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RESEARCH ARTICLE

On $[p,q]_{,\varphi}$-Order and Complex Differential Equations

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Abstract

The fast growing solutions of the following linear differential equation $(*)$ is investigated by using a more general scale $[p,q]_{,\varphi}$-order,

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \qquad (*)$$

where $A_i(z)$ are entire functions in the complex plane, $i = 0, 1, \ldots, k-1$. The growth relationships between entire coefficients and solutions of the equation $(*)$ is found by using the concepts of $[p,q]_{,\varphi}$-order and $[p,q]_{,\varphi}$-type, which extend and improve some previous results.

Keywords Linear differential equations · Entire functions · $[p,q]_{,\varphi}$-order · $[p,q]_{,\varphi}$-type

Mathematics Subject Classification 34M10 · 30D35

1 Introduction and Main Results

We assume that the reader is familiar with the fundamental results and the standard notation of Nevanlinna theory in the complex plane $\mathbb{C}$, see [8, 15] for more details. Considering the linear differential equation

$$L(f) := f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1.1}$$

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where $A_0(z), \dots, A_{k-1}(z)$ are entire functions in $\mathbb{C}$ and $k(\geq 2)$ is integer. Usually, order and hyper order are used to study the growth of solutions of Eq. (1.1), for example, see [7, 10, 14, 15, 18, 19, 20, 21] and therein references. For the fast growing entire function, the iterated order is defined to measure their growing. It is well-known that Kinnunen firstly used the idea of iterated order to study the fast growing of solutions of Eq. (1.1) in [13]. Since then, the iterated order of solutions of Eq. (1.1) is very interesting topic, many results concerning iterated order of solutions of Eq. (1.1) have been obtained, for example [3, 9] and therein references. To estimate precisely the fast growing of entire functions, the concept of $[p, q]$-order is defined in [12]. From then, many results concerning $[p, q]$-order of solutions of Eq. (1.1) have been found by different researchers, for example [16, 17] and theirin references.

In [4], Chyzhykov and Semochko have pointed out that the definition of $[p, q]$-order have weaknesses is that it do not cover arbitrary growth, and given Examples 1.4 and 1.7 in [4] to show the case. And the same time, they given more general growth scale of meromorphic function as follows.

Definition 1 ([4]) Let φ be an increasing unbounded function on $[1, +\infty)$, and f be a meromorphic function. The φ-orders of f are defined by

$$\rho_\varphi^0(f) = \lim_{r\to+\infty} \sup \frac{\varphi(e^{T(r,f)})}{\log r},$$

$$\rho_\varphi^1(f) = \lim_{r\to+\infty} \sup \frac{\varphi(T(r,f))}{\log r}.$$

If f is an entire function, then the φ-orders are defined by

$$\tilde{\rho}_\varphi^0(f) = \lim_{r\to+\infty} \sup \frac{\varphi(M(r,f))}{\log r},$$

$$\tilde{\rho}_\varphi^1(f) = \lim_{r\to+\infty} \sup \frac{\varphi(\log M(r,f))}{\log r}.$$

Remark 1 ([4]) Let $\varphi \in \Phi$ and f be an entire function. Then

$$\rho_\varphi^j(f) = \tilde{\rho}_\varphi^j(f), j = 0, 1.$$

The properties of Φ and φ will be shown in the following Sect. 2. Furthermore, Chyzhykov and Semochko studied the growth of solutions of Eq. (1.1) by using the concept of φ-order.

Theorem 1.1 ([4]) *Let $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ be entire functions. Then all non-trivial solutions f of Eq.* (1.1) *satisfy*

$$\sup\left\{\rho_\varphi^1(f) | L(f) = 0\right\} = \sup\left\{\rho_\varphi^0(A_j) | j = 0, \dots, k-1\right\}.$$

Theorem 1.2 ([4]) *Let $\varphi \in \Phi$, and $l = \max\left\{j | \rho_\varphi^0(A_j) \geq \beta, j = 0, \dots, k-1\right\}$. Then Eq.* (1.1) *possesses at most l entire linearly independent solutions f with $\rho_\varphi^1(f) < \beta$.*

Theorem 1.3 ([4]) *Let* $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ *be entire functions such that* $\rho^0_\varphi(A_0) > \max\left\{\rho^0_\varphi(A_j), j = 1, \dots, k-1\right\}$. *Then all nontrivial solutions f of Eq.* (1.1) *satisfy* $\rho^1_\varphi(f) = \rho^0_\varphi(A_0)$.

Recently, Belaïdi defined the concept of φ-type of meromorphic functions which is used to study the growth of solutions of Eq. (1.1), and the following Theorem 1.4 is obtained.

Definition 2 ([2]) Let φ be an increasing unbounded function on $[1, +\infty)$, and f be a meromorphic function with $\rho^i_\varphi(f) \in (0, +\infty), i = 0, 1$. The φ-types of f are defined by

$$\tau^0_\varphi(f) = \lim_{r\to+\infty} \sup \frac{\exp\left\{\varphi(e^{T(r,f)})\right\}}{r^{\rho^0_\varphi(f)}},$$
$$\tau^1_\varphi(f) = \lim_{r\to+\infty} \sup \frac{\exp\{\varphi(T(r,f))\}}{r^{\rho^1_\varphi(f)}}.$$

If f is an entire function, then the φ-types of f are defined by

$$\tilde{\tau}^0_\varphi(f) = \lim_{r\to+\infty} \sup \frac{\exp\{\varphi(M(r,f))\}}{r^{\tilde{\rho}^0_\varphi(f)}},$$
$$\tilde{\tau}^1_\varphi(f) = \lim_{r\to+\infty} \sup \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\tilde{\rho}^1_\varphi(f)}}.$$

Theorem 1.4 ([2]) *Let* $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ *be entire functions. Assume that*

$$\max\left\{\tilde{\rho}^0_\varphi(A_j), j = 1, \dots, k-1\right\} \le \tilde{\rho}^0_\varphi(A_0) = \rho_0 < +\infty,$$

and

$$\max\left\{\tilde{\tau}^0_{M,\varphi}(A_j) : \tilde{\rho}^0_\varphi(A_j) = \tilde{\rho}^0_\varphi(A_0)\right\} < \tilde{\tau}^0_{M,\varphi}(A_0) = \tau.$$

Then all nontrivial solutions f of Eq. (1.1) *satisfy* $\tilde{\rho}^1_\varphi(f) = \tilde{\rho}^0_\varphi(A_0)$.

Motivated to the $[p, q]$-order of meromorphic function. We introduce the concepts of $[p, q]_{,\varphi}$-order and $[p, q]_{,\varphi}$-type, where $p \ge q \ge 1$. For all $r \in (0, +\infty)$, $\exp_1 r = e^r$, $\exp_{n+1} r = \exp(\exp_n r)$ and $\log_1 r = \log r$ and $\log_{n+1} r = \log(\log_n r)$, $n \in N$. We also denote $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log_1 r$. The $[p, q]_{,\varphi}$-order and $[p, q]_{,\varphi}$-type are defined as follows, respectively.

Definition 3 Let φ be an increasing unbounded function on $[1, +\infty)$, and f be a meromorphic function. The $[p, q]_{,\varphi}$-orders of f are defined by

$$\rho^0_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\varphi(e^{\log_{p-1} T(r,f)})}{\log_q r},$$

$$\rho^1_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\varphi(\log_{p-1} T(r,f))}{\log_q r}.$$

If f is an entire function, then the $[p,q]_{,\varphi}$-orders of f are defined by

$$\tilde{\rho}^0_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\varphi(e^{\log_p M(r,f)})}{\log_q r},$$

$$\tilde{\rho}^1_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\varphi(\log_p M(r,f))}{\log_q r}.$$

Definition 4 Let φ be an increasing unbounded function on $[1,+\infty)$, and f be a meromorphic function with $\rho^i_{[p,q],\varphi}(f) \in (0,+\infty), i = 0, 1$. The $[p,q]_{,\varphi}$-types of f are defined by

$$\tau^0_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\exp\left\{\varphi(e^{\log_{p-1} T(r,f)})\right\}}{[\log_{q-1} r]^{\rho^0_{[p,q],\varphi}(f)}},$$

$$\tau^1_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\exp\left\{\varphi(\log_{p-1} T(r,f))\right\}}{[\log_{q-1} r]^{\rho^1_{[p,q],\varphi}(f)}}.$$

If f is an entire function with $\tilde{\rho}^i_{[p,q],\varphi}(f) \in (0,+\infty), i = 0, 1$, then the $[p,q]_{,\varphi}$-types of f are defined by

$$\tilde{\tau}^0_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\exp\left\{\varphi(e^{\log_p M(r,f)})\right\}}{[\log_{q-1} r]^{\tilde{\rho}^0_{[p,q],\varphi}(f)}},$$

$$\tilde{\tau}^1_{[p,q],\varphi}(f) = \lim_{r\to+\infty} \sup \frac{\exp\left\{\varphi(\log_p M(r,f))\right\}}{[\log_{q-1} r]^{\tilde{\rho}^1_{[p,q],\varphi}(f)}}.$$

The following two examples show that $[p,q]_{,\varphi}$-order is indeed superior to φ-order when studying the same fast growth functions.

Example 1 It follows from [5] that $\exp_4(\alpha(\log r)^\beta)$ is convex in $\log r$. Then there exists an entire function f that satisfies

$$\log_4 T(r,f) = (\alpha + o(1))(\log r)^\beta,$$

where $\alpha, \beta > 0$.

For $\varphi(r) = (\log_2 r)^{\frac{1}{\beta}}$, we can get that

$$\rho^1_\varphi(f) = \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\log r} = \limsup_{r\to+\infty} \frac{[\exp_2((\alpha + o(1))(\log r)^\beta)]^{\frac{1}{\beta}}}{\log r} = +\infty,$$

however,

$$\rho^1_{[3,1],\varphi}(f) = \limsup_{r\to+\infty} \frac{\varphi(\log_2 T(r,f))}{\log r} = \limsup_{r\to+\infty} \frac{(\alpha(\log r)^{\beta})^{\frac{1}{\beta}}}{\log r} = \alpha^{\frac{1}{\beta}}.$$

Example 2 It follows from [5] that $\exp_2(\alpha(\log r)^{\beta})$ is convex in $\log r$. Then there exists an entire function f that satisfies

$$\log_2 T(r,f) = (\alpha + o(1))(\log r)^{\beta},$$

where $\alpha, \beta > 0$.

For $\varphi(r) = (\log_2 r)^{\frac{1}{\beta}}$, we can get that

$$\rho^1_{\varphi}(f) = \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\log r} = \alpha^{\frac{1}{\beta}},$$

however,

$$\rho^1_{[3,2],\varphi}(f) = \limsup_{r\to+\infty} \frac{\varphi(\log_2 T(r,f))}{\log_2 r} = \limsup_{r\to+\infty} \frac{(\log_2[(\alpha + o(1))(\log r)^{\beta}])^{\frac{1}{\beta}}}{\log_2 r} = 0.$$

Here, we study the growth of solutions of Eq. (1.1) by using the concepts of $[p,q]_{,\varphi}$-order and $[p,q]_{,\varphi}$-type, Theorems 1.5–1.8 are obtained which are generalization of previous results from Chyzhykov-Semochko [4] and Belaïdi [2].

Theorem 1.5 *Let $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ be entire functions. Then all nontrivial solutions f of Eq. (1.1) satisfy*

$$\sup\left\{ \rho^1_{[p,q],\varphi}(f) | L(f) = 0 \right\} = \sup\left\{ \rho^0_{[p,q],\varphi}(A_j) | j = 0, \dots, k-1 \right\}.$$

Theorem 1.6 *Let $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ be entire functions, $m = \max\left\{ j | \rho^0_{[p,q],\varphi}(A_j) \ge \lambda, j = 0, \dots, k-1 \right\}$. Then Eq. (1.1) possesses at most m entire linearly independent solutions f with $\rho^1_{[p,q],\varphi}(f) < \lambda$.*

Theorem 1.7 *Let $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ be entire functions such that $\rho^0_{[p,q],\varphi}(A_0) > \max\left\{ \rho^0_{[p,q],\varphi}(A_j), j = 1, \dots, k-1 \right\}$. Then all nontrivial solutions f of Eq. (1.1) satisfy $\rho^1_{[p,q],\varphi}(f) = \rho^0_{[p,q],\varphi}(A_0)$.*

Theorem 1.8 *Let $\varphi \in \Phi$, $A_0(z), \dots, A_{k-1}(z)$ be entire functions. Assume that*

$$\max\left\{ \tilde{\rho}^0_{[p,q],\varphi}(A_j), j = 1, \dots, k-1 \right\} \le \tilde{\rho}^0_{[p,q],\varphi}(A_0) = \rho_0 < +\infty,$$

and

$$\max\left\{ \tilde{\tau}^0_{[p,q],\varphi}(A_j) : \tilde{\rho}^0_{[p,q],\varphi}(A_j) = \tilde{\rho}^0_{[p,q],\varphi}(A_0) \right\} < \tilde{\tau}^0_{[p,q],\varphi}(A_0) = \tau.$$

Then all nontrivial solutions f of Eq. (1.1) *satisfy* $\tilde{\rho}^1_{[p,q],\varphi}(f) = \tilde{\rho}^0_{[p,q],\varphi}(A_0)$.

2 Properties of $[p, q]_{,\varphi}$-order

In [4], Chyzhykov and Semochko defined the class of positive unbounded increasing function on $[1, +\infty)$ by Φ such that $\varphi(e^t)$ is slowly growing, i. e.,

$$\forall c > 0: \quad \frac{\varphi(e^{ct})}{\varphi(e^t)} \to 1, \quad t \to +\infty.$$

First, we recall properties of functions from the class Φ.

Proposition 2.1 ([4]) *If* $\varphi \in \Phi$, *then*

$$\forall m > 0, \quad \forall k \geq 0: \ \frac{\varphi^{-1}(\log x^m)}{x^k} \to +\infty, \quad x \to +\infty; \tag{2.1}$$

$$\forall \delta > 0: \ \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \to +\infty, \quad x \to +\infty. \tag{2.2}$$

Remark 2 If φ is non-decreasing, then (2.2) is equivalent to the definition of the class Φ.

Next, we obtain some basic properties of $[p, q]_{,\varphi}$-order by using standard method.

Proposition 2.2 *Let* $\varphi \in \Phi$, *and f be an entire function. Then*

$$\rho^j_{[p,q],\varphi}(f) = \tilde{\rho}^j_{[p,q],\varphi}(f), j = 0, 1.$$

Proof First, we prove that this is true when $j = 1$, and it can be proved for the case of $j = 0$ by using similar reason as the case of $j = 1$.

According to the monotonicity of function φ and the following inequality

$$T(r,f) \leq \log M(r,f) \leq \frac{R+r}{R-r} T(r,f), \ 0 < r < R, \tag{2.3}$$

we get that

$$\rho^1_{[p,q],\varphi}(f) \leq \tilde{\rho}^1_{[p,q],\varphi}(f).$$

Next, by (2.3) and choose $R = kr$, $k > 1$, we have

$$\frac{\varphi(\log_p M(r,f))}{\log_q r} \le \frac{\varphi(\log_{p-1} \frac{R+1}{R-1} T(R,f))}{\log_q r} \le \frac{\varphi(\log_{p-1} \frac{k+1}{k-1} T(kr,f))}{\log_q r}$$
$$\le \frac{(1+o(1))\varphi(\log_{p-1} T(kr,f))}{\log_q kr} \frac{\log_q kr}{\log_q r}, r \to +\infty.$$

In fact, by the properties of function φ,

$$\forall \alpha > 1 : \quad \varphi(\alpha t) \le \varphi(t^\alpha) \le (1+o(1))\varphi(t),\ t \to +\infty.$$

Hence,

$$\frac{\varphi(\log_{p-1} \frac{k+1}{k-1} T(kr,f))}{\log_q r} \le \frac{\varphi(\frac{k+1}{k-1} \log_{p-1} T(kr,f))}{\log_q r}$$
$$\le \frac{(1+o(1))\varphi(\log_{p-1} T(kr,f))}{\log_q kr} \frac{\log_q kr}{\log_q r},\ r \to +\infty.$$

It is implies that

$$\rho^1_{[p,q],\varphi}(f) \ge \tilde{\rho}^1_{[p,q],\varphi}(f).$$

Therefore, this is completely proved. □

Proposition 2.3 *Let $\varphi \in \Phi$, and let f, f_1, f_2 be three meromorphic functions. Then the following statements hold.*

(i) $\rho^j_{[p,q],\varphi}(f_1 + f_2) \le \max\left\{\rho^j_{[p,q],\varphi}(f_1), \rho^j_{[p,q],\varphi}(f_2)\right\}, j = 0, 1.$
(ii) $\rho^j_{[p,q],\varphi}(f_1 f_2) \le \max\left\{\rho^j_{[p,q],\varphi}(f_1), \rho^j_{[p,q],\varphi}(f_2)\right\}, j = 0, 1.$
(iii) $\rho^j_{[p,q],\varphi}(\frac{1}{f}) = \rho^j_{[p,q],\varphi}(f)$ *for* $f \ne 0, j = 0, 1.$
(iv) *for* $a \in \mathbb{C}\backslash\{0\}$, *we have* $\rho^j_{[p,q],\varphi}(af) = \rho^j_{[p,q],\varphi}(f), \tau^j_{[p,q],\varphi}(af) = \tau^j_{[p,q],\varphi}(f), j=0,1.$

Proof (i) We prove that this is true when $j = 1$, and similarly it can be proved for the case of $j = 0$. Let $a = \rho^1_{[p,q],\varphi}(f_1)$, $b = \rho^1_{[p,q],\varphi}(f_2)$. Without loss of generality, suppose that $a \le b < +\infty$. Now by the definition of $\rho^1_{[p,q],\varphi}$-order, for any $\varepsilon > 0$ and sufficiently large r,

$$\frac{\varphi(\log_{p-1} T(r,f_k))}{\log_q r} \le \rho^1_{[p,q],\varphi}(f_k) + \varepsilon,$$
$$\varphi(\log_{p-1} T(r,f_k)) \le (b+\varepsilon)\log_q r,$$
$$T(r,f_k) \le \exp_{p-1}[\varphi^{-1}((b+\varepsilon)\log_q r)], k = 1, 2.$$

It follows from the properties of Nevanlinna characteristic functions that

$$\begin{aligned} T(r,f_1+f_2) &\le T(r,f_1)+T(r,f_2)+O(1)\\ &\le 3\exp_{p-1}\left(\varphi^{-1}[(b+\varepsilon)\log_q r]\right)\\ &\le \exp_{p-1}\left(\varphi^{-1}[(b+3\varepsilon)\log_q r]\right). \end{aligned}$$

Hence,

$$\frac{\varphi(\log_{p-1} T(r,f_1+f_2))}{\log_q r} \le b+3\varepsilon.$$

It is implies that

$$\rho^1_{[p,q],\varphi}(f_1+f_2) \le \max\left\{\rho^1_{[p,q],\varphi}(f_1), \rho^1_{[p,q],\varphi}(f_2)\right\}.$$

The properties (ii), (iii) and (iv) can be proved by using similar way as in the proof of the case (i). □

Proposition 2.4 *Let* $\varphi \in \Phi$, *and* f_1,f_2 *be two meromorphic functions. If* $\rho^j_{[p,q],\varphi}(f_1) < \rho^j_{[p,q],\varphi}(f_2), j=0,1,$ *then*

$$\rho^j_{[p,q],\varphi}(f_1+f_2) = \rho^j_{[p,q],\varphi}(f_1f_2) = \rho^j_{[p,q],\varphi}(f_2), j=0,1. \tag{2.4}$$

Proof Obviously, we can easily conclude that this is true by Proposition 2.3. □

Proposition 2.5 *Let* $\varphi \in \Phi$, *and* f_1,f_2 *be two meromorphic functions. Then the following statements hold.*

(i) *If* $0 < \rho^j_{[p,q],\varphi}(f_1) < \rho^j_{[p,q],\varphi}(f_2) < +\infty, 0 < \tau^j_{[p,q],\varphi}(f_1) < \tau^j_{[p,q],\varphi}(f_2), j=0,1,$ *then*

$$\tau^j_{[p,q],\varphi}(f_1+f_2) = \tau^j_{[p,q],\varphi}(f_1f_2) = \tau^j_{[p,q],\varphi}(f_2).$$

(ii) *If* $0 < \rho^j_{[p,q],\varphi}(f_1) = \rho^j_{[p,q],\varphi}(f_2) = \rho^j_{[p,q],\varphi}(f_1+f_2), j=0,1,$ *then*

$$\tau^j_{[p,q],\varphi}(f_1+f_2) \le \max\left\{\tau^j_{[p,q],\varphi}(f_1), \tau^j_{[p,q],\varphi}(f_2)\right\}.$$

Moreover, if $\tau^j_{[p,q],\varphi}(f_1) \ne \tau^j_{[p,q],\varphi}(f_2)$, then

$$\tau^j_{[p,q],\varphi}(f_1+f_2) = \max\left\{\tau^j_{[p,q],\varphi}(f_1), \tau^j_{[p,q],\varphi}(f_2)\right\}.$$

(iii) *If* $0 < \rho^j_{[p,q],\varphi}(f_1) = \rho^j_{[p,q],\varphi}(f_2) = \rho^j_{[p,q],\varphi}(f_1f_2), j=0,1,$ *then*

$$\tau^j_{[p,q],\varphi}(f_1f_2) \le \max\left\{\tau^j_{[p,q],\varphi}(f_1), \tau^j_{[p,q],\varphi}(f_2)\right\}.$$

Moreover, if $\tau^j_{[p,q],\varphi}(f_1) \ne \tau^j_{[p,q],\varphi}(f_2)$, then

$$\tau^j_{[p,q],\varphi}(f_1f_2) = \max\left\{\tau^j_{[p,q],\varphi}(f_1), \tau^j_{[p,q],\varphi}(f_2)\right\}.$$

Proof We just prove the case of $j = 1$, and the case of $j = 0$ is very similar.

(i) By the definition of the $\tau^1_{[p,q],\varphi}$-type, for any given $\varepsilon > 0$, there exists a sequence $\{r_n\}$ which tending to infinity and $N_1 \in Z^+$, such that for $n > N_1$,

$$T(r_n,f_2) \geq \exp_{p-1}\left\{\varphi^{-1}\left[\log\left((\tau^1_{[p,q],\varphi}(f_2) - \varepsilon\right)[\log_{q-1} r_n]^{\rho^1_{[p,q],\varphi}(f_2)})\right]\right\}.$$

On the other hand, there exists $N_2 \in Z^+$, such that for $n > N_2$,

$$T(r_n,f_1) \leq \exp_{p-1}\left\{\varphi^{-1}\left[\log\left((\tau^1_{[p,q],\varphi}(f_1) + \varepsilon\right)[\log_{q-1} r_n]^{\rho^1_{[p,q],\varphi}(f_1)})\right]\right\}. \quad (2.5)$$

Obviously,

$$T(r,f_1 + f_2) \geq T(r,f_2) - T(r,f_1) - \log 2.$$

Set $N = \max\{N_1, N_2\}$. By the properties of φ and $n > N$, we have

$$T(r_n,f_1 + f_2) \geq \exp_{p-1}\left\{\varphi^{-1}\left[\log\left((\tau^1_{[p,q],\varphi}(f_2) - 2\varepsilon\right)[\log_{q-1} r_n]^{\rho^1_{[p,q],\varphi}(f_2)})\right]\right\}.$$

It follows from Proposition 2.4 that $\rho^1_{[p,q],\varphi}(f_1 + f_2) = \rho^1_{[p,q],\varphi}(f_2)$. By the monotonicity of φ, we have

$$\frac{e^{\varphi(\log_{p-1} T(r_n,f_1+f_2))}}{[\log_{q-1} r_n]^{\rho^1_{[p,q],\varphi}(f_1+f_2)}} \geq \tau^1_{[p,q],\varphi}(f_2) - 2\varepsilon.$$

And then

$$\tau^1_{[p,q],\varphi}(f_1 + f_2) \geq \tau^1_{[p,q],\varphi}(f_2).$$

Since $\rho^1_{[p,q],\varphi}(f_1 + f_2) = \rho^1_{[p,q],\varphi}(f_2) > \rho^1_{[p,q],\varphi}(f_1) = \rho^1_{[p,q],\varphi}(-f_1)$, then

$$\tau^1_{[p,q],\varphi}(f_2) = \tau^1_{[p,q],\varphi}(f_1 + f_2 - f_1) \geq \tau^1_{[p,q],\varphi}(f_1 + f_2).$$

Thus $\tau^1_{[p,q],\varphi}(f_2) = \tau^1_{[p,q],\varphi}(f_1 + f_2)$.

Next we prove that $\tau^1_{[p,q],\varphi}(f_1f_2) = \tau^1_{[p,q],\varphi}(f_2)$. Obviously, $T(r,f_1f_2) \geq T(r,f_2) - T(r,f_1) - \log 2$. By using similar discussion as in the proof above, we obtain easily that

$$\tau^1_{[p,q],\varphi}(f_1f_2) \geq \tau^1_{[p,q],\varphi}(f_2).$$

Since $\rho^1_{[p,q],\varphi}(f_1f_2) = \rho^1_{[p,q],\varphi}(f_2) > \rho^1_{[p,q],\varphi}(f_1) = \rho^1_{[p,q],\varphi}(\frac{1}{f_1})$, then

$$\tau^1_{[p,q],\varphi}(f_2) = \tau^1_{[p,q],\varphi}(f_1f_2\frac{1}{f_1}) \geq \tau^1_{[p,q],\varphi}(f_1f_2).$$

So, $\tau^1_{[p,q],\varphi}(f_2) = \tau^1_{[p,q],\varphi}(f_1f_2)$.

(ii) By (2.5), we have

$$T(r,(f_1+f_2)) \leq T(r,f_1) + T(r,f_2) + O(1)$$
$$\leq \exp_{p-1}\left\{\varphi^{-1}\left[\log\left(\max\left\{\tau^1_{[p,q],\varphi}(f_1), \tau^1_{[p,q],\varphi}(f_2)\right\} + 3\varepsilon\right)[\log_{q-1} r]^{\rho^1_{[p,q],\varphi}(f_1+f_2)})\right]\right\}.$$

Hence, by the monotonicity of φ,

$$\tau^1_{[p,q],\varphi}(f_1+f_2) \leq \max\left\{\tau^1_{[p,q],\varphi}(f_1), \tau^1_{[p,q],\varphi}(f_2)\right\}. \tag{2.6}$$

Without loss of generality, suppose $\tau^1_{[p,q],\varphi}(f_1) < \tau^1_{[p,q],\varphi}(f_2)$. Then, by (2.6) and $\rho^1_{[p,q],\varphi}(f_1+f_2) = \rho^1_{[p,q],\varphi}(f_1) = \rho^1_{[p,q],\varphi}(-f_1)$, we get

$$\begin{aligned}\tau^1_{[p,q],\varphi}(f_2) &= \tau^1_{[p,q],\varphi}(f_1+f_2-f_1)\\ &\leq \max\left\{\tau^1_{[p,q],\varphi}(f_1), \tau^1_{[p,q],\varphi}(f_1+f_2)\right\}\\ &= \tau^1_{[p,q],\varphi}(f_1+f_2).\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), $\tau^1_{[p,q],\varphi}(f_1+f_2)=\max\left\{\tau^1_{[p,q],\varphi}(f_1), \tau^1_{[p,q],\varphi}(f_2)\right\}$.

(iii) is proved by using similar reason as in the proof of (i) and (ii). □

The following Corollary can be obtain from (i) and (ii) of Proposition 2.5.

Corollary 2.6 *Let $\varphi \in \Phi$, and let f_1, f_2 be two meromorphic functions.*

(i) *If* $0 < \rho^j_{[p,q],\varphi}(f_1) = \rho^j_{[p,q],\varphi}(f_2) = \rho^j_{[p,q],\varphi}(f_1+f_2), j = 0, 1,$ *then* $\tau^j_{[p,q],\varphi}(f_1) \leq \max\left\{\tau^j_{[p,q],\varphi}(f_1+f_2), \tau^j_{[p,q],\varphi}(f_2)\right\}$.

(ii) *If* $0 < \rho^j_{[p,q],\varphi}(f_1) = \rho^j_{[p,q],\varphi}(f_2) = \rho^j_{[p,q],\varphi}(f_1f_2), j = 0, 1,$ *then* $\tau^j_{[p,q],\varphi}(f_1) \leq \max\left\{\tau^j_{[p,q],\varphi}(f_1f_2), \tau^j_{[p,q],\varphi}(f_2)\right\}$.

Proposition 2.7 *Let $\varphi \in \Phi$, and f be a meromorphic function. Then*

$$\rho^j_{[p,q],\varphi}(f') = \rho^j_{[p,q],\varphi}(f), j = 0, 1.$$

Proof Set $\rho^1_{[p,q],\varphi}(f) = \alpha$. From the definition of $\rho^1_{[p,q],\varphi}$-order, for any $\varepsilon > 0$, there exists $r_0 > 1$, such that for all $r \geq r_0$,

$$\log_{p-1} T(r,f) = O\left\{\varphi^{-1}[(\alpha+\varepsilon)(\log_q r)]\right\}.$$

Obviously, $T(r,f') \leq 2T(r,f) + m(r, \frac{f'}{f})$. By the Lemma of logarithmic derivative (p.34 in [8]), we have

$$\log_{p-1} T(r,f') \leq \log_{p-1} \{O(\log rT(r,f))\} + \log_{p-1} T(r,f)$$
$$= O\{\varphi^{-1}[(\alpha+\varepsilon)(\log_q r)]\}, r \notin E,$$

where $E \subset [0, +\infty)$ is of finite linear measure. By Lemma 3.2 in Sect. 3 and for all sufficiently large r,

$$\frac{\varphi[\log_{p-1} T(r,f')]}{\log_q r} \leq \alpha + \varepsilon.$$

It is implies that $\rho^1_{[p,q],\varphi}(f) \geq \rho^1_{[p,q],\varphi}(f')$.

On the other hand, we prove the inequality $\rho^1_{[p,q],\varphi}(f) \leq \rho^1_{[p,q],\varphi}(f')$. The definition of $\rho^1_{[p,q],\varphi}(f') = \beta$ implies that for any given above $\varepsilon > 0$, there exists $r_1 > 1$, such that for all $r > r_1$,

$$\log_{p-1} T(r,f') \leq \varphi^{-1}[(\beta+\varepsilon)(\log_q r)].$$

By the properties of φ and

$$T(r,f) \leq O(T(2r,f') + \log r), \ r \to +\infty,$$

we can get that

$$\log_{p-1} T(r,f) \leq O(\log_{p-1} T(2r,f') + \log_p 2r)$$
$$\leq O(\varphi^{-1}[(\beta+\varepsilon)(\log_q 2r) + (\log_p 2r)])$$
$$\leq O(\varphi^{-1}[(\beta+2\varepsilon)(\log_q 2r)]), \ r \to +\infty.$$

By the monotonicity of φ, we get

$$\varphi(\log_{p-1} T(r,f)) \leq (1+o(1))(\beta+2\varepsilon)\log_q 2r \leq (\beta+3\varepsilon)\log_q 2r.$$

It is implies that $\rho^1_{[p,q],\varphi}(f) \leq \rho^1_{[p,q],\varphi}(f')$. □

3 Auxiliary Results

In the proof of Theorems 1.5 and 1.6, the classical reduced order method is adopted for Eq. (1.1), which aims to find the estimation of $m(r, A_j)(j = 0, \ldots, k-1)$ by using the estimation of $m(r, \frac{f^{(k)}}{f})(k \geq 1)$. The following lemma is an estimation of $m(r, \frac{f^{(k)}}{f})$.

Lemma 3.1 *Let f be a meromorphic function of order $\rho^1_{[p,q],\varphi}(f) = \rho$, $k \in \mathbb{N}$, and $\varphi \in \Phi$. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\right\},$$

outside, possibly, an exceptional set E of finite linear measure.

Proof Let $k = 1$. The definition of $\rho^1_{[p,q],\varphi}$-order implies that for any $\varepsilon > 0$, there exists $r_0 > 1$, such that for all $r > r_0$,

$$T(r,f) = O\{\exp_{p-1}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\}. \tag{3.1}$$

It follows from (3.1) and the lemma of logarithmic derivative that

$$\begin{aligned} m\left(r,\frac{f'}{f}\right) &= O(\log T(r,f) + \log r) \\ &= O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\}, r \notin E, \end{aligned} \tag{3.2}$$

where $E \subset (0,+\infty)$ is of finite linear measure.

Now, we assume that for some $k \in \mathbb{N}$,

$$m\left(r,\frac{f^{(k)}}{f}\right) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\}, r \notin E.$$

Since $N(r,f^{(k)}) \le (k+1)N(r,f)$, we deduce

$$\begin{aligned} T(r,f^{(k)}) &= m(r,f^{(k)}) + N(r,f^{(k)}) \\ &\le m\left(r,\frac{f^{(k)}}{f}\right) + m(r,f) + (k+1)N(r,f) \\ &\le (k+1)T(r,f) + O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\} \\ &= O\{\exp_{p-1}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\}. \end{aligned}$$

It follows from (3.2) and (3.3) that $m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) = O(\exp_{p-2}[\varphi^{-1}\log_q r^{\rho+\varepsilon}])$, $r \notin E$. Thus,

$$\begin{aligned} m\left(r,\frac{f^{(k+1)}}{f}\right) &\le m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r,\frac{f^{(k)}}{f}\right) \\ &= O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\rho+\varepsilon})]\}, r \notin E. \end{aligned}$$

□

The following lemma is needed to prove Theorems 1.5 and 1.6.

Lemma 3.2 ([1]) *Let $g : [0,+\infty) \to \mathbb{R}$ and $h : [0,+\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \le h(r)$ outside an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Wiman-Valiron theory is needed in proving our results, which can be found [15]. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function. Then

$$\mu(r,f) = \max\left\{|a_n|r^n : n \geq 0\right\}, \quad \nu(r,f) = \max\left\{n : |a_n|r^n = \mu(r,f)\right\}$$

are called the maximal term and the central index of f, respectively.

Lemma 3.3 ([15, p. 51]) *Let f be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z such that $|z| = r$ and $|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta}$. Then there exists a set $E \subset \mathbb{R}_+$ of finite logarithmic measure such that*

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m (1 + o(1))f(z)$$

holds for integer $m \geq 0$ and $r \notin E$.

The following estimation of the radius r of the polynomial $P(z)$ is used in the proof of Theorem 1.5.

Lemma 3.4 ([15, p.10]) *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial, where $a_n \neq 0$. Then all zero of $P(z)$ lie in the discs $D(0, r)$ of radius*

$$r \leq 1 + \max_{0 \leq k \leq n-1}\left(\left|\frac{a_k}{a_n}\right|\right).$$

We need the following two lemmas to get estimations of $T(r,f)$ and $m(r,f)$, which is used in proving Theorems 1.6 and 1.8.

Lemma 3.5 *Let f be a meromorphic function with $\rho^0_{[p,q],\varphi}(f) = \rho_0 \in (0,+\infty)$. Then, for all $\mu(< \rho_0)$, there exists a set $E \in [1,+\infty)$ of infinite logarithmic measure, such that $\varphi(e^{\log_{p-1} T(r,f)}) > \mu \log_q r$ holds for all $r \in E$.*

Proof The definition of $\rho^0_{[p,q],\varphi}$-order implies that there exists a sequence $(R_j)_{j=1}^{+\infty}$ satisfying

$$\left(1 + \frac{1}{j}\right)R_j < R_{j+1}, \quad \lim_{j\to+\infty}\frac{\varphi(e^{\log_{p-1} T(R_j,f)})}{\log_q R_j} = \rho_0.$$

From the equality above, for any $\varepsilon \in (0, \rho_0 - \mu)$, there exists an integer j_1 such that for $j \geq j_1$,

$$\varphi(e^{\log_{p-1} T(R_j,f)}) > (\rho_0 - \varepsilon)\log_q R_j. \tag{3.3}$$

Since $\mu < \rho_0 - \varepsilon$, there exists an integer j_2 such that for $j \geq j_2$,

$$\frac{\rho_0 - \varepsilon}{\mu}\log_q R_j > \log_q\left(1 + \frac{1}{j}\right)R_j.$$

It follows from this inequality and (3.4) that for $j \geq j_3 = \max\left\{j_1, j_2\right\}$ and for any $r \in [R_j, (1 + \frac{1}{j})R_j]$,

$$\begin{aligned}\varphi(e^{\log_{p-1} T(r,f)}) &\ge \varphi(e^{\log_{p-1} T(R_j,f)}) > (\rho_0-\varepsilon)\log_q R_j\\ &= \frac{\rho_0-\varepsilon}{\mu}\mu\frac{\log_q R_j}{\log_q r}\log_q r\\ &\ge \frac{\rho_0-\varepsilon}{\mu}\frac{\log_q R_j}{\log_q\left(1+\frac{1}{j}\right)R_j}\mu\log_q r\\ &> \mu\log_q r.\end{aligned}$$

Set $E=\bigcup_{j=j_3}^{+\infty}[R_j,(1+\frac{1}{j})R_j]$. It is easy to show that E is of infinite logarithmic measure,

$$m_l E := \int_E \frac{dr}{r} = \sum_{j=j_3}^{+\infty}\int_{R_j}^{(1+\frac{1}{j})R_j}\frac{dr}{r} = \sum_{j=j_3}^{+\infty}\log\left(1+\frac{1}{j}\right) = +\infty.$$

□

We can also prove the following result by using similar reason as in the proof of Lemma 3.5.

Lemma 3.6 *Let $\varphi\in\Phi$, and f be an entire function with $\tilde{\rho}^0_{[p,q],\varphi}(f)=\rho_0\in(0,+\infty)$ and $\tilde{\tau}^0_{[p,q],\varphi}(f)\in(0,+\infty)$. Then for any given $\beta<\tilde{\tau}^0_{[p,q],\varphi}(f)$, there exists a set $E\in[1,+\infty)$ of infinite logarithmic measure such that for all $r\in E$,*

$$\exp\left\{\varphi(e^{\log_p M(r,f)})\right\} > \beta(\log_{q-1} r)^{\rho_0}.$$

The following lemma is used to prove Theorem 1.7 for the case of $q=1$.

Lemma 3.7 ([9]) *Let f be a solution of Eq. (1.1), and let $1\le\gamma<+\infty$. Then for all $0<r<R$, where $0<R<+\infty$,*

$$m_\gamma(r,f)^\gamma \le C\left(\sum_{j=0}^{k-1}\int_0^{2\pi}\int_0^r |A_j(se^{i\theta})|^{\frac{\gamma}{k-j}}ds d\theta + 1\right),$$

where $C>0$ is a constant which depends on γ and the initial value of f in a point z_0, where $A_j\not\equiv 0$ for some $j=0,\dots,k-1$, and where

$$m_\gamma(r,f)^\gamma = \frac{1}{2\pi}\int_0^{2\pi}(|\log^+|f(re^{i\theta})||)^\gamma d\theta.$$

The following logarithmic derivative estimation was found in [6] from Gundersen.

Lemma 3.8 ([6]) *Let f be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exists a set* $E \subset [1, +\infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α, *and* $i, j, 0 \leq i < j \leq k-1$, *such that for all z satisfying* $|z| = r \notin [0, 1] \bigcup E$,

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left\{\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right\}^{j-i}.$$

Lemma 3.9 *Let* $\varphi \in \Phi$ *and* $A_0(z), \ldots, A_{k-1}(z)$ *be entire functions. Then, every non-trivial solution f of Eq.* (1.1) *satisfies*

$$\tilde{\rho}^{1}_{[p,q],\varphi}(f) \leq \max\left\{\tilde{\rho}^{0}_{[p,q],\varphi}(A_j) : j = 0, 1, \ldots, k-1\right\}.$$

Proof Set

$$\beta = \max\left\{\tilde{\rho}^{0}_{[p,q],\varphi}(A_j) : j = 0, 1, \ldots, k-1\right\}.$$

By the definition of $\tilde{\rho}^{0}_{[p,q],\varphi}(A_j)$, for any $\varepsilon > 0$ and for sufficiently large r,

$$M(r, A_j) \leq \exp_{p-1}\left\{\varphi^{-1}((\beta + \varepsilon)\log_q r)\right\}, j = 0, \ldots, k-1. \tag{3.4}$$

By Lemma 3.7 for $\gamma = 1$, we have

$$T(r,f) = m(r,f) \leq 2\pi C(1 + \sum_{j=0}^{k-1} rM(r, A_j)). \tag{3.5}$$

It follows from (3.5), (3.6) and Proposition 2.2 that

$$\tilde{\rho}^{1}_{[p,q],\varphi}(f) \leq \max\left\{\tilde{\rho}^{0}_{[p,q],\varphi}(A_j) : j = 0, 1, \ldots, k-1\right\}.$$

□

4 Proofs of Theorems 1.5 and 1.6

The classical way of reducing the order is adopted for Eq. (1.1) in proofs of Theorems 1.5 and 1.6, and $T(r, A_j)(j = 0, 1, \ldots, k-1)$ is estimated by $T(r, \frac{f^{(k)}}{f})(k \geq 1)$ in reducing the order.

To state our proving concisely, let E represents the finite logarithmic measure, I represents the infinite logarithmic measure and F represents the finite linear measure in the proofs of Theorems 1.5–1.8. Next we start prove our results by using the similar way as in the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.5 Set $\gamma_{[p,q],\varphi} = \sup\left\{\rho^1_{[p,q],\varphi}(f)|L(f)=0\right\}$, and $\alpha_{[p,q],\varphi} = \sup\left\{\rho^0_{[p,q],\varphi}(A_j)|j=0,1,\dots,k-1\right\}$.

First, we prove that $\alpha_{[p,q],\varphi} \le \gamma_{[p,q],\varphi}$. If $\gamma_{[p,q],\varphi} = +\infty$, it is trivial. Hence we just consider the case of $\gamma_{[p,q],\varphi} < +\infty$. Let $f_1,\dots,f_k$ be a solution base of Eq. (1.1) with $\rho^1_{[p,q],\varphi}(f_j) < +\infty, j=1,\dots,k$. It is clear that $W = W(f_1,\dots,f_k) \neq 0$ by the properties of the Wronsky determinant.

It follows from Propositions 2.3 and 2.7 that $\rho^1_{[p,q],\varphi}(W) < \infty$. By properties of the Wronsky determinant ([15, p.55]),

$$A_{k-s}(z) = -W_{k-s}(f_1,\dots,f_k)\cdot W^{-1}, s\in\{1,\dots,k\},$$

where

$$W_j(f_1,\dots,f_k) = \begin{vmatrix} f_1 & \cdots & f_k \\ \vdots & \vdots & \vdots \\ f_1^{(j-1)} & \cdots & f_k^{(j-1)} \\ f_1^{(k)} & \cdots & f_k^{(k)} \\ f_1^{(j+1)} & \cdots & f_k^{(j+1)} \\ \vdots & \vdots & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{vmatrix}.$$

In view of Proposition 2.3 we can conclude that $\rho^1_{[p,q],\varphi}(A_i) < \infty, i=0,1,\dots,k-1$.

By Lemma 3.1 to $f_i, i=1,\dots,k$,

$$m\left(r,\frac{f_i^{(l)}}{f_i}\right) = O\left\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\right\}, r\notin F, l=1,2,\dots,k.$$

We now apply the standard order reduction procedure ([15, p.53–57]). Denote

$$v_1(z) := \frac{d}{dz}\left(\frac{f(z)}{f_1(z)}\right),$$

$A_k = 1$, and $v_1^{(-1)} := \frac{f}{f_1}$, i.e., $(v_1^{(-1)})' := v_1$. Hence,

$$f^{(l)} = \sum_{m=0}^{l}\binom{l}{m}f_1{}^{(m)}v_1{}^{(k-1-m)}, l=0,\dots,k. \tag{4.1}$$

Substituting (4.1) into (1.1) and using the fact that f_1 solves (1.1), we obtain

$$v_1{}^{(k-1)} + A_{1,k-2}(z)v_1{}^{(k-2)} + \cdots + A_{1,0}(z)v_1 = 0, \tag{4.2}$$

where

$$A_{1,j} = A_{j+1} + \sum_{m=1}^{k-j-1}\binom{j+1+m}{m}A_{j+1+m}\frac{f_1^{(m)}}{f_1}, j=0,\dots,k-2.$$

By $\gamma_{[p,q],\varphi} < +\infty$ and Proposition 2.7, the meromorphic functions

$$v_{1,j}(z) = \frac{d}{dz}\left(\frac{f_{j+1}(z)}{f_1(z)}\right), j = 1, \ldots, k-1, \tag{4.3}$$

are solutions of (4.2) of finite $\rho^1_{[p,q],\varphi}$-order.

Next, we claim that

$$m(r, A_i) = O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}, r \notin F, i = 0, \ldots, k-1, \tag{4.4}$$

when

$$m(r, A_{1,j}) = O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}, r \notin F, j = 0, \ldots, k-2, \tag{4.5}$$

In fact, we prove it by induction on i following [15]. By equality (4.2) for $j = k-2$, we have $A_{1,k-2} = A_{k-1} + k\frac{f'}{f}$. By Lemma 3.1 and (4.4),

$$\begin{aligned} m(r, A_{k-1}) &\le m(r, A_{1,k-2}) + m(r, \frac{f'}{f}) + O(1) \\ &= O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}. \end{aligned}$$

We assume that

$$m(r, A_i) = O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}, i = k-1, \ldots, k-l. \tag{4.6}$$

Since

$$A_{1,k-(l+2)} = A_{k-(l+1)} + \sum_{m=1}^{l+1} \binom{m+k-l-1}{m} A_{m+k-l-1}\frac{f_1^{(m)}}{f_1},$$

by Lemma 3.1, (4.4) and (4.6), we have

$$\begin{aligned} m(r, A_{k-(l+1)}) &\le m(r, A_{1,k-(l+2)}) + m(r, A_{k-1}) + \cdots + m(r, A_{k-l}) \\ &\quad + m(r, \frac{f'}{f}) + \cdots + m(r, \frac{f_1^{(l+1)}}{f_1}) + O(1) \\ &= O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}, r \notin F. \end{aligned} \tag{4.7}$$

We may now proceed as above the order reduction procedure for (4.2). In each reduction step, we obtain a solution base of meromorphic functions of finite $\rho^1_{[p,q],\varphi}$-order according to (4.3), and the implication (4.4) and (4.5) remains valid. Hence, we finally obtain an equation of the form $w' + B(z)w = 0$, and w is any solution of the equation with $\rho^1_{[p,q],\varphi}(w) < \infty$. Then

$$m(r, B) = m\left(r, \frac{w'}{w}\right) = O\big\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\big\}, r \notin F.$$

Observing the reasoning corresponding to (4.4) and (4.5) in the subsequent reduction steps,

$$m(r, A_j) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\}, r \notin F, j = 0, \dots, k-1.$$

It implies that

$$T(r, A_j) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+\varepsilon})]\}, r \notin F, j = 0, 1, \dots, k-1.$$

By Lemma 3.2 and Proposition 2.1, for sufficiently large r, $j = 0, \dots, k-1$,

$$\begin{aligned} T(r, A_j) &= O\{\exp_{p-2}[\varphi^{-1}(\log_q (2r)^{\gamma_{[p,q],\varphi}+\varepsilon})]\} \\ &\leq O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\gamma_{[p,q],\varphi}+2\varepsilon})]\}. \end{aligned}$$

Hence, $\frac{\varphi(e^{\log_{p-1} T(r,A_j)})}{\log_q r} \leq \gamma_{[p,q],\varphi} + 2\varepsilon$. This implies that $\alpha_{[p,q],\varphi} \leq \gamma_{[p,q],\varphi}$.

We next prove the converse inequality under the assumption that $\alpha_{[p,q],\varphi} < +\infty$.

By Lemma 3.3, there exists a set $E \subset \mathbb{R}_+$ of finite logarithmic measure, such that for all z satisfies $|f(z)| = M(r,f)$ and $|z| = r \notin E$,

$$f^{(i)}(z) = \left(\frac{\nu(r,f)}{z}\right)^i (1 + o(1))f(z), i = 0, \dots, k. \tag{4.8}$$

Substituting (4.8) into (1.1),

$$\begin{aligned} &\nu(r,f)^k + zA_{k-1}(z)\nu(r,f)^{k-1}(1 + o(1)) + \cdots \\ &\quad + z^{k-1}A_1(z)\nu(r,f)(1 + o(1)) + z^k A_0(z)(1 + o(1)) = 0. \end{aligned}$$

The definition of $\tilde{\rho}^0_{[p,q],\varphi}$-order and Proposition 2.2 yields that for any $\varepsilon > 0$ there exists $r_0 > 1$, such that for all $r \geq r_0$,

$$M(r, A_j) < \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+\varepsilon})], j = 0, 1, \dots, k-1.$$

By Lemma 3.4 and Proposition 2.1,

$$\begin{aligned} \nu(r,f) &\leq 1 + \max_{0\leq j\leq k-1} |z^{k-j}A_j(z)(1 + o(1))| \\ &\leq 1 + \max_{0\leq j\leq k-1} 2r^{k-j} \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+\varepsilon})] \\ &\leq 1 + 2r^k \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+\varepsilon})] \\ &\leq \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+2\varepsilon})], r \notin E. \end{aligned}$$

It follows from [11, p.36–37] that

$$\begin{aligned}T(r,f) &\le \log M(r,f) \le \log \mu(r,f) + \log(\nu(2r,f)+2)\\ &\le \nu(r,f)\log r + \log(2\nu(2r,f))\\ &\le \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+2\varepsilon})]\log r + \log(2\exp_{p-1}[\varphi^{-1}(\log_q (2r)^{\alpha_{[p,q],\varphi}+2\varepsilon})])\\ &\le \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+3\varepsilon})] + \log 2 + \exp_{p-2}[\varphi^{-1}(\log_q (2r)^{\alpha_{[p,q],\varphi}+2\varepsilon})]\\ &\le \exp_{p-1}[\varphi^{-1}(\log_q r^{\alpha_{[p,q],\varphi}+4\varepsilon})].\end{aligned}$$

This implies that $\gamma_{[p,q],\varphi} \le \alpha_{[p,q],\varphi}$.

□

Proof of Theorem 1.6 By the assumption there exist two numbers λ_1 and λ such that $\rho^0_{[p,q],\varphi}(A_m) \ge \lambda$ and $\rho^0_{[p,q],\varphi}(A_l) \le \lambda_1 < \lambda$ for $l = m+1, \dots, k-1$.

Let $f_1, \dots, f_{m+1}$ be linearly independent solutions of (1.1) such that $\rho^1_{[p,q],\varphi}(f_i) < \lambda$, $i = 1, \dots, m+1$. If $m = k-1$, then all $f_1, \dots, f_k$ are of $\rho^1_{[p,q],\varphi}(f_i) < \lambda$, this contradict with Theorem 1.5. Hence, $m < k-1$. Applying the order reduction procedure as in the proof of Theorem 1.5. We use the notation v_0 instead of f, and $A_{0,0}, \dots, A_{0,k-1}$ instead of $A_0, \dots, A_{k-1}$. On the general reduction step, we obtain an equation of the form

$$v_j^{(k-j)} + A_{j,k-j-1}(z)v_j^{(k-j-1)} + \cdots + A_{j,0}(z)v_j = 0, j = 1, \dots, k-1, \tag{4.9}$$

where

$$A_{j,l} = A_{j-1,l+1} + \sum_{n=1}^{k-l-j} \binom{l+1+n}{n} A_{j-1,l+1+n}\frac{v^{(n)}_{j-1,1}}{v_{j-1,1}}, \tag{4.10}$$

and the functions

$$v_{j,l}(z) = \frac{d}{dz}\left(\frac{v_{j-1,l+1}(z)}{v_{j-1,1}(z)}\right), l = 1, \dots, k-j, v_0 = f, v_j(z) = \frac{d}{dz}\left(\frac{v_{j-1}(z)}{v_{0,j-1}(z)}\right),$$

determine at each reduction step a solution base of (4.9) in terms of the preceding solution base. We may express (1.1) and the mth reduction steps by the following Table. The rows correspond to (4.9) for $v_0, \dots, v_m$, i.e., the first row corresponds to (1.1), and columns from k to 0 give the coefficients of these equations, while the last column lists those solutions with $\rho^1_{[p,q],\varphi}(f) < \lambda$.

	k	k-1	.	**m**	m-1	.	0	$\rho^1_{[p,q],\varphi}(f) < \lambda$
v_0	1	$A_{0,k-1}$	.	$\mathbf{A_{0,m}}$	$A_{0,m-1}$	.	$A_{0,0}$	$v_{0,1}, \dots, v_{0,m+1}$
v_1		1	.	$A_{1,m}$	$\mathbf{A_{1,m-1}}$	.	$A_{1,0}$	$v_{1,1}, \dots, v_{1,m}$
.				.	.	.	.	.
.				.	.	.	.	.
.				.	.	.	.	.

k	k-1	.	**m**	m-1	.	0	$\rho^1_{[p,q],\varphi}(f) < \lambda$
v_{m-1}			$A_{m-1,m}$	$A_{m-1,m-1}$	.	$A_{m,0}$	$v_{m-1,1}, v_{m-1,1}$
v_m			$A_{m,m}$	$A_{m,m-1}$	.	$\mathbf{A_{m,0}}$	$v_{m,1}$

By Lemma 3.1 and (4.10), we see that in the second row, corresponding to the first reduction step, $m(r, A_{1,l}) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F, l = m, \ldots, k-2$, while $\lambda_1 + \varepsilon < \lambda$ and $m(r, A_{1,m-1}) \neq O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F$.

Similarly, in each reduction step (4.10) implies that

$$m(r, A_{j,l}) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F, \tag{4.11}$$

when $l = m + 1 - j, \ldots, k - (j + 1)$, i.e., for all coefficients to the left from the boldface coefficient $A_{j,m-j}$, while for $j = 1, \ldots, m$,

$$m(r, A_{j,m-j}) \neq O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F.$$

In particular,

$$m(r, A_{m,0}) \neq O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda+\varepsilon})]\}, r \notin F.$$

Applying Lemma 3.5 to the coefficient $A_{m,0}$ with the constant λ, and obtain that

$$T(r, A_{m,0}) > \exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda+\varepsilon})], r \to +\infty, r \in I. \tag{4.12}$$

On the other hand, after the mth reduction step, by (4.10), (4.11) and Lemma 3.1, we have

$$A_{m,0} = -\frac{v_{m,1}^{(k-m)}}{v_{m,1}} - A_{m,k-m-1}\frac{v_{m,1}^{(k-m-1)}}{v_{m,1}} - \cdots - A_{m,1}\frac{v'_{m,1}}{v_{m,1}}.$$

That implies that

$$m(r, A_{m,0}) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F.$$

Since $\rho^0_{[p,q],\varphi}(v_{m,1}) < \lambda_1$, in view of Propositions 2.3 and 2.7,

$$N(r, A_{m,0}) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F.$$

Therefore,

$$T(r, A_{m,0}) = O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+\varepsilon})]\}, r \notin F.$$

By Lemma 3.2, for sufficiently large r,

$$\begin{aligned} T(r, A_{m,0}) &= O\{\exp_{p-2}[\varphi^{-1}(\log_q (2r)^{\lambda_1+\varepsilon})]\} \\ &= O\{\exp_{p-2}[\varphi^{-1}(\log_q r^{\lambda_1+2\varepsilon})]\}. \end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we obtain the contradiction with our assumption. Hence, there exists at most m linearly independent solutions Eq. (1.1) with $\rho^1_{[p,q],\varphi}(f) < \lambda$. □

5 Proofs of Theorems 1.7 and 1.8

Proof of Theorem 1.7 Let f be a nontrivial solution of Eq. (1.1). We denote $\rho^1_{[p,q],\varphi}(f) = \rho_1$ and $\rho^0_{[p,q],\varphi}(A_0) = \rho_0$. The inequality $\rho_0 \le \rho_1$ follows from Theorem 1.6 when $m = 0$ and $\lambda = \rho_0$.

To prove the conserve inequality, by Lemma 3.7 for $\gamma = 1$, Proposition 2.1 and the definition of $\rho^0_{[p,q],\varphi}$-order, for any $\varepsilon > 0$,

$$\begin{aligned} m(r,f) &\le C\left(\sum_{j=0}^{k-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds d\theta + 1\right) \\ &\le C\left(k \max_{0\le j\le k-1} \int_0^{2\pi} \int_0^r |A_j(se^{i\theta})|^{\frac{1}{k-j}} ds d\theta + 1\right) \\ &\le C \max_{0\le j\le k-1} \int_0^r (\exp_{p-1}[\varphi^{-1}(\log_q s^{\rho_0+\varepsilon})])^{\frac{1}{k-j}} ds \\ &\le C \int_0^r \exp_{p-1}[\varphi^{-1}(\log_q s^{\rho_0+\varepsilon})] ds \\ &\le C r \exp_{p-1}[\varphi^{-1}(\log_q r^{\rho_0+\varepsilon})] \\ &\le \exp_{p-1}[\varphi^{-1}(\log_q r^{\rho_0+2\varepsilon})]. \end{aligned}$$

Therefore,

$$\frac{\varphi(\log_{p-1} T(r,f))}{\log_q r} \le \rho_0 + 2\varepsilon.$$

It is implies that $\rho_1 \le \rho_0$, and then Theorem 1.7 is proved. □

Proof of Theorem 1.8 Suppose that f is a nontrivial solution of Eq. (1.1). From (1.1), we can write

$$|A_0(z)| \le \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{5.1}$$

If $\max\left\{\tilde{\rho}^0_{[p,q],\varphi}(A_j), j = 1, \dots, k-1\right\} < \tilde{\rho}^0_{[p,q],\varphi}(A_0) = \rho_0 < +\infty$, and by Theorem 1.7, then

$$\tilde{\rho}^1_{[p,q],\varphi}(f) = \tilde{\rho}^0_{[p,q],\varphi}(A_0).$$

Suppose that

$$\max\left\{\tilde{\rho}^0_{[p,q],\varphi}(A_j), j=1,\dots,k-1\right\} = \tilde{\rho}^0_{[p,q],\varphi}(A_0) = \rho_0 < +\infty$$

and

$$\max\left\{\tilde{\tau}^0_{[p,q],\varphi}(A_j) : \tilde{\rho}^0_{[p,q],\varphi}(A_j) = \tilde{\rho}^0_{[p,q],\varphi}(A_0)\right\} < \tilde{\tau}^0_{[p,q],\varphi}(A_0) = \tau < +\infty.$$

First, we prove that $\rho_1 = \tilde{\rho}^1_{[p,q],\varphi}(f) \geq \tilde{\rho}^0_{[p,q],\varphi}(A_0) = \rho_0$. By assumption there exists a set $K \subseteq \{1,2,\dots,k-1\}$ such that

$$\tilde{\rho}^1_{[p,q],\varphi}(A_j) = \tilde{\rho}^0_{[p,q],\varphi}(A_0) = \rho_0, j \in K,$$

and

$$\tilde{\rho}^0_{[p,q],\varphi}(A_j) < \tilde{\rho}^0_{[p,q],\varphi}(A_0), j \in \{1,2,\dots,k-1\} \setminus K.$$

Thus, we choose λ_1 and λ_2 satisfying

$$\max\left\{\tilde{\tau}^0_{[p,q],\varphi}(A_j) : j \in K\right\} < \lambda_1 < \lambda_2 < \tilde{\tau}^0_{[p,q],\varphi}(A_0) = \tau.$$

For sufficiently large r,

$$\left|A_j(z)\right| \leq \exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_1(\log_{q-1} r)^{\rho_0})]\right\}, j \in K, \tag{5.2}$$

and

$$\begin{aligned}\left|A_j(z)\right| &\leq \exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_1(\log_{q-1} r)^{\alpha})]\right\} \\ &\leq \exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_1(\log_{q-1} r)^{\rho_0})]\right\}, j \in \{1,2,\dots,k-1\} \setminus K,\end{aligned} \tag{5.3}$$

where $0 < \alpha < \rho_0$. By Lemma 3.6, there exists a set $I \subset [1,+\infty)$ with infinite logarithmic measure, such that for all $r \in I$,

$$\left|A_0(z)\right| > \exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_2(\log_{q-1} r)^{\rho_0})]\right\}. \tag{5.4}$$

By Lemma 3.8, there exists a constant $B > 0$ and a set $E \subset [1,+\infty)$ having finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \bigcup [0,1]$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B[T(2r,f)]^{k+1}, j = 1,2,\dots,k.$$

Set $\rho_1 = \tilde{\rho}^1_{[p,q],\varphi}(f)$. By Proposition 2.2, for any given $\varepsilon \in (0, \max\left\{\frac{\lambda_2-\lambda_1}{2}, \rho_0 - \rho_1\right\})$ and sufficiently large $|z| = r \notin E \bigcup [0,1]$,

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\le B(T(2r,f))^{k+1} \\ &\le B\left\{\exp_{p-1}[\varphi^{-1}(\log_q(2r)^{\rho_1+\varepsilon})]\right\}^{k+1}, j=1,2,\ldots,k.\end{aligned} \tag{5.5}$$

Hence, substituting (5.2),(5.3), (5.4) and (5.5) into (5.1), for sufficiently large $|z|=r\in I\setminus(E\cup[0,1])$,

$$\begin{aligned}&\exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_2(\log_{q-1}r)^{\rho_0})]\right\} \\ &\le kB\exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_1(\log_{q-1}r)^{\rho_0})]\right\} * \left\{\exp_{p-1}[\varphi^{-1}(\log_q(2r)^{\rho_1+\varepsilon})]\right\}^{k+1} \\ &\le \exp_{p-1}\left\{\varphi^{-1}[\log((\lambda_1+2\varepsilon)(\log_{q-1}r)^{\rho_0})]\right\}.\end{aligned} \tag{5.6}$$

Obviously, $I\setminus(E\cup[0,1])$ is of infinite logarithmic measure. By (5.6), there exists a sequence of points $\{|z_n|\}=\{r_n\}\subset I\setminus(E\cup[0,1])$ tending to $+\infty$, such that

$$\exp_{p-1}\left\{\varphi^{-1}[\log(\lambda_2(\log_{q-1}r_n)^{\rho_0})]\right\} \le \exp_{p-1}\left\{\varphi^{-1}[\log((\lambda_1+2\varepsilon)(\log_{q-1}r_n)^{\rho_0})]\right\}.$$

By the monotonicity of the function φ^{-1}, we obtain that $\lambda_1\ge\lambda_2$. This contradiction implies

$$\tilde{\rho}^1_{[p,q],\varphi}(f)\ge\tilde{\rho}^0_{[p,q],\varphi}(A_0).$$

On the other hand, by Lemma 3.9, we have

$$\tilde{\rho}^1_{[p,q],\varphi}(f)\le\max\left\{\tilde{\rho}^0_{[p,q],\varphi}(A_j) : j=1,\ldots,k-1\right\}=\tilde{\rho}^0_{[p,q],\varphi}(A_0).$$

Hence, every nontrivial solution f of Eq. (1.1) satisfies $\tilde{\rho}^1_{[p,q],\varphi}(f)=\tilde{\rho}^0_{[p,q],\varphi}(A_0)$. □

6 Conclusions

We define new measure $[p,q]_{,\varphi}$-order to describe the growing of meromorphic function, and the new measure is used to study the growth of solutions of complex differential equations.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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