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RESEARCH ARTICLE

# Generalized Ricci Solitons of Three-Dimensional Lorentzian Lie Groups Associated Canonical Connections and Kobayashi-Nomizu Connections

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## Abstract
In this paper, we study the affine generalized Ricci solitons on three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections and we classifying these left-invariant affine generalized Ricci solitons with some product structure.

**Keywords** Generalized Ricci soliton · Lie group · Three-dimensional Lorentzian

**Mathematics Subject Classification** 58C40 · 53E20 · 53C21

## 1 Introduction

The notion of generalized Ricci soliton or Einstein-type manifolds is introduced by Catino et al. as a generalization of Einstein spaces [5]. Study of the generalization Ricci soliton, over different geometric spaces is one of interesting topics in geometry and normalized physics. A pseudo-Riemannian manifold $(M, g)$ is called an generalized Ricci soliton if there exists a vector field $X \in \mathcal{X}(M)$ and a smooth function $\lambda$ on $M$ such that

$$\alpha Ric + \frac{\beta}{2}\mathcal{L}_X g + \mu X^\flat \otimes X^\flat = (\rho S + \lambda)g, \tag{1}$$

for some constants $\alpha, \beta, \mu, \rho \in \mathbb{R}$, with $(\alpha, \beta, \mu) \neq (0, 0, 0)$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of $X$, $X^\flat$ denotes a 1-form such that $X^\flat(Y) = g(X, Y)$, $S$ is the scalar curvature, and $Ric$ is the Ricci tensor. The generalized Ricci soliton becomes

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(i) the homothetic vector field equation when $\alpha = \mu = \rho = 0$ and $\beta \neq 0$,
(ii) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,
(iii) the Ricci-Bourguignon soliton ( or $\rho$-Einstein soliton equation when $\alpha = 1$ and $\mu = 0$.

In the special case that $(M, g)$ is a Lie group and $g$ is a left-invariant metric, we say that $g$ is a left-invariant generalized Ricci soliton on $M$ if the Eq. (1) holds.

In [11, 14, 16, 17, 21, 22], Einstein manifolds associated to affine connections were studied and affine Ricci solitons had been studied in [7, 10, 12, 13, 15]. In [4], Calvaruso studied the Eq. (1) for $\rho = 0$ on three-dimensional generalized Lie groups. Also, in [20] Wang classified affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups. In [8], Etayo and Santamaria investigated the canonical connection and the Kobayashi-Nomizu connection for a product structure. Motivated by [1, 19, 23, 24], we consider the distribution $V = span\{e_1, e_2\}$ and $V^{\perp} = span\{e_3\}$ for the three dimensional Lorentzian Lie group $G_i$, $i = 1, ....., 7$, with product structure $J$ such that $Je_1 = e_1$, $Je_2 = e_2$, and $Je_3 = -e_3$. Then we obtain affine generalized Ricci solitons associated to the canonical connection and the Kobayashi-Nomizu connection.

The paper is organaized as follows. In Sect. 2 we review some necessary concepts on three-dimensional Lie groups which be used throughout this paper. In the Sect. 3 we state the main results and their proof.

## 2 Three-Dimensional Lorentzian Lie Groups

In the following we give a brief description of all three-dimensional unimodular and non-unimodular Lie groups. Complete and simply connected three-dimensional Lorentzian homogeneous manifolds are either symmetric or a Lie group with left-invariant Lorentzian metric [3].

### 2.1 Unimodular Lie Groups

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of signature $(+ + -)$. We denote the Lorentzian vector product on $\mathbb{R}^3_1$ induced by the product of the para-quaternions by $\times$ i.e.,

$$e_1 \times e_2 = -e_3, \quad e_2 \times e_3 = -e_1, \quad e_3 \times e_1 = -e_2.$$

Then the Lie bracket $[\,,\,]$ defines the corresponding Lie algebra $\mathfrak{g}$, which is unimodular if and only if the endomorphism $L$ defined by $[Z, Y] = L(Z \times Y)$ is self-adjoint and non-unimodular if $L$ is not self-adjoint [18]. By assuming the different types of $L$, we get the following four classes of unimodular three-dimensional Lie algebra [9].

$\mathfrak{g}_1$: If $L$ is diagonalizable with eigenvalues $\{a, b, c\}$ with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$, then the corresponding Lie algebra is given by

$$[e_1, e_2] = -ce_3, \quad [e_1, e_3] = -be_2, \quad [e_2, e_3] = ae_1.$$

$\mathfrak{g}_2$: Assume $L$ has a complex eigenvalues. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$, one has

$$L = \begin{pmatrix} a & 0 & 0 \\ 0 & c & -b \\ 0 & b & c \end{pmatrix}, \qquad b \neq 0,$$

then the corresponding Lie algebra is given by

$$[e_1, e_2] = be_2 - ce_3, \quad [e_1, e_3] = -ce_2 - be_3, \quad [e_2, e_3] = ae_1.$$

$\mathfrak{g}_3$: Assume $L$ has a triple root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$, the corresponding Lie algebra is given by

$$[e_1, e_2] = ae_1 - be_3, \quad [e_1, e_3] = -ae_1 - be_2, \quad [e_2, e_3] = be_1 + ae_2 + ae_3, \quad a \neq 0.$$

$\mathfrak{g}_4$: Assume $L$ has a double root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$, the corresponding Lie algebra is given by

$$[e_1, e_2] = -e_2 - (2d - b)e_3, \quad [e_1, e_3] = -be_2 + e_3, \quad [e_2, e_3] = ae_1, \quad d = \pm 1.$$

### 2.2 Non-unimodular Lie Groups

Next we treat the non-unimodular case. Let $\mathfrak{G}$ denotes a special class of the solvable Lie algebra $\mathfrak{g}$ such that $[x, y]$ is a linear combination of $x$ and $y$ for any $x, y \in \mathfrak{g}$. From [6], the non-unimodular Lorentzian Lie algebras of non-constant sectional curvature not belonging to class $\mathfrak{G}$ with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with $e_3$ time-like are one of the following:

$\mathfrak{g}_5$:
$$[e_1, e_2] = 0, \quad [e_1, e_3] = ae_1 + be_2,$$
$$[e_2, e_3] = ce_1 + de_2, \quad a + d \neq 0, \quad ac + bd = 0.$$

$\mathfrak{g}_6$:
$$[e_1, e_2] = ae_2 + be_3, \quad [e_1, e_3] = ce_2 + de_3,$$
$$[e_2, e_3] = 0, \quad a + d \neq 0, \quad ac - bd = 0.$$

$\mathfrak{g}_7$:
$$[e_1, e_2] = -ae_1 - be_2 - be_3, \quad [e_1, e_3] = ae_1 + be_2 + be_3,$$
$$[e_2, e_3] = ce_1 + de_2 + de_3, \quad a + d \neq 0, \quad ac = 0.$$

Throughout this paper, we assume that $G_i, i = 1, 2, ....., 7$ are the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric $g$ and having Lie algebra $g_i, i = 1, 2, ....., 7$, respectively. Let $\nabla$ be the Levi-Civita connection of $G_i$ and $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ be its curvature tensor. The Ricci tensor of $(G_i, g)$ with respect to orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$ is defined by

$$Ric(X, Y) = -g(R(X, e_1)Y, e_1) - g(R(X, e_2)Y, e_2) + g(R(X, e_3)Y, e_3).$$

We consider a product structure $J$ on $G_i$ by $Je_1 = e_1$, $Je_2 = e_2$, $Je_3 = -e_3$. Similar [8], we consider the canonical connection and the Kobayashi-Nomizu connection as

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \qquad \nabla^1_X Y = \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY} J)X],$$

respectively. We define

$$R^i(X, Y)Z = [\nabla^i_X, \nabla^i_Y]Z - \nabla^i_{[X,Y]}Z, \quad i = 0, 1,$$

and the Ricci tensors of $(G_i, g)$ associated to the canonical connection and the Kobayashi-Nomizu connection are defined by

$$Ric^i(X, Y) = -g(R^i(X, e_1)Y, e_1) - g(R^i(X, e_2)Y, e_2) + g(R^i(X, e_3)Y, e_3), \quad i = 0, 1.$$

Let

$$\widetilde{Ric}^i(X, Y) = \frac{Ric^i(X, Y) + Ric^i(Y, X)}{2}, \quad i = 0, 1.$$

Similar to definition of $(\mathcal{L}_V g)$ where $(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y V, Z) + g(Y, \nabla_Z V)$, we define

$$(\mathcal{L}^i_V g)(Y, Z) := g(\nabla^i_Y V, Z) + g(Y, \nabla^i_Z V), \quad i = 0, 1.$$

**Definition 1** The Lie group $(G, g, J)$ is called the affine generalized Ricci soliton associated to the connection $\nabla^i$, $i = 0, 1$ if it satisfies

$$\alpha\widetilde{Ric}^i(Y, Z) + \frac{\beta}{2}\mathcal{L}^i_X g(Y, Z) + \mu X^\flat \otimes X^\flat(Y, Z) = (\rho\widetilde{S}^i + \lambda)g(Y, Z), \quad i = 0, 1, \tag{2}$$

where $\widetilde{S}^i = g^{jk}\widetilde{Ric}^i_{jk}$.

Throughout this paper for prove of our results we use the results of [19, 20].

## 3 Lorentzian Affine Generalized Ricci Solitons on 3D Lorentzian Lie Groups

In this section, we investigate the existence of left-invariant solutions to Eq. (2) on the Lorentzian Lie groups discussed in Sect. 2. We completely solve the corresponding equations and obtain a complete description of all left-invariant affine generalized Ricci solitons.

**Theorem 1** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^0$ on the Lie group $(G_1, g, J, X)$ are the following*:

(i) $\mu = \lambda = 0, a + b - c = 0,$ *and for all* $x_1, x_2, x_3, \alpha, \beta, \rho$ *such that* $(\alpha, \beta, \mu) \neq (0, 0, 0)$ ,

(ii) $\mu = 0,\ a + b - c \neq 0,\ \alpha = 0,\ \beta \neq 0,\ x_1 = x_2 = 0,\ \lambda = \rho c(a + b - c)$, and for all $x_3, \rho$,

(iii) $\mu = 0, a + b - c \neq 0, \alpha \neq 0, c = \beta = \lambda = 0$, and for all $x_1, x_2, x_3, \rho$,

(iv) $\mu = 0, a + b - c \neq 0, \alpha \neq 0, c = \lambda = 0, \beta \neq 0, x_1 = x_2 = 0$, and for all $x_3, \rho$,

(v) $\mu \neq 0,\ x_1 = x_2 = 0,\ \lambda = (\rho - \frac{1}{2}\alpha)c(a + b - c),\ x_3^2 = \frac{\rho c(a+b-c)-\lambda}{\mu}$, *and for all* $x_3, \alpha, \beta, \rho, a, b, c$ *such that* $\frac{\rho c(a+b-c)-\lambda}{\mu} \geq 0$.

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^0 = \begin{pmatrix} -\frac{1}{2}c(a+b-c) & 0 & 0 \\ 0 & -\frac{1}{2}c(a+b-c) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 0 & 0 & -\frac{1}{2}x_2(a+b-c) \\ 0 & 0 & \frac{1}{2}x_1(a+b-c) \\ -\frac{1}{2}x_2(a+b-c) & \frac{1}{2}x_1(a+b-c) & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -c(a + b - c)$ and $X^\flat \otimes X^\flat(e_i, e_j) = \epsilon_i\epsilon_j x_i x_j$ where $(\epsilon_1, \epsilon_2, \epsilon_3) = (1, 1, -1)$. Hence, by Eq. (2) there exists a affine generalized Ricci soliton associated to the connection $\nabla^0$ if and only if the following system of equations is satisfied

$$\begin{cases} -\frac{1}{2}\alpha c(a+b-c)+\mu x_1^2 = -\rho c(a+b-c)+\lambda, \\ \mu x_1 x_2 = 0, \\ -\frac{\beta}{4}x_2(a+b-c)-\mu x_1 x_3 = 0, \\ -\frac{1}{2}\alpha c(a+b-c)+\mu x_2^2 = -\rho c(a+b-c)+\lambda, \\ \frac{\beta}{4}x_1(a+b-c)-\mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho c(a+b-c)-\lambda. \end{cases} \tag{3}$$

Using the first and fourth equations of the system Eq. (3) we have $\mu(x_1^2 - x_2^2) = 0$. From the third and fiveth equations of the system Eq. (3) we get

$$\frac{\beta}{4}(x_1 - x_2)(a+b-c) - \mu x_3(x_1 + x_2) = 0.$$

Multiplying both sides of last equality by $(x_1 - x_2)$ we conclude

$$\beta(x_1 - x_2)^2(a+b-c) = 0. \tag{4}$$

The second equation of the system Eq. (3) implies that $\mu = 0$, or $x_1 = 0$ or $x_2 = 0$. Suppose that $\mu = 0$. In this case, the system Eq. (3) reduces to

$$\begin{cases} \alpha c(a+b-c) = 0, \\ \beta x_2(a+b-c) = 0, \\ \beta x_1(a+b-c) = 0, \\ \rho c(a+b-c) = \lambda. \end{cases} \tag{5}$$

If $a+b-c=0$ then the system Eq. (5) holds for any $x_1, x_2$, and $x_3$. If $a+b-c \neq 0$ for the cases (ii)–(iv) the sytem Eq. (5) holds. Now we assume that $\mu \neq 0$ and $x_1 = 0$, then $x_2 = 0$ and the system Eq. (3) becomes

$$\begin{cases} -\frac{1}{2}\alpha c(a+b-c) = -\rho c(a+b-c)+\lambda, \\ \mu x_3^2 = \rho c(a+b-c)-\lambda. \end{cases} \tag{6}$$

This shows that the case (v) holds. □

**Theorem 2** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_1, g, J, X)$ are the following*:

(i) $\mu = 0, c = 0, \lambda = 0, \beta = 0$, *and for all* $a, b, x_1, x_2, x_3, \alpha, \rho$ *such that* $\alpha \neq 0$,
(ii) $\mu = 0, c = 0, \lambda = 0, \beta \neq 0, a = b = 0$, *and for all* $x_1, x_2, x_3, \alpha, \rho$,
(iii) $\mu = 0, c = 0, \lambda = 0, \beta \neq 0, a = x_1 = 0$, *and for all* $b, x_2, x_3, \alpha, \rho$,
(iv) $\mu = 0, c = 0, \lambda = 0, \beta \neq 0, a \neq 0, x_2 = b = 0$, *and for all* $x_1, x_3, \alpha, \rho$,

(v) $\mu = 0, c = 0, \lambda = 0, \beta \neq 0, a \neq 0, x_2 = x_1 = 0,$ *and for all* $b, x_3, \alpha, \rho,$
(vi) $\mu = 0, c \neq 0, \lambda = \rho c(a+b), b = 0, \beta = a = 0,$ *and for all* $x_1, x_2, x_3, \alpha, \rho$ *such that* $\alpha \neq 0,$
(vii) $\mu = 0, c \neq 0, \lambda = \rho c(a+b), b = 0, \beta \neq 0, a = 0,$ *and for all* $x_1, x_2, x_3, \alpha, \rho,$
(viii) $\mu = 0, c \neq 0, \lambda = \rho c(a+b), b = 0, \beta \neq 0, a \neq 0, x_2 = 0,$ *and for all* $x_1, x_3, \alpha, \rho,$
(ix) $\mu = 0, c \neq 0, \lambda = \rho c(a+b), b \neq 0, \alpha = 0, a = x_1 = 0,$ *and for all* $x_2, x_3, \beta, \rho,$ *such that* $\beta \neq 0,$
(x) $\mu = 0,\ c \neq 0,\ \lambda = \rho c(a+b),\ b \neq 0,\ \alpha = 0,\ a \neq 0,\ x_2 = x_1 = 0,$ *and for all* $x_3, \beta, \rho,$ *such that* $\beta \neq 0,$
(xi) $\mu \neq 0, x_1 = 0, x_3 = 0, c = 0, \lambda = x_2 = 0,$ *for all* $a, b, \alpha, \beta, \rho,$
(xii) $\mu \neq 0, x_1 = 0, x_3 = 0, c \neq 0, \alpha = 0, \lambda = \rho c(a+b), x_2 = 0,$ *for all* $a, b, \beta, \rho,$
(xiii) $\mu \neq 0, x_1 = 0, x_3 = 0, c \neq 0, \alpha \neq 0, b = 0, \lambda = x_2 = a = 0,$ *for all* $\beta, \rho,$
(xiv) $\mu \neq 0, x_1 = 0, x_3 = 0, c \neq 0, \alpha \neq 0, b = 0, x_2 \neq 0, \lambda = \rho ca, \beta = 0, x_2^2 = \frac{a\alpha c}{\mu}$ *for all* $a, \rho,$
(xv) $\mu \neq 0, x_1 = 0, x_3 \neq 0, x_2 = 0, x_3^2 = \frac{ac\alpha}{\mu} > 0,$ *for all* $a, b, c, \alpha, \beta, \rho, \lambda$ *such that* $bc\alpha = ac\alpha = \rho c(a+b) - \lambda,$
(xvi) $\mu \neq 0,\ x_1 \neq 0,\ x_2 = x_3 = 0,\ \lambda = \rho c(a+b),$ *for all* $a, b, c, \alpha, \rho, \beta$ *such that* $\beta b = ac\alpha = 0.$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1 = \begin{pmatrix} -bc & 0 & 0 \\ 0 & -ac & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^1 g) = \begin{pmatrix} 0 & 0 & -ax_2 \\ 0 & 0 & bx_1 \\ -ax_2 & bx_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -c(a+b)$ and the Eq. (2) becomes

$$\begin{cases} -bc\alpha + \mu x_1^2 = -\rho c(a+b) + \lambda, \\ \mu x_1 x_2 = 0, \\ -\frac{\beta}{2} a x_2 - \mu x_1 x_3 = 0, \\ -ac\alpha + \mu x_2^2 = -\rho c(a+b) + \lambda, \\ \frac{\beta}{2} b x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho c(a+b) - \lambda. \end{cases} \tag{7}$$

The second equation of the system Eq. (7) implies that $\mu = 0$ or $x_1 = 0$ or $x_2 = 0$. We consider $\mu = 0$, then the first equation yields $bc\alpha = 0$. If $c = 0$ then we get $\lambda = 0$ and the cases (i)-(v) hold. If we assume that $c \neq 0$ and $\lambda = \rho c(a+b)$ and in this we obtain the cases (vi)-(x). Now, we consider the case $\mu \neq 0$ and $x_1 = 0$. In this case the system Eq. (7) reduces to

$$\begin{cases} -bc\alpha = -\rho c(a+b) + \lambda, \\ \beta a x_2 = 0, \\ -ac\alpha + \mu x_2^2 = -\rho c(a+b) + \lambda, \\ x_2 x_3 = 0, \\ \mu x_3^2 = \rho c(a+b) - \lambda. \end{cases} \tag{8}$$

The fourth equation of the system Eq. (8) implies that $x_2 = 0$ or $x_3 = 0$. If $x_3 = 0$ then we obtain the cases (xi)-(xiv). If $x_3 \neq 0$ and $x_2 = 0$ then the case (xv) holds. Also, if we consider $\mu \neq 0$ and $x_1 \neq 0$ then $x_2 = 0$ and the case (xvi) is true. □

**Theorem 3** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^0$ on the Lie group $(G_2, g, J, X)$ are the following*:

- (i) $\mu = 0, \beta \neq 0, x_1 = x_2 = \alpha = 0, \lambda = \rho(2b^2 + ac)$, *for all* $a, b, c, x_3, \rho$ *such that* $b \neq 0$,
- (ii) $\mu \neq 0, x_2 = 0, x_1 = 0, \alpha = 0, \lambda = \rho(2b^2 + ac), x_3 = 0,$ *for all* $\beta, \rho, a, b, c$ *such that* $b \neq 0$.
- (iii) $\mu \neq 0, x_2 = 0, x_1 = 0, \alpha \neq 0, a = 2c, \lambda = (2\rho - \alpha)(b^2 + c^2), x_3^2 = \frac{\alpha}{\mu}(b^2 + c^2)$, *for all* $\beta, \rho, b$ *such that* $b \neq 0$ *and* $\alpha\mu \geq 0$,
- (iv) $\mu \neq 0,\ x_2 = 0,\ x_1 = -\frac{\beta b}{\mu} \neq 0,\ x_3 = 0,\ \lambda = \rho(2b^2 + ac)$, *for all* $a, b, c, \alpha, \beta, \rho$ *such that* $\beta \neq 0, \alpha\mu(2b^2 + ac) + \beta^2 b^2 = 0,$ *and* $\alpha\mu(2c + a) - \beta^2 a = 0$.

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^0 = \begin{pmatrix} -(b^2 + \frac{ac}{2}) & 0 & 0 \\ 0 & -(b^2 + \frac{ac}{2}) & \frac{bc}{2} - \frac{ab}{4} \\ 0 & \frac{bc}{2} - \frac{ab}{4} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 0 & bx_2 & -\frac{a}{2}x_2 \\ bx_2 & -2bx_1 & \frac{a}{2}x_1 \\ -\frac{a}{2}x_2 & \frac{a}{2}x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = -(2b^2 + ac)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(b^2+\frac{ac}{2})+\mu x_1^2=-\rho(2b^2+ac)+\lambda, \\ \frac{\beta}{2}bx_2+\mu x_1x_2=0, \\ -\frac{\beta a}{4}x_2-\mu x_1x_3=0, \\ -\alpha(b^2+\frac{ac}{2})-\beta bx_1+\mu x_2^2=-\rho(2b^2+ac)+\lambda, \\ \alpha(\frac{bc}{2}-\frac{ab}{4})+\frac{\beta a}{4}x_1-\mu x_2x_3=0, \\ \mu x_3^2=\rho(2b^2+ac)-\lambda. \end{cases} \tag{9}$$

At the first we assume $\mu=0$. In this case, the system Eq. (9) reduces to

$$\begin{cases} -\alpha(b^2+\frac{ac}{2})=0, \\ \beta x_2=0, \\ \beta x_1=0, \\ \alpha(\frac{bc}{2}-\frac{ab}{4})+\frac{\beta a}{4}x_1=0, \\ \rho(2b^2+ac)=\lambda. \end{cases} \tag{10}$$

The second equation of Eq. (10) implies that $\beta=0$ or $x_2=0$. If $\beta=0$ then $\alpha\neq 0$ and the fourth equation of the system Eq. (10) yields $a=2c$ and replacing it in the first equation we obtain $b^2+c^2=0$ which is a contradiction. Thus $\beta\neq 0$ and $x_1=x_2=\alpha=0$.

Now we consider $\mu\neq 0$. Using the first and fourth equations of Eq. (9) we obtain

$$\mu x_1^2+\beta bx_1=\mu x_2^2. \tag{11}$$

The second equation of the system Eq. (9) implies that $x_2=0$ or $x_1=-\frac{\beta b}{2\mu}$. If $x_2\neq 0$ then $x_1=-\frac{\beta b}{2\mu}$ and plugging it in Eq. (11) we get $x_2^2+\frac{\beta^2b^2}{4\mu^2}=0$ which is a contradiction. Therefore $x_2=0$ and in this case we have

$$\begin{cases} -\alpha(b^2+\frac{ac}{2})+\mu x_1^2=-\rho(2b^2+ac)+\lambda, \\ \mu x_1^2+\beta bx_1=0, \\ x_1x_3=0, \\ -\alpha(b^2+\frac{ac}{2})-\beta bx_1=-\rho(2b^2+ac)+\lambda, \\ \alpha(\frac{bc}{2}-\frac{ab}{4})+\frac{\beta a}{4}x_1=0, \\ \mu x_3^2=\rho(2b^2+ac)-\lambda. \end{cases} \tag{12}$$

The third equation of the system Eq. (12) implies that $x_1=0$ or $x_3=0$. If $x_1=0$ then $\alpha(2c-a)=0$. Thus $\alpha=0$ or $a=2c$. In the case $\alpha=0$ we have $\lambda=\rho(2b^2+ac)$ and $x_3=0$. In the case $\alpha\neq 0$ and $a=2c$ we get $\lambda=(2\rho-\alpha)(b^2+c^2)$ and $x_3^2=\frac{\alpha}{\mu}(b^2+c^2)$. Now we assume that $\mu\neq 0$, $x_2=0$, $x_1\neq 0$, and $x_3=0$. In this case we have (iv). □

**Theorem 4** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_2,g,J,X)$ are the following*:

(i) $\mu = 0, \alpha = 0, \beta \neq 0, x_1 = x_2 = x_3 = 0, \lambda = \rho(2b^2 + c^2 + ac)$, *for all* $\rho, a, b, c$ *such that* $b \neq 0$,

(ii) $\mu \neq 0, x_2 = x_3 = 0, \beta = 0, \alpha = 0, x_1 = 0, \lambda = \rho(2b^2 + c^2 + ac)$, *for all* $\rho, a, b, c$ *such that* $b \neq 0$,

(iii) $\mu \neq 0, x_2 = x_3 = 0, \beta \neq 0, c = 0, \alpha = 0, x_1 = 0, \lambda = \rho(2b^2)$, *for all* $\rho, a, b, c$ *such that* $b \neq 0$,

(iv) $\mu \neq 0, x_2 = x_3 = 0, \beta \neq 0, c = 0, \alpha \neq 0, a = 0, x_1 = -\frac{\alpha b}{\beta}, \lambda = \rho(2b^2 + c^2)$, *for all* $\rho, a, b, c$ *such that* $b \neq 0$, $\mu\alpha b^2 = \beta^2(b^2 + c^2)$,

(v) $\mu \neq 0,\ x_2 = x_3 = 0,\ \beta \neq 0,\ c \neq 0,\ x_1 = \frac{\alpha ab}{\beta c},\ \lambda = \rho(2b^2 + c^2 + ac)$, *for all* $\alpha, \rho, a, b, c$ *such that* $b \neq 0, \alpha ab^2 = -\alpha c(b^2 + ac), \mu(\alpha ab)^2 = \alpha\beta^2 c^2(b^2 + c^2)$,

(vi) $\mu \neq 0,\ x_2 = 0,\ x_3^2 = \frac{\alpha}{\mu}(b^2 + c^2) - \frac{\beta^2 b^2}{4\mu^2} \neq 0,\ x_1 = \frac{\beta b}{2\mu},\ \lambda = \rho(2b^2 + c^2 + ac) -\alpha(b^2 + c^2) - \frac{\beta^2 b^2}{2\mu}$, *for all* $\alpha, \beta, \rho, a, b, c$ *such that* $b \neq 0,\ 2a\alpha\mu = \beta^2 c$, $3\beta^2 b^2 - 4\mu\alpha c(c - a) = 0$,

(vii) $\mu \neq 0,\ x_2^2 = -\frac{\alpha}{\mu}c(c - a) \neq 0,\ x_1 = 0,\ \beta = 0,\ x_3 = \frac{\alpha}{\mu}(b^2 + c^2)$, $\lambda = -\alpha(b^2 + c^2) + \rho(2b^2 + c^2 + ac)$ *for all* $\alpha, \rho, a, b, c$ *such that* $b \neq 0$, $-4c(c - a)(b^2 + c^2) = a^2 b^2, \frac{\alpha}{\mu} \geq 0, -c(c - a) \geq 0$,

(viii) $\mu \neq 0,\ x_2 \neq 0,\ x_1 = -\frac{\beta b}{2\mu},\ \beta \neq 0,\ x_3 = \frac{a}{2b}x_2 = -\frac{2\alpha\mu ab + c\beta^2 b}{4\mu^2}$, $\lambda = -\alpha(b^2 + c^2) + \frac{\beta^2 b^2}{4\mu} + \rho(2b^2 + c^2 + ac)$, *for all* $\alpha, \rho, a, b, c$ *such that* $x_2^2 = -(\frac{\beta b}{2\mu})^2 - \frac{\alpha}{\mu}c(c - a), x_3^2 = \frac{\alpha}{\mu}(b^2 + c^2) - \frac{\beta^2 b^2}{4\mu^2} > 0$.

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1 = \begin{pmatrix} -(b^2 + c^2) & 0 & 0 \\ 0 & -(b^2 + ac) & -\frac{ab}{2} \\ 0 & -\frac{ab}{2} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^1 g) = \begin{pmatrix} 0 & bx_2 & -ax_2 + bx_3 \\ bx_2 & -2bx_1 & cx_1 \\ ax_2 + bx_3 & cx_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -(2b^2 + c^2 + ac)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(b^2 + c^2) + \mu x_1^2 = -\rho(2b^2 + c^2 + ac) + \lambda, \\ \frac{\beta}{2}bx_2 + \mu x_1 x_2 = 0, \\ \frac{\beta}{2}(-ax_2 + bx_3) - \mu x_1 x_3 = 0, \\ -\alpha(b^2 + ac) - \beta bx_1 + \mu x_2^2 = -\rho(2b^2 + c^2 + ac) + \lambda, \\ -\alpha\frac{ab}{2} + \frac{\beta}{2}cx_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2b^2 + c^2 + ac) - \lambda. \end{cases} \tag{13}$$

We first consider $\mu = 0$. In this case, the system Eq. (13) becomes

$$\begin{cases} -\alpha(b^2 + c^2) = 0, \\ \beta b x_2 = 0, \\ \beta(-a x_2 + b x_3) = 0, \\ -\alpha(b^2 + ac) - \beta b x_1 = 0, \\ -\alpha ab + \beta c x_1 = 0, \\ \rho(2b^2 + c^2 + ac) = \lambda. \end{cases} \tag{14}$$

Since $b \neq 0$, the first equation of Eq. (14) implies that $\alpha = 0$. Due to $(\alpha, \beta, \mu) \neq (0, 0, 0)$ we conclude $\beta \neq 0$. Then the second equation of the system Eq. (14) yields $x_2 = 0$. Using, the third and fourth equations of Eq. (14) we obtain $x_1 = x_3 = 0$.

Now we consider $\mu \neq 0$. The second equation of the system Eq. (13) implies that $x_2 = 0$ or $x_1 = -\frac{\beta b}{2\mu}$. If $x_2 = 0$ then we get

$$\begin{cases} -\alpha(b^2 + c^2) + \mu x_1^2 + \mu x_3^2 = 0, \\ \frac{\beta}{2} b x_3 - \mu x_1 x_3 = 0, \\ -\alpha(b^2 + ac) - \beta b x_1 + \mu x_3^2 = 0, \\ -\alpha \frac{ab}{2} + \frac{\beta}{2} c x_1 = 0, \\ \mu x_3^2 = \rho(2b^2 + c^2 + ac) - \lambda. \end{cases} \tag{15}$$

From the second equation of the system Eq. (15) we obtain $x_3 = 0$ or $x_1 = \frac{\beta b}{2\mu}$. If $x_3 = 0$ then the cases (ii)-(v) hold. If $x_3 \neq 0$ and $x_1 = \frac{\beta b}{2\mu}$ then the case (vi) holds. Now we assume that $\mu \neq 0$, $x_2 \neq 0$ and $x_1 = -\frac{\beta b}{2\mu}$. In this cases, the system Eq. (13) reduces to

$$\begin{cases} -\alpha(b^2 + c^2) + \frac{\beta^2 b^2}{4\mu} = -\rho(2b^2 + c^2 + ac) + \lambda, \\ \frac{\beta}{2} b x_2 + \mu x_1 x_2 = 0, \\ \beta(-a x_2 + 2b x_3) = 0, \\ -\alpha(b^2 + ac) + \frac{\beta^2 b^2}{2\mu} + \mu x_2^2 = -\rho(2b^2 + c^2 + ac) + \lambda, \\ -\alpha \frac{ab}{2} - \frac{c\beta^2 b}{4\mu} - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2b^2 + c^2 + ac) - \lambda. \end{cases} \tag{16}$$

Thus the cases (vii)-(viii) are true. □

**Theorem 5** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^0$ on the Lie group $(G_3, g, J, X)$ are the following*:

(i) $\mu = 0, \alpha = 0, \beta \neq 0, x_1 = x_2 = 0,$ *for all* $\rho, a, b, c, x_3$ *such that* $a \neq 0$,
(ii) $\mu \neq 0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \alpha = 0,\ \lambda = \rho(2a^2 + b^2),$ *for all* $\beta, a, b, c$ *such that* $a \neq 0$,

(iii) $\mu \neq 0,\ x_1 = 0,\ x_2 = \frac{\beta a}{\mu} \neq 0,\ x_3 = \frac{a\alpha}{2\beta},\ \lambda = (2\rho - \alpha)(a^2 + \frac{b^2}{2}) + \frac{\beta^2 a^2}{\mu}$, *for all* $\alpha, \beta, a, b, c, \rho$ *such that* $a \neq 0$, $\mu\alpha b = \beta^2 b$, $\frac{\alpha^2 a^2}{4\beta^2} = \frac{\alpha}{\mu}(a^2 + \frac{b^2}{2}) - \frac{\beta^2 a^2}{\mu^2}$.

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^0 = \begin{pmatrix} -(a^2+\frac{b^2}{2}) & 0 & \frac{ab}{4} \\ 0 & -(a^2+\frac{b^2}{2}) & \frac{a^2}{2} \\ \frac{ab}{4} & \frac{a^2}{2} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 2ax_2 & -ax_1 & -\frac{b}{2}x_2 \\ -ax_1 & 0 & \frac{b}{2}x_1 \\ -\frac{b}{2}x_2 & \frac{b}{2}x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = -(2a^2 + b^2)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(a^2+\frac{b^2}{2}) + \beta a x_2 + \mu x_1^2 = -\rho(2a^2+b^2) + \lambda, \\ -\frac{\beta}{2}ax_1 + \mu x_1 x_2 = 0, \\ \frac{\alpha ab}{4} - \frac{\beta b}{4}x_2 - \mu x_1 x_3 = 0, \\ -\alpha(a^2+\frac{b^2}{2}) + \mu x_2^2 = -\rho(2a^2+b^2) + \lambda, \\ \alpha\frac{a^2}{2} + \frac{\beta b}{4}x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2+b^2) - \lambda. \end{cases} \tag{17}$$

Let $\mu = 0$. In this case, the system Eq. (17) reduces to

$$\begin{cases} \beta x_2 = 0, \\ \beta x_1 = 0, \\ \alpha = 0, \\ \lambda = \rho(2a^2+b^2). \end{cases} \tag{18}$$

Since $(\alpha, \beta, \mu) \neq (0,0,0)$ we get $\beta \neq 0$ and $x_1 = x_2 = 0$. Thus the case (i) holds. Using of the first and fourth equations of the system Eq. (17) we get

$$\beta a x_2 + \mu x_1^2 = \mu x_2^2. \tag{19}$$

Now, we consider $\mu \neq 0$, in this case, the second equation of the system Eq. (17) implies that $x_1 = 0$ or $x_2 = \frac{\beta a}{2\mu}$. If $x_1 \neq 0$ then $x_2 = \frac{\beta a}{2\mu}$. Substituting it in Eq. (19) we have $(\frac{\beta a}{2\mu})^2 + x_1^2 = 0$ which is a contradiction. Hence $x_1 = 0$ and the system Eq. (17) and Eq. (19) become

$$\begin{cases} -\alpha(a^2+\frac{b^2}{2})+\beta a x_2=-\rho(2a^2+b^2)+\lambda, \\ \frac{\alpha ab}{4}-\frac{\beta b}{4}x_2=0, \\ -\alpha(a^2+\frac{b^2}{2})+\mu x_2^2=-\rho(2a^2+b^2)+\lambda, \\ \alpha\frac{a^2}{2}-\mu x_2 x_3=0, \\ \mu x_3^2=\rho(2a^2+b^2)-\lambda, \\ \beta a x_2=\mu x_2^2. \end{cases} \tag{20}$$

The sixth equation of Eq. (20) yields $x_2=0$ or $x_2=\frac{\beta a}{\mu}$. If $x_2=0$ then the case (ii) is true. If $x_2\neq 0$ and $x_2=\frac{\beta a}{\mu}$ then the case (iii) holds. □

**Theorem 6** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_3, g, J, X)$ are the following*:

(*i*) $\mu=0, \alpha=0, \beta\neq 0, x_1=x_2=x_3=0, \lambda=2\rho(a^2+b^2)$, *for all* $\rho, a, b, c$ *such that* $a\neq 0$,
(ii) $\mu\neq 0, \alpha b=0, x_1=\beta=x_2=\alpha=x^3=0, \lambda=2\rho(a^2+b^2)$,
(iii) $\mu\neq 0, \alpha b=0, x_1=0, \beta=0, x_2=0, x_3=-\frac{\alpha a}{\beta}, \lambda=(2\rho-\alpha)(a^2+b^2)$, $\alpha^2a^2\mu=\beta^2\alpha(a^2+b^2)$,
(iv) $\mu\neq 0, \alpha b=0, x_1=0, \beta=0, x_2=\frac{\beta a}{\mu}, b=0, \alpha\mu=-\beta^2, x_3=\beta^2a^2\frac{\mu-1}{\mu^2}$, $\lambda=(2\rho-\alpha)(a^2+b^2)+\frac{\beta^2a^2}{\mu}$,
(*v*) $\mu\neq 0, \alpha b\neq 0$,

$$x_1=\epsilon_1\sqrt{\frac{-\beta^2a^2+\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}{8\mu^2}}, \quad \epsilon_1=\pm 1,$$

$$x_2=\frac{-\beta a+\epsilon_2\sqrt{\frac{1}{2}\beta^2a^2+\frac{1}{2}\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}}{-2\mu}, \quad \epsilon_2=\pm 1,$$

$$\lambda=(2\rho-\alpha)(a^2+b^2)+\mu\left(\frac{-\beta a+\epsilon_2\sqrt{\frac{1}{2}\beta^2a^2+\frac{1}{2}\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)^2,$$

and

$$x_3 = \epsilon_3 \sqrt{\frac{\alpha}{\mu}(a^2+b^2) - \left(\frac{-\beta a + \epsilon_2\sqrt{\frac{1}{2}\beta^2a^2 + \frac{1}{2}\sqrt{\beta^4a^4 + 64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)^2},$$

where $\epsilon_3 = \pm 1$, $\alpha ab = \epsilon_1\epsilon_2\|\alpha ab\|$,

$$\frac{\alpha ab + \beta b\left(\frac{-\beta a+\epsilon_2\sqrt{\frac{1}{2}\beta^2a^2+\frac{1}{2}\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)}{\epsilon_1\sqrt{\frac{-\beta^2a^2+\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}{8\mu^2}}} - \beta a$$

$$= 2\mu\epsilon_3 \sqrt{\frac{\alpha}{\mu}(a^2+b^2) - \left(\frac{-\beta a + \epsilon_2\sqrt{\frac{1}{2}\beta^2a^2 + \frac{1}{2}\sqrt{\beta^4a^4 + 64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)^2},$$

and

$$\frac{2\mu\alpha a^2 - 2\mu\beta b\epsilon_1\sqrt{\frac{-\beta^2a^2+\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}{8\mu^2}}}{-\beta a + \epsilon_2\sqrt{\frac{1}{2}\beta^2a^2 + \frac{1}{2}\sqrt{\beta^4a^4 + 64\mu^2\alpha^2b^2a^2}}}$$

$$+ \frac{2\mu a\beta\epsilon_3\sqrt{\frac{\alpha}{\mu}(a^2+b^2) - \left(\frac{-\beta a+\epsilon_2\sqrt{\frac{1}{2}\beta^2a^2+\frac{1}{2}\sqrt{\beta^4a^4+64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)^2}}{-\beta a + \epsilon_2\sqrt{\frac{1}{2}\beta^2a^2 + \frac{1}{2}\sqrt{\beta^4a^4 + 64\mu^2\alpha^2b^2a^2}}} - a\beta$$

$$= 2\mu\epsilon_3 \sqrt{\frac{\alpha}{\mu}(a^2+b^2) - \left(\frac{-\beta a + \epsilon_2\sqrt{\frac{1}{2}\beta^2a^2 + \frac{1}{2}\sqrt{\beta^4a^4 + 64\mu^2\alpha^2b^2a^2}}}{-2\mu}\right)^2}.$$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1 = \begin{pmatrix} -(a^2+b^2) & ab & -\frac{ab}{2} \\ ab & -(a^2+b^2) & \frac{a^2}{2} \\ -\frac{ab}{2} & \frac{a^2}{2} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}^1_X g) = \begin{pmatrix} 2ax_2 & -ax_1 & ax_1 - bx_2 \\ -ax_1 & 0 & bx_1 - ax_2 - ax_3 \\ ax_1 - bx_2 & bx_1 - ax_2 - ax_3 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -2(a^2 + b^2)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(a^2 + b^2) + \beta a x_2 + \mu x_1^2 = -2\rho(a^2 + b^2) + \lambda, \\ ab\alpha - \frac{\beta}{2} a x_1 + \mu x_1 x_2 = 0, \\ -\frac{\alpha ab}{2} + \frac{\beta}{2}(ax_1 - bx_2) - \mu x_1 x_3 = 0, \\ -\alpha(a^2 + b^2) + \mu x_2^2 = -2\rho(a^2 + b^2) + \lambda, \\ -\alpha\frac{a^2}{2} + \frac{\beta}{2}(bx_1 - ax_2 - ax_3) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = 2\rho(a^2 + b^2) - \lambda. \end{cases} \tag{21}$$

Let $\mu = 0$ then we have $\alpha = 0$ and $\beta \neq 0$. Thus $x_1 = x_2 = x_3 = 0$ and the case (i) holds. Now, we assume that $\mu \neq 0$. The first and fourth equations of the system Eq. (21) imply that

$$\beta a x_2 + \mu x_1^2 - \mu x_2^2 = 0, \tag{22}$$

and the fourth and sixth equations imply that

$$x_2^2 + x_3^2 = \frac{\alpha}{\mu}(a^2 + b^2). \tag{23}$$

From the second equation we have

$$(2\alpha ba)^2 = x_1^2(\beta a - 2\mu x_2)^2 = x_1^2(\beta^2 a^2 + 4\mu(\mu x_2^2 - \beta a x_2)).$$

Plugging Eq. (22) into last equality we get

$$4\mu^2 x_1^4 + \beta^2 a^2 x_1^2 - (2\alpha ba)^2 = 0. \tag{24}$$

If $\alpha b = 0$ then $x_1 = 0$ and we obtain three cases (ii)-(iv). If $\alpha b \neq 0$, then $x_1 \neq 0$ and the case (v) is true. □

**Theorem 7** *The left-invariant affine generalized Ricci soliton associated to the connection* $\nabla^0$ *on the Lie group* $(G_4, g, J, X)$ *are the following*:

(i) $\mu = 0,\ \alpha = 0,\ \beta \neq 0,\ x_1 = x_2 = 0,\ \lambda = -\rho((2d - b)(a + 2d) - 2)$ *for all* $a, b, c, \rho, x_3$ *such that* $d = \pm 1$.
(ii) $\mu = 0, \alpha \neq 0, d = b, a = 0\ \beta = 0, \lambda = 0,$ *for all* $x_1, x_2, x_3, \rho$ *such that* $d = \pm 1$,
(iii) $\mu = 0, \alpha \neq 0, d = b, a = 0\ \beta \neq 0, x_1 = x_2 = 0, \lambda = 0,$ *for all* $x_3, \rho$ *such that* $d = \pm 1$,

(iv) $\mu \neq 0,\ x_2 = 0,\ x_3 = 0,\ x_1 = 0,\ \alpha = 0,\ \lambda = -\rho((2d-b)(a+2d)-2)$, *for all* $a, b, \rho, \beta, d = \pm 1$,

(v) $\mu \neq 0, x_2 = 0, x_3 = 0, x_1 = 0, \alpha \neq 0, b = d, a = 0, \lambda = 0$, *for all* $\rho, \beta, d = \pm 1$,

(vi) $\mu \neq 0,\ x_2 = 0,\ x_3 = 0,\ x_1 = \frac{\beta}{\mu} \neq 0,\ \lambda = -\rho((2d-b)(a+2d)-2)$ *for all* $a, b, \rho, \alpha$ *such that* $d = \pm 1$, $\beta^2 = -\alpha\mu\big((2d-b)(\frac{a}{2}+d)-1\big) > 0$, $\alpha\big(a+b-(2d-b)(\frac{a}{2}+d)^2\big) = 0$,

(vii) $\mu \neq 0$, $x_2 = 0$, $x_3 \neq 0$, $x_1 = 0$, $x_3^2 = -\frac{\alpha}{\mu}\big((2d-b)(\frac{a}{2}+d)-1\big) > 0$, $\lambda = (\alpha-2\rho)\big((2d-b)(\frac{a}{2}+d)-1\big)$, *for all* $\rho, \beta$ *such that* $d = \pm 1$, $\alpha(a+2b-2d) = 0$.

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^{0} = \begin{pmatrix} (2d-b)(\frac{a}{2}+d)-1 & 0 & 0 \\ 0 & (2d-b)(\frac{a}{2}+d)-1 & \frac{a}{4}+\frac{d}{2}-\frac{b}{2} \\ 0 & \frac{a}{4}+\frac{d}{2}-\frac{b}{2} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 0 & -x_2 & -(\frac{a}{2}+d)x_2 \\ -x_2 & 2x_1 & (\frac{a}{2}+d)x_1 \\ -(\frac{a}{2}+d)x_2 & (\frac{a}{2}+d)x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = (2d-b)(a+2d)-2$ and the Eq. (2) becomes

$$\begin{cases} \alpha\big((2d-b)(\frac{a}{2}+d)-1\big) + \mu x_1^2 = \rho((2d-b)(a+2d)-2) + \lambda, \\ -\frac{\beta}{2}x_2 + \mu x_1 x_2 = 0, \\ -\frac{\beta}{2}(\frac{a}{2}+d)x_2 - \mu x_1 x_3 = 0, \\ \alpha\big((2d-b)(\frac{a}{2}+d)-1\big) + \beta x_1 + \mu x_2^2 = \rho((2d-b)(a+2d)-2) + \lambda, \\ \alpha\big(\frac{a}{4}+\frac{d}{2}-\frac{b}{2}\big) + \frac{\beta}{2}(\frac{a}{2}+d)x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = -\rho((2d-b)(a+2d)-2) - \lambda. \end{cases} \tag{25}$$

We consider $\mu = 0$, then the system Eq. (25) reduces to

$$\begin{cases} \alpha\big((2d-b)(\frac{a}{2}+d)-1\big) = 0, \\ \beta x_2 = 0, \\ \beta x_1 = 0, \\ \alpha\big(\frac{a}{4}+\frac{d}{2}-\frac{b}{2}\big) = 0, \\ \lambda = -\rho((2d-b)(a+2d)-2). \end{cases} \tag{26}$$

If $\alpha = 0$ then $\beta \neq 0$ and $x_1 = x_2 = 0$. Thus the case (i) holds. If $\alpha \neq 0$ then $d = b$, $a = 0$ and the cases (ii)-(iii) hold. Now we consider $\mu \neq 0$. The first and third equations of the system Eq. (25) yield

$$\mu x_1^2 = \beta x_1 + \mu x_2^2. \tag{27}$$

In this case the second eqution of the system Eq. (25) implies that $x_2 = 0$ or $x_1 = \frac{\beta}{2\mu}$. If $x_2 \neq 0$ then $x_1 = \frac{\beta}{2\mu}$ and substutiting it in Eq. (27) we get $x_2^2 + \frac{\beta^2}{4\mu^2} = 0$ which is a cotradiction. Hence, $x_2 = 0$ and the system Eq. (25) becomes

$$\begin{cases} \alpha\big((2d-b)(\frac{a}{2}+d)-1\big) + \mu x_1^2 = \rho((2d-b)(a+2d)-2) + \lambda, \\ x_1x_3 = 0, \\ \alpha\big((2d-b)(\frac{a}{2}+d)-1\big) + \beta x_1 = \rho((2d-b)(a+2d)-2) + \lambda, \\ \alpha\big(\frac{a}{4}+\frac{d}{2}-\frac{b}{2}\big) + \frac{\beta}{2}(\frac{a}{2}+d)x_1 = 0, \\ \mu x_3^2 = -\rho((2d-b)(a+2d)-2) - \lambda. \end{cases} \tag{28}$$

In this cases (iv)-(vi) hold. □

**Theorem 8** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_4, g, J, X)$ are the following*:

(i) $\mu = 0, \alpha = 0, x_1 = x_2 = x_3 = 0, \lambda = 2\rho[1 + (b-2d)b]$, *for all* $a, b, \rho, d = \pm 1$,

(ii) $\mu = 0,\ \alpha \neq 0,\ \beta \neq 0,\ x_2 = x_3 = 0,\ b = d,\ a = 2b,\ x_1 = \frac{\alpha}{2\beta},\ \lambda = 0$, *for all* $\rho$, $d = \pm 1$,

(iii) $\mu \neq 0, x_2 = 0, x_3 = 0, \alpha = 0, x_1 = 0, \lambda = 2\rho[1 + (b-2d)b]$, *for all* $a, b, \rho, \beta$ *such that* $d = \pm 1$,

(iv) $\mu \neq 0, x_2 = 0, x_3 = 0, \alpha \neq 0, \beta \neq 0, b \neq 0, x_1 = \frac{\alpha a}{\beta b}, \lambda = 2\rho[1 + (b-2d)b]$, *for all* $a, b, \rho$ *such that* $d = \pm 1$, $b[1 + (b-2d)a] = a$,

$$-\beta^2[1 + (b-2d)b] + \mu\alpha[1 + (b-2d)a]^2 = 0,$$

(v) $\mu \neq 0$ , $x_2 = 0$ , $x_3^2 = \frac{\alpha}{\mu}[1 + (b-2d)b] - \frac{\beta^2}{4\mu^2} > 0$ , $x_1 = -\frac{\beta}{2\mu}$ , $\lambda = (2\rho - \alpha)[1 + (b-2d)b] + \frac{\beta^2}{4\mu}$, *for all* $a, b, \rho, \alpha$ *such that*

$$d = \pm 1, \quad \alpha(b-2d)(a-b) = -\frac{3\beta^2}{4\mu}, \quad 2\mu\alpha a = -b\beta^2,$$

(vi) $\mu \neq 0$ , $x_2 \neq 0$ , $x_1 = \frac{\beta}{2\mu}$ , $\beta = 0$ , $\lambda = (2\rho - \alpha)[1 + (b-2d)b]$ , $x_2 = \epsilon_1\sqrt{-\frac{\alpha}{\mu}[(b-2d)(b-a)]}, x_3 = \epsilon_2\sqrt{\frac{\alpha}{\mu}[1 + (b-2d)b]}$, *for all* $\rho$ *such that* $d = \pm 1, \frac{\alpha}{\mu}[(b-2d)(b-a)] \leq 0, \frac{\alpha}{\mu}[1 + (b-2d)b] \geq 0, \epsilon_1 = \pm 1, \epsilon_2 = \pm 1$,

$$-\frac{a\alpha}{\mu} = \epsilon_1\epsilon_2\sqrt{-\frac{\alpha}{\mu}[(b-2d)(b-a)]}\sqrt{\frac{\alpha}{\mu}[1 + (b-2d)b]}$$

(vii) $\mu \neq 0, x_2 \neq 0, x_1 = \frac{\beta}{2\mu}, \beta \neq 0, x_3 = -\frac{a}{2}x_2, \lambda = (2\rho - \alpha)[1 + (b-2d)b] + \frac{\beta^2}{4\mu}$, $x_2^2 = \frac{-2\mu\alpha[1+(b-2d)a]+\beta^2}{-2\mu^2(1+\frac{a^2}{4})} > 0$,

$$d = \pm 1, \quad -a\alpha + \frac{\beta^2}{2\mu}b + \frac{a\mu}{2}\frac{-2\mu\alpha[1+(b-2d)a]+\beta^2}{-2\mu^2(1+\frac{a^2}{4})} = 0,$$

$$-\alpha[1+(b-2d)b] + \frac{\beta^2}{4\mu} = -\mu\frac{a^2}{4}\frac{-2\mu\alpha[1+(b-2d)a]+\beta^2}{-2\mu^2(1+\frac{a^2}{4})}.$$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1 = \begin{pmatrix} -[1+(b-2d)b] & 0 & 0 \\ 0 & -[1+(b-2d)a] & \frac{a}{2} \\ 0 & \frac{a}{2} & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^1 g) = \begin{pmatrix} 0 & -x_2 & -ax_2 - x_3 \\ -x_2 & 2x_1 & bx_1 \\ -ax_2 - x_3 & bx_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -2[1+(b-2d)b]$ and the Eq. (2) becomes

$$\begin{cases} -\alpha[1+(b-2d)b] + \mu x_1^2 = -2\rho[1+(b-2d)b] + \lambda, \\ -\frac{\beta}{2}x_2 + \mu x_1 x_2 = 0, \\ \frac{\beta}{2}(-ax_2 - x_3) - \mu x_1 x_3 = 0, \\ -\alpha[1+(b-2d)a] + \beta x_1 + \mu x_2^2 = -2\rho[1+(b-2d)b] + \lambda, \\ -\alpha\frac{a}{2} + \frac{\beta}{2}(bx_1) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = 2\rho[1+(b-2d)b] - \lambda. \end{cases} \tag{29}$$

Let $\mu = 0$, then the system Eq. (29) becomes

$$\begin{cases} \alpha[1+(b-2d)b] = 0, \\ \beta x_2 = 0, \\ \beta x_3 = 0, \\ -\alpha[1+(b-2d)a] + \beta x_1 = 0, \\ -a\alpha + \beta b x_1 = 0, \\ \lambda = 2\rho[1+(b-2d)b]. \end{cases} \tag{30}$$

and the cases (i)-(ii) holds. Now we consider $\mu \neq 0$. In this case the second equation of the system Eq. (29) implies that $x_2 = 0$ or $x_1 = \frac{\beta}{2\mu}$. If $x_2 = 0$ then the system Eq. (29) gives

$$\begin{cases} -\alpha[1+(b-2d)b]+\mu x_1^2 = -2\rho[1+(b-2d)b]+\lambda, \\ \beta x_3 + 2\mu x_1 x_3 = 0, \\ -\alpha[1+(b-2d)a]+\beta x_1 = -2\rho[1+(b-2d)b]+\lambda, \\ -\alpha a + \beta(bx_1) = 0, \\ \mu x_3^2 = 2\rho[1+(b-2d)b]-\lambda. \end{cases} \tag{31}$$

The second equation of the system Eq. (31) implies that $x_3 = 0$ or $x_1 = -\frac{\beta}{2\mu}$. We assume that $x_3 = 0$, thus

$$\begin{cases} -\alpha[1+(b-2d)b]+\mu x_1^2 = 0, \\ -\alpha[1+(b-2d)a]+\beta x_1 = 0, \\ -\alpha a + \beta b x_1 = 0, \\ \lambda = 2\rho[1+(b-2d)b], \end{cases}$$

and the cases (iii)-(iv) are true. If $x_3 \neq 0$ and $x_1 = -\frac{\beta}{2\mu}$ then the case (v) is true. Now, we consider $x_2 \neq 0$ and $x_1 = \frac{\beta}{2\mu}$. In this case the system Eq. (29) yields

$$\begin{cases} -\alpha[1+(b-2d)b]+\frac{\beta^2}{4\mu} = -2\rho[1+(b-2d)b]+\lambda, \\ \beta a x_2 + 2\beta x_3 = 0, \\ -\alpha[1+(b-2d)a]+\frac{\beta^2}{2\mu}+\mu x_2^2 = -2\rho[1+(b-2d)b]+\lambda, \\ -a\alpha + \frac{\beta^2}{2\mu}b - \mu x_2 x_3 = 0, \\ \mu x_3^2 = 2\rho[1+(b-2d)b]-\lambda. \end{cases} \tag{32}$$

The second equation of Eq. (32) implies that $\beta = 0$ or $x_3 = -\frac{a}{2}x_2$. If $\beta = 0$ then we obtain the case (vi). If $\beta \neq 0$ then $x_3 = -\frac{a}{2}x_2$ and the case (vii) holds.

□

**Theorem 9** *The left-invariant affine generalized Ricci soliton associated to the connection* $\nabla^0$ *on the Lie group* $(G_5, g, J, X)$ *are the following*:

(i) $\mu = \beta = \lambda = 0$ *and for all* $\alpha, \rho, x_1, x_2, x_3, a, b, c, d$ *such that* $a + d \neq 0$ *and* $ac + bd = 0$,
(ii) $\mu = \lambda = 0, \beta \neq 0, b = c$, *and for all* $\alpha, \rho, x_1, x_2, x_3, a, d$ *such that* $a + d \neq 0$ *and* $ac + bd = 0$,
(iii) $\mu \neq 0, x_1 = x_2 = x_3 = \lambda = 0$ *and for all* $\alpha, \rho, a, b, c, d$ *such that* $a + d \neq 0$ *and* $ac + bd = 0$.

***Proof*** From [19, 20], we have $\widetilde{Ric}^0 = 0$ and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 0 & 0 & \frac{b-c}{2}x_2 \\ 0 & 0 & -\frac{b-c}{2}x_1 \\ \frac{b-c}{2}x_2 & -\frac{b-c}{2}x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = 0$ and the Eq. (2) becomes

$$\begin{cases} \mu x_1^2 = \lambda, \\ \mu x_1 x_2 = 0, \\ \frac{\beta}{2}\frac{b-c}{2}x_2 - \mu x_1 x_3 = 0, \\ \mu x_2^2 = \lambda, \\ -\frac{\beta}{2}\frac{b-c}{2}x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = -\lambda. \end{cases} \tag{33}$$

The first, fourth and sixth equations of system Eq. (33) imply that

$$\mu(x_1^2 + x_3^2) = \mu(x_2^2 + x_3^2) = 0.$$

We consider $\mu = 0$, then $\lambda = 0$. If $\beta = 0$ or $b = c$ then the system Eq. (33) holds for any $x_1, x_2$, and $x_3$. Now, if $\mu \neq 0$ then $x_1 = x_2 = x_3 = \lambda = 0$. □

**Theorem 10** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_5, g, J, X)$ are the following*:

(i)$\mu = \beta = \lambda = 0$ *and for all* $\alpha, \rho, x_1, x_2, x_3, a, b, c, d$ *such that* $a + d \neq 0$ *and* $ac + bd = 0$,
(ii)$\mu = \lambda = 0, \beta \neq 0, a = b = x_2 = 0$, and for all $\alpha, \rho, x_1, x_3, c, d$ such that $d \neq 0$,
(iii)$\mu = \lambda = 0, \beta \neq 0, a \neq 0, x_1 = x_2 = 0$, and for all $\alpha, \rho, x_3, c, d$ such that $d \neq 0$,
(iv)$\mu = \lambda = 0, \beta \neq 0, a \neq 0, x_1 = c = d = 0, x_2 \neq 0$, and for all $\alpha, \rho, x_3$,
(v)$\mu \neq 0,\ x_1 = x_2 = x_3 = \lambda = 0$ *and for all* $\alpha, \rho, a, b, c, d$ *such that* $a + d \neq 0$ *and* $ac + bd = 0$.

***Proof*** From [19, 20], we have $\widetilde{Ric}^1 = 0$ and

$$(\mathcal{L}_X^1 g) = \begin{pmatrix} 0 & 0 & -ax_1 - cx_2 \\ -0 & 0 & -bx_1 - dx_2 \\ -ax_1 - cx_2 & -bx_1 - dx_2 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = 0$ and the Eq. (2) becomes

$$\begin{cases} \mu x_1^2 = \lambda, \\ \mu x_1 x_2 = 0, \\ \frac{\beta}{2}(-a x_1 - c x_2) - \mu x_1 x_3 = 0, \\ \mu x_2^2 = \lambda, \\ \frac{\beta}{2}(-b x_1 - d x_2) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = -\lambda. \end{cases} \tag{34}$$

The first, fourth and sixth equations of system Eq. (34) imply that

$$\mu(x_1^2 + x_3^2) = \mu(x_2^2 + x_3^2) = 0.$$

We consider $\mu = 0$, then $\lambda = 0$. Let $\beta = 0$, then the system Eq. (34) holds for any $x_1, x_2$, and $x_3$. If $\beta \neq 0$ then the third and fiveth equations of Eq. (34) given

$$\begin{cases} a x_1 + c x_2 = 0, \\ b x_1 + d x_2 = 0. \end{cases}$$

Since $ac + bd = 0$ we get $(a^2 + b^2)x_1 = 0$ and $(c^2 + d^2)x_2 = 0$. We consider $a = 0$. In this case we obtain $d \neq 0$ and $b = x_2 = 0$. if $a \neq 0$ then $x_1 = 0$ and $cx_2 = dx_2 = 0$. For case $x_2 \neq 0$ we have $c = d = 0$. Now, we assume that $\mu \neq 0$, then $x_1 = x_2 = x_3 = \lambda = 0$. □

**Theorem 11** *The left-invariant affine generalized Ricci soliton associated to the connection* $\nabla^0$ *on the Lie group* $(G_6, g, J, X)$ *are the following*:

- (i) $\mu = 0,\ a = 0,\ d \neq 0,\ b = 0,\ c = 0,\ \lambda = 0,$ *for all* $\alpha, \beta, \rho, x_1, x_2, x_3$, *such that* $(\alpha, \beta) \neq (0, 0)$,
- (ii) $\mu = 0, a = 0, d \neq 0, b = 0, c \neq 0, \beta \neq 0, \lambda = 0, x_2 = 0, x_1 = -\frac{\alpha d}{\beta}$ *for all* $\alpha, \rho, x_3$,
- (iii) $\mu = 0,\ a \neq 0,\ \beta = 0,\ \alpha \neq 0,\ \lambda = 0,\ c = \frac{bd}{a}$, *for all* $b, d, \rho, x_1, x_2, x_3$, *such that* $b^2(a - d) = 2a^3, d(a - d) = 2\alpha$,
- (iv) $\mu = 0, a \neq 0, \beta \neq 0, x_1 = x_2 = 0, \alpha = 0, c = \frac{bd}{a}, \lambda = -\rho(b(b - c) - 2a^2)$ *for all* $b, d, x_3, \rho$ *such that* $a + d \neq 0$,
- (v) $\mu \neq 0, x_2 = 0, x_1 = 0, x_3^2 = -\frac{\alpha}{\mu}\left(\frac{1}{2}b^2 - a^2\right) \geq 0, \lambda = (-2\rho + \alpha)\left(\frac{1}{2}b(b - c) - a^2\right)$, *for all* $a, b, c, d, \rho, \beta, \alpha$ *such that* $\alpha c = 0, ac - bd = 0, a + d \neq 0$,
- (vi) $\mu \neq 0,\ x_2 = 0,\ x_3 = 0,\ x_1 = -\frac{\beta a}{\mu} \neq 0,\ \lambda = -\rho(b(b - c) - 2a^2)$, *for all* $a, b, c, d, \rho, \beta, \alpha$ *such that*

$$\alpha\left(\frac{1}{2}b(b - c) - a^2\right) + \frac{\beta^2 a^2}{\mu} = 0, \quad \alpha\frac{1}{2}[-ac + \frac{1}{2}d(b - c)] - \frac{\beta^2 a}{4\mu}(b - c) = 0,$$

$$ac - bd = 0, \quad a + d \neq 0.$$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^0 = \begin{pmatrix} \frac{1}{2}b(b-c)-a^2 & 0 & 0 \\ 0 & \frac{1}{2}b(b-c)-a^2 & \frac{1}{2}[-ac+\frac{1}{2}d(b-c)] \\ 0 & \frac{1}{2}[-ac+\frac{1}{2}d(b-c)] & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} 0 & ax_2 & \frac{c-b}{2}x_2 \\ ax_2 & -2ax_1 & \frac{b-c}{2}x_1 \\ \frac{c-b}{2}x_2 & \frac{b-c}{2}x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = b(b-c) - 2a^2$ and the Eq. (2) becomes

$$\begin{cases} \alpha\left(\frac{1}{2}b(b-c)-a^2\right) + \mu x_1^2 = \rho(b(b-c)-2a^2) + \lambda, \\ \frac{\beta}{2}ax_2 + \mu x_1 x_2 = 0, \\ \frac{\beta}{4}(c-b)x_2 - \mu x_1 x_3 = 0, \\ \alpha\left(\frac{1}{2}b(b-c)-a^2\right) - \beta a x_1 + \mu x_2^2 = \rho(b(b-c)-2a^2) + \lambda, \\ \alpha\frac{1}{2}[-ac+\frac{1}{2}d(b-c)] + \frac{\beta}{4}(b-c)x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = -\rho(b(b-c)-2a^2) - \lambda. \end{cases} \tag{35}$$

Let $\mu = 0$, then the system Eq. (35) becomes

$$\begin{cases} \alpha\left(\frac{1}{2}b(b-c)-a^2\right) = 0, \\ \beta a x_2 = 0, \\ \beta(c-b)x_2 = 0, \\ \beta a x_1 = 0, \\ 2\alpha[-ac+\frac{1}{2}d(b-c)] + \beta(b-c)x_1 = 0, \\ \lambda = -\rho(b(b-c)-2a^2). \end{cases} \tag{36}$$

If $a = 0$ then $d \neq 0$, $b = 0$, and $\beta c x_2 = 0$. If $c = 0$ then the case (i) holds. Now, if $c \neq 0$ then $\beta \neq 0$ and the case (ii) holds. For $a \neq 0$ the cases (iii)-(iv) hold.

Now we assume that $\mu \neq 0$. The first and fourth equations of the system Eq. (35) give

$$\mu x_1^2 = -\beta a x_1 + \mu x_2^2. \tag{37}$$

The second equation of the system Eq. (35) yields $x_2 = 0$ or $x_1 = -\frac{\beta a}{2\mu}$. The Eq. (36) implies that $x_1 \neq -\frac{\beta a}{2\mu}$ thus $x_2 = 0$. The third equation of the system Eq. (35) implies that $x_1 = 0$ or $x_3 = 0$. If $x_1 = 0$ then we have

$$\begin{cases} \alpha\left(\frac{1}{2}b(b-c)-a^2\right)=\rho(b(b-c)-2a^2)+\lambda, \\ ac-bd=0, \\ \alpha[-ac+\frac{1}{2}d(b-c)]=0, \\ \mu x_3^2=-\rho(b(b-c)-2a^2)-\lambda. \end{cases} \tag{38}$$

Hence, the case (v) holds. If $x_1 \neq 0$ and $x_3 = 0$ then the Eq. (37) gives $x_1 = -\frac{\beta a}{\mu}$ and we get

$$\begin{cases} \alpha\left(\frac{1}{2}b(b-c)-a^2\right)+\frac{\beta^2 a^2}{\mu}=0, \\ \alpha\frac{1}{2}[-ac+\frac{1}{2}d(b-c)]-\frac{\beta^2 a}{4\mu}(b-c)=0, \\ \lambda=-\rho(b(b-c)-2a^2). \end{cases} \tag{39}$$

Therefore the case (vi) holds. □

**Theorem 12** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_6, g, J, X)$ are the following*:

- (i) $\mu=0, a=0, b=0, d\neq 0, \beta=0, \lambda=0,$ *for all* $c, \rho, \alpha, x_1, x_2, x_3$ *such that* $\alpha \neq 0$ ,
- (ii) $\mu=0, a=0, b=0, d\neq 0, \beta\neq 0, \lambda=0, x_3=0,$ *for all* $c, \rho, \alpha, x_1, x_2$ *such that* $cx_1=0,$
- (iii) $\mu=0, a\neq 0, \beta\neq 0, x_2=0, \alpha=0, x_1=0, \lambda=\rho(2a^2+bc),$ *for all* $x_3, b, c, d, \rho$ *such that* $a+d\neq 0, c=\frac{bd}{a},$
- (iv) $\mu\neq 0, x_2=0, x_3=0, a=0, b=0, d\neq 0, x_1=0, \lambda=0,$ *for all* $c, d, \rho, \alpha, \beta$ *such that* $(\alpha,\beta)\neq(0,0), d\neq 0,$
- (v) $\mu\neq 0, x_2=0, x_3=0, a\neq 0, c=\frac{bd}{a}, d\neq 0, \alpha=0, x_1=0, \lambda=\rho(2a^2+bc),$ *for all* $d, \rho, \alpha, \beta$ *such that* $(\alpha,\beta)\neq(0,0),$
- (vi) $\mu\neq 0, x_2=0, x_3=0, a\neq 0, d\neq 0, \alpha\neq 0, c=0, x_1=-\frac{\beta a}{\mu}, b=0, \lambda=2\rho a^2,$ *for all* $\rho, \beta,$ *such that* $a+d\neq 0,$
- (vii) $\mu\neq 0$ , $x_2=0$ , $x_3^2=\frac{\alpha a^2}{\mu}+\frac{\beta^2 ad}{2\mu^2}>0$ , $x_1=\frac{\beta d}{2\mu}$ , $c=\frac{bd}{a}$ , $\lambda=-\alpha a^2-\frac{\beta^2 ad}{2\mu}+\rho(2a^2+bc),$ *for all* $a, b, \rho, \alpha, \beta$ *such that* $a+d\neq 0,$ $\beta^2 cd=0, \beta^2 d^2=-2\beta^2 ad,$
- (viii) $\mu\neq 0, x_2^2=\frac{\alpha}{\mu}a^2-\frac{\beta^2 a^2}{2\mu^2}>0, x_1=-\frac{\beta a}{2\mu}, x_3=0, \lambda=\rho(2a^2+bc), c=\frac{bd}{a},$ *for all* $b, d, a, \rho, \alpha, \beta$ *such that* $a+d\neq 0, \beta ac=0, 4\mu\alpha(a^2+bc)+\beta^2 a^2=0.$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1=\begin{pmatrix} -(a^2+bc) & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^1 g) = \begin{pmatrix} 0 & ax_2 & dx_3 \\ ax_2 & -2ax_1 & -cx_1 \\ dx_3 & -cx_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Therefore $\widetilde{S} = -(2a^2 + bc)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(a^2 + bc) + \mu x_1^2 = -\rho(2a^2 + bc) + \lambda, \\ \frac{\beta}{2} ax_2 + \mu x_1 x_2 = 0, \\ \frac{\beta}{2} dx_3 - \mu x_1 x_3 = 0, \\ -\alpha a^2 - \beta a x_1 + \mu x_2^2 = -\rho(2a^2 + bc) + \lambda, \\ -\frac{\beta}{2} cx_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2 + bc) - \lambda. \end{cases} \tag{40}$$

Let $\mu = 0$, then

$$\begin{cases} \alpha(a^2 + bc) = 0, \\ a\beta x_2 = 0, \\ d\beta x_3 = 0, \\ -\alpha a^2 - \beta a x_1 = 0, \\ \beta c x_1 = 0, \\ \lambda = \rho(2a^2 + bc). \end{cases} \tag{41}$$

If we assume that $a = 0$ then the cases (i)-(ii) hold. If we consider $a \neq 0$ then $\beta \neq 0$ and $x_2 = 0$, $\alpha = 0$ and the case (iii) holds.

Now we consider $\mu \neq 0$. The second equation of the system Eq. (40) implies that $x_2 = 0$ or $x_1 = -\frac{\beta a}{2\mu}$. If $x_2 = 0$ then the system Eq. (40) becomes

$$\begin{cases} -\alpha(a^2 + bc) + \mu x_1^2 = -\rho(2a^2 + bc) + \lambda, \\ \frac{\beta}{2} dx_3 - \mu x_1 x_3 = 0, \\ -\alpha a^2 - \beta a x_1 = -\rho(2a^2 + bc) + \lambda, \\ \beta c x_1 = 0, \\ \mu x_3^2 = \rho(2a^2 + bc) - \lambda. \end{cases} \tag{42}$$

The second equation of the system Eq. (42) yields $x_3 = 0$ or $x_1 = \frac{\beta d}{2\mu}$. We consider $x_3 = 0$, then $\lambda = \rho(2a^2 + bc)$ and

$$\begin{cases} -\alpha(a^2 + bc) + \mu x_1^2 = 0, \\ -\alpha a^2 - \beta a x_1 = 0, \\ \beta c x_1 = 0. \end{cases} \tag{43}$$

Thus, the cases (iv)-(vi) hold. If $x_3 \neq 0$ then $x_1 = \frac{\beta d}{2\mu}$ and

$$\begin{cases} -\alpha(a^2+bc)+\frac{\beta^2 d^2}{4\mu}=-\rho(2a^2+bc)+\lambda, \\ -\alpha a^2-\frac{\beta^2 ad}{2\mu}=-\rho(2a^2+bc)+\lambda, \\ \beta^2 cd=0, \\ \mu x_3^2=\rho(2a^2+bc)-\lambda. \end{cases} \tag{44}$$

Hence, the case (vii) holds. Now we assume that $x_2 \neq 0$, then $x_1=-\frac{\beta a}{2\mu}$ and we have

$$\begin{cases} -\alpha(a^2+bc)+\frac{\beta^2 a^2}{4\mu}=-\rho(2a^2+bc)+\lambda, \\ \beta x_3=0, \\ -\alpha a^2+\frac{\beta^2 a^2}{2\mu}+\mu x_2^2=-\rho(2a^2+bc)+\lambda, \\ \frac{\beta^2 ac}{4\mu}-\mu x_2 x_3=0, \\ \mu x_3^2=\rho(2a^2+bc)-\lambda. \end{cases} \tag{45}$$

Therefore $x_3=0$ and the case (viii) holds.

□

**Theorem 13** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^0$ on the Lie group $(G_7, g, J, X)$ are the following*:

(i) $\mu=0, \beta=0, \alpha\neq 0, a=0, d\neq 0, c=0, \lambda=0,$ *for all* $\rho, x_1, x_2, x_3,$
(ii) $\mu=0, \beta\neq 0, a=0, d\neq 0, b=0, c=0, \lambda=0,$ *for all* $\rho, \alpha, x_1, x_2, x_3,$
(iii) $\mu=0, \beta\neq 0, a=0, d\neq 0, b=0, c\neq 0, \lambda=0, x_1=0, x_2=-\frac{\alpha dc}{\beta c}$ *for all* $\rho, \alpha, x_3,$
(iv) $\mu=0, \beta\neq 0, a=0, d\neq 0, b\neq 0, x_1=x_2=0, \lambda=\rho bc,$ *for all* $c, \rho, \alpha, x_3,$ *such that* $\alpha b=\alpha c=0,$
(v) $\mu=0, \beta\neq 0, a\neq 0, c=0, \alpha=x_1=x_2=0, \lambda=2\rho a^2,$ *for all* $\rho, x_3, b, d$ *such that* $a+d\neq 0,$
(vi) $\mu\neq 0, a=0, d\neq 0, x_2=0, x_1=0, \lambda=\rho bc, x_3=0,$ *for all* $b, c, \rho, \alpha, \beta$ *such that* $\alpha c=0,$
(vii) $\mu\neq 0, a=0, d\neq 0, x_2=0, x_1=\frac{\beta b}{\mu}\neq 0, x_3=-\frac{\alpha dc}{4\beta\mu}, \lambda=-\alpha\frac{bc}{2}+\frac{\beta^2 b^2}{\mu}+\rho bc,$ *for all* $\rho, c, \alpha$ *such that* $\mu\alpha c+\beta^2(c-2b)=0, \frac{\alpha^2 d^2 c^2}{16\beta^2\mu}=\frac{\alpha bc}{2}-\frac{\beta^2 b^2}{\mu},$
(viii) $\mu\neq 0, a\neq 0, c=0, x_1=x_2=x_3=0, \alpha=0, \lambda=2\rho a^2,$ *for all* $\rho, \beta, d$ *such that* $a+d\neq 0,$
(ix) $\mu\neq 0, a\neq 0, c=0, x_3=x_2=-\frac{\alpha a}{2\beta}\neq 0, x_1=0, \lambda=(2\rho-\frac{\alpha}{2})a^2,$ *for all* $b, d, \rho, \alpha, \beta$ *such that* $2\alpha\beta^2=\alpha^2\mu, a+d\neq 0.$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^0 = \begin{pmatrix} -(a^2+\frac{bc}{2}) & 0 & -\frac{1}{2}(ac+\frac{dc}{2}) \\ 0 & -(a^2+\frac{bc}{2}) & \frac{1}{2}(a^2+\frac{bc}{2}) \\ -\frac{1}{2}(ac+\frac{dc}{2}) & \frac{1}{2}(a^2+\frac{bc}{2}) & 0 \end{pmatrix}$$

and

$$(\mathcal{L}_X^0 g) = \begin{pmatrix} -2ax_2 & ax_1-bx_2 & (b-\frac{c}{2})x_2 \\ ax_1-bx_2 & 2bx_1 & (\frac{c}{2}-b)x_1 \\ (b-\frac{c}{2})x_2 & (\frac{c}{2}-b)x_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Then $\widetilde{S} = -(2a^2+bc)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha(a^2+\frac{bc}{2}) - \beta a x_2 + \mu x_1^2 = -\rho(2a^2+bc)+\lambda, \\ \frac{\beta}{2}(ax_1 - bx_2) + \mu x_1 x_2 = 0, \\ -\frac{\alpha}{2}(ac+\frac{dc}{2}) + \frac{\beta}{2}(b-\frac{c}{2})x_2 - \mu x_1 x_3 = 0, \\ -\alpha(a^2+\frac{bc}{2}) + \beta b x_1 + \mu x_2^2 = -\rho(2a^2+bc)+\lambda, \\ \frac{\alpha}{2}(a^2+\frac{bc}{2}) + \frac{\beta}{2}(\frac{c}{2}-b)x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2+bc) - \lambda. \end{cases} \tag{46}$$

Let $\mu = 0$ then the system Eq. (46) becomes

$$\begin{cases} -\alpha(a^2+\frac{bc}{2}) - \beta a x_2 = 0, \\ \frac{\beta}{2}(ax_1 - bx_2) = 0, \\ -\frac{\alpha}{2}(ac+\frac{dc}{2}) + \frac{\beta}{2}(b-\frac{c}{2})x_2 = 0, \\ -\alpha(a^2+\frac{bc}{2}) + \beta b x_1 = 0, \\ \frac{\alpha}{2}(a^2+\frac{bc}{2}) + \frac{\beta}{2}(\frac{c}{2}-b)x_1 = 0, \\ \lambda = \rho(2a^2+bc). \end{cases} \tag{47}$$

If $\beta = 0$ then $\alpha \neq 0$ and the case (i) is true. If $\beta \neq 0$ and $a = 0$ then $d \neq 0$ and we have

$$\begin{cases} \alpha bc = 0, \\ bx_2 = 0, \\ \alpha dc + \beta c x_2 = 0, \\ bx_1 = 0, \\ cx_1 = 0, \\ \lambda = \rho bc. \end{cases} \tag{48}$$

Hence the cases (ii)-(iv) hold.

For the case $\beta \neq 0$ and $a \neq 0$ we have $c = 0$ and $\alpha = x_1 = x_2 = 0$. Therefore the case (v) holds.

Now, we assume that $\mu \neq 0$. The first and the fourth equations of the system Eq. (46) imply

$$-\beta a x_2 + \mu x_1^2 = \beta b x_1 + \mu x_2^2. \tag{49}$$

Since $ac = 0$ then $a = 0$ or $c = 0$. If $a = 0$ then the system Eq. (46) becomes

$$\begin{cases} -\alpha \frac{bc}{2} + \mu x_1^2 = -\rho bc + \lambda, \\ -\frac{\beta}{2} b x_2 + \mu x_1 x_2 = 0, \\ -\frac{\alpha}{2}\frac{dc}{2} + \frac{\beta}{2}(b - \frac{c}{2}) x_2 - \mu x_1 x_3 = 0, \\ -\alpha \frac{bc}{2} + \beta b x_1 + \mu x_2^2 = -\rho bc + \lambda, \\ \frac{\alpha}{2}\frac{bc}{2} + \frac{\beta}{2}(\frac{c}{2} - b) x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho bc - \lambda. \end{cases} \tag{50}$$

The second equation of Eq. (50) yields $x_2 = 0$ or $x_1 = \frac{\beta b}{2\mu}$. If $x_2 \neq 0$ then $x_1 = \frac{\beta b}{2\mu}$ and substituting it in Eq. (49) we get $\frac{\beta^2 b^2}{4\mu^2} + x_2^2 = 0$ and this is a contradiction. Thus $x_2 = 0$ and from the Eq. (49) we have $x_1 = 0$, or $x_1 = \frac{\beta b}{\mu}$. If $x_1 = 0$ then $\lambda = \rho bc$, $\alpha c = 0$, $x_3 = 0$, and the case (vi) is true. Also, if $x_1 = \frac{\beta b}{\mu} \neq 0$ then

$$\begin{cases} -\alpha \frac{bc}{2} + \frac{\beta^2 b^2}{\mu} = -\rho bc + \lambda, \\ -\frac{\alpha}{2}\frac{dc}{2} - \beta b x_3 = 0, \\ \frac{\alpha}{2}\frac{bc}{2} + \frac{\beta^2 b}{2\mu}(\frac{c}{2} - b) = 0, \\ \mu x_3^2 = \rho bc - \lambda. \end{cases} \tag{51}$$

Thus, the case (vii) holds. Now for case $\mu \neq 0$ and $a \neq 0$ we have $c = 0$ and

$$\begin{cases} -\alpha a^2 - \beta a x_2 + \mu x_1^2 = -2\rho a^2 + \lambda, \\ \frac{\beta}{2}(a x_1 - b x_2) + \mu x_1 x_2 = 0, \\ \frac{\beta}{2} b x_2 - \mu x_1 x_3 = 0, \\ -\alpha a^2 + \beta b x_1 + \mu x_2^2 = -2\rho a^2 + \lambda, \\ \frac{\alpha}{2} a^2 - \frac{\beta}{2} b x_1 - \mu x_2 x_3 = 0, \\ \mu x_3^2 = 2\rho a^2 - \lambda. \end{cases} \tag{52}$$

The fourth, fiveth and sixth equations of Eq. (52) imply that $x_2 = x_3$. The second and third equations imply that $\beta x_1 = 0$. Then

$$\begin{cases} -\alpha a^2 - \beta a x_2 + \mu x_1^2 = -2\rho a^2 + \lambda, \\ -\frac{\beta}{2} b x_2 + \mu x_1 x_2 = 0, \\ -\alpha a^2 + \mu x_2^2 = -2\rho a^2 + \lambda, \\ \frac{\alpha}{2} a^2 - \mu x_2^2 = 0, \\ \mu x_2^2 = 2\rho a^2 - \lambda. \end{cases} \tag{53}$$

Using the second equation of Eq. (53) we have $x_2 = 0$ or $x_1 = \frac{\beta b}{2\mu}$. If $x_2 = 0$ then $\alpha = 0$ and the case (viii) holds. If $x_2 \neq 0$ then $x_1 = \frac{\beta b}{2\mu} = 0$ and

$$\begin{cases} -\beta a x_2 = \frac{\alpha}{2} a^2, \\ \frac{\alpha}{2} a^2 - \mu x_2^2 = 0, \\ \frac{\alpha}{2} a^2 = 2\rho a^2 - \lambda. \end{cases} \tag{54}$$

Thus the case (ix) is true. □

**Theorem 14** *The left-invariant affine generalized Ricci soliton associated to the connection $\nabla^1$ on the Lie group $(G_7, g, J, X)$ are the following*:

(i) $\mu = 0, a = 0, d \neq 0, \beta \neq 0, b = 0, x_2 = 0, x_3 = \frac{2\alpha d}{\beta}, \lambda = 0,$ *for all* $c, \rho, x_1,$

(ii) $\mu = 0, a = 0, d \neq 0, \beta \neq 0, b = 0, x_2 \neq 0, c = 0, x_2 + x_3 = \frac{2\alpha d}{\beta}, \lambda = 0,$ *for all* $\rho, x_1,$

(iii) $\mu = 0,\ a = 0,\ d \neq 0,\ \beta \neq 0,\ b \neq 0,\ x_1 = \frac{\alpha(b+c)}{\beta},\ x_2 = \frac{\alpha b d}{\beta b},\ x_3 = \frac{2\alpha d}{\beta} - \frac{c\alpha d}{\beta b},$ $\lambda = \rho(b^2 + bc),$ *for all* $c, \rho, \alpha$ *such that* $c\alpha d^2 - b^3\alpha - \alpha b d^2 = 0,$

(iv) $\mu = 0, a \neq 0, c = 0, \beta \neq 0, b = 0, \alpha = 0, x_1 = x_2 = 0, \lambda = 2\rho a^2,$ *for all* $\rho, d$ *such that* $a + d \neq 0, dx_3 = 0,$

(v) $\mu = 0, a \neq 0, c = 0, \beta \neq 0, b \neq 0, \alpha = 0, x_1 = x_2 = x_3 = 0, \lambda = \rho(2a^2 + b^2),$ *for all* $\rho, d$ *such that* $a + d \neq 0,$

(vi) $\mu \neq 0, a = 0, d \neq 0, x_1 = x_3 = 0, \lambda = \rho(b^2 + bc), \beta = 0, \alpha = 0, x_2 = 0,$ *for all* $\rho, b, c,$

(vii) $\mu \neq 0, a = 0, d \neq 0, x_1 = x_3 = 0, \lambda = 0, \beta \neq 0, x_2 = \alpha = 0,$ *for all* $\rho, b, c,$

(viii) $\mu \neq 0$ , $a \neq 0$ , $c = 0$ , $\beta = 0$ , $x_1 = \epsilon_1 \sqrt{\frac{\alpha a^2 - \rho(2a^2+b^2)+\lambda}{\mu}}$ , $x_2 = \epsilon_2 \sqrt{\frac{\alpha(a^2+b^2)-\rho(2a^2+b^2)+\lambda}{\mu}}, x_3 = \epsilon_3 \sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}},$ *for all* $b, d, \rho, \lambda, \alpha$ *such that* $a + d \neq 0,$

$$\frac{\alpha a^2 - \rho(2a^2 + b^2) + \lambda}{\mu} \geq 0, \quad \frac{\rho(2a^2 + b^2) - \lambda}{\mu} \geq 0,$$

$$\frac{\alpha}{2}(bd - ab) + \mu\epsilon_1\epsilon_2 \sqrt{\frac{\alpha a^2 - \rho(2a^2 + b^2) + \lambda}{\mu}} \sqrt{\frac{\alpha(a^2 + b^2) - \rho(2a^2 + b^2) + \lambda}{\mu}} = 0,$$

$$\alpha b(a + d) - \mu\epsilon_1\epsilon_3 \sqrt{\frac{\alpha a^2 - \rho(2a^2 + b^2) + \lambda}{\mu}} \sqrt{\frac{\rho(2a^2 + b^2) - \lambda}{\mu}} = 0,$$

$$\frac{\alpha}{2}(ad + 2d^2) - \mu\epsilon_2\epsilon_3 \sqrt{\frac{\alpha(a^2 + b^2) - \rho(2a^2 + b^2) + \lambda}{\mu}} \sqrt{\frac{\rho(2a^2 + b^2) - \lambda}{\mu}} = 0.$$

(ix) $\mu \neq 0, a \neq 0, c = 0, \beta \neq 0, x_1 = F, x_2 = \frac{\rho(2a^2+b^2)-\lambda-\alpha a^2+\mu F^2}{\beta a}, x_3 = \epsilon \sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}},$ *for all* $b, d, \rho, \alpha$ *such that* $a + d \neq 0, \frac{\rho(2a^2+b^2)-\lambda}{\mu} \geq 0,$

$$\alpha b(a+d)+\frac{\beta}{2}\left(-aF-b\epsilon\sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}}\right)-\mu F\epsilon\sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}}=0,$$

$$-\alpha(a^2+b^2)+\beta bF+\mu\Big(\frac{\rho(2a^2+b^2)-\lambda-\alpha a^2+\mu F^2}{\beta a}\Big)^2=-\rho(2a^2+b^2)+\lambda,$$

$$\frac{\alpha}{2}(ad+2d^2)+\frac{\beta}{2}\left(-bF-d\frac{\rho(2a^2+b^2)-\lambda-\alpha a^2+\mu F^2}{\beta a}-d\epsilon\sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}}\right)$$

$$-\mu\epsilon\frac{\rho(2a^2+b^2)-\lambda-\alpha a^2+\mu F^2}{\beta a}\sqrt{\frac{\rho(2a^2+b^2)-\lambda}{\mu}}=0,$$

$$\frac{\rho(2a^2+b^2)-\lambda}{\mu}\geq 0,$$

$$27C^2+(4A^3-18AB)C+4B^3-A^2B^2\geq 0,$$

where $\epsilon=\pm 1$,

$$F=E-\frac{\frac{B}{3}-\frac{A^2}{9}}{E}-\frac{A}{3},$$

$$E=\left(\frac{\sqrt{27C^2+(4A^3-18AB)C+4B^3-A^2B^2}}{2(3^{\frac{3}{2}})}+\frac{AB-3C}{6}-\frac{A^3}{27}\right)^{\frac{1}{3}}\neq 0,$$

$$A=-\frac{b\beta}{2\mu},$$

$$B=\frac{\beta^2a^2}{2\mu^2}+\frac{1}{\mu}\Big(\rho(2a^2+b^2)-\lambda-\alpha a^2\Big),$$

$$C=\frac{\beta a}{2\mu^2}\Big(\alpha(bd-ab)-\frac{b}{a}\Big(\rho(2a^2+b^2)-\lambda-\alpha a^2\Big)\Big).$$

***Proof*** From [19, 20], we have

$$\widetilde{Ric}^1=\begin{pmatrix}-a^2 & \frac{1}{2}(bd-ab) & b(a+d)\\ \frac{1}{2}(bd-ab) & -(a^2+b^2+bc) & \frac{1}{2}(bc+ad+2d^2)\\ b(a+d) & \frac{1}{2}(bc+ad+2d^2) & 0\end{pmatrix}$$

and

$$(\mathcal{L}_X^1 g)=\begin{pmatrix}-2ax_2 & ax_1-bx_2 & -ax_1-cx_2-bx_3\\ ax_1-bx_2 & 2bx_1 & -bx_1-dx_2-dx_3\\ -ax_1-cx_2-bx_3 & -bx_1-dx_2-dx_3 & 0\end{pmatrix}$$

with respect to the basis $\{e_1,e_2,e_3\}$. Therefore $\widetilde{S}=-(2a^2+b^2+bc)$ and the Eq. (2) becomes

$$\begin{cases} -\alpha a^2 - \beta a x_2 + \mu x_1^2 = -\rho(2a^2 + b^2 + bc) + \lambda, \\ \frac{\alpha}{2}(bd - ab) + \frac{\beta}{2}(a x_1 - b x_2) + \mu x_1 x_2 = 0, \\ \alpha b(a + d) + \frac{\beta}{2}(-a x_1 - c x_2 - b x_3) - \mu x_1 x_3 = 0, \\ -\alpha(a^2 + b^2 + bc) + \beta b x_1 + \mu x_2^2 = -\rho(2a^2 + b^2 + bc) + \lambda, \\ \frac{\alpha}{2}(bc + ad + 2d^2) + \frac{\beta}{2}(-b x_1 - d x_2 - d x_3) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2 + b^2 + bc) - \lambda. \end{cases} \tag{55}$$

Let $\mu = 0$, then the system Eq. (55) becomes

$$\begin{cases} -\alpha a^2 - \beta a x_2 = 0, \\ \frac{\alpha}{2}(bd - ab) + \frac{\beta}{2}(a x_1 - b x_2) = 0, \\ \alpha b(a + d) + \frac{\beta}{2}(-a x_1 - c x_2 - b x_3) = 0, \\ -\alpha(a^2 + b^2 + bc) + \beta b x_1 = 0, \\ \frac{\alpha}{2}(bc + ad + 2d^2) + \frac{\beta}{2}(-b x_1 - d x_2 - d x_3) = 0, \\ \lambda = \rho(2a^2 + b^2 + bc). \end{cases} \tag{56}$$

Since $ac = 0$ we get $a = 0$ or $c = 0$. If $a = 0$ then $a + d \neq 0$ implies that $d \neq 0$ and we have

$$\begin{cases} \alpha bd - \beta b x_2 = 0, \\ \alpha bd + \frac{\beta}{2}(-c x_2 - b x_3) = 0, \\ -\alpha(b^2 + bc) + \beta b x_1 = 0, \\ \frac{\alpha}{2}(bc + 2d^2) + \frac{\beta}{2}(-b x_1 - d x_2 - d x_3) = 0, \\ \lambda = \rho(b^2 + bc). \end{cases} \tag{57}$$

If $\beta = 0$ then $(\alpha, \beta, \mu) \neq (0, 0, 0)$ yields $\alpha \neq 0$. Also, the first equation of Eq. (57) gives $b = 0$ and the fourth equation of Eq. (57) implies that $\alpha d^2 = 0$ which is a contradiction. Hence, $\beta \neq 0$ and the cases (i)–(iii) hold.

Now, we consider $\mu = 0$ and $a \neq 0$, then $c = 0$ and we get

$$\begin{cases} \alpha a + \beta x_2 = 0, \\ \alpha(bd - ab) + \beta(a x_1 - b x_2) = 0, \\ \alpha b(a + d) + \frac{\beta}{2}(-a x_1 - b x_3) = 0, \\ -\alpha(a^2 + b^2) + \beta b x_1 = 0, \\ \frac{\alpha}{2}(ad + 2d^2) + \frac{\beta}{2}(-b x_1 - d x_2 - d x_3) = 0, \\ \lambda = \rho(2a^2 + b^2). \end{cases} \tag{58}$$

If $\beta = 0$ then the first equation of the system Eq. (58) yields $\alpha = 0$ which is a contradiction, then $\beta \neq 0$. If $b = 0$ then the case (iv) holds. If $b \neq 0$ then from the first three equations of the system Eq. (58) we obtain $x_1 = -\frac{\alpha bd}{\beta a}$, $x_2 = -\frac{\alpha a}{\beta}$, $x_3 = \frac{2\alpha a + 3\alpha d}{\beta}$. Hence using the fourth and fiveth equations of the system Eq. (58) we have

$$\begin{cases} \alpha(a^2 + b^2 + \frac{b^2 d}{a}) = 0, \\ \alpha(-d^2 + \frac{b^2 d}{a}) = 0. \end{cases} \tag{59}$$

If $\alpha \neq 0$ then $a^2 + b^2 + d^2 = 0$ which is a contradiction, then $\alpha = 0$, $x_1 = x_2 = x_3 = 0$ and the case (v) is true. Now, we consider $\mu \neq 0$. If $a = 0$ then $d \neq 0$ and the system Eq. (55) gives

$$\begin{cases} \mu x_1^2 = -\rho(b^2 + bc) + \lambda, \\ \alpha bd - \beta bx_2 + 2\mu x_1 x_2 = 0, \\ \alpha bd + \frac{\beta}{2}(-cx_2 - bx_3) - \mu x_1 x_3 = 0, \\ -\alpha(b^2 + bc) + \beta bx_1 + \mu x_2^2 = -\rho(b^2 + bc) + \lambda, \\ \frac{\alpha}{2}(bc + 2d^2) + \frac{\beta}{2}(-bx_1 - dx_2 - dx_3) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(b^2 + bc) - \lambda. \end{cases} \tag{60}$$

The first and sixth equations of the system Eq. (60) imply that $x_1 = x_3 = 0$ and $\lambda = \rho(b^2 + bc)$. Thus the system Eq. (60) gives

$$\begin{cases} \alpha bd - \beta bx_2 = 0, \\ 2\alpha bd - \beta cx_2 = 0, \\ -\alpha(b^2 + bc) + \mu x_2^2 = 0, \\ \alpha(bc + 2d^2) - \beta dx_2 = 0. \end{cases} \tag{61}$$

If $\beta = 0$ then $\alpha = 0$, $x_2 = 0$ and the case (vi) holds. If $\beta \neq 0$ and $b = 0$ then $x_2 = \alpha = 0$ and the case (vii) is true. Notice if $b \neq 0$ and $\alpha \neq 0$ then from the first two equations of the system Eq. (61) we infer $c = 2b$ and Replacing it with $x_2 = \frac{\alpha d}{\beta}$ in the fourth equation of the system Eq. (61) we obtain $2b^2 + d^2 = 0$ which is a contradiction. Let $\mu \neq 0$ and $a \neq 0$ then $c = 0$ and the system Eq. (55) gives

$$\begin{cases} -\alpha a^2 - \beta ax_2 + \mu x_1^2 = -\rho(2a^2 + b^2) + \lambda, \\ \frac{\alpha}{2}(bd - ab) + \frac{\beta}{2}(ax_1 - bx_2) + \mu x_1 x_2 = 0, \\ \alpha b(a + d) + \frac{\beta}{2}(-ax_1 - bx_3) - \mu x_1 x_3 = 0, \\ -\alpha(a^2 + b^2) + \beta bx_1 + \mu x_2^2 = -\rho(2a^2 + b^2) + \lambda, \\ \frac{\alpha}{2}(ad + 2d^2) + \frac{\beta}{2}(-bx_1 - dx_2 - dx_3) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2 + b^2) - \lambda. \end{cases} \tag{62}$$

If $\beta = 0$ then the system Eq. (62) reduces to

$$\begin{cases} -\alpha a^2 + \mu x_1^2 = -\rho(2a^2 + b^2) + \lambda, \\ \frac{\alpha}{2}(bd - ab) + \mu x_1 x_2 = 0, \\ \alpha b(a + d) - \mu x_1 x_3 = 0, \\ -\alpha(a^2 + b^2) + \mu x_2^2 = -\rho(2a^2 + b^2) + \lambda, \\ \frac{\alpha}{2}(ad + 2d^2) - \mu x_2 x_3 = 0, \\ \mu x_3^2 = \rho(2a^2 + b^2) - \lambda. \end{cases} \tag{63}$$

Then the case (viii) is true. If $\beta \neq 0$, then the first and second equations of the system Eq. (62) imply that

$$x_1^3 + Ax_1^2 + Bx_1 + C = 0. \tag{64}$$

Thus the case (ix) holds.

□

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## Declarations

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