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RESEARCH ARTICLE

Dependence of Eigenvalues of $(2n+1)$ th Order Boundary Value Problems with Transmission Conditions

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Abstract

This paper deals with some boundary value problems generated by $(2n+1)$ th order differential equation with transmission conditions. After showing that these problems generate self-adjoint operators and the eigenvalues of the problems are real, we introduce the continuous dependence and differentiable dependence of eigenvalues on parameters: coefficient functions and weight function, boundary conditions, transmission conditions, as well as the endpoints and transmission points. In addition, we obtain the differential expressions of all given parameters respectively.

Keywords Dependence of eigenvalue · Odd order differential equation · Self-adjoint operator · Transmission conditions · Boundary value problem

1 Introduction

The dependence of eigenvalues play an important role in the theory of differential operators, it provides theoretical support for the numerical calculation of eigenvalues [1–3]. As early as 1987, Poeschel and Trubowitz in [4] considered the $\lambda = \lambda_n(q)$ as a function of potential function q, and showed that λ is Frechet differentiable of q by using the asymptotic form of the solutions for $|\lambda| \to \infty$. Then Dauge and Helffer in [5] investigated the Neumann eigenvalues $\lambda = \lambda_n(q)$ with respect to the domain, and proved that the eigenvalues of the regular Sturm–Liouville problems with the Neumann boundary conditions are differentiable functions of the right endpoint. On this basis, Kong and Zettl in [6] studied these problems by using more simple methods, and proved that the eigenvalues of Sturm–Liouville problems not only continuously but also differentiably depend on the end points. In the same year, they investigated these problems more systematically in [7], and showed that the

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eigenvalues of Sturm–Liouville problems not only differentiably depend on the end points, but also differentiably depend on all the given parameters: boundary conditions, coefficient and weight functions. They also obtained the differential expressions of the given parameters by using the similar methods in [6]. For a special class of regular boundary value problems studied by Naimark [8] and Weidmann [9], Kong and Zettl in [10] showed that the eigenvalues differentiably depend on the problem data and obtained the differential expressions. Their results extended theorems in [7] from the second-order case to the general even order case.

It is well known that boundary value transmission problems have important applications in physics and engineering, such as heat conduction and mass transfer, string vibration problems with nodes located internally [11], etc., and their physical applications connected with these problems are also found in literature [12–14]. In recent years, more and more researchers are interested in the study of problems with interior discontinuities, in particular, the dependence of eigenvalues of even order boundary value transmission problems, and some meaningful results have been obtained [15–27]. To deal with interior discontinuities, some conditions are imposed on the discontinuous points, which are often called transmission conditions (see [15, 17, 22, 23, 25–27, 31]) or interface conditions (see [21, 24]). Among them, Zhang and Wang [21] investigated the dependence of eigenvalues of Strum-Liouville problems with interface conditions for second order case and gave the differential expressions of eigenvalues with respect to the given parameters. Li et al. in [22] generalized the results of [21] in fourth order case in 2017. In the same year, Li et al. further generalized these results to the general even order differential operators in [23]. Although the general theory and methods for such even order boundary value problems have been highly developed, little is known about the odd order case, especially in the general odd order case.

In 1975, Walker in [28] investigated a vector–matrix formulation for formally symmetric ordinary differential equations. By defining the quasi-derivative $y^{[k]}$ and corresponding matrices, they showed that the differential expression

$$M[y] = \lambda wy \text{ on interval } I,$$

has a first-order vector–matrix formulation as show below:

$$JY' = [\lambda A + B]\, Y \text{ on } I. \tag{1}$$

Here $M[\cdot]$ is a formally symmetric ordinary differential expression of order m(m can be even or odd),I is an interval of the real line,λ is a complex number and w is a weight function.A, B and J are $m \times m$ matrices.

However, according to the different symmetric expressions in which m is even or odd, the definitions of the quasi-derivatives are different, and the corresponding matrices are also different. For $m = 2n + 1$, the formally symmetric ordinary differential expressions can be expressed as show below:

$$M\,[y] = \sum_{k=0}^{n} (-1)^k \left\{ i \left[\left(q_{n-k} y^{(k)}\right)^{(k+1)} + \left(q_{n-k} y^{(k+1)}\right)^{(k)} \right] + \left(p_{n-k} y^{(k)}\right)^{(k)} \right\},$$

and corresponding quasi-derivatives are defined as follows:

$$\begin{aligned}
&y^{[0]} = y,\\
&y^{[k+1]} = \left(y^{[k]}\right)', \; if\; 0 \le k \le n-2,\\
&Let\, \theta = (1/\sqrt{2})(1+i)\\
&y^{[n]} = -\theta q_0 \left(y^{[n-1]}\right)'\\
&y^{[n+1]} = -(\theta q_0)\left(y^{[n]}\right)' + (i\theta p_0/q_0) y^{[n]} - i q_1 y^{[n-1]}\\
&y^{[n+2]} = -(y^{[n+1]})' - (\theta q_1/q_0)(y^{[n]}) + p_1 y^{[n-1]} - i q_2 y^{[n-2]}, if\; n+2 \le 2n\\
&y^{[n+k+1]} = -(y^{[n+k]})' + p_k y^{[n-k]} - i\left[q_k y^{[n-k+1]} - q_{k+1} y^{[n-k-1]}\right], if\; 2 \le k \le n-1
\end{aligned} \tag{2}$$

On this basis, Hinton in [29] studied the deficiency indices of odd order differential operators. By using the form of symmetric differential Eq. (1), they got the Lagrange identity for odd order as follows:

$$-w\left\{L[y]\bar{z} - yL[z]\right\} = \frac{d}{dx}\left[\sum_{k=0}^{n-1}\left(y^{[2n-k]}\bar{z}^{[k]} - y^{[k]}\bar{z}^{[2n-k]}\right) + i y^{[n]}\bar{z}^{[n]}\right],$$

where $L[y] = w^{-1}M[y]$. These results provided an important foundation for us to further study the self-adjointness and boundary value problems of odd order differential operators.

In recent years, Uğurlu in [30] considered a class of formally symmetric boundary value problems generated by the third-order differential equations studied by Walker in [28]. After showing that these problems generate self-adjoint operators, the dependence of eigenvalues on the data for these problems was studied and the derivatives of the eigenvalues with respect to some elements of data were introduced. Then they generalized these results to differential operators with transmission conditions in [31]. In spired by [30], Li et al. in [32] had studied the self-adjoint of a class of third-order differential operators with an eigenparameter contained in the boundary conditions. At the same year, Bai et al. investigated the dependence of eigenvalues for these problems in [33]. Other studies on third-order differential operators were found in literature [34–36].

However, up to now, we have not found any study on the dependence of eigenvalues of general odd order boundary value problems with transmission conditions. To further develop the odd order differential operators theory, in this paper, we study the symmetric operators generated by a class of $(2n+1)$ th order differential equations with transmission conditions. Combining the quasi-derivatives and the matrices defined in [28] and using the methods in [7], we prove the self-adjointness of the operators, on this basis, we further introduce the continuous dependence of eigenvalues on the problems. In addition, we show the differential properties of the eigenvalues on the given parameters, not only including the boundary conditions and transmission conditions, coefficient functions and weight function, two endpoints,but also including the interior discontinuities points. In particular, we also gave the details of the proof of the differentiability of eigenvalues with respect to the weight function.

The rest of this investigation is arranged as follows. In Sect. 2, some notations and preliminaries are gave. In Sect. 3, we construct an operator T associated with the problems (3–5), and prove that T is a self-adjoint operator. Then we introduce the continuity results of eigenvalues and eigenfunctions in Sect. 4. In Sect. 5, we obtain differential expressions of the eigenvalues with respect to the given parameters.

2 Notations and Preliminaries

Consider the $(2n+1)$ th order symmetric differential equations which were studied in [28]

$$Ly = \frac{1}{w}\left\{\sum_{k=0}^{n}(-1)^k\left\{i\left[(q_{n-k}y^{(k)})^{(k+1)} + (q_{n-k}y^{(k+1)})^{(k)}\right] + \left(p_{n-k}y^{(k)}\right)^{(k)}\right\}\right\} = \lambda y$$
$$on\, J' = (a', c)\cup(c, b').\, Let\, J = [a, c)\cup(c, b],\ a' < a < c < b < b'. \tag{3}$$

Consider the boundary conditions

$$AY(a) + BY(b) = 0, \tag{4}$$

and transmission conditions

$$Y(c-) = CY(c+), \tag{5}$$

where $q_0^{-1}(x), q_1(x), \cdots, q_n(x), p_0(x), \cdots, p_n(x), w(x) \in L_{loc}(J', \mathbf{R})$, $w(x) > 0$ a.e. on J', $\lambda \in \mathbf{C}$ is the spectral parameter, $Y(x) = (y^{[0]}(x), y^{[1]}(x), \cdots, y^{[2n]}(x))^T$. $Y(c\pm) = \lim_{x\to c\pm} Y(x)$. Here $y^{[0]}(x), y^{[1]}(x), \cdots, y^{[2n]}(x)$ are call quasi-derivatives of y as defined in Eq. (2). A and B are $(2n+1)\times(2n+1)$ complex matrices, C is $(2n+1)\times(2n+1)$ real matrix, they can be ordered to satisfy

$$\begin{aligned}&\mathrm{rank}\,(A|B) = 2n+1,\ \det(C) = \rho^{\frac{1}{2}(2n+1)},\ \rho > 0,\\ &\rho A Q_{2n+1} A^* = B Q_{2n+1} B^*,\ \rho Q_{2n+1} = C^* Q_{2n+1} C,\end{aligned} \tag{6}$$

where $Q_{2n+1} = \left(q_{kj}\right)$ are $(2n+1)\times(2n+1)$ matrices defining as follows (see [28]):

$$q_{kj} = \begin{cases} 0,\ j+k \neq 2n+2 \\ -1, j+k = 2n+2, j = 1, 2, \dots, n. \\ i,\ j = k = n+1 \\ 1,\ j+k = 2n+2, j = n+2, n+3, \dots, 2n+1 \end{cases}.$$

Clearly, we have $Q^*_{2n+1} = -Q_{2n+1}, Q^{-1}_{2n+1} = -Q_{2n+1}$.

Let $H_w = L^2_{w_1}[a, c) \oplus L^2_{w_2}(c, b]$ be the weighted Hilbert space with the inner product

$<f,g>=\int_a^c f_1(x)\overline{g_1}(x)w_1(x)dx+\rho\int_c^b f_2(x)\overline{g_2}(x)w_2(x)dx,$ where $f(x)=\begin{cases} f_1(x), x\in[a,c) \\ f_2(x), x\in(c,b]\end{cases}$, $g(x)=\begin{cases} g_1(x), x\in[a,c) \\ g_2(x), x\in(c,b]\end{cases}$, $w(x)=\begin{cases} w_1(x), x\in[a,c) \\ w_2(x), x\in(c,b]\end{cases}$.

We set maximal operators as follows:

$L_{\max}y=Ly$, $y\in D_{\max}$, $x\in J=[a,c)\cup(c,b]$, with the domain

$D_{\max}=\{y\in L_w^2(J)|y^{[0]},\cdots,y^{[2n]}\in AC_{loc}(J), Ly\in L_w^2(J)\}$.

Here $AC_{loc}(J)$ denotes the set of complex valued functions which are absolutely continuous on all compact subintervals of J.

For $y,z\in D_{\max}$, integration by parts yields the Lagrange identity as show below:

$$<Ly,z>-<y,Lz>=[y\overline{z}]|_a^{c-}+\rho[y\overline{z}]|_{c+}^{b} \tag{7}$$

where

$$[y\overline{z}]|_{t_1}^{t_2}=[y\overline{z}](t_2)-[y\overline{z}](t_1), \tag{8}$$

$$[yz]=W(y,\overline{z};x)=\sum_{k=0}^{n-1}\left(y^{[2n-k]}\overline{z}^{[k]}-y^{[k]}\overline{z}^{[2n-k]}\right)+iy^{[n]}\,\overline{z}^{[n]}\ =Z^*(x)Q_{2n+1}Y(x). \tag{9}$$

Here $Z(x)=\left(z(x),z^{[1]}(x),\ldots,z^{[2n]}(x)\right)^T$ and $Z^*(x)$ denotes the complex conjugate transpose of $Z(x)$.

By the definition of quasi-derivatives, we can transfer the Eq. (3) to the following first-order system

$$Y'+DY=\lambda WY,\ x\in J, \tag{10}$$

where $W=\left(w_{ij}\right)$ is $(2n+1)\times(2n+1)$ matrix which is given by $w_{2n+1,1}=-w$ and $w_{i,j}=0$ for $(i,j)\neq(2n+1,1)$, $D=\left(d_{ij}\right)$ is $(2n+1)\times(2n+1)$ matrix as defined in [28] when $m=2n+1$.

3 Operator Theoretic Formulation and Self-Adjointness

According to the above analysis, we shall construct the operators related with boundary value problems (3–5). Consider the operator T defined by

$Ty=Ly, y\in D(T)$,with the domain

$D(T)=\left\{y\in H_w|y^{[0]},\ldots,y^{[2n]}\in AC_{loc}(J), AY(a)+BY(b)=0,\ Y(c-)=CY(c+),\ Ty\in H_w\right\}$.

Then we have the following Lemma.

Lemma 3.1 *$D(T)$ is dense in H_w.*

Proof Let $C_0^\infty(J)$ be all the following functions:

$\gamma(x)=\begin{cases} \gamma_1(x), x\in[a,c) \\ \gamma_2(x), x\in(c,b]\end{cases}$, where $\gamma_1(x)\in C_0^\infty[a,c)$,$\gamma_2(x)\in C_0^\infty(c,b]$.

Let $f(x) \in H_w$ where $f(x) = \begin{cases} f_1(x), x \in [a, c) \\ f_2(x), x \in (c, b] \end{cases}$, since $C_0^\infty(J) \subset D(T)$ and $C_0^\infty[a, c)$ is dense in $L_w^2[a, c)$, hence for any $\varepsilon > 0$, there exists $g_1(x) \in C_0^\infty[a, c)$ satisfying

$$\int_a^c |f_1 - g_1|^2 dx < \frac{\varepsilon}{2}$$

and there exists $g_2(x) \in C_0^\infty(c, b]$ satisfying

$$\rho \int_c^b |f_2 - g_2|^2 dx < \frac{\varepsilon}{2}.$$

Let $g(x) = \begin{cases} g_1(x), x \in [a, c) \\ g_2(x), x \in (c, b] \end{cases}$, then we have $\int_a^c |f_1 - g_1|^2 dx + \rho \int_c^b |f_2 - g_2|^2 dx < \varepsilon$. Therefore $C_0^\infty(J)$ is dense in H_w, so $D(T)$ is dense in H_w.

Lemma 3.2 *The operator T is symmetric.*

Proof For any $y, z \in D(T)$, from (7–9), we get that

$$\begin{aligned} < Ly, z > - < y, Lz >&= [y\bar{z}]|_a^{c-} + \rho[y\bar{z}]|_{c+}^b \\ &= [y\bar{z}](c-) - [y\bar{z}](a) + \rho[y\bar{z}](b) - \rho[y\bar{z}](c+) \\ &= W(y, \bar{z}; c-) - W(y, \bar{z}; a) + \rho W(y, \bar{z}; b) - \rho W(y, \bar{z}; c+), \end{aligned} \tag{11}$$

from conditions (4–6), we have

$$\begin{aligned} W(y, \bar{z}; c-) &= Z^*(c-)Q_{2n+1}Y(c-) = (CZ(c+))^* Q_{2n+1}(CY(c+)) \\ &= Z^*(c+)C^* Q_{2n+1} CY(c+) = \rho Z^*(c+)Q_{2n+1}Y(c+) = \rho W(y, \bar{z}; c+), \end{aligned} \tag{12}$$

$$\begin{aligned} W(y, \bar{z}; a) &= Z^*(a)Q_{2n+1}Y(a) = (A^{-1}BZ(b))^* Q_{2n+1}(A^{-1}BY(b)) \\ &= Z^*(b)B^T(A^{-1})^T Q_{2n+1}A^{-1}BY(b) = \rho Z^*(b)Q_{2n+1}Y(b) = \rho W(y, \bar{z}; b). \end{aligned} \tag{13}$$

According to (11–13), we have $< Ly, z > - < y, Lz >= 0$. This completes the proof.

Theorem 3.1 *The operator T is self-adjoint in H_w.*

Proof Since T is symmetric, it suffices to prove that for any $y \in D(T)$, $z \in D(T^*)$, $u \in H_w$ satisfying

$< Ly, z >=< y, u >$, then $z \in D(T)$ and $Lz = u$, i.e.

(i) $z^{[j]}(x) \in AC(J), j = 0, 1, \cdots, 2n, Lz \in H_w$;

(ii) $u(x) = Lz$;

(iii) $AZ(a) + BZ(b) = 0$ and $Z(c-) = CZ(c+)$.

Assume that for any $z \in C_0^\infty \subset D(T^*)$ satisfying $< Ly, z >=< y, u >$, using the classical differential operator theory (see[37]), we have (i) hold. Since T is symmetric, we have $< Ly, z >=< y, Lz >$, thus (ii) also hold. Next, we need to prove that (iii) holds.

According to the above analysis, for all $y \in D(T)$,$< Ly, z >=< y, u >=< y, Lz >$, we have

$$< Ly, z >= \int_a^c y\overline{Lz}w_1(x)dx + \rho \int_c^b y\overline{Lz}w_2(x)dx, \tag{14}$$

from (7)-(9), we have

$$\begin{aligned} < Ly, z > &= \int_a^c y\overline{Lz}w_1(x)dx + \rho \int_c^b y\overline{Lz}w_2(x)dx + [y\overline{z}]|_a^{c-} + \rho[y\overline{z}]|_{c+}^b \\ &= \int_a^c y\overline{Lz}w_1(x)dx + \rho \int_c^b y\overline{Lz}w_2(x)dx + W(y, \overline{z}; c-) \\ &\quad - W(y, \overline{z}; a) + \rho W(y, \overline{z}; b) - \rho W(y, \overline{z}; c+), \end{aligned} \tag{15}$$

combining (14) and (15), we get that

$$W(y, \overline{z}; c-) - W(y, \overline{z}; a) + \rho W(y, \overline{z}; b) - \rho W(y, \overline{z}; c+) = 0. \tag{16}$$

Using Naimark Patching Lemma (see [8]), there exists $y_1, y_2, \cdots, y_{2n+1} \in D(T)$ such that. $y_i(c+) = y_i^{[1]}(c+) = \cdots = y_i^{[2n]}(c+) = 0, i = 1, \cdots, 2n + 1$, then we have $W(y, \overline{z}; c+) = 0$.

For $y_1, y_2, \dots, y_{2n+1} \in D(T)$ and satisfying $Y_i(c-) = CY_i(c+)(i = 1, \dots, 2n + 1)$, because the matrix C is a nonsingular matrix, So $Y_i(c-) = 0$, therefore we have $W(y, \overline{z}; c-) = 0$, from (16), we can get that $W(y, \overline{z}; a) = \rho W(y, \overline{z}; b)$.

Let

$$F(a) = \begin{pmatrix} y_1(a) & y_2(a) & \cdots & y_{2n+1}(a) \\ y_1^{[1]}(a) & y_2^{[1]}(a) & \cdots & y_{2n+1}^{[1]}(a) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n]}(a) & y_2^{[2n]}(a) & \cdots & y_{2n+1}^{[2n]}(a) \end{pmatrix}, F(b) = \begin{pmatrix} y_1(b) & y_2(b) & \cdots & y_{2n+1}(b) \\ y_1^{[1]}(b) & y_2^{[1]}(b) & \cdots & y_{2n+1}^{[1]}(b) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n]}(b) & y_2^{[2n]}(b) & \cdots & y_{2n+1}^{[2n]}(b) \end{pmatrix},$$

then we get that $Z^*(a)Q_{2n+1}F(a) = \rho Z^*(b)Q_{2n+1}F(b)$.

Let

$$Q_{2n+1}F(a) = A^*, \ \rho Q_{2n+1}F(b) = -B^*, \tag{17}$$

then we have $Z^*(a)A^* = -Z^*(b)B^*$, thus one obtains $AZ(a) + BZ(b) = 0$.

Further, it follows from (17) that

$$F(a) = -Q_{2n+1}A^*, \ F(b) = \frac{1}{\rho}Q_{2n+1}B. \tag{18}$$

Since $y_1, y_2, \dots, y_{2n+1} \in D(T)$, therefore $AY_i(a) + BY_i(b) = 0(i = 1, \dots, 2n + 1)$, thus one obtains

$$AF(a) + BF(b) = 0. \tag{19}$$

Combining (18) and (19), we have $\rho A Q_{2n+1} A^* = B Q_{2n+1} B^*$, thus, the matrices A and B determined by (17) satisfy the assumptions of the problems (3–5).

Similarly, there exist $y_1, y_2, \dots, y_{2n+1} \in D(T)$ such that $y_i(a) = y_i^{[1]}(a) = \cdots = y_i^{[2n]}(a) = 0, i = 1, \cdots, 2n+1$, then we have $W(y, \bar{z}; a) = 0$. For $y_1, y_2, \dots, y_{2n+1} \in D(T)$ and satisfying $AY_i(a) + BY_i(b) = 0$ $(i = 1, \dots, 2n+1)$, and because the matrices A and B are nonsingular matrices, thus $Y_i(b) = 0$, therefore one gets $W(y, \bar{z}; b) = 0$, then by (16), we can obtain

$$W(y, \bar{z}; c-) = \rho W(y, \bar{z}; c+). \tag{20}$$

Similarly, Let

$$\begin{aligned} F(c-) &= \begin{pmatrix} y_1(c-) & y_2(c-) & \cdots & y_{2n+1}(c-) \\ y_1^{[1]}(c-) & y_2^{[1]}(c-) & \cdots & y_{2n+1}^{[1]}(c-) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n]}(c-) & y_2^{[2n]}(c-) & \cdots & y_{2n+1}^{[2n]}(c-) \end{pmatrix}, \\ F(c+) &= \begin{pmatrix} y_1(c+) & y_2(c+) & \cdots & y_{2n+1}(c+) \\ y_1^{[1]}(c+) & y_2^{[1]}(c+) & \cdots & y_{2n+1}^{[1]}(c+) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{[2n]}(c+) & y_2^{[2n]}(c+) & \cdots & y_{2n+1}^{[2n]}(c+) \end{pmatrix}, \end{aligned} \tag{21}$$

where $F(c-) = \lim_{x \to c^-} F(x)$, $F(c+) = \lim_{x \to c^+} F(x)$.

From (20) and (21), we have $Z^*(c-) Q_{2n+1} F(c-) = \rho Z^*(c+) Q_{2n+1} F(c+)$.

Let

$$Q_{2n+1} F(c-) = I^*, \ \rho Q_{2n+1} F(c+) = C^*, \tag{22}$$

where I is the identity matrix of order $(2n+1)$, then we have

$$Z^*(c-) = C^* Z^*(c+),$$

so we get that $Z(c-) = CZ(c+)$.

In addition, from (22), one obtains

$$F(c-) = -Q_{2n+1}, \ F(c+) = -\frac{1}{\rho} Q_{2n+1} C^*. \tag{23}$$

Because $y_1, y_2, \dots, y_{2n+1} \in D(T)$, therefore $Y_i(c-) = CY_i(c+) (i = 1, \dots, 2n+1)$, so we have

$$F(c-) = CF(c+). \tag{24}$$

Combining (23) and (24), we can get that $\rho Q_{2n+1} = C^* Q_{2n+1} C$, therefore, the matrix C determined by (22) satisfies the assumptions of the problems (3–5).

Therefore, (iii) hold. This completes the proof.

By the self-adjointness of the operator T, we have the following Corollary.

Corollary 3.1 *The eigenvalues of T are real, and they are finite or countably infinite without finite accumulation point.*

4 Continuity of Eigenvalues and Eigenfunctions

In this section, we introduce the continuity of eigenvalues and eigenfunctions.

Suppose that $\varphi_1(x,\lambda), \cdots, \varphi_{2n+1}(x,\lambda)$ are the solutions of Eq. (3) on the interval $[a,c)$ and satisfy the initial conditions

$$(C_{\varphi_1}, \dots, C_{\varphi_{2n+1}})(a,\lambda) = I, \tag{25}$$

where $C_{\varphi_i}(a,\lambda) = \begin{pmatrix} \varphi_i(a,\lambda) & \varphi_i^{[1]}(a,\lambda) & \cdots & \varphi_i^{[2n]}(a,\lambda) \end{pmatrix}^T$ $(i = 1, 2, \dots, 2n+1)$, I is the identity matrix of order $2n+1$.

Clearly, the above solutions are linearly independent.

Let $\tau_1(x,\lambda), \cdots, \tau_{2n+1}(x,\lambda)$ be the solutions of Eq. (3) on the interval $(c,b]$ and satisfy the initial conditions $(C_{\varphi_1}, \dots, C_{\varphi_{2n+1}})(c-,\lambda) = C\,(C_{\tau_1}, \dots, C_{\tau_{2n+1}})(c+,\lambda)$.

According to the properties of dependence of the solutions on the parameters, the Wronskians

$W_1(\lambda) = W(\varphi_1(x,\lambda), \dots, \varphi_{2n+1}(x,\lambda))$ and $W_2(\lambda) = W(\tau_1(x,\lambda), \dots, \tau_{2n+1}(x,\lambda))$ are independent of the variable x and are entire functions of parameter λ, short calculation yields that $W_2(\lambda) = \frac{1}{\rho^{\frac{2n+1}{2}}} W_1(\lambda)$, this implies that $\tau_1(x,\lambda), \dots, \tau_{2n+1}(x,\lambda)$ are linearly independent on the interval $(c,b]$.

Lemma 4.1 *Let* $u(x) = \begin{cases} u_1(x), & x \in [a,c) \\ u_2(x), & x \in (c,b] \end{cases}$ *be an arbitrary solution of Eq.* (3) *and can be expressed as follows* $u(x) = \begin{cases} c_1\varphi_1(x) + \cdots + c_{2n+1}\varphi_{2n+1}(x), & x \in [a,c) \\ d_1\tau_1(x) + \cdots + d_{2n+1}\tau_{2n+1}(x), & x \in (c,b] \end{cases}$, $c_1, \dots, c_{2n+1}, d_1, \dots, d_{2n+1} \in \mathbf{C}$, *assume that* $u(x)$ *satisfy the transmission conditions* (5)*, then we have* $c_1 = d_1, \dots, c_{2n+1} = d_{2n+1}$.

Proof The proof can be given similarly as in [17] Lemma 5.3.1, thus is omitted here.

Let

$$\Phi_1(x,\lambda) = (C_{\varphi_1}, \dots, C_{\varphi_{2n+1}})(x,\lambda),\ x \in [a,c), \tag{26}$$

and

$$\Phi_2(x,\lambda) = (C_{\tau_1}, \dots, C_{\tau_{2n+1}})(x,\lambda),\ x \in (c,b], \tag{27}$$

where $\Phi_1(c,\lambda)$ and $\Phi_2(c,\lambda)$ are defined by left and right limits.

Let $\Phi(x,\lambda) = \begin{cases} \Phi_1(x,\lambda), & x \in [a,c) \\ \Phi_2(x,\lambda), & x \in (c,b] \end{cases}$ and $\Phi(c-,\lambda) = \Phi_1(c,\lambda), \Phi(c+,\lambda) = \Phi_2(c,\lambda)$.

It is clear, for any $x \in J$, $\Phi(x, \lambda)$ is an entire function of λ.

Lemma 4.2 *A complex number λ is an eigenvalue of the operator T if and only if*

$$\Delta(\lambda) = \det(A + B\Phi(b, \lambda)) = 0.$$

Proof Let λ be an eigenvalue of the operator T and $u(x, \lambda)$ be the corresponding eigenfunction.

By Lemma 4.1, $u(x, \lambda)$ can be expressed by

$$u(x, \lambda) = \begin{cases} c_1\varphi_1(x, \lambda) + \cdots + c_{2n+1}\varphi_{2n+1}(x, \lambda), & x \in [a, c) \\ c_1\tau_1(x, \lambda) + \cdots + c_{2n+1}\tau_{2n+1}(x, \lambda), & x \in (c, b] \end{cases},$$

where at least one of coefficients $c_i (i = 1, \cdots, 2n + 1)$ is not zero.

Substituting $u(x, \lambda)$ into boundary conditions (4), one obtains that $A(C_{\varphi_1}, \dots, C_{\varphi_{2n+1}})(a, \lambda)(c_1, \dots, c_{2n+1})^T + B(C_{\tau_1}, \dots, C_{\tau_{2n+1}})(b, \lambda)(c_1, \dots, c_{2n+1})^T = 0.$

By (25–27), one gets that

$$(A + B\Phi(b, \lambda))(c_1, \dots, c_{2n+1})^T = 0, \tag{28}$$

since $c_1, \cdots, c_{2n+1}$ are not all zero, so we have $\det(A + B\Phi(b, \lambda)) = 0$.

Conversely, if $\det(A + B\Phi(b, \lambda)) = 0$, then the homogeneous system of the linear Eq. (28) for the constants $c_1, \dots, c_{2n+1}$ has non-zero solution $(c'_1, \dots, c'_{2n+1})^T$.

Let

$$u(x, \lambda) = \begin{cases} c'_1\varphi_1(x, \lambda) + \cdots + c'_{2n+1}\varphi_{2n+1}(x, \lambda), & x \in [a, c) \\ c'_1\tau_1(x, \lambda) + \cdots + c'_{2n+1}\tau_{2n+1}(x, \lambda), & x \in (c, b] \end{cases},$$

then $u(x, \lambda)$ is the non-trivial solution of equation $Mu = \lambda u$ satisfying conditions (4) and (5), therefore, λ is an eigenvalue of T.

In the following, we want to show that a small change of the problem results in only a small change in the eigenvalues and eigenfunctions. To this we introduce the Banach space.

Define

$$\Omega = \left\{\omega = (a, b, c-, c+, A, B, C, w, p_0, \dots, p_n)\right\},$$

and

$$\Omega_1 = \left\{\omega_1 = (a, b, c-, c+, A, B, C, \tilde{w}, \tilde{p}_0, \dots, \tilde{p}_n)\right\}$$

where $\tilde{p}_0 = \begin{cases} p_0, & x \in J \\ 0, & x \in J' \backslash J \end{cases}$, and $\tilde{p}_1, \cdots, \tilde{p}_n, \tilde{w}$ have similar definitions.

Consider the Banach space with the norm

$$X = \mathbf{R}^4 \times M_{2n+1}(\mathbf{C}) \times M_{2n+1}(\mathbf{C}) \times M_{2n+1}(\mathbf{C}) \times \underbrace{L(a', b') \times \cdots \times L(a', b')}_{n+2},$$

$$||\omega|| = ||\omega_1|| = |a| + |b| + |c-| + |c+| + ||A|| + ||B|| + ||C|| + \int_{a'}^{b'} \left(\tilde{w} + \sum_{i=0}^{n} |\tilde{p}_i| \right), \tag{29}$$

where $M_{2n+1}(\mathbf{C})$ denotes the set of $(2n+1) \times (2n+1)$ matrices with complex entries.

It is clear that Ω is not a subset of X, but Ω_1 is. And with Ω_1 as a subset of X to inherit the norm from X and convergence in Ω that is determined by the norm (29). Then based on the space X, the set Ω and Lemma 4.2, we introduce the following theorems.

Theorem 4.1 *Let* $\omega_0 = (a_0, b_0, c_0-, c_0+, A_0, B_0, C_0, w_0, p_{00}, \cdots, p_{n0}) \in \Omega$. *Assume that* $\lambda_0 = \lambda(\omega_0)$ *is an eigenvalue of the operator* T *determined by* ω_0. *Then* $\lambda = \lambda(\omega)$ *is continuous at* ω_0. *That is, given any* $\varepsilon > 0$, *there exists* $\delta > 0$, *such that* $||\omega - \omega_0|| < \delta$ *for any* $\omega \in \Omega$, *then* $|\lambda(\omega) - \lambda(\omega_0)| < \varepsilon$.

Proof From Lemma 4.2, we know that for any $\omega \in \Omega$, $\lambda = \lambda(\omega)$ is an eigenvalue of the operator T if and only if $\Delta(\lambda, \omega) = 0$. Note that $\Delta(\omega, \lambda)$ is an entire function of λ and is continuous in ω(see [38]). Since the operator T is self-adjoint, we know that $\lambda_0 = \lambda(\omega_0)$ is an isolated eigenvalue, then $\Delta(\lambda, \omega_0)$ is not constant in λ. Thus, there exists $\rho_0 > 0$ such that for $\lambda \in S_{\rho_0} := \left\{\lambda \in \mathbf{C} : |\lambda - \lambda_0| = \rho_0\right\}$, we have $\Delta(\lambda, \omega_0) \neq 0$. By the theorem on continuity of the roots of an equation as a function of parameters(see [39]), the proof for Theorem 4.1 is completed.

Remark 4.1 Theorem 4.1 implies that for any fixed eigenvalue $\lambda_0 = \lambda(\omega_0)$, there exists a continuous eigenvalue branch $\lambda(\omega)$ satisfying $\lambda_0 = \lambda(\omega_0)$. However, this result does not mean that for each fixed n, the *nth* eigenvalue $\lambda_n(\omega)$ is always continuous in ω(see [7]). Below we will consider that each eigenvalue $\lambda(\omega)$ of (10) for $\omega \in \Omega$ is embedded in a continuous branch.

Lemma 4.3 (*see*[30]) *Let* y *be a solution of Eq.* (3) *and* (10) *satisfying the initial conditions.* $y^{[j]}(\varsigma, \lambda) = k_j$, $\varsigma \in [a, b]$, $k_j \in \mathbf{C}$, $j = 0, 1, \cdots, 2n$, *then the solution* $y = (\cdot, \varsigma, k_0, \cdots, k_{2n}, p_0, \ldots, p_{2n}, w)$ *is continuous of all its variables.*

Lemma 4.4 *Let* $\omega_0 = (a_0, b_0, c_0-, c_0+, A_0, B_0, C_0, w_0, p_{00}, \ldots, p_{n0}) \in \Omega$. *Let* $\lambda = \lambda(\omega)$ *be an eigenvalue of the operator* T. *Assume the multiplicity of* $\lambda(\omega_0)$ *is* 1, *then there exists a neighborhood* N *of* ω_0 *belonging to* Ω *such that the multiplicity of* $\lambda(\omega)$ *is* 1 *for every* ω *in* N.

Proof If $\lambda(\omega_0)$ is simple, then $\Delta'(\lambda(\omega_0)) \neq 0$. Since $\Delta(\lambda)$ is an entire function of λ, then the conclusion follows from Theorem 4.1.

A normalized eigenfunction u of the operator T means an eigenfunction u satisfies

$$< u, u >= \int_a^c u\bar{u}w_1 dx + \rho \int_c^b u\bar{u}w_2(x)dx = 1.$$

Theorem 4.2 *Let the notation and hypotheses of Theorem 4.1 hold.*

(i)Assume the eigenvalue $\lambda(\omega)$ is simple for all $\omega \in N$, and $N \in \Omega$ is a neighborhood of ω_0. Let $u_1(\cdot, \omega_0)$ be any normalized eigenfunctions of $\lambda(\omega_0)$. Then there exist normalized eigenfunctions $u_1(\cdot, \omega)$ of $\lambda(\omega)$, as $\omega \to \omega_0$, we have $u_1(\cdot, \omega) \to u_1(\cdot, \omega_0)$, $u_1^{[j]}(\cdot, \omega) \to u_1^{[j]}(\cdot, \omega_0)$, $j = 1, 2, \dots, 2n$, uniformly on the interval J.

(ii) Assume the multiplicity of eigenvalue $\lambda(\omega)$ is $l\,(l = 2, \dots, 2n+1)$ for all $\omega \in N$, and $N \in \Omega$ is a neighborhood of ω_0. Let $u_k(\cdot, \omega_0)$ be any normalized eigenfunctions of $\lambda(\omega_0)$. Then there exist l linearly independent normalized eigenfunctions $u_k(\cdot, \omega)$ of $\lambda(\omega)$. As $\omega \to \omega_0$, we have$u_k(\cdot, \omega) \to u_k(\cdot, \omega_0)$, $u_k^{[j]}(\cdot, \omega) \to u_k^{[j]}(\cdot, \omega_0)$, $k = 1, \dots, l, j = 1, 2, \dots, 2n$, uniformly on the interval J.

Proof The proof can be given similarly as in [23], with the aid of Theorem 4.1 and Lemma 4.3.

5 Differential Expressions of Eigenvalues on the Problems

In this section we introduce the derivatives of eigenvalues with respect to the given parameters.

Definition 5.1. (see [10]) Let X, Y be Banach space. A map $\Gamma : X \to Y$ is Fréchet differentiable at a given point $x \in X$, if a bounded linear operator $d\Gamma_x : X \to Y$ satisfies for $h \in X$, $|\Gamma(x+h) - \Gamma(x) - d\Gamma(h)| = o(h)$ as $h \to 0$.

Lemma 5.1 (*see* [6]) *Assume a real-valued function $f \in L_{loc}(a, b)$, then*

$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f = f(x)$ *a.e. in* (a, b).

Theorem 5.1 *Let $\omega = (a, b, c-, c+, A, B, C, w, p_0, \dots, p_n) \in \Omega$, $\lambda = \lambda(\omega)$ be an eigenvalue of operator T connected with ω, and let $u = u(\cdot, \omega)$ be the corresponding normalized eigenfunction of $\lambda(\omega)$. Assume that $\lambda(\omega)$ has constant geometric multiplicity in some neighborhood of ω in Ω. Then λ is continuously differentiable with respect to all the parameters in ω. More precisely, we have the following.*

(i) *Fix all the parameters of* ω *except* $p_{n-k}(k=0,1,\cdots,n)$. *Let* $\lambda=\lambda(p_{n-k})$ *and* $u=u(\cdot,p_{n-k})$, *then* λ *is Fréchet differentiable at* $p_{n-k}(k=0,1,\cdots,n)$ *and*

$$d\lambda_{p_{n-k}}(h)=(-1)^k\left(\int_a^c h|u^{(n-k)}|^2dx+\rho\int_c^b h|u^{(n-k)}|^2dx\right),\ h\in L(J).$$

(ii) *Fix all the parameters of* ω *except* w. *Let* $\lambda=\lambda(w)$ *and* $u=u(\cdot,w)$, *then* λ *is Frechet differentiable at* w *and*

$$d\lambda_w(h)=-\lambda(\int_a^c h|u|^2dx+\rho\int_c^b h|u|^2dx),\ h\in L(J).$$

(iii) *Fix all the parameters of* ω *except* A. *For small* $K\in M_{2n+1}(\mathbf{C})$ *satisfying*

$\rho(A+K)Q_{2n+1}(A+K)^*=BQ_{2n+1}B^*$ *in the neighborhood of* A. *Let* $\lambda=\lambda(A)$ *and* $u=u(\cdot,A)$, *then* λ *is Fréchet differentiable at* A *and*

$$d\lambda_A(K)=-U^*(a)K^*(A^{-1})^*Q_{2n+1}U(a).$$

(iv) *Fix all the parameters of* ω *except* B. *For small* $K\in M_{2n+1}(\mathbf{C})$ *satisfying*

$\rho AQ_{2n+1}A^*=(B+K)Q_{2n+1}(B+K)^*$ *in the neighborhood of* B. *Let* $\lambda=\lambda(B)$ *and* $u=u(\cdot,B)$, *then* λ *is Fréchet differentiable at* B *and*

$$d\lambda_B(K)=\rho U^*(b)K^*(B^{-1})^*Q_{2n+1}U(b).$$

(v) *Fix all the parameters of* ω *except* C. *For small* $K\in M_{2n+1}(\mathbf{C})$ *satisfying*

$\rho Q_{2n+1}=(C+K)^*Q_{2n+1}(C+K)$ *and* $\det(C+K)=\det(C)=\rho^{\frac{2n+1}{2}}$ *in the neighborhood of* C. *Let* $\lambda=\lambda(C)$ *and* $u=u(\cdot,C)$, *then* λ *is Fréchet differentiable at* C *and*

$$d\lambda_C(K)=-U^*(c+)K^*Q_{2n+1}CU(c+).$$

(vi) *Fix all the parameters of* ω *except* a. *Let* $\lambda=\lambda(a)$ *and* $u=u(\cdot,a)$, *then* λ *is Fréchet differentiable at* a *and*

$\lambda'_a(h)=-(U^*)'(a,a)Q_{2n+1}U(a,a)$, *a.e. in* (a',c).

(vii) *Fix all the parameters of* ω *except* b. *Let* $\lambda=\lambda(b)$ *and* $u=u(\cdot,b)$, *then* λ *is Fréchet differentiable at* b *and*

$\lambda'_b(h)=\rho\,(U^*)'\,(b,b)\,Q_{2n+1}U(b,b)$, *a.e. in* (c,b').

(viii) *Fix all the parameters of* ω *except* c_1, *here* $c_1=c-$. *Let* $\lambda=\lambda(c_1)$ *and* $u=u(\cdot,c_1)$, *then* λ *is Fréchet differentiable at* c_1 *and*

$\lambda'_{c_1}(h)=(U^*)'(c_1,c_1)Q_{2n+1}U(c_1,c_1)$, *a.e. in* (a,b).

(ix) *Fix all the parameters of* ω *except* c_2, *here* $c_2 = c+$. *Let* $\lambda = \lambda(c_2)$ *and* $u = u(\cdot, c_2)$, *then* λ *is Fréchet differentiable at* c_2 *and* $\lambda'_{c_2}(h) = -\rho(U^*)'(c_2, c_2)Q_{2n+1}U(c_2, c_2)$, *a.e. in* (a, b).

Proof First at all, we should emphasize that by $\lambda(\omega)$ we mean a continuous eigenvalue branch, further be a normalized eigenfunction $u(\cdot, \omega)$ we mean a uniformly convergent normalized eigenfunction branch.

(i) Let $u = u(\cdot, p_{n-k}), v = u(\cdot, p_{n-k} + h)$ such that $u(\cdot, p_{n-k} + h) \to u(\cdot, p_{n-k})$ $(k = 0, 1, \cdots, n)$ uniformly on J as $h \to 0$, integrating by parts, we have

$$
\begin{aligned}
&[\lambda(p_{n-k} + h) - \lambda(p_{n-k})] < u, v > \\
&= \left[\lambda(p_{n-k} + h) - \lambda(p_{n-k})\right]\left(\int_a^c u\bar{v}w_1 dx + \rho \int_c^b u\bar{v}w_2 dx\right) \\
&= (-1)^k\left\{\int_a^c \left[(p_{n-k} + h) - p_{n-k}\right] u^{(n-k)}\overline{v}^{(n-k)} dx + \rho \int_c^b \left[(p_{n-k} + h) - p_{n-k}\right] u^{(n-k)}\overline{v}^{(n-k)}]dx\right\} \\
&= (-1)^k\left(\int_a^c h|u^{(n-k)}|^2 dx + \rho \int_c^b |u^{(n-k)}|^2 dx\right),
\end{aligned}
$$

then by Theorem 4.1 and 4.2 we have

$$
[\lambda\,(p_{n-k} + h) - \lambda\,(p_{n-k})]\,(1 + o(1)) = (-1)^k\left(\int_a^c h|u^{(n-k)}|^2 dx + \rho \int_c^b |u^{(n-k)}|^2 dx\right) + o(h),
$$

and consequently

$$
\begin{aligned}
\lambda(p_{n-k} + h) - \lambda(p_{n-k}) = &\left[(-1)^k\left(\int_a^c h|u^{(n-k)}|^2 dx + \rho \int_c^b |u^{(n-k)\,2}|dx\right) + o(h)\right] \\
&(1 + o(1))^{-1} = (-1)^k\left(\int_a^c h|u^{(n-k)}|^2 dx + \rho \int_c^b |u^{(n-k)}|^2 dx\right) \\
&+ o(h),\ as\, h \to 0.
\end{aligned}
$$

Therefore, we have $d\lambda_{p_{n-k}}(h) = (-1)^k\left(\int_a^c h|u^{(n-k)}|^2 dx + \rho \int_c^b h|u^{(n-k)}|^2 dx\right)$. Thus (i) hold.

(ii) Let $My = \sum_{k=0}^{n} (-1)^k \left\{i[(q_{n-k}y^{(k)})^{(k+1)} + (q_{n-k}y^{(k+1)})^{(k)}] + (p_{n-k}y^{(k)})^{(k)}\right\} = \lambda wy$, then Eq. (3) can be expressed as

$$
Ly = \frac{1}{w}My. \tag{30}
$$

Let $u = u(\cdot, w)$, $v = u(\cdot, w + h)$, from (30) we get that

$$Lu=\frac{1}{w}Mu,\ Lv=\frac{1}{w+h}Mu, \tag{31}$$

hence, it follows that

$$[\lambda(w+h)-\lambda(w)]<u,v>$$

$$=\left(\int_a^c u\overline{Lv}w_1\,dx+\rho\int_c^b u\overline{Lv}w_2\,dx\right)-\left(\int_a^c Lu\bar{v}w_1\,dx+\rho\int_c^b Lu\bar{v}w_2\,dx\right)$$

$$=\left(\int_a^c \frac{w_1}{w_1+h}u\overline{Mu}\,dx+\rho\int_c^b \frac{w_2}{w_2+h}u\overline{Mu}\,dx\right)-\left(\int_a^c Mu\bar{v}dx+\rho\int_c^b Mu\bar{v}dx\right)$$

$$=\int_a^c \frac{-h}{w_1+h}u\overline{Mu}\,dx+\rho\int_c^b \frac{-h}{w_2+h}u\overline{Mu}\,dx$$

$$=-\left(\int_a^c hu\,\overline{\lambda u}\,dx+\rho\int_c^b hu\,\overline{\lambda u}\,dx\right)$$

$$=-\lambda(\int_a^c h|u|^2dx+\rho\int_c^b h|u|^2dx),$$

then we can get the result as follows by using similar discussion to that of (i)

$$d\lambda_w(h)=-\lambda(\int_a^c h|u|^2dx+\rho\int_c^b h|u|^2dx).$$

(iii) Let $u=u(\cdot,A)$, $v=u(\cdot,A+K)$ such that $u(\cdot,A+K)\to u(\cdot,A)$ as $K\to 0$, then by (7)–(9) and (12) we have

$$[\lambda(A+K)-\lambda(A)]<u,v>$$

$$=-[u\bar{v}]|_a^c-\rho[u\bar{v}]|_c^b$$

$$=[u\bar{v}](a)-\rho[u\bar{v}](b)$$

$$=V^*(a)Q_{2n+1}U(a)-\rho V^*(b)Q_{2n+1}U(b)$$

$$=U^*(a)Q_{2n+1}U(a)-\rho U^*(a)(B^{-1}(A+K))^*Q_{2n+1}(B^{-1}A)U(a), \tag{32}$$

from (6), we get that

$$(B^{-1}A)^*Q_{2n+1}(B^{-1}A) = \frac{1}{\rho}Q_{2n+1},\ \rho(B^{-1})^*Q_{2n+1}B^{-1} = (A^{-1})^*Q_{2n+1}A^{-1}. \tag{33}$$

Combining (32) and (33) and letting $K \to 0$, we obtain that

$$\lambda(A+K) - \lambda(A) = -U^*(a)K^*(A^{-1})^*Q_{2n+1}U(a) + o(K), \tag{34}$$

thus the result follows from the (34).

The proof for (iv) is similar to this proof, thus is omitted here.

(v) Let $u = u(\cdot, C)$, $v = u(\cdot, C+K)$, then by (7–9) and (13) we have

$$[\lambda(C+K) - \lambda(C)] < u, v >= -[u\bar{v}]|_a^c - \rho[u\bar{v}]|_c^b = \rho[u\bar{v}](c+) - [u\bar{v}](c-)$$

$$= \rho V^*(c+)Q_{2n+1}U(c+) - V^*(c-)Q_{2n+1}U(c-)$$

$$= \rho U^*(c+)Q_{2n+1}U(c+) - U^*(c+)(C+K)^*Q_{2n+1}CU(c+)$$

$$= -U^*(c+)K^*Q_{2n+1}CU(c+),$$

let $K \to 0$, the desired result can be obtained by Theorem 4.2.

(vi) For small h, let $u = u(\cdot, a)$, $v = u(\cdot, a+h)$, then by condition (4) and (5) we have

$$[\lambda(a+h) - \lambda(a)] < u, v >= -[u\bar{v}]|_a^c - \rho[u\bar{v}]|_c^b = [u\bar{v}](a) - \rho[u\bar{v}](b)$$

$$= V^*(a)Q_{2n+1}U(a) - \rho V^*(b)Q_{2n+1}U(b)$$

$$= U^*(a, a+h)Q_{2n+1}U(a,a) - \rho\,(B^{-1}AU(a+h, a+h))^*Q_{2n+1}\,(B^{-1}AU(a,a))$$

$$= U^*(a, a+h)Q_{2n+1}U(a,a) - \rho U^*(a+h, a+h)(B^{-1}A)^*Q_{2n+1}(B^{-1}A)U(a,a), \tag{35}$$

substituting (33) into (35), we obtain that

$$[\lambda(a+h) - \lambda(a)] < u, v >$$

$$= U^*(a, a+h)Q_{2n+1}U(a,a) - U^*(a+h, a+h)Q_{2n+1}U(a,a)$$

$$= [U^*(a, a+h) - U^*(a+h, a+h)]Q_{2n+1}U(a,a)$$

$$= -\left[\int_a^{a+h} (\,U^*)'(s, a+h)ds\right] Q_{2n+1} U\,(a, a)$$

$$= -\left[\int_a^{a+h} (U^*)'(s, a)ds + o(h)\right] Q_{2n+1} U(a, a),$$

from Lemma 5.1, we get that

$[\lambda(a+h) - \lambda(a)] = -h(U^*)'(a, a)Q_{2n+1}U(a, a) + o(h)$, a.e. in (a', c).

Dividing both sides of the above equality by h and letting $h \to 0$, then we obtain

$\lambda_a'(h) = -(U^*)'(a, a)Q_{2n+1}U(a, a)$.

Therefore, (vi) hold. Using the same methods of (vi), one can prove that (vii) is also true.

(viii) For small h, let $u = u(\cdot, c_1)$, $v = u(\cdot, c_1 + h)$, then by condition (5) and (13) we have

$$[\lambda(c_1+h) - \lambda(c_1)] < u, v >= -[u\bar{v}]|_a^c - \rho[u\bar{v}]|_c^b = \rho[u\bar{v}](c_2) - [u\bar{v}](c_1)$$

$$= \rho V^*(c_2)Q_{2n+1}U(c_2) - V^*(c_1)Q_{2n+1}U(c_1)$$

$$= \rho(C^{-1}U(c_1+h, c_1+h))^*Q_{2n+1}(C^{-1}U(c_1, c_1)) - U^*(c_1, c_1+h)Q_{2n+1}U(c_1, c_1)$$

$$= \rho U^*(c_1+h, c_1+h)(C^{-1})^*Q_{2n+1}C^{-1}U(c_1, c_1) - U^*(c_1, c_1+h)Q_{2n+1}U(c_1, c_1). \tag{36}$$

Further, it follows from (6) that

$$(C^{-1})^*Q_{2n+1}C^{-1} = \frac{1}{\rho}Q_{2n+1}, \tag{37}$$

substituting (37) into (36), we can obtain

$$[\lambda(c_1+h) - \lambda(c_1)] < u, v >$$

$$= U^*(c_1+h, c_1+h)Q_{2n+1}U(c_1, c_1) - U^*(c_1, c_1+h)Q_{2n+1}U(c_1, c_1)$$

$$= [U^*(c_1+h, c_1+h) - U^*(c_1, c_1+h)]Q_{2n+1}U(c_1, c_1)$$

$$= \left[\int_{c_1}^{c_1+h} (U^*)'(s, c_1+h)ds\right] Q_{2n+1} U\,(c_1, c_1)$$

$$= \left[\int_{c_1}^{c_1+h} (U^*)'(s,c_1)ds + o(h)\right] Q_{2n+1}U(c_1,c_1),$$

hence, from Lemma 5.1, we further obtain

$$\lambda(c_1+h) - \lambda(c_1) = h(U^*)'(c_1,c_1)Q_{2n+1}U(c_1,c_1) + o(h)\, a.e. \text{ in } (a,b).$$

Dividing both sides of the above equality by h and letting $h \to 0$, then we obtain. $\lambda'_{c_1}(h) = (U^*)'(c_1,c_1)Q_{2n+1}U(c_1,c_1)$.

Hence, (viii) hold. Using the same methods of (viii), one can prove that (ix) hold.

6 Conclusion

This paper investigate the eigenvalues dependence of a class of $(2n+1)$ th order differential equations with transmission conditions. We obtain that the eigenvalues of the problems not only continuously but also differentiably depend on the given parameters of the problems and obtain some new differential expressions of the eigenvalues with respect to the given parameters. This extend the theorems in [31] from the third-order case to general order case with general boundary conditions and transmissions conditions. This further develops the theory of boundary value problems of odd order differential operators.

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References

1. Bailey, P., Gordon, M., Shampine, L.: Automatic solution of the Sturm-Liouville problem. ACM Trans. Math. Softw. **4**(3), 193–208 (1978)
2. Bailey, P., Everitt, W., Zettl, A.: The SLEIGN2 Sturm-Liouville code. ACM Trans. Math. Softw. **21**, 143–192 (2001)
3. Greenberg, L., Marletta, M.: The code SLEUTH for solving fourth order Sturm-Liouville problems. ACM Trans. Math. Softw. **23**, 453–493 (1997)
4. Poeschel, J., Trubowitz, E.: Inverse spectral theory. Academic Press, New York (1987)
5. Dauge, M., Helffer, M.: Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators. J. Differ. Equ. **104**, 243–262 (1993)
6. Kong, Q., Zettl, A.: Dependence of eigenvalues of Sturm-Liouville problems on the boundary. J. Differ. Equ. **126**, 389–407 (1996)
7. Kong, Q., Zettl, A.: Eigenvalues of regular Sturm-Liouville problems. J. Differ. Equ. **131**, 1–19 (1996)
8. Naimark, M.A.: Linear differential operators. Frederick Ungar Publishing Co, London (1968)
9. Weidmann, J.: Spectral Theory of Ordinary Differential Operators. In: Lecture Notes in Mathematics V1268. Springer Verlag, Berlin (1987)
10. Kong, Q., Wu, H., Zettl, A.: Dependence of eigenvalues on the problem. Math. Nachr. **188**, 173–201 (1997)
11. Tikhonov, A.N., Samarskii, A.A.: Equations of mathematical physics. Dover Publications Inc., New York (1990)
12. Cao, X., Diao, H., Liu, H., Zou, J.: On novel geometric structures of Laplacian eigenfunctions in R^3 and applications to inverse problems. SIAM J. Mathemat. Anal. **53**, 1263–1294 (2021)
13. Diao, H., Cao, X., Liu, H.: On the geometric structures of transmission eigenfunctions with a conductive boundary condition and applications. Comm. Partial Different. Equat. **46**(4), 630–679 (2021)
14. Diao, H., Liu, H., Sun, B.: On a local geometric structure of generalized elastic transmission eigenfunctions and application. Inverse Prob. **37**, 105015 (2021)
15. Tunc, E., Mukhtarov, O.: Fundamental solutions and eigenvalues of one boundary value problem with transmission conditions. Appl. Math. Comput. **157**, 347–355 (2004)
16. Mukhtarov, O., Kadakal, M.: Some spectral properties of one Sturm-Liouville type problem with discontinuous weight. Sib. Math. J. **46**(4), 860–875 (2005)
17. Wang, A.: Research on Weimann conjecture and differential operators with transmission conditions[D]. Inner Mongolia University, Inner Mongolia (2006). ((**Chinese**))
18. Kadakal, M., Mukhtarov, O.: Sturm-Liouville problems with discontinuities at two points. Comput. Math. Appl. **54**, 1367–1379 (2007)
19. Yang, Q., Wang, W.: Asymptotic behavior of a differential operator with discontinuities at two points. Math. Methods Appl. **34**, 373–383 (2011)
20. Zhang, M., Sun, J., Zettl, A.: Eigenvalues of limit-point Sturm-Liouville problems. J. Math. Anal. Appl. **419**, 627–642 (2014)
21. Zhang, M., Wang, Y.: Dependence of eigenvalues of Sturm-Liouville problems with interface conditions. Appl. Math. Comput. **265**, 31–39 (2015)
22. Li, K., Sun, J., Hao, X.: Eigenvalues of regular fourth order Sturm-Liouville problems with transmission conditions. Math. Methods Appl. **40**, 3538–3551 (2017)
23. Li, K., Sun, J., Hao, X.: Dependence of eigenvalues of 2nth order boundary value transmission problems. Bound. Value Probl. **2017**(1), 143 (2017)

24. Zhang, H., Ao, J., Li, M.: Dependence of Eigenvalues of Sturm-Liouville Problems with Eigenparameter-Dependent Boundary Conditions and Interface Conditions. Mediterr. J. Math. **19**(2), 1–17 (2022)
25. Zinsou, B.: Dependence of eigenvalues of fourth-order boundary value problems with transmission conditions. Rocky Mt. J. Math. **50**, 369–381 (2020)
26. Mukhtarov, O., Aydemir, K.: Two-linked periodic Sturm-Liouville problems with transmission conditions. Math. Methods Appl. **44**, 1–13 (2021)
27. Ao, J., Zhang, L.: An inverse spectral problem of Sturm-Liouville problems with transmission conditions. Mediter. J. Math. **17**, 160 (2020)
28. Walker, P.: A vector-matrix formulation for formally symmetric ordinary differential equations with applications to solutions of integrable square. J. London Math. Soc. **9**, 151–159 (1974)
29. Hinton, D.: Defificiency indices of odd-order differential operators. Rocky Mt. J. Math. **8**(4), 627–640 (1978)
30. Uğurlu, E.: Regular third-order boundary value problems. Appl. Math. Comput. **343**, 247–257 (2019)
31. Uğurlu, E.: Third-order boundary value transmission problems. Turk. J. Math. **43**, 1518–1532 (2019)
32. Li, K., Bai, Y., Wang, W., Meng, F.: Self-adjoint realization of a class of third-order differential operators with an eigenparameter contained in the boundary conditions. J. Appl. Anal. Comput. **10**(6), 2631–2643 (2020)
33. Bai, Y., Wang, W., Li, K., Zheng, Z.: Eigenvalues of a class of eigenparameter dependent third-order differential operators. J. Nonlinear Math. Phys. **29**, 477–492 (2022)
34. Niu, T., Hao, X., Sun, J., Li, K.: Canonical forms of self-adjoint boundary conditions for regular differential operators of order three. Oper. Matrices. **14**(1), 207–220 (2020)
35. Sun, K., Gao, Y.: 2022 A Class of Differential Operators with Eigenparameter Dependent Boundary Conditions. Journal of Mathematical Physics:Series A
36. Zolotarev, V.A.: Inverse spectral problem for a third-order differential operator with non-local potential. J. Differ. Equ. **303**, 456–481 (2021)
37. Cao, Z.J.: Ordinary differential operators (in Chinese). Science Press, Beijing (2016)
38. Kong, Q., Zettl, A.: Linear ordinary differential equations. In: Agarwal, R.P. (ed.) Inequalities and Applications, WSSIAA, vol. 3. World Scientific, Singapore, pp. 381–397 (1994)
39. Dieudonné, J.: Foundations of Modern Analysis. Academic Press, New York (1969)