \mathrm{Ric}_\phi(\nabla f,\nabla f) e^{-\phi}\, d\Omega.
\end{aligned}
\tag{2.7}
$$

We easily infer from (2.7) the following:

$$
\begin{aligned}
&\int_M V\left(\left(\mathbb{L}_\phi f\right)^2 - |\nabla^2 f|^2\right) d\sigma \\
&= \int_{\partial M}\left(-V\sum_{i=1}^n f_i f_{i\nu} + \frac{1}{2}V_\nu|\nabla f|^2 + V(\mathbb{L}_\phi f) f_\nu\right)e^{-\phi}\, d\Sigma \\
&\quad + \int_M\left(-\frac{1}{2}(\mathbb{L}_\phi V)|\nabla f|^2 - (\mathbb{L}_\phi f)\sum_{i=1}^n V_i f_i + V\mathrm{Ric}_\phi(\nabla f,\nabla f)\right)e^{-\phi}\, d\Omega.
\end{aligned}
\tag{2.8}
$$

Applying the integration by parts, we get

$$
\begin{aligned}
&-\int_M (\mathbb{L}_\phi f) \sum_{i=1}^n V_i f_i e^{-\phi}\, d\Omega \\
&\quad = -\int_{\partial M} f_\nu \sum_{i=1}^n V_i f_i e^{-\phi}\, d\Sigma + \int_M \left(\sum_{i,j=1}^n V_{ij} f_i f_j + \sum_{i=1}^n V_i \left(\frac{1}{2}|\nabla f|^2\right)_i \right) e^{-\phi}\, d\Omega \\
&\quad = \int_{\partial M} \left(-f_\nu \sum_{i=1}^n V_i f_i + \frac{1}{2}|\nabla f|^2 V_\nu \right) e^{-\phi}\, d\Sigma + \int_M \left(\sum_{i,j=1}^n V_{ij} f_i f_j - \frac{1}{2}(\mathbb{L}_\phi V)|\nabla f|^2 \right) e^{-\phi}\, d\Omega.
\end{aligned}
\tag{2.9}
$$

Taking (2.9) into (2.8), we have

$$
\begin{aligned}
&\int_M V\left(\left(\mathbb{L}_\phi f\right)^2 - |\nabla^2 f|^2\right) d\sigma \\
&\quad = \int_{\partial M} \left(-V \sum_{i=1}^n f_{i\nu} f_i + V_\nu |\nabla f|^2 + V(\mathbb{L}_\phi f) f_\nu - f_\nu \sum_{i=1}^n V_i f_i \right) e^{-\phi}\, d\Sigma \\
&\qquad + \int_M \left(-\frac{1}{2}(\mathbb{L}_\phi V)|\nabla f|^2 + \sum_{i,j=1}^n V_{ij} f_i f_j - \frac{1}{2}(\mathbb{L}_\phi V)|\nabla f|^2 + V \mathrm{Ric}_\phi(\nabla f, \nabla f) \right) e^{-\phi}\, d\Omega.
\end{aligned}
\tag{2.10}
$$

Note that $H_\phi = H - \langle \nabla\phi, \boldsymbol{\nu} \rangle$. From the Gauss–Weingarten formula

$$
\Delta f = H f_\nu + \overline{\Delta} f + f_{\nu\nu} \tag{2.11}
$$

and

$$
\langle \nabla f, \nabla f_\nu \rangle = f_\nu f_{\nu\nu} - II(\overline{\nabla} f, \overline{\nabla} f) + \langle \overline{\nabla} f, \overline{\nabla} f_\nu \rangle, \tag{2.12}
$$

we have

$$
\begin{aligned}
&\int_{\partial M} V\left((\mathbb{L}_\phi f) f_\nu - \sum_{i=1}^n f_{i\nu} f_i \right) e^{-\phi}\, d\Sigma \\
&\quad = \int_{\partial M} V\left(\sum_{i=1}^n (f_{ii} - f_i \phi_i) f_\nu - \sum_{i=1}^n f_{i\nu} f_i \right) e^{-\phi}\, d\Sigma \\
&\quad = \int_{\partial M} V\left(\sum_{i=1}^{n-1} \left(f_{ii} - f_i \phi_i\right) f_\nu - f_\nu^2 \phi_\nu - \sum_{i=1}^{n-1} f_{i\nu} f_i \right) e^{-\phi}\, d\Sigma.
\end{aligned}
\tag{2.13}
$$

Putting (2.11) and (2.12) into (2.13), we get

$$
\begin{aligned}
&\int_{\partial M} V\left((\mathbb{L}_\phi f)f_\nu - \sum_{i=1}^{n} f_{i\nu}f_i\right)e^{-\phi}\,d\Sigma \\
&= \int_{\partial M} V\left(\left(Hf_\nu + \overline{\Delta} f + f_{\nu\nu} - \sum_{i=1}^{n-1} f_i\phi_i\right)f_\nu - f_\nu^2\phi_\nu - f_\nu f_{\nu\nu}\right)e^{-\phi}\,d\Sigma \\
&\quad + \int_{\partial M} V\left(II(\overline{\nabla} f, \overline{\nabla} f) - \langle \overline{\nabla} f, \overline{\nabla} f_\nu\rangle\right)e^{-\phi}\,d\Sigma \\
&= \int_{\partial M} V\left(\left(\overline{\Delta} f - \sum_{i=1}^{n-1} f_i\phi_i\right)f_\nu + (Hf_\nu^2 - f_\nu^2\phi_\nu) + II(\overline{\nabla} f, \overline{\nabla} f) - \langle \overline{\nabla} f, \overline{\nabla} f_\nu\rangle\right)e^{-\phi}\,d\Sigma \\
&= \int_{\partial M} V\left(f_\nu(\overline{\mathbb{L}_\phi} f) + H_\phi f_\nu^2 - \langle \overline{\nabla} f_\nu, \overline{\nabla} f\rangle + II(\overline{\nabla} f, \overline{\nabla} f)\right)e^{-\phi}\,d\Sigma.
\end{aligned}
\tag{2.14}
$$

Again applying the integration by parts shows that

$$
\begin{aligned}
&\int_{\partial M}\left(|\nabla f|^2 V_\nu - f_\nu \sum_{i=1}^{n} V_i f_i\right)e^{-\phi}\,d\Sigma \\
&= \int_{\partial M}\left(|\overline{\nabla} f|^2 V_\nu - f_\nu\langle \overline{\nabla} V, \overline{\nabla} f\rangle\right)e^{-\phi}\,d\Sigma \\
&= \int_{\partial M}\left(|\overline{\nabla} f|^2 V_\nu + V\langle \overline{\nabla} f_\nu, \overline{\nabla} f\rangle + Vf_\nu(\overline{\mathbb{L}_\phi} f)\right)e^{-\phi}\,d\Sigma.
\end{aligned}
\tag{2.15}
$$

Then combining (2.14) and (2.15), we obtain

$$
\begin{aligned}
&\int_{\partial M}\left(-V\sum_{i=1}^{n} f_{i\nu}f_i + V_\nu|\nabla f|^2 + V\mathbb{L}_\phi f f_\nu - f_\nu\sum_{i=1}^{n} V_i f_i\right)dv \\
&= \int_{\partial M} V\left(2f_\nu(\overline{\mathbb{L}_\phi} f) + H_\phi f_\nu^2 + II(\overline{\nabla} f, \overline{\nabla} f)\right)dv + \int_{\partial M} V_\nu|\overline{\nabla} f|^2\,dv.
\end{aligned}
\tag{2.16}
$$

Substituting (2.16) into (2.10), we can get

$$
\begin{aligned}
&\int_M V\left(\left(\mathbb{L}_\phi f\right)^2 - |\nabla^2 f|^2 - \mathrm{Ric}_\phi(\nabla f, \nabla f)\right)d\sigma \\
&= \int_{\partial M} V\left(2f_\nu\overline{\mathbb{L}_\phi} f + H_\phi f_\nu^2 + II(\overline{\nabla} f, \overline{\nabla} f)\right)dv \\
&\quad + \int_{\partial M} V_\nu|\overline{\nabla} f|^2\,dv + \int_M\left(\nabla^2 V - \left(\mathbb{L}_\phi V\right)g\right)(\nabla f, \nabla f)\,d\sigma,
\end{aligned}
\tag{2.17}
$$

which is equivalent to

$$
\begin{aligned}
&\int_M V\Big(\frac{1}{m}\big(\mathbb{L}_\phi f\big)^2 - |\nabla^2 f|^2 - \mathrm{Ric}_\phi(\nabla f, \nabla f)\Big)\, d\sigma \\
&\quad + \int_M V\Big(\mathrm{Ric}_\phi^m(\nabla f, \nabla f) + \Big(1 - \frac{1}{m}\Big)\big(\mathbb{L}_\phi f\big)^2\Big)\, d\sigma \\
&\qquad = \int_{\partial M} V\big(2 f_\nu \overline{\mathbb{L}_\phi} f + H_\phi f_\nu^2 + II(\overline{\nabla} f, \overline{\nabla} f)\big)\, dv + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) V\, d\sigma \\
&\quad + \int_{\partial M} V_\nu |\overline{\nabla} f|^2\, dv + \int_M \big(\nabla^2 V - (\mathbb{L}_\phi V) g\big)(\nabla f, \nabla f)\, d\sigma.
\end{aligned}
\tag{2.18}
$$

Therefore, by the definition of J, we conclude the proof of Lemma 2.1. □

Let $(M, g, d\sigma = e^{-\phi}\, d\Omega)$ be a smooth compact metric measure space with boundary ∂M and $f \in C^\infty(M)$ be a *positive* function. Let ω be another smooth function on M satisfying the following boundary conditions

$$
\omega|_{\partial M} = 0, \qquad \omega_\nu|_{\partial M} = -1. \tag{2.19}
$$

We take $V = f^b \omega$, for any $b \in \mathbb{R}$.

Lemma 2.2

$$
\begin{aligned}
&\int_M \Big(\Big(1 - \frac{1}{m}\Big)(\mathbb{L}_\phi f)^2 - J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\Big) f^b \omega\, d\sigma \\
&\quad + \int_M \frac{b}{2} \omega f^{b-2} |\nabla f|^2 \big(3 f \mathbb{L}_\phi f + (b-1)|\nabla f|^2\big)\, d\sigma \\
&\qquad = \int_M \Big(f^b \nabla^2 \omega(\nabla f, \nabla f) - |\nabla f|^2 f^b \mathbb{L}_\phi \omega - \frac{b}{2} |\nabla f|^2 f^{b-1} \langle \nabla f, \nabla \omega\rangle\Big)\, d\sigma \\
&\quad + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b \omega\, d\sigma - \int_{\partial M} f^b |\overline{\nabla} f|^2\, dv.
\end{aligned}
\tag{2.20}
$$

Proof A direct calculation gives

$$
\begin{aligned}
\nabla V &= f^b \nabla \omega + b \omega f^{b-1} \nabla f, \\
\nabla^2 V &= f^b \nabla^2 \omega + b f^{b-1}(df \otimes d\omega + d\omega \otimes df) + b \omega f^{b-1} \nabla^2 f \\
&\quad + b(b-1) \omega f^{b-2} df \otimes df, \\
\mathbb{L}_\phi V &= f^b \mathbb{L}_\phi \omega + 2 b f^{b-1} \langle \nabla f, \nabla \omega \rangle + b \omega f^{b-1} \mathbb{L}_\phi f + b(b-1) \omega f^{b-2} |\nabla f|^2, \\
\nabla^2 V(\nabla f, \nabla f) &= f^b \nabla^2 \omega(\nabla f, \nabla f) + 2 b f^{b-1} |\nabla f|^2 \langle \nabla f, \nabla \omega \rangle \\
&\quad + b \omega f^{b-1} \nabla^2 f(\nabla f, \nabla f) + b(b-1) \omega f^{b-2} |\nabla f|^4.
\end{aligned}
$$

Plugging these equations into (2.1), we have

$$
\begin{aligned}
&\int_M \Big(1-\frac{1}{m}\Big)(\mathbb{L}_\phi f)^2 f^b \omega\, d\sigma + \int_M \Big(-J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\Big) f^b \omega\, d\sigma \\
&\quad = \int_M f^b \nabla^2 \omega(\nabla f, \nabla f) + b\omega f^{b-1} \nabla^2 f(\nabla f, \nabla f)\, d\sigma \\
&\qquad - \int_M |\nabla f|^2 \big(f^b \mathbb{L}_\phi \omega + b\omega f^{b-1} \mathbb{L}_\phi f\big)\, d\sigma + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b \omega\, d\sigma \\
&\qquad + \int_{\partial M} f^b \omega \Big(2 f_\nu \overline{\mathbb{L}_\phi} f + H_\phi f_\nu^2 + II(\overline{\nabla} f, \overline{\nabla} f)\Big)\, dv + \int_{\partial M} (f^b \omega_\nu + b\omega f^{b-1} f_\nu) |\overline{\nabla} f|^2\, dv.
\end{aligned} \tag{2.21}
$$

Using the boundary conditions for ω in (2.19) yields

$$
\begin{aligned}
&\int_M \Big(1-\frac{1}{m}\Big)(\mathbb{L}_\phi f)^2 f^b \omega\, d\sigma + \int_M \Big(-J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\Big) f^b \omega\, d\sigma \\
&\quad = \int_M \big(f^b \nabla^2 \omega(\nabla f, \nabla f) + b\omega f^{b-1} \nabla^2 f(\nabla f, \nabla f)\big)\, d\sigma + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b \omega\, d\sigma \\
&\qquad - \int_M |\nabla f|^2 \big(f^b \mathbb{L}_\phi \omega + b\omega f^{b-1} \mathbb{L}_\phi f\big)\, d\sigma - \int_{\partial M} f^b |\overline{\nabla} f|^2 dv.
\end{aligned} \tag{2.22}
$$

Using the integration by parts and $\omega|_{\partial M} = 0$, we get

$$
\begin{aligned}
&b\int_M \omega f^{b-1}\nabla^2 f(\nabla f,\nabla f)\,d\sigma\\
&\quad= b\int_M \omega f^{b-1}\sum_{i,j=1}^n f_{ij}f_i f_j e^{-\phi}\,d\Omega\\
&\quad= \frac{b}{2}\int_M \omega f^{b-1}\sum_{j=1}^n \left(|\nabla f|^2\right)_j f_j e^{-\phi}\,d\Omega\\
&\quad= \frac{b}{2}\int_M \left(\sum_{j=1}^n \left(\omega f^{b-1}f_j|\nabla f|^2 e^{-\phi}\right)_j - \sum_{j=1}^n (\omega f^{b-1}f_j)_j|\nabla f|^2 e^{-\phi}\right)d\Omega\\
&\qquad+ \frac{b}{2}\int_M \sum_{j=1}^n \phi_j e^{-\phi}(\omega f^{b-1}f_j)|\nabla f|^2\,d\Omega\\
&\quad= \frac{b}{2}\int_{\partial M}\omega f^{b-1}f_\nu|\nabla f|^2\,dv - \frac{b}{2}\int_M\sum_{j=1}^n\left(\omega_j f^{b-1}f_j + (b-1)\omega f^{b-2}f_j^2\right)|\nabla f|^2 e^{-\phi}\,d\Omega\\
&\qquad- \frac{b}{2}\int_M\sum_{j=1}^n \omega f^{b-1}f_{jj}|\nabla f|^2 e^{-\phi}\,d\Omega + \frac{b}{2}\int_M |\nabla f|^2\omega f^{b-1}\langle\nabla\phi,\nabla f\rangle\,d\sigma\\
&\quad= \frac{b}{2}\int_{\partial M}\omega f^{b-1}f_\nu|\nabla f|^2\,dv - \frac{b}{2}\int_M f^{b-1}|\nabla f|^2\langle\nabla\omega,\nabla f\rangle + (b-1)f^{b-2}\omega|\nabla f|^4\,d\sigma\\
&\qquad- \frac{b}{2}\int_M \omega f^{b-1}|\nabla f|^2\Delta f - f^{b-1}\omega|\nabla f|^2\langle\nabla\phi,\nabla f\rangle\,d\sigma\\
&\quad= -\frac{b}{2}\int_M\left(\omega f^{b-1}|\nabla f|^2\mathbb{L}_\phi f + (b-1)f^{b-2}\omega|\nabla f|^4 + f^{b-1}|\nabla f|^2\langle\nabla\omega,\nabla f\rangle\right)d\sigma.
\end{aligned}
\tag{2.23}
$$

Substituting (2.23) into (2.22), we attain

$$
\begin{aligned}
&\int_M\left(1-\frac{1}{m}\right)(\mathbb{L}_\phi f)^2 f^b\omega d\sigma + \int_M\left(-J + \mathrm{Ric}_\phi^m(\nabla f,\nabla f)\right)f^b\omega\,d\sigma\\
&\quad= \int_M\Bigg(f^b\nabla^2\omega(\nabla f,\nabla f) - |\nabla f|^2 f^b\mathbb{L}_\phi\omega - \frac{b}{2}\omega f^{b-2}|\nabla f|^2\left(3f\mathbb{L}_\phi f + (b-1)|\nabla f|^2\right)\\
&\qquad- \frac{b}{2}|\nabla f|^2 f^{b-1}\langle\nabla f,\nabla\omega\rangle + \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\omega\Bigg)d\sigma - \int_{\partial M} f^b|\overline{\nabla} f|^2 dv.
\end{aligned}
$$

Reorganizing yields the desired equality (2.20). Therefore the proof of Lemma 2.2 is completed. □

Lemma 2.3 (*Weighted Pohozave identity*)

$$
\begin{aligned}
&\int_M\left[f^b\nabla^2\omega(\nabla f,\nabla f) + f^{b-1}(f\mathbb{L}_\phi f + \frac{b}{2}|\nabla f|^2)\langle\nabla f,\nabla\omega\rangle - \frac{1}{2}f^b|\nabla f|^2\mathbb{L}_\phi\omega\right]d\sigma\\
&\quad= \frac{1}{2}\int_{\partial M} f^b(|\overline{\nabla} f|^2 - f_\nu^2)\,dv.
\end{aligned}
\tag{2.24}
$$

Proof For any smooth vector field $\nabla\omega$, we can get

$$\begin{aligned}&\operatorname{div}\Big(e^{-\phi}\Big(\langle\nabla f,\nabla\omega\rangle\nabla f-\frac{1}{2}|\nabla f|^2\nabla\omega\Big)\Big)\\&\quad=e^{-\phi}\Big(\nabla^2\omega(\nabla f,\nabla f)+(\mathbb{L}_\phi f)\langle\nabla f,\nabla\omega\rangle-\frac{1}{2}|\nabla f|^2\mathbb{L}_\phi\omega\Big).\end{aligned}$$

Note that $\nabla\omega=-\boldsymbol{\nu}$ on ∂M. Multiplying both sides of the above identity by f^b and integrating yields

$$\begin{aligned}&\int_M f^b e^{-\phi}\Big(\nabla^2\omega(\nabla f,\nabla f)+(\mathbb{L}_\phi f)\langle\nabla f,\nabla\omega\rangle-\frac{1}{2}|\nabla f|^2\mathbb{L}_\phi\omega\Big)\,d\Omega\\&=\int_M f^b\operatorname{div}\Big(e^{-\phi}\Big(\langle\nabla f,\nabla\omega\rangle\nabla f-\frac{1}{2}|\nabla f|^2\nabla\omega\Big)\Big)\,d\Omega\\&=\int_M f^b\sum_{j=1}^n\left(e^{-\phi}\sum_{i=1}^n\Big(f_i\omega_i f_j-\frac{1}{2}f_i^2\omega_j\Big)\right)_j d\Omega\\&=-\int_M bf^{b-1}\sum_{j=1}^n f_j\left(e^{-\phi}\sum_{i=1}^n f_i\omega_i f_j-\frac{1}{2}\sum_{i=1}^n f_i^2\omega_j e^{-\phi}\right)d\Omega\\&\quad+\int_M\sum_{j=1}^n\left(f^b\sum_{i=1}^n\Big(e^{-\phi}f_i\omega_i f_j-\frac{1}{2}f_i^2\omega_j e^{-\phi}\Big)\right)_j d\Omega\\&=\int_{\partial M}\Big[f^b f_\nu\langle\nabla f,\nabla\omega\rangle-\frac{1}{2}f^b|\nabla f|^2\omega_\nu\Big]\,dv-\int_M bf^{b-1}\langle\nabla f,\nabla\omega\rangle|\nabla f|^2\,d\sigma\\&\quad+\frac{1}{2}\int_M|\nabla f|^2\langle\nabla f,\nabla\omega\rangle\,d\sigma\\&=\int_{\partial M}f^b\Big(-f_\nu^2+\frac{1}{2}|\nabla f|^2\Big)\,dv-\frac{b}{2}\int_M f^{b-1}\langle\nabla f,\nabla\omega\rangle|\nabla f|^2\,d\sigma.\end{aligned}\tag{2.25}$$

By a straightforward computation, we obtain

$$\frac{1}{2}\int_{\partial M}f^b|\nabla f|^2\,dv=\frac{1}{2}\int_{\partial M}f^b(|\overline{\nabla}f|^2+f_\nu^2)\,dv.\tag{2.26}$$

Substituting (2.26) into (2.25), we get

$$\begin{aligned}&\int_M f^b\operatorname{div}\Big(e^{-\phi}\Big(\langle\nabla f,\nabla\omega\rangle\nabla f-\frac{1}{2}|\nabla f|^2\nabla\omega\Big)\Big)\,d\Omega\\&\quad=\frac{1}{2}\int_{\partial M}f^b(|\overline{\nabla}f|^2-f_\nu^2)\,dv-\frac{b}{2}\int_M f^{b-1}\langle\nabla f,\nabla\omega\rangle|\nabla f|^2\,d\sigma.\end{aligned}$$

Thus, we have

$$\int_M f^b\nabla^2\omega(\nabla f,\nabla f)+(f\mathbb{L}_\phi f+\frac{b}{2}|\nabla f|^2)f^{b-1}\langle\nabla f,\nabla\omega\rangle-\frac{1}{2}f^b|\nabla f|^2\mathbb{L}_\phi\omega\,d\sigma$$
$$=\frac{1}{2}\int_{\partial M}f^b(|\overline{\nabla}f|^2-f_\nu^2)\,dv,$$

which completes the proof of Lemma 2.3. □

Lemma 2.4 [17] *Let u be a smooth function on M. For every $m\geq n$, we have*

$$|\nabla^2u|^2+\mathrm{Ric}_\phi(\nabla u,\nabla u)\geq\frac{1}{m}(\mathbb{L}_\phi u)^2+\mathrm{Ric}_\phi^m(\nabla u,\nabla u).\tag{2.27}$$

Moreover, the equality in (2.27) holds if and only if

$$\nabla^2u=-\frac{1}{n}(\Delta u)g\quad\text{and}\quad\mathbb{L}_\phi u=\frac{m}{m-n}\langle\nabla\phi,\nabla u\rangle.$$

3 Proof of Theorem 1.5

In this section, we shall prove Theorem 1.5.

Proof of Theorem 1.5 Let u be a positive solution of (1.11). Let a be a nonzero real number to be determined later and take $u=f^{-a}$. Then $f>0$ satisfies the following equation

$$\begin{cases}\mathbb{L}_\phi f=(a+1)f^{-1}|\nabla f|^2 & in\ M,\\ f_\nu=\frac{1}{a}(\lambda f-f^{1+a-aq}) & on\ \partial M.\end{cases}\tag{3.1}$$

For any $s\in\mathbb{R}$, multiplying the first equation in (3.1) by f^s and integrating over M yields

$$(a+s+1)\int_M|\nabla f|^2f^{s-1}d\sigma=\int_{\partial M}f^sf_\nu\,dv.\tag{3.2}$$

Inserting (3.1) into (2.20), we obtain

$$\begin{aligned}&\int_M\left((1-\frac{1}{m})(a+1)^2+\frac{b(3a+b+2)}{2}\right)f^{b-2}|\nabla f|^4\omega d\sigma\\&+\int_M\left(-J+\mathrm{Ric}_\phi^m(\nabla f,\nabla f)\right)f^b\omega\,d\sigma\\&\quad=\int_M\left(f^b\nabla^2\omega(\nabla f,\nabla f)-|\nabla f|^2f^b\mathbb{L}_\phi\omega-\frac{b}{2}f^{b-1}|\nabla f|^2\langle\nabla f,\nabla\omega\rangle\right)d\sigma\\&+\int_M\mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\omega\,d\sigma-\int_{\partial M}f^b|\overline{\nabla}f|^2\,dv.\end{aligned}\tag{3.3}$$

Taking (3.1) into (2.24), we have

$$\int_M \Big(f^b\nabla^2\omega(\nabla f,\nabla f)+(a+1+\frac{b}{2})f^{b-1}|\nabla f|^2\langle\nabla f,\nabla\omega\rangle-\frac{1}{2}f^b|\nabla f|^2\mathbb{L}_\phi\omega\Big)\,d\sigma$$
$$=\frac{1}{2}\int_{\partial M} f^b(|\overline{\nabla}f|^2-f_\nu^2)\,dv. \tag{3.4}$$

In order to eliminate the term

$$\frac{b}{2}\int_M |\nabla f|^2 f^{b-1}\langle\nabla f,\nabla\omega\rangle\,d\sigma,$$

we multiply (3.4) by $\frac{\frac{b}{2}}{a+1+\frac{b}{2}}$ to get

$$\begin{aligned}
&\frac{b}{2}\int_M f^{b-1}|\nabla f|^2\langle\nabla f,\nabla\omega\rangle\,d\sigma\\
&\quad=\frac{\frac{b}{2}}{a+1+\frac{b}{2}}\left(\frac{1}{2}\int_{\partial M} f^b(|\overline{\nabla}f|^2-f_\nu^2)\,dv+\frac{1}{2}\int_M f^b|\nabla f|^2\mathbb{L}_\phi\omega\,d\sigma\right)\\
&\qquad-\frac{\frac{b}{2}}{a+1+\frac{b}{2}}\int_M f^b\nabla^2\omega(\nabla f,\nabla f)\,d\sigma\\
&\quad=\frac{\frac{b}{4}}{a+1+\frac{b}{2}}\int_{\partial M} f^b(|\overline{\nabla}f|^2-f_\nu^2)\,dv-\frac{\frac{b}{2}}{a+1+\frac{b}{2}}\int_M f^b\nabla^2\omega(\nabla f,\nabla f)\,d\sigma\\
&\qquad+\frac{\frac{b}{4}}{2\left(a+1+\frac{b}{2}\right)}\int_M f^b|\nabla f|^2\mathbb{L}_\phi\omega\,d\sigma.
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.3), we can get

$$\begin{aligned}
&\left[\left(1-\frac{1}{m}\right)(a+1)^2+\frac{b(3a+b+2)}{2}\right]\int_M f^{b-2}|\nabla f|^4\omega\,d\sigma-\int_M f^b\omega J\,d\sigma\\
&\quad+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\omega\,d\sigma\\
&\quad=\int_M\left(\frac{a+1+b}{a+1+\frac{b}{2}}f^b\nabla^2\omega(\nabla f,\nabla f)-\frac{a+1+\frac{3}{4}b}{a+1+\frac{b}{2}}|\nabla f|^2f^b\mathbb{L}_\phi\omega\right)d\sigma\\
&\quad+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\omega\,d\sigma+\int_{\partial M}\left(\frac{\frac{1}{4}b}{a+1+\frac{b}{2}}f^bf_\nu^2-\frac{a+1+\frac{3}{4}b}{a+1+\frac{b}{2}}f^b|\overline{\nabla}f|^2\right)dv.
\end{aligned}$$

We choose $b=-\frac{4}{3}(a+1)$. Then

$$\frac{[5m-9-(m+9)a](a+1)}{9m}\int_M f^{b-2}|\nabla f|^4\omega\,d\sigma+\int_M\left(-J+\mathrm{Ric}_\phi^m(\nabla f,\nabla f)\right)f^b\omega\,d\sigma$$
$$=-\int_M f^b\nabla^2\omega(\nabla f,\nabla f)\,d\sigma+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\omega\,d\sigma-\int_{\partial M}f^bf_\nu^2\,dv. \tag{3.6}$$

□

By arguing as in [23], we consider a weight function $\psi := \rho - c\frac{\rho^2}{2}$, where $\rho = d(\cdot,\partial M)$ denotes the distance function to the boundary ∂M. Notice that ψ is smooth near ∂M and satisfies

$$\psi|_{\partial M}=0, \qquad \psi_\nu=-1.$$

From now on we assume that M has nonnegative sectional curvature and $II \geq c$ for a positive constant c on ∂M. By the Hessian comparison theorem [18] $\rho \leq \frac{1}{c}$ hence $\psi \geq 0$ and $-\nabla^2\psi \geq cg$ in the support sense. To overcome the difficulty that ψ is not smooth, we also need

Proposition 3.1 [23] *Fix a neighborhood C of $Cut(\partial M)$ in the interior of M, with $Cut(\partial M)$ the cut-locus of points at the boundary ∂M. Then for any $\varepsilon > 0$, there exists a smooth nonnegative function ψ_ε on M such that $\psi_\varepsilon = \psi$ on $M\backslash C$ and $-\nabla^2\psi_\varepsilon \geq (c-\varepsilon)g$.*

Taking the weight $\omega = \psi_\varepsilon$ in (3.6) yields

$$\begin{aligned}
&\frac{[5m-9-(m+9)a](a+1)}{9m}\int_M f^{b-2}|\nabla f|^4\psi_\varepsilon\,d\sigma+\int_M\left(-J+\mathrm{Ric}_\phi^m(\nabla f,\nabla f)\right)f^b\psi_\varepsilon\,d\sigma\\
&=-\int_C f^b\nabla^2\psi_\varepsilon(\nabla f,\nabla f)\,d\sigma-\int_{M\backslash C}f^b\nabla^2\psi_\varepsilon(\nabla f,\nabla f)\,d\sigma\\
&\quad+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\psi_\varepsilon\,d\sigma-\int_{\partial M}f^bf_\nu^2\,dv\\
&\geq (c-\varepsilon)\int_C f^b|\nabla f|^2\,d\sigma-\int_{M\backslash C}f^b\nabla^2\psi(\nabla f,\nabla f)\,d\sigma\\
&\quad+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\psi_\varepsilon\,d\sigma-\int_{\partial M}f^bf_\nu^2\,dv.
\end{aligned}$$

By letting $\varepsilon \to 0$ and shrinking the neighborhood, we get the following

$$\begin{aligned}
&\frac{[5m-9-(m+9)a](a+1)}{9m}\int_M f^{b-2}|\nabla f|^4\psi\,d\sigma+\int_M\left(-J+\mathrm{Ric}_\phi^m(\nabla f,\nabla f)\right)f^b\psi\,d\sigma\\
&\geq c\int_C f^b|\nabla f|^2\,d\sigma-\int_{M\backslash C}f^b\nabla^2\psi(\nabla f,\nabla f)\,d\sigma+\int_M \mathrm{Ric}_\phi^m(\nabla f,\nabla f)f^b\psi\,d\sigma\\
&\quad-\int_{\partial M}f^bf_\nu^2\,dv.
\end{aligned}$$

Since the function ψ is smooth and $-\nabla^2\psi \geq cg$ on $M\backslash C$, we obtain

$$\frac{[5m-9-(m+9)a](a+1)}{9m}\int_M f^{b-2}|\nabla f|^4\psi\, d\sigma + \int_M \left(-J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\right) f^b\psi\, d\sigma$$
$$\geq c\int_M f^b|\nabla f|^2\, d\sigma + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b\psi\, d\sigma - \int_{\partial M} f^b f_\nu^2\, dv.$$

Applying (3.2) and the boundary condition for f in (3.1), we have

$$\begin{aligned}
&\frac{[5m-9-(m+9)a](a+1)}{9m}\int_M f^{b-2}|\nabla f|^4\psi\, d\sigma + \int_M \left(-J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\right) f^b\psi\, d\sigma\\
&\geq \int_M cf^b|\nabla f|^2\, d\sigma + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b\psi\, d\sigma - \frac{1}{a}\int_{\partial M}\left(\lambda f^{b+1} - f^{b+1+a-aq}\right) f_\nu\, dv\\
&= \int_M \left(cf^b|\nabla f|^2 - \frac{(a+b+2)\lambda}{a} f^b|\nabla f|^2 + \frac{2a+b+2-aq}{a} f^{b+a-aq}|\nabla f|^2\right) d\sigma\\
&\quad + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b\psi\, d\sigma\\
&= \int_M \left(\left(c - \frac{\lambda(2-a)}{3a}\right) f^b|\nabla f|^2 + \left(\frac{2}{3} - q + \frac{2}{3a}\right) f^{b+a-aq}|\nabla f|^2\right) d\sigma\\
&\quad + \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b\psi\, d\sigma,
\end{aligned} \tag{3.7}$$

which is equivalent to

$$\begin{aligned}
&A\int_M f^{b-2}|\nabla f|^4\psi\, d\sigma + B\int_M f^b|\nabla f|^2\, d\sigma\\
&\quad + C\int_M f^{b+a-aq}|\nabla f|^2\, d\sigma + \int_M \left(-J + \mathrm{Ric}_\phi^m(\nabla f, \nabla f)\right) f^b\psi\, d\sigma\\
&\qquad \geq \int_M \mathrm{Ric}_\phi^m(\nabla f, \nabla f) f^b\psi\, d\sigma,
\end{aligned} \tag{3.8}$$

where, with $x = a^{-1}$

$$\begin{aligned}
A &= \frac{[5m-9-(m+9)a](a+1)}{9m} = \frac{[(5m-9)x-(m+9)](x+1)}{9mx^2},\\
B &= \frac{\lambda(2-a)}{3a} - c = \frac{\lambda}{3}(2x-1) - c,\\
C &= q - \frac{2}{3} - \frac{2}{3a} = q - \frac{2}{3} - \frac{2}{3}x.
\end{aligned}$$

By choosing a such that $A, B, C \leq 0$, we can get

$$\begin{aligned}\left(x-\frac{m+9}{5m-9}\right)(x+1)&\le 0,\\ \frac{\lambda}{3}(2x-1)-c&\le 0,\\ q-\frac{2}{3}-\frac{2}{3}x&\le 0.\end{aligned}$$

Direct computation gives

$$\begin{aligned}-1\le x&\le \frac{m+9}{5m-9},\\ \frac{3}{2}q-1\le x&\le \frac{3}{2}\frac{c}{\lambda}+\frac{1}{2}.\end{aligned}$$

The selection is possible when $\frac{3}{2}q-1\le x\le \frac{3}{2}\frac{c}{\lambda}+\frac{1}{2}$ and $\frac{3}{2}q-1\le \frac{m+9}{5m-9}\le 0$, that is, $(q-1)\lambda\le c$ and $q\le \frac{4m}{5m-9}$. Since $q>1$ we must have $2\le m\le 8$. Then when $q\le \frac{4m}{5m-9}$ and $(q-1)\lambda\le c$, take $\frac{1}{a}=\frac{3}{2}q-1$, we get

$$\begin{aligned}&C=0,\\ &B=(q-1)\lambda-c\le 0,\\ &A=\frac{5m-9}{6m}q\left(\frac{3}{2}q-1\right)^{-2}\left(\frac{3}{2}q-1-\frac{m+9}{5m-9}\right)\le 0.\end{aligned}$$

Therefore, the left hand side of (3.8) is nonpositive while the right hand side is nonnegative. Thus, both sides of (3.8) are zero and we must have

$$\begin{aligned}&\nabla^2 f=-\frac{1}{n}(\Delta f)g,\\ &\mathbb{L}_\phi f=\frac{m}{m-n}\langle\nabla f,\nabla\phi\rangle,\\ &\mathrm{Ric}_\phi^m(\nabla f,\nabla f)=0.\end{aligned}\tag{3.9}$$

From (3.9), we obtain

$$\begin{aligned}\nabla^2 f&=-\frac{1}{n}(\Delta f)g=-\frac{1}{n}(\mathbb{L}_\phi f+\langle\nabla f,\nabla\phi\rangle)g\\ &=-\frac{1}{n}\left(\mathbb{L}_\phi f+\frac{m-n}{m}\mathbb{L}_\phi f\right)g\\ &=-\frac{2m-n}{mn}(a+1)f^{-1}|\nabla f|^2 g.\end{aligned}\tag{3.10}$$

If $q<\frac{4m}{5m-9}$ or $(q-1)\lambda<c$ we have $A<0$ or $B<0$, respectively and thus f must be constant. It needs to verify that f must also be constant when

$$q=\frac{4m}{5m-9},\quad (q-1)\lambda=c.\tag{3.11}$$

With the assumption (3.11), we can get

$$a = \frac{1}{\frac{3}{2}q - 1} = \frac{5m-9}{m+9}.$$

Since $\mathrm{Ric}_\phi^m = 0$ and $\nabla^2\phi - \frac{1}{m-n} d\phi \otimes d\phi \geq 0$ on M, we have $\mathrm{Ric}(\nabla f, \cdot) = 0$. We denote

$$\xi := -\frac{2m-n}{mn}(a+1)f^{-1}|\nabla f|^2 = -\frac{12m-6n}{(m+9)n}f^{-1}|\nabla f|^2.$$

By (3.10), we obtain $\nabla^2 f = \xi g$. Working with a local orthonormal frame we differentiate

$$\begin{aligned} \xi_j &= f_{ij,i} = f_{ii,j} - R_{jiil}f_l \\ &= (\Delta f)_j + R_{jl}f_l \\ &= n\xi_j. \end{aligned}$$

Thus $\xi_j = 0$ and ξ is constant. To continue, recall that we have

$$|\nabla f|^2 = -\frac{(m+9)n}{12m-6n}\xi f.$$

Differentiating both sides we get

$$-\frac{(m+9)n}{12m-6n}\xi f_j = 2f_i f_{ij} = 2f_i \xi g_{ij} = 2\xi f_j.$$

Thus, we have

$$\frac{25m-12n+9}{12m-6n}\xi\nabla f = 0.$$

By taking inner product on both sides with ∇f and applying $f > 0$, we obtain $\frac{(25m-12n+9)(m+9)n}{(12m-6n)^2}\xi^2 = 0$. Since $m \geq n$, we have $\xi = 0$ and thus $\nabla f = 0$ and f must be a constant function. This finishes the proof of Theorem 1.5.

4 Proof of Theorem 1.3 and Corollary 1.4

In this section, we shall prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3 We suppose $m > 2$, and consider the following family of functionals $\mathcal{J}_q$, which is defined by

$$\mathcal{J}_q(\eta) = \int_M |\nabla\eta|^2\, d\sigma + \frac{c}{q-1}\int_{\partial M} \eta^2\, dv$$

and consider $\mu_q := \inf\{\mathcal{J}_q(\eta), \eta \in \mathcal{H}_q\}$, where

$$\mathcal{H}_q := \left\{ \eta \in \mathcal{H}_1^2(d\sigma) : \int_{\partial M} \eta^{q+1}\, dv = 1 \right\}.$$

Another important key here is that the real-valued function

$$\widetilde{g} : x \longmapsto \frac{4x}{5x-9}$$

is decreasing. So

$$\frac{4m}{5m-9} = \widetilde{g}(m) < \widetilde{g}(n) = \frac{4n}{5n-9}$$

as $n < m$. Using the compactness of the inclusions

$$H_1^2(d\sigma) \hookrightarrow L^2(d\sigma) \text{ and } H_1^2(d\sigma) \hookrightarrow L^{q+1}(d\sigma)$$

for any $q+1 < \frac{2m}{m-2}$, we can confirm that μ_q is realized by a positive function $\psi_q \in \mathcal{H}_q$ and hence we can easily check that ψ_q verifies weakly the following system:

$$\begin{cases} \mathbb{L}_\phi \psi_q = 0 & \textit{in } M, \\ \partial_\nu \psi_q + \frac{c}{q-1}\psi_q = \mu_q \psi_q^q & \textit{on } \partial M. \end{cases} \tag{4.1}$$

The regularity result of [3, Theorem 1] indicates that ψ_q is smooth, so by applying Theorem 1.5, we can infer that ψ_q is constant. Since $\psi_q \in \mathcal{H}_q$, we have

$$\psi_q = \mathrm{vol}_\phi(\partial M)^{-\frac{1}{q+1}} \text{ and } \mu_q = \frac{c}{q-1}\psi_q^{1-q} = \frac{c}{q-1}\mathrm{vol}_\phi(\partial M)^{\frac{q-1}{q+1}}.$$

Hence, recalling the definition of μ_q, we can get

$$\frac{c}{q-1}\mathrm{vol}_\phi(\partial M)^{\frac{q-1}{q+1}} \leq \int_M |\nabla \eta|^2\, d\sigma + \frac{c}{q-1}\int_{\partial M} \eta^2\, dv.$$

Simple calculation can be obtained,

$$1 \leq \mathrm{vol}_\phi(\partial M)^{-\frac{q-1}{q+1}} \left(\frac{q-1}{c} \int_M |\nabla \eta|^2\, d\sigma + \int_{\partial M} \eta^2\, dv \right).$$

Considering any $\eta \in H_1^2(d\sigma)$ satisfies

$$\left(\int_{\partial M} \eta^{q+1}\, dv \right)^{\frac{2}{q+1}} \leq \mathrm{vol}_\phi(\partial M)^{-\frac{q-1}{q+1}} \left(\frac{q-1}{c} \int_M |\nabla \eta|^2\, d\sigma + \int_{\partial M} \eta^2\, dv \right). \tag{4.2}$$

Hence,

$$\left(\frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} |u|^{q+1}\, dv \right)^{\frac{2}{q+1}} \leq \frac{q-1}{c(\mathrm{vol}_\phi(\partial M))} \int_M |\nabla u|^2\, d\sigma + \frac{1}{\mathrm{vol}_\phi(\partial M)} \int_{\partial M} u^2\, dv. \tag{4.3}$$

Thus we complete the proof of Theorem 1.3 for $m > 2$. The case where $m = 2$ (i.e., $n = 2$ and ϕ is constant), (1.10) can be obtained from [9, Corollary 1]. Therefore the proof of Theorem 1.3 is completed. □

Using Theorem 1.3 we can prove Corollary 1.4.

Proof of Corollary 1.4 Under the assumption on u (4.3) can be written as

$$\frac{\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{\frac{2}{q+1}}-1}{q-1}\le\frac{1}{c(\mathrm{vol}_\phi(\partial M))}\int_M|\nabla u|^2\,d\sigma. \tag{4.4}$$

Let $F(q) := \left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{\frac{2}{q+1}}-1$. We can get

$$\begin{aligned}
F'(q) =&F(q)\left(\left(\frac{2}{q+1}\right)'\ln\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)+\frac{2}{q+1}\frac{\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)'}{\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)}\right)\\
=&F(q)\left(\frac{2}{q+1}\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\ln u\,dv\right)\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{-1}\right)\\
&+F(q)\left(-\frac{2}{(q+1)^2}\left[\ln\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)\right]\right)\\
=&-\frac{2}{(q+1)^2}\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{\frac{2}{q+1}}\left[\ln\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)\right]\\
&+\frac{2}{q+1}\frac{1}{\mathrm{vol}_\phi(\partial M)}\left(\int_{\partial M}|u|^{q+1}\ln u\,dv\right)\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{\frac{2}{q+1}-1}.
\end{aligned}$$

Taking limit $q \downarrow 1$ and applying L'Hospital's rule yields

$$\begin{aligned}
&\lim_{q\to1}\frac{\left(\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^{q+1}\,dv\right)^{\frac{2}{q+1}}-1}{q-1}\\
&=\lim_{q\to1}\frac{F(q)}{q-1}=\lim_{q\to1}F'(q)\\
&=\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^2\ln u\,dv\\
&=\frac{1}{2}\frac{1}{\mathrm{vol}_\phi(\partial M)}\int_{\partial M}|u|^2\ln u^2\,dv.
\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4), we get the desired inequality. we complete the proof. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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