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RESEARCH ARTICLE

On the Existence and Multiplicity of Classical and Weak Solutions of a Hamiltonian Integro-Differential System and Their Equivalence Relation

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Abstract
This paper is devoted to the study of existence and multiplicity of weak solutions to a Hamiltonian integro-differential system. The main tool used is the theory of min–max based on Mountain-Pass theorem. Hamiltonian integro-differential considered system is of Fredholm type and the imposed Dirichlet boundary conditions are occurred at the integral bounds. Furthermore, we demonstrate some cases in which the weak solutions are equivalent with classical solutions

Keywords Hamiltonian system · Fredholm integro-differential equations · Weak solution · Classical solution · Variational method

Mathematics Subject Classification 34Bxx · 34Kxx

1 Introduction

A Hamiltonian system is in fact a mathematical formality introduced by Hamilton for studying the evolution equations in physical systems. The advantage of this explanation is that it uncover important insight about the dynamics. It plays important role in classical and celestial mechanics by studying periodic solutions of Hamiltonian systems.

Mawhin and Willem [1] studied the periodic solutions of the following convex Hamiltonian system

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$$\begin{cases} \ddot{q} + \nabla H(t, q(t)) = 0, \quad t \in (0, T) \\ q(0) - q(T) = \dot{q}(0) - \dot{q}(T) = 0, \end{cases} \tag{1}$$

where $H : [0, T] \times R^N \to R$ is measurable in t for every $q \in R^n$ and continuously differentiable and convex in q for a.e. $t \in [0, T]$. The readers are refereed to see [2–11] for more information about Hamiltonian systems and related problems. Ge and Zhao [12] have studied the existence of multiple solutions of the following four point boundary value problem:

$$\begin{cases} \big(P(t)x'(t)\big)' = \nabla H(t, x(t)), \quad t \in (0, 1) \\ x'(0) = \alpha x(\xi), \ x'(1) = \beta x(\eta), \end{cases} \tag{2}$$

where, the matrix P is symmetric. Also, Wen Lian et. al. [13] have studied the existence of multiple solutions to the following three-point differential Hamiltonian system

$$\begin{cases} \big(P(t)x'(t)\big)' + \nabla H(t, x(t)) = 0, \quad a.e. t \in (0, 1) \\ x(0) = 0, \ x(1) = \beta x(\eta), \end{cases} \tag{3}$$

with some conditions on $P(t)$ (that is a matrix) and $H(t, x(t))$.

Research on integro-differential equations of the integer order and even fractional order in scientific sources shows that they are used in many fields. Readers can study [14, 15] and references therein. On the other hand, in general, there are different methods for checking the existence of the solutions to the boundary value problems, the most important of which are the fixed point method based on the Green function through the upper and lower solutions, and the variational methods [1, 16–32].

In this paper, we study the existence of weak solutions for the following system of integro-differential equations

$$\begin{cases} \big(A(t)u'(t)\big)' = \int_0^1 K(t, s)\nabla H(s, u(t))ds, \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

where $t \in (0, 1)$, $A : [0, 1] \to \mathbb{R}^{n\times n}$ is a continuously symmetric matrix, i.e., $A^T(t) = A(t)$ being continuous in t, the kernel $K(., .) \in C^1(]0, 1[)$, $H : \mathbb{R}^n \to \mathbb{R}$ is differentiable scaler function such that its partial derivatives are continuously differentiable, and

$$\nabla H(t, x) = \begin{bmatrix} \partial_{x_1} H(t, x) \\ \partial_{x_2} H(t, x) \\ \dots \\ \partial_{x_n} H(t, x) \end{bmatrix}.$$

First of all, to achieve our purpose, we recall one basic optimization principle for extreme eigenvalues of symmetric matrixes:

Lemma 1.1 ([33]) *If A is a real symmetric $n \times n$ matrix, then*

$$\lambda_1 = \max\{x^T Ax \mid \|x\| = 1\}, \quad \lambda_n = \min\{x^T Ax \mid \|x\| = 1\}.$$

where λ_1 and λ_n are respectively, its largest and smallest eigenvalues.

As a result, we can extend preceding lemma for $\|x\| \neq 1$, by normalization in the form of below:

$$\lambda_1 = \max\left\{\frac{x^T Ax}{\|x\|^2} \mid x \neq 0\right\}, \quad \lambda_n = \min\left\{\frac{x^T Ax}{\|x\|^2} \mid x \neq 0\right\}. \tag{5}$$

2 Variational Structure on the Problem

Consider the sobolev Banach space

$$H^1([0,1],\mathbb{R}^n) = \left\{u \in L^2([0,1],\mathbb{R}^n) \mid \int_0^1 |u(t)|^2 dt + \int_0^1 |u'(t)|^2 dt < \infty\right\},$$

with respect to the norm

$$\|u\|_{H^1([0,1],\mathbb{R}^n)} = \left(\int_0^1 |u(t)|^2 dt + \int_0^1 |u'(t)|^2 dt\right)^{\frac{1}{2}}. \tag{6}$$

In confronting with our boundary values, we consider its closed subspace

$$X := H_0^1([0,1],\mathbb{R}^n) = \left\{u \in H^1([0,1],\mathbb{R}^n) \mid u(0) = u(1) = 0\right\}.$$

$H^1([0,1],\mathbb{R}^n)$ is reflexive, so is X.

Lemma 2.1 ([34]) *If $x \in C^1[0,1]$ and $x(0) = 0$, then we have*

$$\int_0^1 |x(t)|^2 dt \leq \frac{4}{\pi^2} \int_0^1 |x'(t)|^2.$$

By preceding lemma, the norm (11) is equivalent to

$$\|u\| = \left(\int_0^1 |u'(t)|^2 dt\right)^{\frac{1}{2}}. \tag{7}$$

After that, we apply this norm.

Lemma 2.2 ([13]) *X is compactly embedded into $C([0,1],\mathbb{R}^n)$ there exists a positive embedding constant k such that $\|u\|_\infty \leq k\|u\|$ $\left(\|u\|_\infty = \max_{t\in[0,1]} |u(t)|\right)$.*

Definition 2.3 A function $u : [0,1] \to \mathbb{R}^n$ is said to be a weak solution of problem (4), if $u \in X$ and

$$\int_0^1 \prec (A(t)u'(t))', \phi(t) \succ dt - \int_0^1 \prec \int_0^1 K(t,s)\nabla H(s,u(t))ds, \phi(t) \succ dt = 0, \tag{8}$$

for all $C_0^\infty(]0,1[,\mathbb{R}^n)$.

Now define the functional $\Psi(u) : \mathbb{R}^n \to \mathbb{R}$ as

$$\Psi(u) = \int_0^1 \big[\frac{1}{2} \prec A(t)u'(t), u'(t) \succ + \int_0^1 K(t,s)H(s,u(t))ds\big]dt. \tag{9}$$

It is clear that Ψ is well defined and we can say more:

Lemma 2.4 *The critical points of the functional* $\Psi(u)$ *are exactly weak solutions of* (4).

Proof Since A as function is continuous at t, and K and H are continuously differentiable, so is $\Psi(u)$, and whose Gâteaux derivative at point u in direction v, given by

$$\begin{aligned}
&\prec \Psi'(u), v \succ = \frac{d}{d\theta}[\Psi(u+\theta v)|_{\theta=0} = \frac{d}{d\theta}\big[\int_0^1 \big(\frac{1}{2} \prec A(t)u'(t)+\theta v'(t)), (u'(t)+\theta v'(t))\\
&\succ + \int_0^1 K(t,s)H(s,u(t)+\theta v(t))ds\big)dt\big]_{\theta=0}\\
&= \prec \int_0^1 \big[\frac{1}{2}\big(\prec A(t)v'(t), u'(t)+\theta v'(t) \succ + \prec A(t)(u'(t)+\theta v'(t)), v'(t) \succ \big)+\\
&\quad \int_0^1 K(t,s)\nabla H(s,u(t)+\theta v(t)), v(t) \succ ds\big]dt \,|_{\theta=0}\\
&= \int_0^1 \big[\frac{1}{2}\big(\prec A(t)v'(t), u'(t) \succ + \prec A(t)u'(t), v'(t) \succ \big)+\\
&\quad \int_0^1 K(t,s) \prec \nabla H(s,u(t), v(t) \succ ds\big]dt =\\
&\quad \int_0^1 \big[\prec A(t)u'(t), v'(t) \succ + \int_0^1 K(t,s) \prec \nabla H(s,u(t)), v(t) \succ ds\big]dt.
\end{aligned} \tag{10}$$

By applying divergence theorem, we conclude:

$$\begin{aligned}
&\prec \Psi'(u), y \succ =\\
&- \int_0^1 \prec (A(t)u'(t))', v'(t) \succ dt + \int_0^1 \int_0^1 K(t,s) \prec \nabla H(s,u(t), v(t) \succ dsdt =\\
&- \Big(\int_0^1 \prec (A(t)u'(t))', v'(t) \succ dt - \int_0^1 \prec \int_0^1 K(t,s)\nabla H(s,u(t), v(t)ds \succ dt\Big).
\end{aligned}$$

On the other hand the assumption, u is a critical point of Ψ, implies

$$\prec \Psi'(u), y \succ = 0$$

This means u is exactly a weak solution of (4).□

From now on, we will focus on finding critical points for functional Ψ. Our main tool is a famous min–max Theorem named Mountain Pass as follows:

Theorem 2.5 ([35]) *Assume X be a real banach space and $I \in C^1(X, R)$ satisfies the Palais–Smale condition. Suppose also*

1. $I(0) = 0$,
2. *there exist constants $r, \alpha > 0$ such that $I(u) > \alpha$ if $\|u\| = r$,*
3. *there exists an element $v \in X$ with $\|v\| > r$, $I(v) \leq 0$.*

define

$$\Gamma := \{g \in (C[0,1], X) |\ g(0) = 0, g(1) = v\}.$$

Then

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t))$$

is a critical value of I.

3 Main Result

Now we can turn to the main theorem:

Theorem 3.1 *Assume that the following conditions hold:*

($\mathbf{H_1}$) *$\{\lambda_i(t)\}$ is the eigenvalue of $A(t)$ and* $0 < a \leq \min_{0 \leq t \leq 1} \min_{1 \leq j \leq n} \lambda_j(t) \leq \max_{0 \leq t \leq 1} \max_{1 \leq j \leq n} \lambda_j(t) \leq b$;

($\mathbf{H_2}$) *for partial derivatives H_{x_i}, we have* $\lim_{|x| \to 0} \min_{0 \leq t \leq 1} \frac{\int_0^1 K(t,s) H_{x_i}(s,x) ds}{|x|} = 0$;

($\mathbf{H_3}$) *there is constant $\mu > 0$ such that $\frac{a}{2} - \frac{b}{\mu} > 0$ and $\alpha \geq 0$ such that for $|x| \geq \alpha$, we have*

$$\int_0^1 \mu K(t,s)H(s,x(t))ds - \int_0^1 K(t,s)(x,\nabla H(s,x(t)))ds \geq 0.$$

Then problem (4) *has a unique nontrivial weak solution.*

Proof It is sufficient to show that functional Ψ, defined in (9), satisfies all conditions of Theorem 2.5. The first step is to show (PS)-condition holds. In this order, assume $\{u_k\} \subset X$ be a sequence such that $\{\Psi(u_k)\}$ is bounded, i.e. there exists $L > 0$ such that $|\Psi(u_k)| \leq L$, and $\Psi'(u_k) \to 0$ as $k \to \infty$. We should prove that $\{u_k\}$ admits a convergence subsequence in X. By definition of symmetric matrix, $\prec A(t)u'(t), u'(t) \succ = \prec u'(t), A(t)u'(t) \succ$. So by definition of Ψ, and taking into account (H_1) and (5) one has

$$\begin{aligned}\Psi(u) = &\int_0^1 \big[\frac{1}{2} \prec A(t)u'(t), u'(t) \succ + \int_0^1 K(t,s)H(s,u(t))ds\big]dt \\ \geq &\frac{a}{2}\|u\|^2 - \int_0^1 \int_0^1 K(t,s)H(s,u(t))dsdt;\end{aligned} \tag{11}$$

In similar way, by (10) and (H_1) we get

$$\begin{aligned}\frac{1}{\mu} \prec \Psi'(u), u \succ = &\frac{1}{\mu} \int_0^1 \big[\prec A(t)u'(t), u'(t) \succ + \int_0^1 K(t,s) \prec \nabla H(s,u(t), u(t) \succ ds\big]dt \\ \leq &\frac{1}{\mu}\big[\int_0^1 b|u'(t)|^2 dt - \int_0^1 \int_0^1 K(t,s) \prec \nabla H(s,u(t), u(t) \succ dsdt\big] \\ = &\frac{1}{\mu}\big[b\|u\|^2 - \int_0^1 \int_0^1 K(t,s) \prec \nabla H(s,u(t), u(t) \succ dsdt\big].\end{aligned} \tag{12}$$

For k large enough, put $u := u_k$, and $E := \{t \in [0,1] |\ |u(t)| \geq \alpha\}$, combining (11) and (12) implies:

$$\begin{aligned}L \geq &\Psi(u) \\ \geq &\frac{1}{\mu} \prec \Psi'(u), u \succ + (\frac{a}{2} - \frac{b}{\mu})\|u\|^2 + \\ &\frac{1}{\mu}\int_0^1 \int_0^1 \big[K(t,s)\big(H(s,u(t)) - \prec \nabla H(s,u(t)), u(t) \succ\big)\big]dsdt \\ = &\frac{1}{\mu} \prec \Psi'(u), u \succ + (\frac{a}{2} - \frac{b}{\mu})\|u\|^2 + \\ &\int_E \frac{1}{\mu} \int_0^1 \big[K(t,s)\big(H(s,u(t)) - \prec \nabla H(s,u(t)), u(t) \succ\big)\big]dsdt \\ &+ \int_{[0,1]\setminus E} \int_0^1 \big[K(t,s)\big(H(s,u(t)) - \prec \nabla H(s,u(t)), u(t) \succ\big)\big]dsdt.\end{aligned} \tag{13}$$

The first term on the right side of (13) tend to zero, by $(\mathbf{H_3})$, $\frac{a}{2}-\frac{b}{\mu}>0$ and the third term is positive. For fourth term, since $|u(t)|<\alpha$, it is clear that is bounded by a constant independently of k. So $\{u_k\}$ is bounded in X. Whereas X is reflexive, $\{u_k\}$ has a weakly convergence subsequence. Going to a subsequence, Assume $u_k \overset{w}{\to} u$ in X. According to compactly embedding in lemma 2.2, $u_k \to u$ in $C_0^\infty(]0,1[,\mathbb{R}^n)$. By properties of inner product, we conclude

$$\prec \Psi'(u_k)-\Psi'(u), u_k-u \succ = \prec \Psi'(u_k), u_k-u \succ - \prec \Psi'(u), u_k-u \succ \to 0,\ (k\to\infty) \tag{14}$$

Due to $(\mathbf{H_1})$,

$$\begin{aligned} \prec \Psi'(u_k)-\Psi'(u), u_k-u \succ &= \prec A(t)u_k'(t)-u'(t), u'(t) \succ dt \\ &+\int_0^1\int_0^1 K(t,s) \prec \nabla H(s,u_k(t))-\nabla H(s,u(t)), u_k(t)-u(t) \succ dsdt \\ &\geq a\int_0^1 |u_k'(t)-u'(t)|^2 dt \\ &+\int_0^1\int_0^1 K(t,s) \prec \nabla H(s,u_k(t))-\nabla H(s,u(t)), u_k(t)-u(t) \succ dsdt. \end{aligned} \tag{15}$$

On the other hand $u_k \to u$ in $C_0^\infty(]0,1[,\mathbb{R}^n)$. Hence $|u_k(t)-u(t)|\to 0$ for $t=0,1$ and

$$\int_0^1\int_0^1 K(t,s) \prec \nabla H(s,u_k(t))-\nabla H(s,u(t)), u_k(t)-u(t) \succ dsdt \to 0$$

as $k\to\infty$. Consequently, (15) implies

$$\int_0^1 |u_k'(t)-u'(t)|^2 dt \to 0\ \ (k\to\infty)$$

But $u_k\in X$, so $u_k(0)=u_k(1)=0$, hence $u(0)=u(1)=0$, i.e. $u\in X$. That is $\|u_k-u\|\to 0$ in X. Then (PS)-condition holds. It is obvious that $\Psi(0)=0$. For the second step, according to (H_2), $\lim\limits_{u\to 0}\frac{\int_0^1 K(t,s)H_{u_i}(s,u)ds}{|u|}=0$ that means for all $\epsilon_i>0 (1\leq i\leq n)$ there is a $0<\delta\leq 1$ such that $|u|\leq\delta$ concludes $|\int_0^1 K(t,s)H_{u_i}(s,u)ds|\leq \epsilon_i|u|$. Equivalently, for all $\epsilon=\max\limits_{1\leq i\leq n}\epsilon_i>0$ we have

$$|\int_0^1 K(t,s)\nabla H(s,u(t))ds| \leq \frac{\epsilon}{2}|u|^2 \tag{16}$$

By integrating both side of above inequality and applying lemma 2.1 we have

$$\int_0^1\int_0^1 K(t,s)\nabla H(s,u(t))dsdt \leq \frac{\epsilon}{2}\int_0^1 |u(t)|^2 dt \leq \frac{2\epsilon}{\pi^2}\int_0^1 |u'(t)|^2 dt, \tag{17}$$

Therefore

$$\Psi(u) \geq \frac{a}{2}\|u\|^2 - \frac{2\epsilon}{\pi^2}\int_0^1 |u'(t)|^2 dt = \left(\frac{a}{2} - \frac{2\epsilon}{\pi^2}\right)\|u\|^2 \geq \left(\frac{a}{2} - \frac{2\epsilon}{\pi^2}\right)\|u\|_\infty^2. \quad (18)$$

By setting $\epsilon = \frac{a\pi^2}{8}$, $r = \delta$, $\alpha = \frac{a\delta^2}{4}$, we conclude that $\Psi(u) \geq \alpha$. So condition (2) in theorem 2.5 holds. To complete the proof, it is sufficient to show that condition (3) holds. Notice that by (H_3), there exist constants $c, M > 0$ such that

$$\int_0^1 K(t,s)H(s,x(t))ds \leq M - c|x|^\mu \quad \forall t \in [0,1], x \in \mathbb{R}^n. \quad (19)$$

Thus

$$\int_0^1 \int_0^1 K(t,s)H(s,u(t))dsdt \leq M - c\int_0^1 |u(t)|^\mu dt \quad (20)$$

$$\begin{aligned}
\Psi(uz) &= \int_0^1 \left[\frac{1}{2} \prec A(t)zu'(t), zu'(t) \succ + \int_0^1 K(t,s)H(s,zu(t)), \phi(t)ds\right]dt \\
&\leq \frac{bz^2}{2}\int_0^1 |u'(t)|^2 dt + \int_0^1 \int_0^1 K(t,s)H(s,zu(t)), \phi(t)dsdt \quad (21)\\
&\leq \frac{bz^2}{2}\|u\|^2 + M - cz^\mu \int_0^1 |u(t)|^\mu dt
\end{aligned}$$

which yields $\Psi(uz) \to -\infty$ as $z \to \infty$. So the desired result is achieved.□

Now consider the following condition:
$(\mathbf{H_4})$ ∇H is locally Lipschitz continuous function in $(0,1) \times \mathbb{R}$.

Theorem 3.2 *Under the hypothesis of theorem 3.1, if H also satis (H_4), then system (4) possess a positive and a negative solution.*

Proof Define

$$\widetilde{H}(t,x) = \begin{cases} 0, & x \leq 0; \\ H(t,x), & x > 0. \end{cases} \quad (22)$$

The arguments of theorem 3.1 proof show that

$$\widetilde{\Psi(u)} = \frac{1}{2}\|u\|^2 - \int_0^1 \int_0^1 K(t,s)H(s,u(t))dsdt$$

satisfies the hypotheses of Mountain Pass theorem. Indeed ∇H satisfies (H_2). Also (H_3) holds for $x > 0$ while $\widetilde{H} = \nabla\widetilde{H} = 0$ for $x \geq 0$. Assumption (H_3) was required to verify (PS) and (3) conditions. But $\widetilde{H} \geq \frac{1}{\mu}\nabla\widetilde{H}$ for large enough $|x|$ and this is sufficient to get (PS)-condition. Notice that (19) holds for $x \geq 0$ so choose $u \in X\backslash\{0\}$ in (21) to be nonnegative function, condition (3) in theorem 2.5 holds. Consequently by the Mountain Pass theorem the system

$$\begin{cases} \big(A(t)u'(t)\big)' = \int_0^1 K(t,s)\nabla\widetilde{H}(s,u(t))ds, \\ u(0) = u(1) = 0. \end{cases} \tag{23}$$

has a weak solution, $u \not\equiv 0$. By (H_4) and [remark 2.22,[27]], u is a classical solution of (4). Let $\mathcal{A} = \{t \in [0,1] | u(t) < 0\}$. Then by definition of $\widetilde{H}$ we have

$$\big(A(t)u'(t)\big)' = 0 \;\; \forall t \in [0,1]$$

Therefore the maximum principle shows $u \equiv 0$ in $\mathcal{A}$. This means $\mathcal{A} = \emptyset$. Hence $u \geq 0$ in [0, 1]. The negative solution of (4) is produced in similar way. □

Theorem 3.3 *Assume that conditions H_1 and H_3 holds. Furthermore the following hypotheses holds*:

($\mathbf{H_5}$) $\int_0^1 K(t,s)H(s,-x)ds = \int_0^1 K(t,s)H(s,x)ds$;

($\mathbf{H_6}$) $\lim\limits_{|x|\to 0} \min\limits_{0\leq t\leq 1} \frac{\int_0^1 K(t,s)H_{x_i}(s,x)ds}{|x|^2} = -\infty$.

Then BVP (4) has infinity many pairs of nontrivial weak solutions.

Proof Condition (H_6) means for any $m \in \mathbb{N}$ there is $\delta > 0$ such that

$$\prec \int_0^1 K(t,s)\nabla H(s,x)ds, x \succ \leq -4b(m+1)^2\pi^2|x(t)|^2 < -4bm^2\pi^2|x(t)|^2, \;\; 0 < |x| \leq \delta.$$

So the conclusion comes from [theorem 1.1,[12]].

4 Concluding Remarks

In this paper, we have studied and researched on a class of important model namely Hamiltonian integro-differential system of Fredholm type with Dirichlet boundary conditions at the integral bounds (also bounds of the problem domain). It have been given and proved some good results on the existence and multiplicity of weak solutions to a Hamiltonian integro-differential system by applying the theory of min–max based on Mountain-Pass theorem. Furthermore, we showed that it is possible to extract also classical solutions through weak results.

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