Eigenvalues of a Class of Eigenparameter Dependent Third-Order Differential Operators

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Abstract

The present paper deals with a class of third-order differential operators with eigenparameter dependent boundary conditions. The continuity of eigenvalues concerning a given parameter is proved. Moreover, the derivative formulas of eigenvalues concerning the parameters, in particular, the eigenparameter dependent boundary condition matrix, are given.

Keywords Third-order differential operators · Eigenparameter dependent boundary conditions · Continuity of eigenvalues · Differentiability of eigenvalues

1 Introduction

Boundary value problem (BVP) with an eigenparameter contained in equations and boundary conditions is a significant part of differential operator theory for its wide applications in various fields. And it has attracted more and more attentions from mathematicians and physicists in recent decades. Various applications in physics and other fields such as the vibration of loaded strings, diffusion processes in probability theory and so on yield such problems [1]. A large number of literature have devoted to the study of such problems for Sturm–Liouville (S–L) problems and fourth-order beam equations, and numerous significance results are obtained (see, for example [2–12]).

The dependence of eigenvalues of classical regular S–L problems has been well investigated by Kong, Wu and Zettl. They showed that each eigenvalue of regular S–L problems can be inserted into a continuous eigenvalue branch, moreover, it is

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shown that each eigenvalue is differentiable with regards to a given parameter [13, 14] (the coefficients of differential expression, the endpoints and the boundary conditions). Such dependence has been generalized by Shi, Zettl etc. into singular S–L problems, especially in the case of limit point S–L problems [15, 16]. The dependence of eigenvalues of eigenparameter dependent S–L problems was investigated by Zhang etc. [17]. And in [18–20] fourth-order and higher-order differential operators were studied. In [21–24], the dependence of eigenvalues of discontinuous differential operators were investigated. These results play a significant role in the eigenvalue theory of differential operators and it is fundamental from the numerical computation of spectrum, for example, the codes SLEIGN2 [25] and SLEUTH [26].

Boundary condition plays a crucial role in the differential operators theory since it carries the information of the spectrum. As is well-known, the self-adjoint boundary conditions of differential operators can be classified into three cases: mixed, separated and coupled, which was shown by Wang, Sun and Zettl [27]. For the second-order S–L operators, there only exist two cases: separated and coupled. However, with regard to third-order differential operators, there only coupled and mixed self-adjoint boundary conditions exist [28] and a complete characterization of self-adjoint domain of odd order differential operators was given in [29]. Also, third-order differential equations have numerous physical backgrounds, for example, three-layer beam, the deflection of a curved beam varying cross-section, and so on [30, 31]. Based on these facts, Uğurlu investigated the continuity and differentiability of eigenvalues concerning some data for third-order self-adjoint differential operators without eigenparameter dependent boundary conditions, moreover, the differential expressions of eigenvalues concerning some parameters are given [32]. Therefore, a natural question is rising: does a small change of third-order differential operators with eigenparameter dependent boundary conditions results in small change of each eigenvalue and eigenfunction of the problem? In the present paper, we will solve this problem.

In this paper, we will consider a class of eigenparameter dependent third-order differential operators with mixed boundary conditions. We characterize the continuity of eigenvalues with respect to given data by using the continuity theorem for analytic functions with isolated zeros, moreover, the differential expressions of eigenvalues are also presented. The paper is organized as follows. In Sect. 2, we investigate some notations and preliminaries associated with the problem. The continuity and differentiability of eigenvalues are presented in Sects. 3 and 4, respectively.

2 Preliminaries

We study third-order symmetric differential equation

$$\ell f := \frac{1}{w} \{ -i[q_0(q_0 f')']' - (p_0 f')' + i[q_1 f' + (q_1 f)'] + p_1 f \} = \lambda f, \quad \text{on } [a, b], \quad (1)$$

with boundary conditions:

$$L_1 f := (\alpha_1 \lambda + \tilde{\alpha}_1) f(a) - (\alpha_2 \lambda + \tilde{\alpha}_2) f^{[2]}(a) = 0, \quad (2)$$

$$L_2 f := (\gamma_1 \lambda + \tilde{\gamma}_1) f(b) + (\gamma_2 \lambda + \tilde{\gamma}_2) f^{[2]}(b) = 0, \quad (3)$$

$$L_3 f := (\sin \theta + i) f^{[1]}(a) + (i \sin \theta + 1) f^{[1]}(b) = 0, \quad (4)$$

where λ is the spectral parameter, $\theta \in (0, \pi]$, q_0, q_1, p_0, p_1, w satisfy the conditions

$$q_0^{-1}, q_0^{-2}, p_0, q_1, p_1, w \in L^1([a, b], \mathbb{R}), \quad q_0 > 0, \quad w > 0. \quad (5)$$

$\alpha_k, \tilde{\alpha}_k, \gamma_k, \tilde{\gamma}_k$ ($k = 1, 2$) are arbitrary real numbers, and satisfy

$$\rho_1 = \tilde{\alpha}_1 \alpha_2 - \alpha_1 \tilde{\alpha}_2 > 0, \quad \rho_2 = \tilde{\gamma}_1 \gamma_2 - \gamma_1 \tilde{\gamma}_2 > 0. \quad (6)$$

Third-order BVP with an eigenparameter contained in the boundary conditions consisting of (1)–(4) is considered here. Firstly, we investigate some basic preliminaries.

The quasi-derivatives of f are defined as [33]

$$f^{[0]} = f, \quad f^{[1]} = -\frac{1+i}{\sqrt{2}} q_0 f', \quad f^{[2]} = i q_0 (q_0 f')' + p_0 f' - i q_1 f.$$

and $H_w = L_w^2[a, b]$ is a weighted Hilbert space equipped with the inner product $\langle f, g \rangle_w = \int_a^b f \bar{g} w dx$ consisting of functions f which satisfy $\int_a^b |f|^2 w dx < \infty$.

Maximal operator $L_{\max}$ is defined as

$$L_{\max} f = \ell f, \quad f \in H_w$$

with the domain

$$D_{\max} = \{f \in L_w^2[a, b] \mid f, f^{[1]}, f^{[2]} \in AC[a, b], \ell f \in L_w^2[a, b]\}.$$

Then for arbitrary $f, g \in D_{\max}$, integration by parts yields Lagrange identity

$$\langle L_{\max} f, g \rangle_w - \langle f, L_{\max} g \rangle_w = [f, \bar{g}]_a^b,$$

where

$$\begin{aligned} [f, \bar{g}]_a^b &= [f, \bar{g}](b) - [f, \bar{g}](a), \\ [f, \bar{g}](x) &= f(x) \overline{g^{[2]}(x)} - f^{[2]}(x) \overline{g(x)} + i f^{[1]}(x) \overline{g^{[1]}(x)}. \end{aligned}$$

Let $\mathcal{H} = L_w^2[a, b] \oplus \mathbb{C}^2$ equipped with inner product

$$\langle F, G \rangle = \int_a^b f \bar{g} w dx + \frac{1}{\rho_1} f_1 \bar{g}_1 + \frac{1}{\rho_2} f_2 \bar{g}_2, \quad (7)$$

where $F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}$, $G = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}$. It is easy verified that $\mathcal{H}$ is a Hilbert space.

Define the operator T as follows

$$\begin{aligned} D(T) &= \{F = (f(x), f_1, f_2)^T \in \mathcal{H} \mid L_3 f = 0, \\ f_1 &= \alpha_1 f(a) - \alpha_2 f^{[2]}(a), f_2 = \gamma_1 y(b) + \gamma_2 f^{[2]}(b), f \in D_{\max}\}, \end{aligned} \quad (8)$$

and

$$F(x) = \begin{pmatrix} f(x) \\ \alpha_1 f(a) - \alpha_2 f^{[2]}(a) \\ \gamma_1 f(b) + \gamma_2 f^{[2]}(b) \end{pmatrix} \in D(T), \quad TF = \begin{pmatrix} \ell f \\ \tilde{\alpha}_2 f^{[2]}(a) - \tilde{\alpha}_1 f(a) \\ -[\tilde{\gamma}_1 f(b) + \tilde{\gamma}_2 f^{[2]}(b)] \end{pmatrix}. \quad (9)$$

By the definition of the operator T , the eigenvalue problem of BVP (1)–(4) is transferred to the spectra problem of the operator T .

Considering the operator T , the following properties hold.

Lemma 2.1 T is self-adjoint in $\mathcal{H}$, its eigenvalues are discrete, real-valued and have no finite point of accumulation. Moreover, the multiplicity of each eigenvalue at most 3.

Proof For detail proof, we refer to [34]. □

Let

$$\begin{aligned} A_\lambda &= \begin{pmatrix} \alpha_1 \lambda + \tilde{\alpha}_1 & 0 & -(\alpha_2 \lambda + \tilde{\alpha}_2) \\ 0 & 0 & 0 \\ 0 & \sin \theta + i & 0 \end{pmatrix}, \\ B_\lambda &= \begin{pmatrix} 0 & 0 & 0 \\ \gamma_1 \lambda + \tilde{\gamma}_1 & 0 & \gamma_2 \lambda + \tilde{\gamma}_2 \\ 0 & i \sin \theta + 1 & 0 \end{pmatrix}. \end{aligned}$$

Let $\vartheta_1(x, \lambda), \vartheta_2(x, \lambda), \vartheta_3(x, \lambda)$ be the system of linearly independent fundamental solutions of Eq. (1) satisfy initial condition

$$\Theta(x, \lambda) = \begin{pmatrix} \vartheta_1^{[0]}(x, \lambda) & \vartheta_2^{[0]}(x, \lambda) & \vartheta_3^{[0]}(x, \lambda) \\ \vartheta_1^{[1]}(x, \lambda) & \vartheta_2^{[1]}(x, \lambda) & \vartheta_3^{[1]}(x, \lambda) \\ \vartheta_1^{[2]}(x, \lambda) & \vartheta_2^{[2]}(x, \lambda) & \vartheta_3^{[2]}(x, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

Lemma 2.2 $\lambda \in \mathbb{C}$ is an eigenvalue of BVP (1)–(4) (also of the operator T) if and only if

$$\Delta(\lambda) = \det[A_\lambda + B_\lambda \Theta(b, \lambda)] = 0.$$

Proof The proof is routine by substituting the general solution consisting of the independent solutions $\vartheta_1(x, \lambda), \vartheta_2(x, \lambda), \vartheta_3(x, \lambda)$ into boundary conditions (2)–(4),

then we can get a system of homogeneous linear equations, using ordinary differential equation theory, we can get the conclusion. $\square$

3 The Banach Space

For purpose of investigating the continuity of eigenvalues, we introduce the following Banach space.

Consider the Banach space

$$\mathfrak{B} = L^1[a, b] \oplus L^1[a, b] \oplus L^1[a, b] \oplus \mathbb{R}^9$$

with the norm

$$\|\omega\| = \int_a^b (|p_0| + |p_1| + |w|)dx + |\Delta_1| + |\Delta_2| + |\theta|$$

for any $W = (p_0, p_1, w, \Delta_1, \Delta_2, \theta) \in \mathfrak{B}$, where

$$|\Delta_1| = |\alpha_1| + |\alpha_2| + |\tilde{\alpha}_1| + |\tilde{\alpha}_2|, \quad |\Delta_2| = |\gamma_1| + |\gamma_2| + |\tilde{\gamma}_1| + |\tilde{\gamma}_2|.$$

Let

$$\Omega = \{\omega \in \mathfrak{B} : (2.5), (2.6) \text{ hold, and } \theta \in (0, \pi)\}.$$

Lemma 3.1 *The solution $f = f(x, c_0, d_1, d_2, d_3, p_0, p_1, w)$ of Eq. (1) satisfying the conditions*

$$f(c_0, \lambda) = d_1, f^{[1]}(c_0, \lambda) = d_2, f^{[2]}(c_0, \lambda) = d_3, c_0 \in [a, b]$$

is continuous with respect to all its variables.

Proof By transferring the Eq. (1) to a first-order system, and based on the existence and uniqueness of solutions and Theorem 2.7 in [35], the conclusion holds. $\square$

Theorem 3.2 *Let $\omega^* = (p_0^*, p_1^*, w^*, \Delta_1^*, \Delta_2^*, \theta^*) \in \Omega$, and $\lambda = \lambda(\omega)$ be an eigenvalue of BVP (1)–(4). Then for any $\varepsilon > 0$, there exists a $\delta > 0$ so that if $\omega \in \Omega$ satisfies*

$$\begin{aligned} \|\omega - \omega^*\| &= \int_a^b (|p_0 - p_0^*| + |p_1 - p_1^*| + |w - w^*|)dx \\ &\quad + |\alpha_1 - \alpha_1^*| + |\alpha_2 - \alpha_2^*| + |\tilde{\alpha}_1 - \tilde{\alpha}_1^*| + |\tilde{\alpha}_2 - \tilde{\alpha}_2^*| \\ &\quad + |\gamma_1 - \gamma_1^*| + |\gamma_2 - \gamma_2^*| + |\tilde{\gamma}_1 - \tilde{\gamma}_1^*| + |\tilde{\gamma}_2 - \tilde{\gamma}_2^*| + |\theta - \theta^*| \\ &< \delta, \end{aligned}$$

then

$$|\lambda(\omega) - \lambda(\omega^*)| < \varepsilon.$$

Proof By Lemma 2.2, one can see that λ is an eigenvalue of BVP (1)–(4) if and only if $\Delta(\lambda) = 0$. By the continuous dependence of solutions on the problem ([36], Theorem 2.4.1), $\Delta(\lambda)$ is an entire function with respect to λ and is continuous at $\omega \in \mathfrak{B}$. We can also verified that $\Delta(\lambda)$ is not a constant. According to the continuity theorem for analytic functions with isolated zeros, the conclusion holds. $\square$

Definition 1 We call $F = (f(x), f_1, f_2)^T \in H$ is an normalized eigenvector of the operator T , where $f_1 = \alpha_1 f(a) - \alpha_2 f^{[2]}(a)$, $f_2 = \gamma_1 f(b) + \gamma_2 f^{[2]}(b)$, if the eigenvector $F = (f(x), f_1, f_2)^T$ of T satisfies

$$\begin{aligned}\|F\|^2 &= \langle (f(x), f_1, f_2)^T, (f(x), f_1, f_2)^T \rangle \\ &= \int_a^b |f(x)|^2 w dx + \frac{1}{\rho_1} |f_1|^2 + \frac{1}{\rho_2} |f_2|^2 \\ &= 1.\end{aligned}$$

The continuity of eigenvectors of the operator T is stated as follows.

Theorem 3.3 Let $\lambda(\omega)$ ($\omega \in \Omega$) be an eigenvalue with multiplicity n ($n = 1, 2, 3$) for all ω in some neighborhoods of ω^* in Ω . Let $F_k(x, \omega^*) = (f_k(x, \omega^*), f_{k1}(\omega^*), f_{k2}(\omega^*))^T$ be the normalized eigenvectors of $\lambda(\omega^*)$. Then, there exist n linearly independent normalized eigenvectors $F_k(x, \omega) = (f_k(x, \omega), f_{k1}(\omega), f_{k2}(\omega))^T$ of $\lambda(\omega)$ so that

$$\begin{aligned}f_k(x, \omega) &\rightarrow f_k(x, \omega^*), \quad f_k^{[1]}(x, \omega) \rightarrow f_k^{[1]}(x, \omega^*), \quad f_k^{[2]}(x, \omega) \rightarrow f_k^{[2]}(x, \omega^*) \\ f_{k1}(\omega) &\rightarrow f_{k1}(\omega^*), \quad f_{k2}(\omega) \rightarrow f_{k2}(\omega^*), \quad k = 1, \dots, n, \text{ as } \omega \rightarrow \omega^* \quad \text{in } \Omega\end{aligned}$$

both uniformly on $[a, b]$.

Proof It can be proved by using similar methods in [17], with the aid of Lemma 3.1, Theorem 3.2 as well as Theorem 3.2 in [13]. $\square$

4 Derivative Formulas of Eigenvalues

We give the differentiability of eigenvalues concerning some data, especially the boundary condition parameter matrix in this section.

Definition 2 ^[12] Let $\mathbf{X}, \mathbf{Y}$ be Banach spaces. We call the map $\Gamma : \mathbf{X} \rightarrow \mathbf{Y}$ is Fréchet differentiable at $x \in \mathbf{X}$ provided that a bounded linear operator $d\Gamma_x : \mathbf{X} \rightarrow \mathbf{Y}$ exists, such that for $\tau \in \mathbf{X}$,

$$|\Gamma(x + \tau) - \Gamma(x) - d\Gamma_x(\tau)| = o(\tau), \quad \tau \rightarrow 0.$$

Theorem 4.1 Let $\omega = (p_0, p_1, w, \Delta_1, \Delta_2, \theta) \in \Omega$, $\lambda = \lambda(\omega)$ be an eigenvalue of BVP (1)–(4), $F(\omega) = (f(x, \omega), f_1(\omega), f_2(\omega))^T \in \mathcal{H}$ be the corresponding eigenvector. E is the identity matrix, R is 2×2 real-valued matrix. Assume that for all fixed elements

of ω except one, the geometric multiplicity of $\lambda(\omega)$ is invariant in some neighborhoods $\mathcal{M} \subset \Omega$. Then the following differential expressions hold.

- (1) Let all the elements of ω be fixed except θ and $\lambda = \lambda(\theta)$. Then λ is differentiable and satisfies

$$\lambda'(\theta) = \frac{2 \cos \theta}{1 + \sin^2 \theta} |f^{[1]}(a)|^2 = \frac{2 \cos \theta}{1 + \sin^2 \theta} |f^{[1]}(b)|^2.$$

- (2) Let all the elements of ω be fixed except α_1 and $\lambda = \lambda(\alpha_1)$. Then λ is differentiable and satisfies

$$\lambda'(\alpha_1) = \frac{\lambda}{\alpha_2 \lambda + \tilde{\alpha}_2} |f(a)|^2,$$

where $\alpha_2 \lambda + \tilde{\alpha}_2 \neq 0$.

- (3) Let all the elements of ω be fixed except $\tilde{\alpha}_1$ and $\lambda = \lambda(\tilde{\alpha}_1)$. Then λ is differentiable and satisfies

$$\lambda'(\tilde{\alpha}_1) = \frac{1}{\alpha_2 \lambda + \tilde{\alpha}_2} |f(a)|^2,$$

where $\alpha_2 \lambda + \tilde{\alpha}_2 \neq 0$.

- (4) Let all the elements of ω be fixed except α_2 and $\lambda = \lambda(\alpha_2)$. Then λ is differentiable and satisfies

$$\lambda'(\alpha_2) = -\frac{\lambda}{\alpha_1 \lambda + \tilde{\alpha}_1} |f^{[2]}(a)|^2,$$

where $\alpha_1 \lambda + \tilde{\alpha}_1 \neq 0$.

- (5) Let all the elements of ω be fixed except $\tilde{\alpha}_2$ and $\lambda = \lambda(\tilde{\alpha}_2)$. Then λ is differentiable and satisfies

$$\lambda'(\tilde{\alpha}_2) = -\frac{1}{\alpha_1 \lambda + \tilde{\alpha}_1} |f^{[2]}(a)|^2,$$

where $\alpha_1 \lambda + \tilde{\alpha}_1 \neq 0$.

- (6) Let all the elements of ω be fixed except γ_1 and $\lambda = \lambda(\gamma_1)$. Then λ is differentiable and satisfies

$$\lambda'(\gamma_1) = \frac{\lambda}{\gamma_2 \lambda + \tilde{\gamma}_2} |f(b)|^2,$$

where $\gamma_2 \lambda + \tilde{\gamma}_2 \neq 0$.

- (7) Let all the elements of ω be fixed except $\tilde{\gamma}_1$ and $\lambda = \lambda(\tilde{\gamma}_1)$. Then λ is differentiable and satisfies

$$\lambda'(\tilde{\gamma}_1) = \frac{1}{\gamma_2 \lambda + \tilde{\gamma}_2} |f(b)|^2,$$

where $\gamma_2 \lambda + \tilde{\gamma}_2 \neq 0$.

- (8) Let all the elements of ω be fixed except γ_2 and $\lambda = \lambda(\gamma_2)$. Then λ is differentiable and satisfies

$$\lambda'(\gamma_2) = -\frac{\lambda}{\gamma_1 \lambda + \tilde{\gamma}_1} |f^{[2]}(b)|^2,$$

where $\gamma_1 \lambda + \tilde{\gamma}_1 \neq 0$.

- (9) Let all the elements of ω be fixed except $\tilde{\gamma}_2$ and $\lambda = \lambda(\tilde{\gamma}_2)$. Then λ is differentiable and satisfies

$$\lambda'(\tilde{\gamma}_2) = -\frac{1}{\gamma_1 \lambda + \tilde{\gamma}_1} |f^{[2]}(b)|^2,$$

where $\gamma_1 \lambda + \tilde{\gamma}_1 \neq 0$.

- (10) Let all the elements of ω be fixed except the boundary condition parameter matrix $K = \begin{pmatrix} \alpha_1 & \tilde{\alpha}_1 \\ \alpha_2 & \tilde{\alpha}_2 \end{pmatrix}$ and $\lambda = \lambda(K)$. Then for all R satisfying $\det[K + R] = -\rho_1$, the Fréchet derivative of λ with respect to K is formulated as

$$d\lambda_K(R) = (-f(a), f^{[2]}(a)) [E - (K + R)K^{-1}] (\overline{f^{[2]}(a)}, \overline{f(a)})^T.$$

- (11) Let all the elements of ω be fixed except the boundary condition parameter matrix $\tilde{K} = \begin{pmatrix} \gamma_1 & \tilde{\gamma}_1 \\ \gamma_2 & \tilde{\gamma}_2 \end{pmatrix}$ and $\lambda = \lambda(\tilde{K})$. Then for all R satisfying $\det[\tilde{K} + R] = -\rho_2$, the Fréchet derivative of λ with respect to $\tilde{K}$ is formulated as

$$d\lambda_{\tilde{K}}(R) = (f(b), f^{[2]}(b)) [E - (\tilde{K} + R)\tilde{K}^{-1}] (\overline{f^{[2]}(b)}, -\overline{f(b)})^T.$$

- (12) Let all the elements of ω be fixed except p_0 and $\lambda = \lambda(p_0)$. Then the Fréchet derivative of λ with respect to p_0 is formulated as

$$d\lambda_{p_0}(\tau) = \int_a^b \tau |f^{[1]}|^2 dx, \quad \tau \in L^1[a, b].$$

- (13) Let all the elements of ω be fixed except p_1 and $\lambda = \lambda(p_1)$. Then the Fréchet derivative of λ with respect to p_1 is formulated as

$$d\lambda_{p_1}(\tau) = \int_a^b \tau |f|^2 dx, \quad \tau \in L^1[a, b].$$

(14) Let all the elements of ω be fixed except w and $\lambda = \lambda(w)$. Then the Fréchet derivative of λ with respect to w is formulated as

$$d\lambda_w(\tau) = \lambda_w \int_a^b \tau |f|^2 dx, \quad \tau \in L^1[a, b].$$

Proof Let all the elements of $\omega \in \Omega$ be fixed except one. For sufficiently small $\varepsilon > 0$, when $\|\omega - \omega^*\| < \varepsilon$, let $\lambda(\omega)$ be an eigenvalue of BVP (1)–(4) satisfying Theorem 3.2. For all the above six cases, the $\lambda(\omega)$ is replaced by $\lambda(\theta + \Delta\theta)$, $\lambda(\alpha_1 + \Delta\alpha_1)$, $\lambda(\tilde{\alpha}_1 + \Delta\tilde{\alpha}_1)$, $\lambda(\alpha_2 + \Delta\alpha_2)$, $\lambda(\tilde{\alpha}_2 + \Delta\tilde{\alpha}_2)$, $\lambda(\gamma_1 + \Delta\gamma_1)$, $\lambda(\tilde{\gamma}_1 + \Delta\tilde{\gamma}_1)$, $\lambda(\gamma_2 + \Delta\gamma_2)$, $\lambda(\tilde{\gamma}_2 + \Delta\tilde{\gamma}_2)$, $\lambda(K + R)$, $\lambda(\tilde{K} + R)$, $\lambda(p_0 + \eta)$, $\lambda(p_1 + \eta)$, $\lambda(w + \eta)$, respectively.

(1) Let all the elements of ω be fixed except θ . We adopt the following notations:

$$\begin{aligned} \mathcal{B}_1(f) &= \alpha_1 f(a) - \alpha_2 f^{[2]}(a), & \mathcal{B}_2(f) &= \gamma_1 f(b) + \gamma_2 f^{[2]}(b), \\ \mathcal{D}_1(f) &= \tilde{\alpha}_2 f^{[2]}(a) - \tilde{\alpha}_1 f(a), & \mathcal{D}_2(f) &= -[\tilde{\gamma}_1 f(b) + \tilde{\gamma}_2 f^{[2]}(b)]. \end{aligned}$$

Let

$$\begin{aligned} F(x, \theta) &= (f(x, \theta), f_1(\theta), f_2(\theta))^T, \\ G(x, \theta) &= (g(x, \theta), g_1(\theta), g_2(\theta))^T \end{aligned}$$

be the normalized eigenvectors corresponding to $\lambda(\theta)$ and $\lambda(\theta + \Delta\theta)$ respectively. By the self-adjointness of the operator T and the boundary condition (4), we have

$$\begin{aligned} &[\lambda(\theta + \Delta\theta) - \lambda(\theta)] \langle G, F \rangle \\ &= \langle \lambda(\theta + \Delta\theta)G, F \rangle - \langle G, \lambda(\theta)F \rangle \\ &= [g, \bar{f}]_a^b + \frac{1}{\rho_1} [\mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)}] + \frac{1}{\rho_2} [\mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)}] \\ &= ig^{[1]}(b) \overline{f^{[1]}(b)} - ig^{[1]}(a) \overline{f^{[1]}(a)} \\ &= i \left[\frac{i + \sin(\theta + \Delta\theta)}{1 + i \sin(\theta + \Delta\theta)} \frac{\sin \theta - i}{1 - i \sin \theta} - 1 \right] g^{[1]}(a) \overline{f^{[1]}(a)} \end{aligned} \quad (11)$$

Dividing both sides of $\Delta\theta$, and taking the limit as $\Delta\theta \rightarrow 0$, we have

$$\lambda'(\theta) = \frac{2 \cos \theta}{1 + \sin^2 \theta} |f^{[1]}(a)|^2 \quad (12)$$

by Theorem 3.3. Using the boundary condition (4), we have $|f^{[1]}(a)|^2 = |f^{[1]}(b)|^2$. Hence, (1) holds.

(2) Let all the elements of ω be fixed except α_1 , and

$$F(x, \alpha_1) = (f(x, \alpha_1), f_1(\alpha_1), f_2(\alpha_1))^T,$$

$$G(x, \alpha_1) = (g(x, \alpha_1), g_1(\alpha_1), g_2(\alpha_1))^T$$

be the normalized eigenvectors corresponding to $\lambda(\alpha_1)$ and $\lambda(\alpha_1 + \Delta\alpha_1)$, respectively. By the self-adjointness of the operator T and the boundary condition (2), (4), we have

$$\begin{aligned} & [\lambda(\alpha_1 + \Delta\alpha_1) - \lambda(\alpha_1)] \langle G, F \rangle \\ &= \langle \lambda(\alpha_1 + \Delta\alpha_1)G, F \rangle - \langle G, \lambda(\alpha_1)F \rangle \\ &= [g, \bar{f}]_a^b + \frac{1}{\rho_1} [\mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)}] + \frac{1}{\rho_2} [\mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)}] \\ &= g^{[2]}(a) \overline{f(a)} - g(a) \overline{f^{[2]}(a)} + \frac{1}{\rho_1} [\tilde{\alpha}_2 g^{[2]}(a) - \tilde{\alpha}_1 g(a)] [\alpha_1 \overline{f(a)} - \alpha_2 \overline{f^{[2]}(a)}] \\ &\quad - \frac{1}{\rho_1} [(\alpha_1 + \Delta\alpha_1)g(a) - \alpha_2 g^{[2]}(a)] [\tilde{\alpha}_2 \overline{f^{[2]}(a)} - \tilde{\alpha}_1 \overline{f(a)}] \\ &= \frac{\Delta\alpha_1}{\rho_1} (\tilde{\alpha}_1 g(a) \overline{f(a)} - \tilde{\alpha}_2 g(a) \overline{f^{[2]}(a)}) \\ &= \frac{\Delta\alpha_1}{\rho_1} (\tilde{\alpha}_1 - \tilde{\alpha}_2 \frac{\alpha_1 \lambda + \tilde{\alpha}_1}{\alpha_2 \lambda + \tilde{\alpha}_2}) g(a) \overline{f(a)} \\ &= \frac{\lambda \Delta\alpha_1}{\alpha_2 \lambda + \tilde{\alpha}_2} g(a) \overline{f(a)} \end{aligned} \tag{13}$$

Dividing both sides of $\Delta\alpha_1$, and taking the limit as $\Delta\alpha_1 \rightarrow 0$, we have

$$\lambda'(\alpha_1) = \frac{\lambda}{\alpha_2 \lambda + \tilde{\alpha}_2} |f(a)|^2 \tag{14}$$

by Theorem 3.3. Using the same methods of (2), one can prove that (3), (4) and (5) are also true.

(6) Let all the elements of ω be fixed except γ_1 , and

$$F(x, \gamma_1) = (f(x, \gamma_1), f_1(\gamma_1), f_2(\gamma_1))^T,$$

$$G(x, \gamma_1) = (g(x, \gamma_1), g_1(\gamma_1), g_2(\gamma_1))^T$$

be the normalized eigenvectors corresponding to $\lambda(\gamma_1)$ and $\lambda(\gamma_1 + \Delta\gamma_1)$, respectively. Then By the self-adjointness of the operator T and the boundary condition (3), (4), we have

$$\begin{aligned}
& [\lambda(\gamma_1 + \Delta\gamma_1) - \lambda(\gamma_1)] \langle G, F \rangle \\
&= \langle \lambda(\gamma_1 + \Delta\gamma_1)G, F \rangle - \langle G, \lambda(\gamma_1)F \rangle \\
&= [g, \bar{f}]_a^b + \frac{1}{\rho_1} [\mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)}] + \frac{1}{\rho_2} [\mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)}] \\
&= g(b) \overline{f^{[2]}(b)} - g^{[2]}(b) \overline{f(b)} - \frac{1}{\rho_2} [\tilde{\gamma}_1 g(b) + \tilde{\gamma}_2 g^{[2]}(b)] [\gamma_1 \overline{f(b)} + \gamma_2 \overline{f^{[2]}(b)}] \\
&\quad + \frac{1}{\rho_2} [(\gamma_1 + \Delta\gamma_1)g(b) + \gamma_2 g^{[2]}(b)] [\tilde{\gamma}_1 \overline{f(b)} + \tilde{\gamma}_2 \overline{f^{[2]}(b)}] \\
&= \frac{\Delta\gamma_1}{\rho_2} (\tilde{\gamma}_1 g(b) \overline{f(b)} + \tilde{\gamma}_2 g(b) \overline{f^{[2]}(b)}) \\
&= \frac{\Delta\gamma_1}{\rho_2} (\tilde{\gamma}_1 - \tilde{\gamma}_2 \frac{\gamma_1 \lambda + \tilde{\gamma}_1}{\gamma_2 \lambda + \tilde{\gamma}_2}) g(a) \overline{f(a)} \\
&= \frac{\lambda \Delta\gamma_1}{\gamma_2 \lambda + \tilde{\gamma}_2} g(b) \overline{f(b)}
\end{aligned} \tag{15}$$

Dividing both sides of $\Delta\gamma_1$, and taking the limit as $\Delta\gamma_1 \rightarrow 0$, we have

$$\lambda'(\gamma_1) = \frac{\lambda}{\gamma_2 \lambda + \tilde{\gamma}_2} |f(b)|^2 \tag{16}$$

by Theorem 3.3. Using the similar methods of (6), one can prove that (7), (8) and (9) are also true.

(10) Let all the elements of ω be fixed except K . Let $K + R = \begin{pmatrix} \alpha_{1R} & \tilde{\alpha}_{1R} \\ \alpha_{2R} & \tilde{\alpha}_{2R} \end{pmatrix}$, with $\det(K + R) = -\rho_1$, and

$$\begin{aligned}
F(x, K) &= (f(x, K), f_1(K), f_2(K))^T, \\
G(x, K) &= (g(x, K), g_1(K), g_2(K))^T
\end{aligned}$$

be the normalized eigenvectors corresponding to $\lambda(K)$ and $\lambda(K + R)$, respectively. Then

$$\begin{aligned}
\lambda(K) [\alpha_1 f(a) - \alpha_2 f^{[2]}(a)] &= \tilde{\alpha}_2 f^{[2]}(a) - \tilde{\alpha}_1 f(a), \\
\lambda(K + R) [\alpha_{1R} \overline{g(a)} - \alpha_{2R} \overline{g^{[2]}(a)}] &= \tilde{\alpha}_{2R} \overline{g^{[2]}(a)} - \tilde{\alpha}_{1R} \overline{g(a)}
\end{aligned}$$

by the boundary condition (2). Using the boundary condition (2)–(4), simple calculation yields

$$\begin{aligned}
& [\lambda(K+R) - \lambda(K)]\langle G, F \rangle \\
&= \langle \lambda(K+R)G, F \rangle - \langle G, \lambda(K)F \rangle \\
&= [g, \bar{f}]_a^b + \frac{1}{\rho_1} \mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \frac{1}{\rho_1} \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)} + \frac{1}{\rho_2} \mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \frac{1}{\rho_2} \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)} \\
&= g^{[2]}(a) \overline{f(a)} - g(a) \overline{f^{[2]}(a)} + \frac{1}{\rho_1} \mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \frac{1}{\rho_1} \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)} \\
&= g^{[2]}(a) \overline{f(a)} - g(a) \overline{f^{[2]}(a)} + \frac{1}{\rho_1} [\tilde{\alpha}_{2R} g^{[2]}(a) - \tilde{\alpha}_{1R} g(a)] [\alpha_1 \overline{f(a)} - \alpha_2 \overline{f^{[2]}(a)}] \\
&\quad - \frac{1}{\rho_1} [\alpha_{1R} g(a) - \alpha_{2R} g^{[2]}(a)] [\tilde{\alpha}_2 \overline{f^{[2]}(a)} - \tilde{\alpha}_1 \overline{f(a)}] \\
&= (-g(a), g^{[2]}(a)) (\overline{f^{[2]}(a)}, \overline{f(a)})^T \\
&\quad + \frac{1}{\rho_1} (-g(a), g^{[2]}(a)) (\tilde{\alpha}_{1R}, \tilde{\alpha}_{2R})^T (-\alpha_2, \alpha_1) (\overline{f^{[2]}(a)}, \overline{f(a)})^T \\
&\quad - \frac{1}{\rho_1} (-g(a), g^{[2]}(a)) (-\alpha_{1R}, -\alpha_{2R})^T (\tilde{\alpha}_2, -\tilde{\alpha}_1) (\overline{f^{[2]}(a)}, \overline{f(a)})^T \\
&= (-g(a), g^{[2]}(a)) \left[E + \frac{1}{\rho_1} (\tilde{\alpha}_{1R}, \tilde{\alpha}_{2R})^T (-\alpha_2, \alpha_1) \right. \\
&\quad \left. - \frac{1}{\rho_1} (-\alpha_{1R}, -\alpha_{2R})^T (\tilde{\alpha}_2, -\tilde{\alpha}_1) \right] (\overline{f^{[2]}(a)}, \overline{f(a)})^T \\
&= (-g(a), g^{[2]}(a)) \left[E + \frac{1}{\rho_1} \begin{pmatrix} \alpha_{1R} \tilde{\alpha}_2 - \tilde{\alpha}_{1R} \alpha_2 & \tilde{\alpha}_{1R} \alpha_1 - \alpha_{1R} \tilde{\alpha}_1 \\ \alpha_{2R} \tilde{\alpha}_2 - \tilde{\alpha}_{2R} \alpha_2 & \tilde{\alpha}_{2R} \alpha_1 - \alpha_{2R} \tilde{\alpha}_1 \end{pmatrix} \right] (\overline{f^{[2]}(a)}, \overline{f(a)})^T \\
&= (-g(a), g^{[2]}(a)) [E - (K+R)K^{-1}] (\overline{f^{[2]}(a)}, \overline{f(a)})^T.
\end{aligned}$$

Let $R \rightarrow 0$, we get that

$$[\lambda(K+R) - \lambda(K)](1 + o(1)) = (-f(a), f^{[2]}(a)) [E - (K+R)K^{-1}] (\overline{f^{[2]}(a)}, \overline{f(a)})^T,$$

that is,

$$\lambda(K+R) - \lambda(K) = (-f(a), f^{[2]}(a)) [E - (K+R)K^{-1}] (\overline{f^{[2]}(a)}, \overline{f(a)})^T + o(R).$$

Hence (10) holds. The proof of (11) is similar to this.

(12) Let all the elements of ω be fixed except p_0 , and

$$\begin{aligned}
F(x, p_0) &= (f(x, p_0), f_1(p_0), f_2(p_0))^T, \\
G(x, p_0) &= (g(x, p_0), g_1(p_0), g_2(p_0))^T
\end{aligned}$$

be the normalized eigenvectors corresponding to $\lambda(p_0)$ and $\lambda(p_0 + \tau)$. Then we have

$$\begin{aligned}
& [\lambda(p_0 + \tau) - \lambda(p_0)] \langle G, F \rangle \\
&= [\lambda(p_0 + \tau) - \lambda(p_0)] \left(\int_a^b g \bar{f} w dx + \frac{1}{\rho_1} g_1 \bar{f}_1 + \frac{1}{\rho_2} g_2 \bar{f}_2 \right) \\
&= \int_a^b \ell(g) \bar{f} w dx - \int_a^b g \overline{\ell(f)} w dx + [\lambda(p_0 + \tau) - \lambda(p_0)] \left(\frac{1}{\rho_1} g_1 \bar{f}_1 + \frac{1}{\rho_2} g_2 \bar{f}_2 \right) \\
&= \int_a^b [(-g^{[2]})' + i q_1 g' + p_1 g] \bar{f} dx - \int_a^b \overline{g[(-f^{[2]})' + i q_1 f' + p_1 f]} dx \\
&\quad + \frac{1}{\rho_1} \mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \frac{1}{\rho_1} \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)} + \frac{1}{\rho_2} \mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \frac{1}{\rho_2} \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)} \\
&= [g \bar{f}^{[2]} - g^{[2]} \bar{f}] \Big|_a^b + \int_a^b [i q_0 (q_0 g')' + (p_0 + \tau) g' - i q_1 g] \bar{f}' dx \\
&\quad - \int_a^b g' [-i q_0 (q_0 \bar{f}')' + p_0 \bar{f}' + i q_1 \bar{f}] dx + i \int_a^b [q_1 g' \bar{f} + q_1 g \bar{f}'] dx \\
&\quad + \frac{1}{\rho_1} \mathcal{D}_1(g) \overline{\mathcal{B}_1(f)} - \frac{1}{\rho_1} \mathcal{B}_1(g) \overline{\mathcal{D}_1(f)} + \frac{1}{\rho_2} \mathcal{D}_2(g) \overline{\mathcal{B}_2(f)} - \frac{1}{\rho_2} \mathcal{B}_2(g) \overline{\mathcal{D}_2(f)} \\
&= \int_a^b (p_0 + \tau) g' \bar{f}' dx - \int_a^b p_0 g' \bar{f}' dx \\
&= \int_a^b \tau g' \bar{f}' dx.
\end{aligned}$$

It follows from the above results that (12) holds. Using the similar methods of (12), one can prove that (13) and (14) are also true. $\square$

5 Conclusion

In the present paper, we study the dependence of eigenvalues of third-order differential operators with eigenparameter dependent boundary conditions. It is proved that the eigenvalues of such problem depend not only continuously but also smoothly on the parameter. The dependence of eigenvalues with respect to the data plays an important role in the theory of differential operators. It gives theoretical support for the numerical computation of eigenvalues. Moreover, the properties of monotonicity of eigenvalues with respect to the parameters can be obtained by the derivatives of eigenvalues on the given parameter.

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