Localized Waves for the Coupled Mixed Derivative Nonlinear Schrödinger Equation in a Birefringent Optical Fiber

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Abstract

In this paper, the higher-order localized waves for the coupled mixed derivative nonlinear Schrödinger equation are investigated using generalized Darboux transformation. On the basis of seed solutions and a Lax pair, the first- and second-order localized wave solutions are derived from the N th-order iteration formulas of generalized Darboux transformation. Then, the dynamics of the localized waves are analyzed and displayed via numerical simulation. It is found that the second-order rogue wave split into three first-order rogue waves due to the influence of the separation function. In addition, a series of novel dynamic evolution plots exhibit that rogue waves coexist with dark-bright solitons and breathers.

Keywords Coupled mixed derivative nonlinear Schrödinger equation · Generalized Darboux transformation · Soliton · Rogue wave · Breather

1 Introduction

As is well known, most nonlinear partial differential equations (PDEs) in mathematics and physics are integrable, including the Hirota equation [1], the nonlinear Schrödinger equation [2, 3], the Gerdjikov-Ivanov equation [4], the Korteweg-de Vries equation [5, 6], the Sasa-Satsuma equation [7], and so on. These nonlinear equations play an important role in various fields of nonlinear science such as water waves, nonlinear optics, and Bose-Einstein condensates. Because natural phenomena are more complicated in the real world and in some experimental environments, the study of coupled systems has drawn more attention of many

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researchers than the single-component system [8, 9]. Further, These nonlinear equations can be used to describe and study localized waves, which consist of rogue waves [10, 11], solitons [12, 13], and breathers [14–17]. To date, some methods have been proposed and applied to investigate localized waves of coupled systems, such as the Darboux transformation (DT) [18], the Riemann-Hilbert approach [19], the Hirota bilinear method [20], the generalized DT [21], and so on. Due to the profound theoretical significance and potential applicability, the study of localized waves of the nonlinear PDEs has always been concerned, and great progress has been made [22–29].

In this paper, the coupled mixed derivative nonlinear Schrödinger (CMDNLS) equation is considered [30],

$$\begin{aligned} iq_{1t} + q_{1xx} + \mu(|q_1|^2 + |q_2|^2)q_1 + i\gamma[(|q_1|^2 + |q_2|^2)q_1]_x &= 0, \\ iq_{2t} + q_{2xx} + \mu(|q_1|^2 + |q_2|^2)q_2 + i\gamma[(|q_1|^2 + |q_2|^2)q_2]_x &= 0. \end{aligned} \quad (1)$$

In Eq. (1), q_1 and q_2 are the slowly varying envelopes of two kinds of polarization in the electric fields, which are the functions of the evolution time t and normalized distance x . μ and γ are real constant parameters that denote the measure of cubic nonlinear strength and derivative cubic nonlinearity, respectively. Eq. (1) can be used to describe the propagation of short pulses in the femtosecond or picosecond regions of birefringent optical fibers. Some results of Eq. (1) have been achieved in Refs. [30–34]. Yan obtained the first-order localized wave solutions via DT and Darboux-dressing transformation, including breather wave solution, rogue wave and bright-dark rogue wave [30]. In the present paper, generalized DT is applied to obtain the N th-order localized waves solutions of Eq. (1) and study the dynamics of the higher-order localized waves.

The remainder of this paper is organized as follows. In Sect. 2, a generalized DT for Eq. (1) is constructed, and the iterative formulas of the N th-order solutions are deduced. In Sect. 3, the first- and second-order localized wave solitons are given, and some evolution plots of localized wave solitons are illustrated. Finally, several conclusions are provided in Sect. 4.

2 Generalized Darboux Transformation

In this section, a generalized DT is constructed and applied to deduce the N th-order localized wave solutions of Eq. (1).

Equation (1) is an integrable nonlinear system, and the Lax pair is given as follows

$$\Phi_x = U\Phi = (\lambda^2 U_2 + \lambda U_1 + U_0)\Phi, \quad (2a)$$

$$\Phi_t = V\Phi = (\lambda^4 V_4 + \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + V_0)\Phi, \quad (2b)$$

where

$$\begin{aligned}
U_0 &= -\frac{\mu}{3\gamma}Q_0, U_1 = \frac{1}{2}Q_1, U_2 = \frac{1}{6\gamma}Q_0, V_0 = \frac{\mu^2}{3\gamma^2}Q_0, \\
V_1 &= -\frac{\mu}{2\gamma}Q_1 + \frac{\gamma}{2}Q_2 + \frac{i}{2}Q_{1x}, V_2 = -\frac{\mu}{3\gamma^2}Q_0 + \frac{i}{4}Q_3, V_3 = \frac{1}{4\gamma}Q_1, V_4 = \frac{1}{12\gamma^2}Q_0, \\
Q_0 &= \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} |q_1|^2 + |q_2|^2 & 0 & 0 \\ 0 & -|q_1|^2 & -q_1^*q_2^* \\ 0 & -q_1q_2^* & -|q_2|^2 \end{pmatrix}, \\
Q_2 &= \begin{pmatrix} 0 & -q_1(|q_1|^2 + |q_2|^2) & -q_2(|q_1|^2 + |q_2|^2) \\ q_1^*(|q_1|^2 + |q_2|^2) & 0 & 0 \\ q_2^*(|q_1|^2 + |q_2|^2) & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Suppose that $\Phi = (\phi, \varphi, \chi)^T$ is a vector solution of Eq. (2), λ is a spectral parameter and the asterisk denotes the complex conjugate. Thus, it can be easily proven that U and V satisfy the zero-curvature equation $U_t - V_x + [U, V] = 0$.

Through classical DT constructed in Refs. [31, 35], Eq. (2) can be transformed into

$$\begin{aligned}
\Phi[1]_x &= U\Phi[1], \Phi[1]_t = V\Phi[1], \\
U[1] &= (T_x + TU)T^{-1}, V[1] = (T_t + TV)T^{-1},
\end{aligned}$$

where $U[1]$ and $V[1]$ have the same form as U and V , meanwhile, the old potentials q_1 and q_2 are replaced by the new ones $q_1[1]$ and $q_2[1]$.

The Darboux matrix T is constructed as follows:

$$T = \lambda^2 M_0 + \lambda M_1 - I,$$

where I is a 3×3 identity matrix, and

$$\begin{aligned}
M_0 &= \frac{1}{\lambda_1^{*2}}I + \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1^{*2}\lambda_1} \begin{pmatrix} \frac{\phi_1\phi_1^*}{\Gamma} & 0 & 0 \\ 0 & \frac{\varphi_1\varphi_1^*}{\Gamma^*} & \frac{\varphi_1\chi_1^*}{\Gamma^*} \\ 0 & \frac{\chi_1\varphi_1^*}{\Gamma^*} & \frac{\chi_1\chi_1^*}{\Gamma^*} \end{pmatrix}, M_1 = \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1\lambda_1^*} \begin{pmatrix} 0 & \frac{\phi_1\varphi_1^*}{\Gamma} & \frac{\phi_1\chi_1^*}{\Gamma} \\ \frac{\varphi_1\phi_1^*}{\Gamma^*} & 0 & 0 \\ \frac{\chi_1\phi_1^*}{\Gamma^*} & 0 & 0 \end{pmatrix}, \\
\Gamma &= \lambda_1|\phi_1|^2 + \lambda_1^*|\varphi_1|^2 + \lambda_1^*|\chi_1|^2.
\end{aligned}$$

Assume that $\Phi_k = (\phi_k, \varphi_k, \chi_k)^T$ ($k = 1, 2, \dots$) is an eigenfunction of Eq. (2) with $\lambda = \lambda_k$. Furthermore, DT can be iterated. Thus, the classical DT of Eq. (1) is defined as follows,

$$\begin{aligned}
\Phi_N[N-1] &= T[N-1]T[N-2] \cdots T[1]T[0]\Phi_N, \\
q_1[N] &= q_1[0] + \sum_{k=1}^N 2(\lambda_k^{*2} - \lambda_k^2) \left[\frac{i\mu}{\gamma} \frac{\phi_k[k-1]\varphi_k^*[k-1]}{\Lambda_k} - \left(\frac{\phi_k[k-1]\varphi_k^*[k-1]}{\Lambda_k} \right)_{x,t} \right], \\
q_2[N] &= q_2[0] + \sum_{k=1}^N 2(\lambda_k^{*2} - \lambda_k^2) \left[\frac{i\mu}{\gamma} \frac{\phi_k[k-1]\chi_k^*[k-1]}{\Lambda_k} - \left(\frac{\phi_k[k-1]\chi_k^*[k-1]}{\Lambda_k} \right)_{x,t} \right], \\
T[0] &= I, T[k] = \lambda_{k+1}^2 M_0[k-1] + \lambda_{k+1} M_1[k-1] - I,
\end{aligned}$$

with

$$\begin{aligned}
\Phi_k[k-1] &= T[k-1]T[k-2] \cdots T[1]|_{\lambda=\lambda_k} \Phi_k, \\
\Lambda_k &= |\lambda_k|^2 [\lambda_k |\phi_k[k-1]|^2 + \lambda_k^* |\varphi_k[k-1]|^2 + \lambda_k^* |\chi_k[k-1]|^2], \\
M_0[k-1] &= \frac{1}{\lambda_k^{*2}} I + \frac{\lambda_k^{*2} - \lambda_k^2}{\lambda_k^{*2} \lambda_k} \begin{pmatrix} \frac{\phi_k[k-1]\phi_k^*[k-1]}{\Gamma_k} & 0 & 0 \\ 0 & \frac{\varphi_k[k-1]\varphi_k^*[k-1]}{\Gamma_k} & \frac{\varphi_k[k-1]\chi_k^*[k-1]}{\Gamma_k} \\ 0 & \frac{\chi_k[k-1]\varphi_k^*[k-1]}{\Gamma_k} & \frac{\chi_k[k-1]\chi_k^*[k-1]}{\Gamma_k} \end{pmatrix}, \\
M_1[k-1] &= \frac{\lambda_k^{*2} - \lambda_k^2}{\lambda_k \lambda_k^*} \begin{pmatrix} 0 & \frac{\phi_k[k-1]\varphi_k^*[k-1]}{\Gamma_k} & \frac{\phi_k[k-1]\chi_k^*[k-1]}{\Gamma_k} \\ \frac{\varphi_k[k-1]\phi_k^*[k-1]}{\Gamma_k} & 0 & 0 \\ \frac{\chi_k[k-1]\phi_k^*[k-1]}{\Gamma_k} & 0 & 0 \end{pmatrix}, \\
\Gamma_k &= \lambda_k |\phi_k[k-1]|^2 + \lambda_k^* |\varphi_k[k-1]|^2 + \lambda_k^* |\chi_k[k-1]|^2.
\end{aligned}$$

On the basis of above DT, a generalized DT of Eq. (1) can be derived. Assume that $\Phi_1 = \Phi_1(\lambda_1 + \eta) = (\phi_1, \varphi_1, \chi_1)^T$ is a solution of Eq. (2) with $q_1 = q_1[0]$, $q_2 = q_2[0]$ and $\lambda = \lambda_1 + \eta$, and η is a small parameter. According to the Taylor expansion of the function Φ_1 at $\eta = 0$, Φ_1 can be rewritten as follows

$$\Phi_1 = \Phi_1^{[0]} + \Phi_1^{[1]}\eta + \Phi_1^{[2]}\eta^2 + \cdots + \Phi_1^{[N]}\eta^N + o(\eta^N), \quad (3)$$

where $\Phi_1^{[j]} = (\phi_1^{[j]}, \varphi_1^{[j]}, \chi_1^{[j]})^T = \frac{1}{j!} \frac{\partial^j}{\partial \lambda^j} \Phi_1(\lambda)|_{\lambda=\lambda_1}$ ($j = 0, 1, 2, \dots$).

It is easy to determine that $\Phi_1[0] = \Phi_1^{[0]}$ is a vector solution of Eq. (2) with $q_1 = q_1[0]$, $q_2 = q_2[0]$, and $\lambda = \lambda_1$. Then, the generalized DT is presented as follows,

$$\begin{aligned}
\Phi_1[N-1] &= \Phi_1^{[0]} + \sum_{l=1}^{N-1} T_1[l]\Phi_1^{[1]} + \sum_{l=1}^{N-1} \sum_{k>l}^{N-1} T_1[k]T_1[l]\Phi_1^{[2]} + \cdots \\
&\quad + T_1[N-1]T_1[N-2] \cdots T_1[1]\Phi_1^{[N-1]},
\end{aligned} \quad (4)$$

$$q_1[N] = q_1[0] + \sum_{k=1}^N 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[k-1]\varphi_1^*[k-1]}{\Lambda_k} - \left(\frac{\phi_1[k-1]\varphi_1^*[k-1]}{\Lambda_k} \right)_{x,1} \right], \quad (5a)$$

$$q_2[N] = q_2[0] + \sum_{k=1}^N 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[k-1]\chi_1^*[k-1]}{\Lambda_k} - \left(\frac{\phi_1[k-1]\chi_1^*[k-1]}{\Lambda_k} \right)_{x,1} \right], \quad (5b)$$

$$T_1[k] = \lambda_1^2 M_0[k-1] + \lambda_1 M_1[k-1] - I, \quad (1 \leq k \leq N), \quad (6)$$

with

$$\begin{aligned}
\Phi_1[k-1] &= (\phi_1[k-1], \varphi_1[k-1], \chi_1[k-1])^T, \\
\Lambda_k &= |\lambda_1|^2 [|\lambda_1 \phi_1[k-1]|^2 + \lambda_1^* |\varphi_1[k-1]|^2 + \lambda_1^* |\chi_1[k-1]|^2], \\
M_0[k-1] &= \frac{1}{\lambda_1^{*2}} I + \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1^{*2} \lambda_1} \begin{pmatrix} \frac{\phi_1[k-1] \phi_1^*[k-1]}{\Gamma_k} & 0 & 0 \\ 0 & \frac{\varphi_1[k-1] \varphi_1^*[k-1]}{\Gamma_k^*} & \frac{\varphi_1[k-1] \chi_1^*[k-1]}{\Gamma_k^*} \\ 0 & \frac{\chi_1[k-1] \varphi_1^*[k-1]}{\Gamma_k^*} & \frac{\chi_1[k-1] \chi_1^*[k-1]}{\Gamma_k^*} \end{pmatrix}, \\
M_1[k-1] &= \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1 \lambda_1^*} \begin{pmatrix} 0 & \frac{\phi_1[k-1] \varphi_1^*[k-1]}{\Gamma_k} & \frac{\phi_1[k-1] \chi_1^*[k-1]}{\Gamma_k} \\ \frac{\varphi_1[k-1] \phi_1^*[k-1]}{\Gamma_k^*} & 0 & 0 \\ \frac{\chi_1[k-1] \phi_1^*[k-1]}{\Gamma_k^*} & 0 & 0 \end{pmatrix}, \\
\Gamma_k &= \lambda_1 |\phi_1[k-1]|^2 + \lambda_1^* |\varphi_1[k-1]|^2 + \lambda_1^* |\chi_1[k-1]|^2.
\end{aligned}$$

3 Dynamics of Localized Waves

In this section, according to Eqs. (3)–(6), the first- and the second-order localized wave solutions of Eq. (1) are obtained and the dynamics of localized waves are analyzed via numerical simulation plots.

Setting plane waves $q_1[0] = d_1 e^{i\theta}$ and $q_2[0] = d_2 e^{i\theta}$ as the seed solutions of Eq. (1), where $\theta = \mu(d_1^2 + d_2^2)t$, and d_1 and d_2 are real constants. The corresponding basic solution with $\lambda = \lambda_1(1 + \eta^2)$ is calculated as follows,

$$\Phi_1 = \begin{pmatrix} (C_1 e^{M_1+M_2} - C_2 e^{M_1-M_2}) e^{\frac{i}{2}\theta} \\ \rho_1 (C_1 e^{M_1-M_2} - C_2 e^{M_1+M_2}) e^{-\frac{i}{2}\theta} + \alpha d_2 e^{M_3} \\ \rho_2 (C_1 e^{M_1-M_2} - C_2 e^{M_1+M_2}) e^{-\frac{i}{2}\theta} - \alpha d_1 e^{M_3} \end{pmatrix},$$

where

$$C_1 = \frac{\left[\frac{\lambda^2 - 2\mu}{\gamma} - \sqrt{\left(\frac{\lambda^2 - 2\mu}{\gamma} \right)^2 + 4\lambda^2(d_1^2 + d_2^2)} \right]^{\frac{1}{2}}}{\sqrt{\left(\frac{\lambda^2 - 2\mu}{\gamma} \right)^2 + 4\lambda^2(d_1^2 + d_2^2)}}, \rho_1 = \frac{d_1}{\sqrt{d_1^2 + d_2^2}},$$

$$C_2 = \frac{\left[\frac{\lambda^2 - 2\mu}{\gamma} + \sqrt{\left(\frac{\lambda^2 - 2\mu}{\gamma} \right)^2 + 4\lambda^2(d_1^2 + d_2^2)} \right]^{\frac{1}{2}}}{\sqrt{\left(\frac{\lambda^2 - 2\mu}{\gamma} \right)^2 + 4\lambda^2(d_1^2 + d_2^2)}}, \rho_2 = \frac{d_2}{\sqrt{d_1^2 + d_2^2}},$$

$$M_1 = -\frac{i(\lambda^2 - 2\mu)}{24\gamma^2} [2\gamma x + (\lambda^2 - 2\mu)t], M_3 = \frac{i(\lambda^2 - 2\mu)}{12\gamma^2} [2\gamma x + (\lambda^2 - 2\mu)t],$$

$$M_2 = \frac{i}{8\gamma} \sqrt{\left(\frac{\lambda^2 - 2\mu}{\gamma} \right)^2 + 4\lambda^2(d_1^2 + d_2^2)} [2\gamma x - (2\gamma^2(d_1^2 + d_2^2 - \lambda^2 + 2\mu))t + \Omega(\eta)],$$

$$\Omega(\eta) = \sum_{k=1}^N s_k \eta^{2k}, s_k = m_k + in_k, \quad (k = 1, 2, \dots, N).$$

Here α, m_k , and n_k are real constants. With $\lambda_1 = i\gamma\sqrt{d_1^2 + d_2^2} + \sqrt{2\mu - \gamma^2(d_1^2 + d_2^2)}$, expanding function $\Phi_1(\eta)$ as Taylor series in the neighborhood of $\eta = 0$,

$$\Phi_1 = \Phi_1^{[0]} + \Phi_1^{[1]}\eta^2 + \Phi_1^{[2]}\eta^4 + \dots + \Phi_1^{[j]}\eta^{2j} + \dots, (j = 0, 1, 2, \dots) \quad (7)$$

where $\Phi_1^{[j]}$ can be obtained with Maple, and

$$\phi_1^{[0]} = -\frac{-\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu}(3\delta\gamma t - x) - 3i\delta^2\gamma^2 t + i\delta\gamma x + 2i\delta\mu t + 1}{(2i\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu} - 2\delta\gamma)^{\frac{1}{2}}} e^{\xi_1},$$

$$\varphi_1^{[0]} = \frac{-\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu}(3\delta\gamma t - x) - 3i\delta^2\gamma^2 t + i\delta\gamma x + 2i\delta\mu t - 1}{\sqrt{\delta}(2i\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu} - 2\delta\gamma)^{\frac{1}{2}}} d_1 e^{\xi_2} + \alpha d_2 e^{\xi_3},$$

$$\chi_1^{[0]} = \frac{-\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu}(3\delta\gamma t - x) - 3i\delta^2\gamma^2 t + i\delta\gamma x + 2i\delta\mu t - 1}{\sqrt{\delta}(2i\sqrt{\delta}\sqrt{-\delta\gamma^2 + 2\mu} - 2\delta\gamma)^{\frac{1}{2}}} d_2 e^{\xi_2} - \alpha d_1 e^{\xi_3},$$

with $\delta = d_1^2 + d_2^2$, and

$$\begin{aligned}\xi_1 &= -\frac{1}{3}\sqrt{\delta}\sqrt{-\delta\gamma^2+2\mu}\left(\delta\gamma t-\frac{1}{2}x\right)+\frac{1}{6}i\delta(-2\delta\gamma^2t+\gamma x+5\mu t), \\ \xi_2 &= -\frac{1}{3}\sqrt{\delta}\sqrt{-\delta\gamma^2+2\mu}\left(\delta\gamma t-\frac{1}{2}x\right)-\frac{1}{6}i\delta\left(2\delta\gamma^2t-\gamma x+\mu t\right), \\ \xi_3 &= \frac{1}{3}i\left(i\sqrt{\delta}\sqrt{-\delta\gamma^2+2\mu}t-\delta\gamma t+x\right)\left(i\sqrt{\delta}\sqrt{-\delta\gamma^2+2\mu}-\delta\gamma\right).\end{aligned}$$

Moreover, the expressions of $\phi_1^{[1]}$, $\varphi_1^{[1]}$, and $\chi_1^{[1]}$ are omitted due to their cumbersome forms.

Obviously, $\Phi_1[0] = \Phi_1^{[0]} = (\phi_1^{[0]}, \varphi_1^{[0]}, \chi_1^{[0]})^T$ is a solution of Eq. (2) with $q_1[0] = d_1 e^{i\theta}$, $q_2[0] = d_2 e^{i\theta}$, and $\lambda = \lambda_1$. According to Eqs. (5)–(7), the first-order localized wave solutions of Eq. (1) are obtained,

$$\begin{aligned}q_1[1] &= q_1[0] + 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[0]\varphi_1^*[0]}{\Lambda_1} - \left(\frac{\phi_1[0]\varphi_1^*[0]}{\Lambda_1} \right)_{x,t} \right], \\ q_2[1] &= q_2[0] + 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[0]\chi_1^*[0]}{\Lambda_1} - \left(\frac{\phi_1[0]\chi_1^*[0]}{\Lambda_1} \right)_{x,t} \right],\end{aligned}$$

with

$$\Lambda_1 = |\lambda_1|^2[|\phi_1[0]|^2 + \lambda_1^*|\varphi_1[0]|^2 + \lambda_1^*|\chi_1[0]|^2].$$

There are five free parameters μ, γ, d_1, d_2 , and α , included in solutions $q_1[1]$ and $q_2[1]$. With the change of free parameters, the dynamic behaviors of the first-order localized wave solutions are analyzed in three cases.

- (1) $d_1 = d_2 = 1$ and $\alpha = 0$. The first-order rogue waves $q_1[1]$ and $q_2[1]$ have the spatial and temporal symmetry, which is similar to those reported in Ref. [30].
- (2) $d_1 = 1, d_2 = 0$ and $\alpha \neq 0$. Figures 1 and 2 exhibit the interaction between a dark-bright soliton and a rogue wave. With the parameter $\alpha = \frac{1}{10}$, the dark-bright

Fig. 1 The first-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = 1, d_2 = 0$, and $\alpha = \frac{1}{10}$

Fig. 2 The first-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = 1, d_2 = 0$, and $\alpha = 1$

Fig. 3 The first-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = d_2 = 1$, and $\alpha = \frac{1}{1000}$

Fig. 4 The first-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = d_2 = 1$, and $\alpha = 1$

soliton together with the rogue wave in the component $q_1[1]$ can be observed, but in the component $q_2[2]$, the rogue wave cannot be easily identified, as demonstrated in Fig. 1. Moreover, it is found from Figs. 1 and 2 that the dark-bright soliton and the rogue wave merge with increases in the value of α .

- (3) $d_1 = d_2 = 1$ and $\alpha \neq 0$. The interaction between a breather and a rogue wave is demonstrated in Figs. 3 and 4. It is found that the propagation direction of a breather is parallel with the positive direction of t -axis. Compared Figs. 4 with 3, from which it is also observed that the breather and the rogue wave merge with increases in the value of α .

Next, considering the following limit,

$$\Phi_1[1] = \lim_{\eta \rightarrow 0} \frac{T[1]|_{\lambda=\lambda_1(1+\eta^2)} \Phi_1}{\eta^2} = \lambda_1 \Phi_1^{[0]} + T_1[1] \Phi_1^{[1]}, \quad (8)$$

where

$$\begin{aligned} \Phi_1[1] &= (\phi_1[1], \varphi_1[1], \chi_1[1])^T, \\ T_1[1] &= \lambda_1^2 M_0[0] + \lambda_1 M_1[0] - I, \\ M_0[0] &= \frac{1}{\lambda_1^{*2}} I + \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1^{*2} \lambda_1} \begin{pmatrix} \frac{\phi_1[0] \phi_1^{*}[0]}{\Gamma_1} & 0 & 0 \\ 0 & \frac{\varphi_1[0] \varphi_1^{*}[0]}{\Gamma_1^{*}} & \frac{\varphi_1[0] \chi_1^{*}[0]}{\Gamma_1^{*}} \\ 0 & \frac{\chi_1[0] \varphi_1^{*}[0]}{\Gamma_1^{*}} & \frac{\chi_1[0] \chi_1^{*}[0]}{\Gamma_1^{*}} \end{pmatrix}, \\ M_1[0] &= \frac{\lambda_1^{*2} - \lambda_1^2}{\lambda_1 \lambda_1^{*}} \begin{pmatrix} 0 & \frac{\phi_1[0] \varphi_1^{*}[0]}{\Gamma_1} & \frac{\phi_1[0] \chi_1^{*}[0]}{\Gamma_1} \\ \frac{\varphi_1[0] \phi_1^{*}[0]}{\Gamma_1^{*}} & 0 & 0 \\ \frac{\chi_1[0] \phi_1^{*}[0]}{\Gamma_1^{*}} & 0 & 0 \end{pmatrix}, \\ \Gamma_1 &= \lambda_1 |\phi_1[0]|^2 + \lambda_1^{*} |\varphi_1[0]|^2 + \lambda_1^{*} |\chi_1[0]|^2. \end{aligned}$$

According Eqs. (5)–(8), the second-order localized wave solutions of Eq. (1) are obtained with the free parameters $\mu, \gamma, d_1, d_2, \alpha, m_1$, and n_1 ,

$$\begin{aligned} q_1[2] &= q_1[1] + 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[1] \varphi_1^{*}[1]}{\Lambda_2} - \left(\frac{\phi_1[1] \varphi_1^{*}[1]}{\Lambda_2} \right)_{x,t} \right], \\ q_2[2] &= q_2[1] + 2(\lambda_1^{*2} - \lambda_1^2) \left[\frac{i\mu}{\gamma} \frac{\phi_1[1] \chi_1^{*}[1]}{\Lambda_2} - \left(\frac{\phi_1[1] \chi_1^{*}[1]}{\Lambda_2} \right)_{x,t} \right], \end{aligned}$$

with

$$\Lambda_2 = |\lambda_1|^2 [\lambda_1 |\phi_1[1]|^2 + \lambda_1^{*} |\varphi_1[1]|^2 + \lambda_1^{*} |\chi_1[1]|^2].$$

Similarly, the dynamic properties of the second-order localized wave solutions with the variations of the free parameters involved are discussed in following cases.

Fig. 5 The second-order rogue waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = d_2 = 1, \alpha = 0$, and $n_1 = 0$

Fig. 6 The second-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = 1, d_2 = 0$, and $n_1 = 0$

- (1) $d_1 = d_2 = 1$ and $\alpha = 0$. Figure 5 presents the contour plots of components $q_1[2]$ and $q_2[2]$. Let $m_1 = 0$, $q_1[2]$ and $q_2[2]$ are the fundamental second-order rogue wave in Fig. 5a. Furthermore, setting $m_1 = 100$, $q_1[2]$ and $q_2[2]$ are consist of three first-order rogue waves in Fig. 5b.
- (2) $d_1 = 1, d_2 = 0$ and $\alpha \neq 0$. The interactions of two dark-bright solitons and a second-order rogue wave are illustrated in Fig. 6. Similarly to dynamics of the second-order localized waves in Figs. 1 and 2, the second-order rogue wave in the component $q_1[2]$ can be clearly observed, but in the component $q_2[2]$, the rogue wave cannot be easily identified after setting $\alpha = \frac{1}{10000}$, as demonstrated in Fig. 6. Decreasing the value of α , the second-order rogue wave separates from two dark-bright solitons. In addition, With the parameter $m_1 = 100$, the funda-

Fig. 7 The second-order localized waves with parameters $\mu = 2, \gamma = \frac{1}{100}, d_1 = d_2 = 1$, and $n_1 = 0$

mental second-order rogue wave can split into three first-order rogue waves in Fig. 6c and f.

- (3) $d_1 = d_2 = 1$ and $\alpha \neq 0$. Figure 7 demonstrates a second-order rogue wave coexisted with breathers. It is found that the propagation directions of a breathers are parallel with the positive direction of t -axis, and also observed that breathers and the second-order rogue wave separate with decreases in the value of α . Moreover, due to the influence of separation function $\Omega(\eta)$, the fundamental second-order rogue wave splits into three first-order rogue waves, as displayed in Fig. 7c and f.

4 Conclusions

The CMDNLS equation was studied in this work. Based on the classical DT and a Lax pair, a generalized DT was constructed and the N th-order localized wave solutions were derived. It is found that the free parameters involved in localized wave solutions are $\mu, \gamma, \alpha, d_1, d_2, m_j$ and $n_j (j = 1, 2, \dots)$. Then, the first- and second-order localized wave solutions were obtained and the corresponding evolution plots were provided. Further, altering the values of related parameters, the

interactions of localized waves were analyzed. The results of this paper present richer dynamics of CMDNLS equation.

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Availability of data and material No data, models, or code were generated or used during the study.

Declarations

Conflict of interests The authors declare that they have no conflict of interest.

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