Gevrey regularity and analyticity for the modified Camassa–Holm equation

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Abstract

This paper deals with the analyticity and Gevrey regularity for the Cauchy problem of a modified Camassa–Holm (mCH) equation in Sobolev–Gevrey space $G_{r,s}^\delta$ with $s > \frac{1}{2}$, $r \geq 1$ and $0 < \delta < 1$, and a lower bound of lifespan and continuity of the data-to-solution map was also obtained.

Keywords Camassa–Holm equation · Analyticity · Gevrey regularity

Mathematics Subject Classification 35Q53 · 35B30 · 35B44 · 35C07 · 35G25

1 Introduction

In this paper, we consider the following modified Camassa–Holm equation (mCH)

$$\begin{cases} \gamma_t = \frac{\gamma^2}{2} + \lambda v - (v\gamma)_x, & t \geq 0, \quad x \in \mathbb{R}, \\ v_{xx} - v = \gamma_x + \frac{\gamma^2}{2\lambda}, & t > 0, \quad x \in \mathbb{R}, \\ \gamma(x, 0) = \gamma_0(x), & t = 0, \quad x \in \mathbb{R}. \end{cases} \quad (1.1)$$

It is known that the Camassa–Holm equation describes pseudo-spherical surfaces and therefore its integrability properties can be studied by geometrical means [21], using this fact, Gorka and Reyes (see (6.1)–(6.2) in [15]) introduced a "Miura transform" and proposed the modified Camassa–Holm (1.1) equation, in analogy with the Korteweg–de Vries theory. They also obtained conservation laws for (1.1) and shown that it describes pseudo-spherical surfaces, so that, in particular, it is the integrability condition of an $sl(2, \mathbb{R})$ -valued linear problem (cf. [15, 16]).

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The original Camassa–Holm equation was obtained by Fuchssteiner and Fokas by using the method of recursion operators [14],

$$m_t + 2u_x m + um_x = 0, m = (1 - \partial_x^2)u, \quad (1.2)$$

and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [1]. Analogous to the famous KdV equation, the Camassa–Holm equation (1.2) also has a bi-Hamiltonian structure [1, 14], which is completely integrable not only in the sense of the existence of a Lax pair [1], but also (by means of inverse scattering and inverse spectral theory) as an infinite-dimensional Hamiltonian flow that can be linearised in suitable action-angle variables, cf. (see [3, 4, 8, 11, 12]). The orbital stability of solitary waves and the stability of the peakons ($k = 0$) for the Camassa–Holm equation are investigated by Constantin and Strauss [9, 10]. The advantage of the Camassa–Holm equation in comparison with the famous KdV equation lies in the fact that the Camassa–Holm models the peculiar wave breaking phenomena, that is, the solution remains bounded but its slope becomes unbounded in finite time (cf. [2, 7]). In addition to wave breaking, one of the most interesting aspects of the equation is the existence of peakon solutions, namely, the travelling wave solutions of greatest height of the governing equations for water waves have a peak at their crest (cf. [5, 6]). The latest progress about the Camassa–Holm type equations can be refer to [20] and references therein.

Recently, Luo, Qiao and Yin [19] established the local well-posedness for the Cauchy problem of the Equ. (1.1) in nonhomogeneous Besov spaces by Littlewood–Paley theory and transport equation theory. In this paper, we show the Gevrey and analytic regularity of the Cauchy problem for mCH system. We first introduce the Sobolev–Gevrey spaces (cf. [13]).

Definition 1.1 Let s be a real number and $r, \delta > 0$. A function $f \in G_{r,s}^\delta(\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R})$ and satisfies

$$\|f\|_{G_{r,s}^\delta(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s e^{2\delta|\xi|^{\frac{1}{r}}} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

Note that the Fourier multiplier $e^{\delta(-\Delta)^{\frac{1}{2r}}}$ by: $e^{\delta(-\Delta)^{\frac{1}{2r}}} f = \mathcal{F}^{-1}(e^{\delta|\xi|^{\frac{1}{r}}} \hat{f})$, which implies that $\|f\|_{G_{r,s}^\delta(\mathbb{R})} = \|e^{\delta(-\Delta)^{\frac{1}{2r}}} f\|_{H^s(\mathbb{R})}$. For $0 < r < 1$, it is called ultra-analytic function. If $r = 1$, it is usual analytic function and δ is called the radius of analyticity. If $r > 1$, it is the Gevrey class function.

It is well known that the solutions to the Camassa–Holm type systems are analytic in both space and time variables for a short time (see [17]). However, it gives no estimate about the size of the analytic lifespan δ . Also, it does not supply information about the evolution of the uniform radius of analyticity. Recently, Barostichi et al. solved these important problems for Camassa–Holm and other nonlocal equations and systems by using an Ovsiannikov type theorem for an autonomous abstract Cauchy problem in a scale of decreasing Banach spaces, and they also studied the stability of their solution map. By taking the advantage of this idea, Luo and Yin [18] proved the Gevrey regularity and analyticity of these systems be a generalized

Ovsyannikov theorem. Our motivation to solve this problem follow the line of [17, 18].

Theorem 1.1 *Let $\gamma_0 \in G_{r,s}^1(\mathbb{R})$ with $r \geq 1$ and $s > 1/2$. Thus, for all $0 < \delta < 1$, there exists a $T_0 > 0$ such that the mCH system has a unique solution γ which is holomorphic in $|t| < D_r \frac{(1-\delta)^r T_0}{2^{r+1}}$ with values in $G_{r,s}^\delta(\mathbb{R})$, where $D_r = \frac{1}{2^r - 2 + \frac{1}{2^{r+1}}}$. Moreover*

$$T_0 = \frac{1}{C_s 2^{2r+7} (e^{-r} r^r + 2) \left[\left(\|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \right) \|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12} \right]}.$$

With the existence and uniqueness at hand, we next show the continuity of the data-to-solution map in $G_{r,s}^1(\mathbb{R})$, which means that:

Definition 1.2 If for a given $\gamma_0^\infty \in G_{r,s}^1(\mathbb{R})$ with $r \geq 1$ and $s > \frac{1}{2}$, there exists a $T(\|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})})$ such that for any sequence $\gamma_0^n \in G_{r,s}^1(\mathbb{R})$, and $\|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} \rightarrow 0$ for $n \rightarrow \infty$, the corresponding solution $\gamma^n(t)$ of mCH system satisfies $\|\gamma^n(t) - \gamma^\infty(t)\|_{E_T} \rightarrow 0$ for $n \rightarrow \infty$, where

$$\|\gamma\|_{E_T} = \sup_{|t| < D_r \frac{T(1-\delta)^r}{2^{r+1}}, 0 < \delta < 1} \left(\|\gamma\|_{G_{r,s}^\delta(\mathbb{R})} (1-\delta)^r \sqrt{1 - \frac{|t|}{T(1-\delta)^r}} \right).$$

Our last result about the continuity of the data-to-solution map was stated as follows:

Theorem 1.2 *If $\gamma_0 \in G_{r,s}^1(\mathbb{R})$ with $r \geq 1$ and $s > \frac{1}{2}$. Then the data-to-solution map $\gamma_0 \rightarrow \gamma$ of the Equ. (1.1) is continuous from $G_{r,s}^1(\mathbb{R})$ into the solutions space.*

Remark 1 In the period case, the Sobolev-Gevrey norm can be defined as follows

$$\|f\|_{G_{r,s}^\delta(\mathbb{T})} = \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^s e^{2\delta|k|^{\frac{1}{r}}} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} = \|e^{\delta(-\Delta)^{\frac{1}{2r}}} f\|_{H^s(\mathbb{T})}.$$

in view of the same argument as in Theorems 1.1 and 1.2, we can get the similar Gevrey regularity and analytic for the mCH (1.1).

The plan of this paper is organized as follows. In the next section, we recall some properties about Sobolev-Gevrey spaces. In the last section, we prove the analyticity and Gevrey regularity of the solutions to the modified Camassa-Holm equation, and prove Theorems 1.1–1.2.

2 Preliminaries

In this section, we will investigate the analytic solutions to mCH system (1.1) by nonlinear Cauchy-Kowalevski theorem, and the Cauchy-Kowalevski theorem is read as follows:

Theorem 2.1 (See Theorem 3.1 in [18, 22]) *Let $(X_\delta, \|\cdot\|_\delta)_{0 < \delta \leq 1}$ be a scale of decreasing Banach spaces, such that for any $0 < \delta' \leq 1$ we have $X_\delta \subset X_{\delta'}$ with $\|\cdot\|_{\delta'} < \|\cdot\|_\delta$. Consider the Cauchy problem*

$$\begin{cases} \frac{d\gamma}{dt} = F(t, \gamma(t)), \\ \gamma(0) = 0. \end{cases} \quad (2.3)$$

Let $T, R > 0$ and $r \geq 1$. For given $\gamma_0 \in X_1$, suppose that F satisfies the following conditions:

- (1) If for any $0 < \delta' < \delta < 1$, the function $t \mapsto \gamma(t)$ is holomorphic on $|t| < T$ and continuous on $|t| \leq T$ with values in X_δ and

$$\sup_{|t| \leq T} \|\gamma(t)\|_\delta < R,$$

then $t \mapsto F(t, \gamma(t))$ is a holomorphic function on $|t| < T$ with values in X_δ .

- (2) For any $0 < \delta' < \delta < 1$ and $\gamma_1, \gamma_2 \in B(\gamma_0, R) \subset X_\delta$, that is, $\|\gamma_1\|_\delta < R, \|\gamma_2\|_\delta < R$, there exists a positive constant L depending on γ_0 and R such that

$$\sup_{|t| \leq T} \|F(t, \gamma_1) - F(t, \gamma_2)\|_{\delta'} \leq \frac{L}{(\delta - \delta')^r} \|\gamma_1 - \gamma_2\|_\delta.$$

- (3) There exists a $M > 0$ depending on γ_0 and R such that for any $0 < \delta < 1$,

$$\sup_{|t| \leq T} \|F(t, \gamma_0)\|_\delta \leq \frac{M}{(1 - \delta)^r}.$$

Then there exists a $T_0 \in (0, T)$ and a unique function $\gamma(t)$ to the Cauchy problem (2.3), which is holomorphic in $|t| < D_r \frac{(1-\delta)^r T_0}{2^{r+1}}$ with value in X_δ for every $\delta \in (0, 1)$.

Remark 2.1 $T_0 = \min \left\{ \frac{1}{2^{2r+4}L}, \frac{2^{r+1}R}{2^{r+1}2^{2r+3}LR+MD_r} \right\}$, $D_r = \frac{1}{2^r-2+\frac{1}{2^{r+1}}}$. (see Lemma 3.7 in [18, 22]).

If $X_\delta = G_\delta^\delta(\mathbb{R})$ (see Definition 1.1), then $G_{r,s}^\delta(\mathbb{R}) \hookrightarrow G_{r,s'}^{\delta'}(\mathbb{R})$ with $0 < \delta' < \delta$, $G_{r',s}^\delta(\mathbb{R}) \hookrightarrow G_{r,s}^\delta(\mathbb{R})$ with $0 < r' < r$, $G_{r,s}^\delta(\mathbb{R}) \hookrightarrow G_{r,s'}^\delta(\mathbb{R})$ with $s' < s$. Moreover, $G_{r,s}^\delta(\mathbb{R})$ has the following three properties:

Proposition 2.1 (See Proposition 2.5-2.6 in [18]) *Let $s > 1/2$, $r \geq 1$ and $\delta > 0$, then $G_{r,s}^\delta(\mathbb{R})$ is an algebra, and for any $\gamma_1 \in G_{r,s-1}^\delta(\mathbb{R})$, $\gamma_2 \in G_{r,s}^\delta(\mathbb{R})$, there is a constant C_s that is only dependent of s such that*

$$\|\gamma_1 \gamma_2\|_{G_{r,s-1}^\delta(\mathbb{R})} \leq C_s \|\gamma_1\|_{G_{r,s-1}^\delta(\mathbb{R})} \|\gamma_2\|_{G_{r,s}^\delta(\mathbb{R})}. \quad (2.4)$$

Moreover, for any $\gamma_1, \gamma_2 \in G_{r,s}^\delta(\mathbb{R})$, there is a constant C'_s that is only dependent of s such that

$$\|\gamma_1 \gamma_2\|_{G_{r,s}^\delta(\mathbb{R})} \leq C'_s \|\gamma_1\|_{G_{r,s}^\delta(\mathbb{R})} \|\gamma_2\|_{G_{r,s}^\delta(\mathbb{R})}. \quad (2.5)$$

Proposition 2.2 (See Proposition 2.4 in [18]) *Let s be a real number and $r > 0$, $s > 0$. For any $0 < \delta' < \delta$, we have*

$$\|\gamma_x\|_{G_{r,s-1}^\delta(\mathbb{R})} \leq \|\gamma\|_{G_{r,s}^\delta(\mathbb{R})}, \quad \|\gamma_x\|_{G_{r,s}^{\delta'}(\mathbb{R})} < \frac{e^{-r} r^r}{(\delta - \delta')^r} \|\gamma\|_{G_{r,s}^\delta(\mathbb{R})}. \quad (2.6)$$

Proposition 2.3 (See Proposition 2.7 in [18]) *For any $s \in \mathbb{R}$, $r, \delta > 0$ and $\gamma \in G_{r,s}^\delta(\mathbb{R})$, we get*

$$\|\Lambda^{-2} \gamma\|_{G_{r,s}^\delta(\mathbb{R})} \leq \|\gamma\|_{G_{r,s}^\delta(\mathbb{R})}, \quad \|\Lambda^{-2} \partial_x \gamma\|_{G_{r,s}^\delta(\mathbb{R})} \leq \|\gamma\|_{G_{r,s-1}^\delta(\mathbb{R})}, \quad (2.7)$$

where $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$.

3 Proof of Theorem 1.1 and 1.2

In this section, we will prove Theorem 1.1 and 1.2 by Theorem 2.1 and Proposition 2.1, 2.2 and 2.3.

Proof of Theorem 1.1 In order to applying the Theorem 2.1, we rewrite (mCH) as follows:

$$\begin{cases} \gamma_t = F(\gamma) \doteq \frac{\gamma^2}{2} + \lambda v - v_x \gamma - v \gamma_x, \\ v = -\Lambda^{-2} \left(\gamma_x + \frac{\gamma^2}{2\lambda} \right), \\ \gamma|_{t=0} = \gamma_0. \end{cases} \quad (3.8)$$

Next, we can prove that function $F(\gamma)$ satisfies the three conditions in Theorem 2.1. According to Propositions 2.1, 2.2 and 2.3, for $0 < \delta' < \delta$, $s > \frac{1}{2}$, we can deduce that

$$\begin{aligned}
\|F(\gamma)\|_{G_{r,s}^{\delta'}(\mathbb{R})} &\leq \frac{1}{2} \|\gamma^2\|_{G_{r,s}^{\delta'}(\mathbb{R})} + \|\lambda v\|_{G_{r,s}^{\delta'}(\mathbb{R})} + \|v_x \gamma\|_{G_{r,s}^{\delta'}(\mathbb{R})} + \|v \gamma_x\|_{G_{r,s}^{\delta'}(\mathbb{R})} \\
&\leq \frac{C_s}{2} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + C_s \|v\|_{G_{r,s}^{\delta}(\mathbb{R})} \\
&\quad + C_s \|v_x\|_{G_{r,s}^{\delta'}(\mathbb{R})} \|\gamma\|_{G_{r,s}^{\delta'}(\mathbb{R})} + C_s \|v\|_{G_{r,s}^{\delta'}(\mathbb{R})} \|\gamma_x\|_{G_{r,s}^{\delta'}(\mathbb{R})} \\
&\leq C_s \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + \frac{C_s}{2} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})} + C_s \|v_x\|_{G_{r,s}^{\delta'}(\mathbb{R})} \|\gamma\|_{G_{r,s}^{\delta'}(\mathbb{R})} \\
&\quad + \frac{C_s e^{-r} r^r}{(\delta - \delta')^r} \|v\|_{G_{r,s}^{\delta}(\mathbb{R})} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})},
\end{aligned} \tag{3.9}$$

where

$$\|v\|_{G_{r,s}^{\delta}(\mathbb{R})} = \Lambda^{-2} \left(\gamma_x + \frac{\gamma^2}{2\lambda} \right) \|_{G_{r,s}^{\delta}(\mathbb{R})} \leq \frac{1}{2} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})} + \frac{C_s}{2} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2. \tag{3.10}$$

Combining (3.9) and (3.10), we can get

$$\begin{aligned}
\|F(\gamma)\|_{G_{r,s}^{\delta'}(\mathbb{R})} &\leq 2C_s \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + \frac{C_s}{2} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})} + \frac{C_s e^{-r} r^r}{2(\delta - \delta')^r} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 \\
&\quad + \frac{C_s (e^{-r} r^r + 2)}{2(\delta - \delta')^r} \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^3 \\
&\leq \frac{C_s (e^{-r} r^r + 2) \left(\|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}^3 \right) + C_s \|\gamma\|_{G_{r,s}^{\delta}(\mathbb{R})}}{2(\delta - \delta')^r}.
\end{aligned}$$

Thus condition (1) in Theorem 1.1 holds.

According to the same token as the above, we can get that

$$\|F(\gamma_0)\|_{G_{r,s}^{\delta}(\mathbb{R})} \leq \frac{C_s (e^{-r} r^r + 2) \left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + \|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}^3 \right) + C_s \|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}}{2(\delta - \delta')^r}.$$

So we see that $F(\gamma)$ satisfies the condition (3) in Theorem 2.1 with

$$M = \frac{C_s (e^{-r} r^r + 2) \left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + \|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}^3 \right) + C_s \|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})}}{2}.$$

Now, we only need to prove that $F(\gamma)$ satisfies the condition (2) in Theorem 2.1. Suppose that $\|\gamma_1 - \gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} \leq R$ and $\|\gamma_2 - \gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} \leq R$, applying the estimates (2.4)-(2.7), we can get

$$\begin{aligned}
& \|F(\gamma_1) - F(\gamma_2)\|_{G_{r,s}^{\delta'}(\mathbb{R})} = \left\| \frac{\gamma_1^2}{2} + \lambda v_1 - (v_1 \gamma_1)_x - \left(\frac{\gamma_2^2}{2} + \lambda v_2 - (v_2 \gamma_2)_x \right) \right\|_{G_{r,s}^{\delta'}(\mathbb{R})} \\
& \leq \frac{C_s(e^{-r}r^r + 2)}{(\delta - \delta')^r} \left(\|\gamma_1\|_{G_{r,s}^{\delta}(\mathbb{R})} + \|\gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})} \right) \|\gamma_1 - \gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})} \\
& \quad + \frac{C_s(e^{-r}r^r + 2)}{(\delta - \delta')^r} \left(\|\gamma_1\|_{G_{r,s}^{\delta}(\mathbb{R})} \|\gamma_1 + \gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})} \right) \|\gamma_1 - \gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})} \\
& \quad + \frac{C_s(e^{-r}r^r + 2)}{(\delta - \delta')^r} \left(\|\gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})}^2 + 1 \right) \|\gamma_1 - \gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})} \\
& \leq \frac{3C_s(e^{-r}r^r + 2)}{(\delta - \delta')^r} \left[\left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} + R + 1 \right) \left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} + R \right) + \frac{1}{3} \right] \|\gamma_1 - \gamma_2\|_{G_{r,s}^{\delta}(\mathbb{R})}.
\end{aligned} \tag{3.11}$$

According to the above inequality, we have proved that $F(\gamma)$ satisfies the condition (2) of Theorem 2.1 with $L = 3C_s(e^{-r}r^r + 2) \left[\left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} + R + 1 \right) \left(\|\gamma_0\|_{G_{r,s}^{\delta}(\mathbb{R})} + R \right) + \frac{1}{3} \right]$, and that $T_0 = \min \left\{ \frac{1}{2^{2r+4}L}, \frac{(2^{r+1})R}{(2^{r+1})2^{2r+3}LR + MD_r} \right\}$, by setting $R = \|\gamma_0\|_{G_{r,s}^1(\mathbb{R})}$, we can obtain that

$$L = 12C_s(e^{-r}r^r + 2) \left[\left(\|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \right) \|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12} \right] \tag{3.12}$$

and $M \leq \frac{2^{2r+3}(2^{r+1})LR}{D_r}$. Thus, we get $T_0 = \frac{1}{C_s 2^{2r+7}(e^{-r}r^r + 2) \left[\left(\|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \right) \|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12} \right]}$. $\square$

Next, we will prove the Theorem 1.2 as follows.

Proof of Theorem 1.2 Applying the proof of Theorem 1.1, for the given $r \geq 1$, $s > 1/2$, we can know that the lifespan of the corresponding solution to the mCH Cauchy problem γ^n and γ^∞ are given by

$$\begin{aligned}
T^n &= \frac{1}{C_s 2^{2r+7}(e^{-r}r^r + 2) \left[\left(\|\gamma_0^n\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \right) \|\gamma_0^n\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12} \right]}, \\
T^\infty &= \frac{1}{C_s 2^{2r+7}(e^{-r}r^r + 2) \left[\left(\|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \right) \|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12} \right]}.
\end{aligned} \tag{3.13}$$

Since $\|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} \rightarrow 0$, thus, there exists a constant N such that, if $n \geq N$, we get

$$\|\gamma_0^n\|_{G_{r,s}^1(\mathbb{R})} \leq \|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + 1. \tag{3.14}$$

So for $n \geq N$, we can infer that

$$\begin{aligned}
T &\doteq \frac{1}{C_s 2^{2r+7}(e^{-r}r^r + 2) \left[\left(\|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + \frac{3}{2} \right) \left(\|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + 1 \right) + \frac{1}{12} \right]} \\
&\leq \min \{T^n, T^\infty\}.
\end{aligned} \tag{3.15}$$

As in proof Theorem 1.1, we see that T^n and T^∞ are the existence time corresponding to $\|\gamma_0^n\|_{G_{r,s}^1(\mathbb{R})}$ and $\|\gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})}$ respectively. Which implies that for any $n \geq N$

$$\gamma^n(t, x) = \gamma_0^n(x) + \int_0^t F(\gamma^n(t, \tau)) d\tau, \quad 0 \leq t \leq D_r \frac{T(1-\delta)^r}{2^{r+1}}, \quad (3.16)$$

$$\gamma^\infty(t, x) = \gamma_0^\infty(x) + \int_0^t F(\gamma^\infty(t, \tau)) d\tau, \quad 0 \leq t \leq D_r \frac{T(1-\delta)^r}{2^{r+1}}, \quad (3.17)$$

where F is given in (3.8). On the basis of the above equation, we affirm that for any $0 \leq t < D_r \frac{T(1-\delta)^r}{2^{r+1}}$ and $0 < \delta < 1$,

$$\begin{aligned} \|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} &\leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^\delta(\mathbb{R})} \\ &+ \int_0^t \|F(\gamma^n(\tau)) - F(\gamma^\infty(\tau))\|_{G_{r,s}^\delta(\mathbb{R})} d\tau. \end{aligned} \quad (3.18)$$

In consideration of (3.11), we have $\|F(\gamma^n(\tau)) - F(\gamma^\infty(\tau))\|_{G_{r,s}^\delta(\mathbb{R})} \leq \frac{L\|\gamma^n - \gamma^\infty\|_{G_{r,s}^{\delta(\tau)}(\mathbb{R})}}{(\delta(\tau) - \delta)^r}$ which $L = 12C_s(e^{-r}r^r + 2)\left[\left(\|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2}\right)\|\gamma_0\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{12}\right]$ with $\delta < \delta(\tau) < 1$ to be confirmed later. Plugging it into (3.18) obtains that

$$\|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} \leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^\delta(\mathbb{R})} + L \int_0^t \frac{\|\gamma^n - \gamma^\infty\|_{G_{r,s}^{\delta(\tau)}(\mathbb{R})}}{(\delta(\tau) - \delta)^r} d\tau. \quad (3.19)$$

Define that $\delta(\tau) = \frac{1+\delta}{2} + \left(\frac{1}{2}\right)^{2+\frac{1}{r}} \left\{ \left[(1-\delta)^r - \frac{\tau}{T} \right]^{\frac{1}{r}} - \left[(1-\delta)^r + (2^{r+1} - 1)\frac{\tau}{T} \right]^{\frac{1}{r}} \right\}$, using the Lemma 3.7 in [22] with $a = T$, we derive that

$$\|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} \leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^\delta(\mathbb{R})} + \frac{2^{2r+3}LT\|\gamma^n - \gamma^\infty\|_{E_T}}{(1-\delta)^r} \sqrt{\frac{T(1-\delta)^r}{T(1-\delta)^r - t}}. \quad (3.20)$$

According to the definition of L and T (see (3.12) and (3.15)), we can obtain that $2^{2r+3}LT < \frac{1}{2}$. So we get

$$\begin{aligned} \|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} &\leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^\delta(\mathbb{R})} \\ &+ \frac{\|\gamma^n - \gamma^\infty\|_{E_T}}{2(1-\delta)^r} \sqrt{\frac{T(1-\delta)^r}{T(1-\delta)^r - t}}, \end{aligned} \quad (3.21)$$

which leads to

$$\begin{aligned}
& (1-\delta)^r \sqrt{1 - \frac{t}{T(1-\delta)^r}} \|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} \\
& \leq (1-\delta)^r \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^\delta(\mathbb{R})} \sqrt{1 - \frac{t}{T(1-\delta)^r}} + \frac{1}{2} \|\gamma^n - \gamma^\infty\|_{E_T} \quad (3.22) \\
& \leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})} + \frac{1}{2} \|\gamma^n - \gamma^\infty\|_{E_T}.
\end{aligned}$$

We can find easily that the right term of the above posture is independence of t and δ . In the light of the definition of E_T , we get

$$\begin{aligned}
& \|\gamma^n(t) - \gamma^\infty(t)\|_{E_T} \\
& = \sup_{|t| < D_r \frac{T(1-\delta)^r}{2^{r+1}}, 0 < \delta < 1} (1-\delta)^r \|\gamma^n(t) - \gamma^\infty(t)\|_{G_{r,s}^\delta(\mathbb{R})} \sqrt{1 - \frac{t}{T(1-\delta)^r}} \quad (3.23) \\
& \leq \|\gamma_0^n - \gamma_0^\infty\|_{G_{r,s}^1(\mathbb{R})}.
\end{aligned}$$

The above inequality holds true for any $n \geq N$, and leads to our desire results. $\square$

4 Conclusions

This paper mainly studies the Cauchy problem of the modified Camassa-Holm equation. According to the Ovsyannikov theorem and the basic properties of Sobolev-Gevrey spaces, we prove the Gevrey regularity and analyticity of the equation (see Theorem 1.1). In addition, we obtain a lower bound of the lifespan and the continuity of the data-to-solution map (see Theorem 2.2).

Abbreviations For any $s \in \mathbb{R}$ we take the operator $\Lambda^s = (1 - \partial_x^2)^{s/2}$ to be defined by $\widehat{\Lambda^s f}(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi)$, where $\hat{f}(\xi)$ is the Fourier transform, $\hat{f}(\xi) \doteq \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$. Then, for $f \in H^s(\mathbb{T})$ we have

$$\|f\|_{H^s} \doteq \|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

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Declarations

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