On Initial Data Problem for a Periodic Two-Component b -Family System

Xiaohuan Wang¹

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Abstract

This paper is concerned with the initial data problem for a periodic two-component b -family system. We prove that the solution map of the Cauchy problem of the b -family system is not uniformly continuous in $H^s(\mathbb{S})$, $s > 5/2$.

Keywords Non-uniform dependence · b -Family system · Energy estimates · Initial value problem

Mathematics Subject Classification 35G25 · 35B30 · 35L05

Abbreviations

$\mathbb{S}$ $(0, 1)$;

$f \gtrsim g$ Means that there exists a positive constant C such that $f \geq Cg$ (similarly for $f \lesssim g$).

1 Introduction

In this paper, we study the periodic case of following nonlinear dispersive system [1]

$$\begin{cases} m_t = um_x + k_1 mu_x + k_2 \rho \rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t = k_3(\rho u)_x, & t > 0, x \in \mathbb{R}, \\ m(0, x) = m_0(x), \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \quad (1)$$

✉ Xiaohuan Wang
003304@nuist.edu.cn

¹ College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, People's Republic of China

where $m = u - u_{xx}$. Lin–Yin [2] obtained the local well-posedness of system (1). Zou [3] considered the system (1) and obtained some properties of the solutions. (1) is also considered by Zhu and Xu [4] in Besov space.

If $\rho \equiv 0$, system (1) will become a number of the b -family equation

$$u_t - u_{xxx} + c_0 u_x + (b + 1)uu_x + \sigma u_{xxx} = bu_x + uu_{xxx}, \quad (2)$$

where c_0, b and σ are constants. For any $b \neq -1$, (2) can be derived as the family of asymptotically equivalent shallow water wave equation that emerge at quadratic order accuracy by an appropriate Kodama transformation, see [5, 6] for the details. For $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated, see [5, 6].

If $b = 2$ and $\sigma = 0$, (2) will become the Camassa–Holm equation [7]. The authors in [8–11] studied the Cauchy problem of the Camassa–Holm equation. If $b = 3$ and $c_0 = \sigma = 0$, similarly, (2) will become the Degasperis–Procesi equation [12], which has been studied by [2, 13].

If $\rho \neq 0$ and if $k_1 = 2$, the system (1) will become the two-component Camassa–Holm system [14, 15]

$$\begin{cases} m_t + um_x + 2mu_x + \sigma\rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \end{cases} \quad (3)$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. When $\sigma = 1$, local well-posedness of system (3) was established by [15, 16].

As for Camassa–Holm equation, it is more important to study the properties of solutions, such as the persistence properties and unique continuation of solutions, non-uniform dependence on initial data, see [17–23] for details. In this paper, we will consider the non-uniform dependence on initial data to periodic system (1), that is

$$\begin{cases} m_t = um_x + k_1 mu_x + k_2 \rho \rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t = k_3 (\rho u)_x, & t > 0, x \in \mathbb{R}, \\ m(0, x) = m_0(x), \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), \rho(t, x + 1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}. \end{cases} \quad (4)$$

Denote $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, and $\bar{\rho}$ follows from the pointwise density ρ by convolution against the Green's function of the Helmholtz operator $1 - \partial_x^2$ (cf. [14]). If $\gamma = \bar{\rho} - \bar{\rho}_0$, then system (1) is called as modified Camassa–Holm system. We remark that there is significant difference between system (1) and (3) with $\rho = \gamma - \gamma_{xx}$. It is easy to see that when $\rho = \gamma - \gamma_{xx}$, there are some similar properties between the two equations in system (3). Thus the proof of non-uniform dependence on initial data to system (3) with $\rho = \gamma - \gamma_{xx}$ is similar to the signal equation, for example, Camassa–Holm equation. But in system (1), ρ and u have different properties, see Theorem 2.1. This needs construct different asymptotic solution, see Sect. 3. On the other hand, in order to obtain the non-uniform dependence on initial data to system (1),

we must modify the earlier method used in [19, 22]. In papers [19, 22], they estimated the H^1 -norm of the difference between the approximate and actual solutions, but it does not work for system (1). It is easy to see that the method used in [19, 22] is available only when $k_1 = 2$ and $k_2 = k_3$. In this paper, there is not any restriction about the parameters k_1, k_2 and k_3 . Comparing with [24], there are some difference. Firstly, in [24], we considered the whole space, which is different from the periodic case. And so the approximate solutions constructed in this paper is different from [24]. Secondly, we take different norm from [24], see [24, Theorems 3.1 and 4.1]. Lastly, the model considered in [24] is a special case of (1).

This paper is organized as follows. In Sect. 2, we obtain the well-posedness of system (1) by using Kato's theory and then use it to prove the basic energy estimate from which we derive a lower bound for the lifespan of the solution as well as an estimate of the $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ norm of the solution $(u(t, x), \rho(t, x))$ in terms of $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ norm of the initial data (u_0, ρ_0) , where $\mathbb{S} = (0, 1)$. In Sect. 3, we construct approximate solutions, compute the error and estimate the H^μ -norm of this error. In Sect. 4, we estimate the difference between approximate and actual solutions, where the exact solution is a solution to system (1) with initial data given by the approximate solutions evaluated at time zero. The non-uniform dependence on initial data for system (1) is established in Sect. 5 by construct two sequences of solutions to (1) in a bounded subset of the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, whose distance at the initial time is converging to zero while at any later time it is bounded below by a positive constant.

2 Local Well-Posedness

In this section we apply the Kato's theory [25] to establish the local well-posedness of (1). Set $\Lambda = (1 - \partial_x^2)^{1/2}$ and $G(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of x . Then the operator Λ^{-2} can be expressed as

$$\Lambda^{-2}f(x) = (G * f)(x) = \frac{1}{2} \int_0^1 \frac{\cosh(x - y - [x - y] - \frac{1}{2})}{\sinh(\frac{1}{2})} f(y) dy, \quad f \in L^2(\mathbb{S}),$$

where $G(x)$ is the Green's function associated with Λ^{-2} . Hence (4) is equivalent to the following system

$$\begin{cases} u_t - uu_x = \partial_x \Lambda^{-2} \left(\frac{k_1}{2} u^2 + \frac{3-k_1}{2} u_x^2 + \frac{k_2}{2} \rho^2 \right), & t > 0, x \in \mathbb{R}, \\ \rho_t - k_3 u \rho_x = k_3 u_x \rho, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad \rho(t, x+1) = \rho(t, x), & t \geq 0, x \in \mathbb{R}. \end{cases}$$

In the rest of this paper, for convenience, we rewrite the above system as

$$\begin{cases} u_t - uu_x = \partial_x \Lambda^{-2} \left(\frac{k_1}{2} u^2 + \frac{3-k_1}{2} u_x^2 + \frac{k_2}{2} \rho^2 \right), & t > 0, x \in \mathbb{S}, \\ \rho_t - k_3 u \rho_x = k_3 u_x \rho, & t > 0, x \in \mathbb{S}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{S}. \end{cases} \quad (5)$$

Let

$$z := \begin{pmatrix} u \\ \rho \end{pmatrix}, \quad A(z) := \begin{pmatrix} -u \partial_x & 0 \\ 0 & -k_3 u \partial_x \end{pmatrix},$$

$$f(z) := \begin{pmatrix} \partial_x \Lambda^{-2} \left(\frac{k_1}{2} u^2 + \frac{3-k_1}{2} u_x^2 + \frac{k_2}{2} \rho^2 \right) \\ k_3 u_x \rho \end{pmatrix},$$

then the first two equations of system (5) can be replaced by

$$\frac{dz}{dt} + A(z)z = f(z).$$

One can mimic the method of [2] to verify that all the assumptions of the Kato's theory holds for system (5). Thus we have the following result.

Theorem 2.1 *Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$. Then there exists a maximal existence time $T = T(\|z_0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $z = (u, \rho)$ to system (5) such that*

$$z = z(\cdot, z_0) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$z_0 \rightarrow z(\cdot, z_0) : H^s \times H^{s-1} \rightarrow C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$$

is continuous.

Next, we want to get the lower bounded of the maximal existence time T . Meanwhile, we will prove that there exists a positive constant T_0 such that the H^s -norm of the solution will be dominated by the H^s -norm of the initial data. The following lemmas are needed.

Lemma 2.1 [26] *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover,*

$$\|fg\|_{H^r} \leq C(\|f\|_\infty \|g\|_{H^r} + \|f\|_{H^r} \|g\|_\infty),$$

where C is a positive constant depending only on r .

Lemma 2.2 [26] *If $r > 0$, then*

$$\|[\Lambda^r, f]g\|_2 \leq C(\|f_x\|_\infty \|\Lambda^{r-1}g\|_2 + \|\Lambda^r f\|_2 \|g\|_\infty),$$

where C is a positive constant depending only on r .

Due to the proof of the following is similar to that in [23], we only give the outline of proof.

Theorem 2.2 *Let $s > \frac{5}{2}$. If $z = (u, \rho)$ is a solution of system (5) with initial data z_0 described in Theorem 2.1, then the maximal existence time T satisfies*

$$T \geq T_0 := \frac{1}{2C_s \|z_0\|_{H^s \times H^{s-1}}}, \quad (6)$$

where C_s is a constant depending only on s . Also, we have

$$\|z(t)\|_{H^s \times H^{s-1}} \leq 2\|z_0\|_{H^s \times H^{s-1}}, \quad 0 \leq t \leq T_0. \quad (7)$$

Proof Similar to [23], it is not hard to get that

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|_{H^s \times H^{s-1}}^2 \leq C_s \|z(t)\|_{H^s \times H^{s-1}}^3, \quad 0 \leq t < T. \quad (8)$$

Integrating (8) from 0 to t , we have

$$\|z(t)\|_{H^s \times H^{s-1}} \leq \frac{\|z_0\|_{H^s \times H^{s-1}}}{1 - C_s \|z_0\|_{H^s \times H^{s-1}} t}.$$

The above inequality implies that $\|z(t)\|_{H^s \times H^{s-1}}$ is finite if $C_s \|z_0\|_{H^s \times H^{s-1}} t < 1$. Let $T_0 = \frac{1}{2C_s \|z_0\|_{H^s \times H^{s-1}}}$, then we get

$$\|z(t)\|_{H^s \times H^{s-1}} \leq \frac{\|z_0\|_{H^s \times H^{s-1}}}{1 - C_s \|z_0\|_{H^s \times H^{s-1}} T_0} = 2\|z_0\|_{H^s \times H^{s-1}}, \quad 0 \leq t \leq T_0.$$

The proof is complete. $\square$

Recall that $\|z(t)\|_{H^s \times H^{s-1}}^2 = \|u(t)\|_{H^s}^2 + \|\rho(t)\|_{H^{s-1}}^2$, where $z(t) = (u(t), \rho(t))$. It follows from Theorem 2.2 that

$$\|u(t)\|_{H^s}, \|\rho(t)\|_{H^{s-1}} \leq \|z(t)\|_{H^s \times H^{s-1}} \leq 2\|z_0\|_{H^s \times H^{s-1}}, \quad 0 \leq t \leq T_0. \quad (9)$$

3 Approximate Solutions

In this section we first construct a two-parameter family of approximate solutions by using a similar method to [20], then compute the error and last estimate the H^μ -norm of the error ($1 < \mu < \frac{3}{2}$). Throughout this paper, we assume that $s > 5/2$ and $1 < \mu < s - 1$.

Following [20], our approximate solutions $u^{\omega,\lambda} = u^{\omega,\lambda}(t, x)$ and $\rho^{\omega,\lambda} = \rho^{\omega,\lambda}(t, x)$ to (5) will be consist of a low frequency and a high frequency part, i.e.

$$u^{\omega,\lambda} = \omega\lambda^{-1} + \lambda^{-s} \cos(\lambda x + \omega t), \quad \rho^{\omega,\lambda} = \omega\lambda^{-1} + \lambda^{-s+1} \cos(\lambda x + k_3\omega t),$$

where ω is in a bounded set of $\mathbb{S}$ and λ is in the set of positive integers $\mathbb{Z}_+$.

Direct calculation shows that

$$\begin{aligned} u_t^{\omega,\lambda} - u^{\omega,\lambda} u_x^{\omega,\lambda} &= \frac{1}{2} \lambda^{-2s+1} \sin(2\lambda x + 2\omega t) \\ &:= E_1, \end{aligned}$$

and

$$\begin{aligned} \partial_x \Lambda^{-2} \left(\frac{k_1}{2} (u^{\omega,\lambda})^2 + \frac{3-k_1}{2} (u_x^{\omega,\lambda})^2 + \frac{k_2}{2} (\rho^{\omega,\lambda})^2 \right) \\ = -\frac{k_1}{2} \lambda^{-2s+1} \Lambda^{-2} \sin(2\lambda x + \omega t) - k_1 \omega \lambda^{-s} \Lambda^{-2} \sin(\lambda x + \omega t) + \frac{3-k_1}{2} \lambda^{-2s+3} \Lambda^{-2} \sin(2\lambda x + 2\omega t) \\ - k_2 \omega \lambda^{-s+1} \Lambda^{-2} \sin(\lambda x + k_3\omega t) - \frac{k_2}{2} \lambda^{-2s+3} \Lambda^{-2} \sin(2\lambda x + 2k_3\omega t) \\ := E_2 + E_3 + E_4 + E_5 + E_6. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \rho_t^{\omega,\lambda} - k_3 u^{\omega,\lambda} \rho_x^{\omega,\lambda} - k_3 u_x^{\omega,\lambda} \rho^{\omega,\lambda} \\ = k_3 \lambda^{-s} \sin(\lambda x + \omega t) + k_3 \lambda^{-2s+2} \sin(2\lambda x + (k_3 + 1)\omega t) \\ := F_1 + F_2. \end{aligned}$$

Let C be a generic positive constant. For any positive quantities P and Q , we write $P \lesssim Q$ ($P \gtrsim Q$) means that $P \leq CQ$ ($P \geq CQ$) in the following.

Next, we estimate the error. We need the following lemma.

Lemma 3.1 [20] *Let $\sigma \in \mathbb{R}$. If $\lambda \in \mathbb{Z}_+$ and $\lambda \gg 1$ then*

$$\|\cos(\lambda x - \alpha)\|_{H^\sigma(\mathbb{S})} \approx \lambda^\sigma, \quad \alpha \in \mathbb{R}.$$

The above relation also holds if $\cos(\lambda x - \alpha)$ is replaced by $\sin(\lambda x - \alpha)$.

Estimating the H^μ -norm of E_1 . By Lemma 3.1 and $\mu \leq s - 1$, we have

$$\begin{aligned} \|E_1\|_{H^\mu} &= \left\| \frac{1}{2} \lambda^{-2s+1} \sin(2\lambda x + 2\omega t) \right\|_{H^\mu} \\ &\lesssim \lambda^{-2s+\mu+1} \lesssim \lambda^{-s}. \end{aligned}$$

Estimating the H^μ -norms of E_2 – E_6 . Also, we have

$$\begin{aligned}
\|E_2\|_{H^\mu} &= \left| \frac{k_1}{2} \right| \cdot \|\lambda^{-2s+1} \Lambda^{-2} \sin(2\lambda x + 2\omega t)\|_{H^\mu} \\
&\lesssim \lambda^{-2s+1} \|\sin(2\lambda x + 2\omega t)\|_{H^{\mu-2}} \lesssim \lambda^{-2s+\mu-1}, \\
\|E_3\|_{H^\mu} &= |k_1| \cdot \|\omega \lambda^{-s} \Lambda^{-2} \sin(\lambda x + \omega t)\|_{H^\mu} \\
&\lesssim \lambda^{-s} \|\sin(\lambda x + \omega t)\|_{H^{\mu-2}} \lesssim \lambda^{-s+\mu-2}, \\
\|E_4\|_{H^\mu} &= \left| \frac{3-k_1}{2} \right| \cdot \left\| \frac{1}{2} \lambda^{-2s+3} \Lambda^{-2} \sin(2\lambda x + 2\omega t) \right\|_{H^\mu} \\
&\lesssim \lambda^{-2s+3} \|\sin(2\lambda x + 2\omega t)\|_{H^{\mu-2}} \lesssim \lambda^{-2s+\mu+1}, \\
\|E_5\|_{H^\mu} &= |k_2| \cdot \|\omega \lambda^{-s+1} \Lambda^{-2} \sin(\lambda x + k_3 \omega t)\|_{H^\mu} \\
&\lesssim \lambda^{-s+1} \|\sin(\lambda x + k_3 \omega t)\|_{H^{\mu-2}} \lesssim \lambda^{-s+\mu-1}, \\
\|E_6\|_{H^\mu} &= \left| \frac{k_2}{2} \right| \cdot \left\| \frac{1}{2} \lambda^{-2s+3} \Lambda^{-2} \sin(2\lambda x + 2k_3 \omega t) \right\|_{H^\mu} \\
&\lesssim \lambda^{-2s+3} \|\sin(2\lambda x + 2k_3 \omega t)\|_{H^{\mu-2}} \lesssim \lambda^{-2s+\mu+1}.
\end{aligned}$$

Similarly, we can estimate the $H^{\mu-1}$ -norm of F_1 and F_2

$$\begin{aligned}
&\|k_3 \lambda^{-s} \sin(\lambda x + \omega t) + k_3 \lambda^{-2s+2} \sin(2\lambda x + (k_3 + 1)\omega t)\|_{H^{\mu-1}} \\
&\lesssim \lambda^{-s+\mu-1} + \lambda^{-2s+\mu+1}.
\end{aligned}$$

Collecting all error estimates together, we have the following theorem.

Theorem 3.1 *Let $s > \frac{5}{2}$ and $1 < \mu < \frac{3}{2}$. When ω is in a bounded set of $\mathbb{R}$ and $\lambda \gg 1$, we have that*

$$\|E\|_{H^\mu}, \|F\|_{H^{\mu-1}} \lesssim \lambda^{-r_s}, \quad \text{for } \lambda \gg 1, \quad 0 < t < T, \quad (10)$$

where $r_s = s - \mu + 1 > 0$.

4 Difference Between Approximate and Actual Solutions

In this section, we will estimate the difference between the approximate and actual solutions.

Let $(u_{\omega,\lambda}(t, x), \rho_{\omega,\lambda}(t, x))$ be the solution to system (5) with initial data the value of the approximate solution $(u^{\omega,\lambda}(t, x), \rho^{\omega,\lambda}(t, x))$ at time zero, that is, $(u_{\omega,\lambda}(t, x), \rho_{\omega,\lambda}(t, x))$ satisfies

$$\begin{cases} \partial_t u_{\omega,\lambda} - u_{\omega,\lambda} \partial_x u_{\omega,\lambda} - \partial_x \Lambda^{-2} \left(\frac{k_1}{2} u_{\omega,\lambda}^2 + \frac{3-k_1}{2} (\partial_x u_{\omega,\lambda})^2 + \frac{k_2}{2} \rho_{\omega,\lambda}^2 \right) = 0, & t > 0, x \in \mathbb{S}, \\ \partial_t \rho_{\omega,\lambda} - k_3 u_{\omega,\lambda} \partial_x \rho_{\omega,\lambda} - k_3 \partial_x u_{\omega,\lambda} \rho_{\omega,\lambda} = 0, & t > 0, x \in \mathbb{S}, \\ u_{\omega,\lambda}(0, x) = u^{\omega,\lambda}(0, x) = \omega \lambda^{-1} + \lambda^{-s} \cos(\lambda x), & x \in \mathbb{S}, \\ \rho_{\omega,\lambda}(0, x) = \rho^{\omega,\lambda}(0, x) = \omega \lambda^{-1} + \lambda^{-s+1} \cos(\lambda x), & x \in \mathbb{S}. \end{cases} \quad (11)$$

Note that $(u_{\omega,\lambda}(0, x), \rho_{\omega,\lambda}(0, x)) \in H^s \times H^{s-1}$, $s \geq 0$. Moreover, we have

$$\begin{aligned} \|u_{\omega,\lambda}(0, x)\|_{H^s} &\lesssim |\omega| \lambda^{-1} + 1, \quad \lambda \gg 1, \\ \|\rho_{\omega,\lambda}(0, x)\|_{H^{s-1}} &\lesssim |\omega| \lambda^{-1} + 1, \quad \lambda \gg 1. \end{aligned}$$

Therefore, if $s > \frac{5}{2}$, by using Theorems 2.1 and 2.2, we have that for any ω in a bounded set and $\lambda \gg 1$, problem (11) has a unique solution $z_{\omega,\lambda} \in C([0, T]; H^s) \times C([0, T]; H^{s-1})$ with

$$T \gtrsim \frac{1}{\|z_{\omega,\lambda}(0)\|_{H^s \times H^{s-1}}} \gtrsim \frac{1}{1 + \lambda^{-1}} \gtrsim 1. \quad (12)$$

To estimate the difference between the approximate and actual solutions, we let

$$v = u^{\omega,\lambda} - u_{\omega,\lambda}, \quad \sigma = \rho^{\omega,\lambda} - \rho_{\omega,\lambda}.$$

Then (v, σ) satisfies

$$\begin{cases} v_t - \frac{1}{2} \partial_x [(u^{\omega,\lambda} + u_{\omega,\lambda})v] - \partial_x \Lambda^{-2} \left[\frac{k_1}{2} (u^{\omega,\lambda} + u_{\omega,\lambda})v \right. \\ \quad \left. + \frac{3-k_1}{2} (u_x^{\omega,\lambda} + \partial_x u_{\omega,\lambda})v_x + \frac{k_2}{2} \sigma(\rho^{\omega,\lambda} + \rho_{\omega,\lambda}) \right] = E, & t > 0, x \in \mathbb{S}, \\ \sigma_t - k_3 \rho_x^{\omega,\lambda} v - k_3 \rho^{\omega,\lambda} v_x - k_3 \sigma \partial_x u_{\omega,\lambda} - k_3 \sigma_x u_{\omega,\lambda} = F, & t > 0, x \in \mathbb{S}, \\ v(0, x) = \sigma(0, x) = 0, & x \in \mathbb{S}, \end{cases} \quad (13)$$

where E and F are defined as in Sect. 3. Now we prove that the $H^\mu \times H^{\mu-1}$ -norm of difference decays.

Theorem 4.1 *Let $1 < \mu < \frac{3}{2}$ and $s > \frac{5}{2} > \mu + 1$, then*

$$\|v(t)\|_{H^\mu}, \|\sigma(t)\|_{H^{\mu-1}} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \quad \lambda \gg 1,$$

where $r_s = s - \mu + 1 - \frac{1}{2}\delta > 0$.

Proof Note that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\mu}^2 = \int_{\mathbb{S}} \Lambda^\mu v_t \cdot \Lambda^\mu v dx, \quad (14)$$

$$\frac{1}{2} \frac{d}{dt} \|\sigma(t)\|_{H^{\mu-1}}^2 = \int_{\mathbb{S}} \Lambda^{\mu-1} \sigma_t \cdot \Lambda^{\mu-1} \sigma dx. \quad (15)$$

Applying the operators Λ^μ and $\Lambda^{\mu-1}$ to both sides of the first and second equations of system (13), respectively, and integrating them, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\mu}^2 &= \int_{\mathbb{S}} \Lambda^\mu E \cdot \Lambda^\mu v dx + \frac{1}{2} \int_{\mathbb{S}} \partial_x \Lambda^\mu [(u^{\omega,\lambda} + u_{\omega,\lambda})v] \Lambda^\mu v dx \\
&\quad + \int_{\mathbb{S}} \partial_x \Lambda^{\mu-2} \left[\frac{k_1}{2} (u^{\omega,\lambda} + u_{\omega,\lambda})v \right. \\
&\quad \left. + \frac{3-k_1}{2} (u_x^{\omega,\lambda} + \partial_x u_{\omega,\lambda})v_x + \frac{k_2}{2} \sigma(\rho^{\omega,\lambda} + \rho_{\omega,\lambda}) \right] \Lambda^\mu v dx,
\end{aligned} \tag{16}$$

$$\frac{1}{2} \frac{d}{dt} \|\sigma(t)\|_{H^{\mu-1}}^2 = \int_{\mathbb{S}} \Lambda^{\mu-1} F \cdot \Lambda^{\mu-1} \sigma dx + k_3 \int_{\mathbb{S}} \Lambda^{\mu-1} \partial_x (u_{\omega,\lambda} \sigma + v \rho^{\omega,\lambda}) \Lambda^{\mu-1} \sigma dx. \tag{17}$$

Applying the Cauchy–Schwarz inequality in the first integral of right side of (16) and (17), we get

$$\begin{aligned}
\left| \int_{\mathbb{S}} \Lambda^\mu E \cdot \Lambda^\mu v dx \right| &\leq \|E\|_{H^\mu} \|v(t)\|_{H^\mu}, \\
\left| \int_{\mathbb{S}} \Lambda^{\mu-1} F \cdot \Lambda^{\mu-1} \sigma dx \right| &\leq \|F\|_{H^{\mu-1}} \|\sigma(t)\|_{H^{\mu-1}}.
\end{aligned}$$

In order to estimate the other terms in (16) and (17), we need the following lemma.

Lemma 4.1 [20, Lemma 3] *If $\sigma + 1 \geq 0$, then*

$$\|[\Lambda^\sigma \partial_x, f]g\|_{L^2} \leq C \|f\|_{H^\rho} \|g\|_{H^\sigma}$$

provided that $\rho > 3/2$ and $\sigma + 1 \leq \rho$.

In [20], the authors obtained that

$$\begin{aligned}
&\left| \int_{\mathbb{S}} \partial_x \Lambda^\mu [(u^{\omega,\lambda} + u_{\omega,\lambda})v] \Lambda^\mu v dx \right| \\
&\lesssim (\|u^{\omega,\lambda}\|_{H^\gamma} + \|u_{\omega,\lambda}\|_{H^\gamma}) \|v(t)\|_{H^\mu},
\end{aligned}$$

where $3/2 < \gamma < s$. We remark that, in paper [20], the authors considered the periodic Camassa–Holm equation, but it is no difference between the periodic and line. Next, we estimate the third term of right side of (16). Firstly, by Lemma 2.2, we have

$$\begin{aligned}
&\left| \int_{\mathbb{S}} \partial_x \Lambda^{\mu-2} \left[\frac{k_1}{2} (u^{\omega,\lambda} + u_{\omega,\lambda})v \right] \Lambda^\mu v dx \right| \\
&\lesssim \|(u^{\omega,\lambda} + u_{\omega,\lambda})v\|_{H^{\mu-1}} \|v\|_{H^\mu} \\
&\lesssim (\|u^{\omega,\lambda} + u_{\omega,\lambda}\|_{L^\infty} \|v\|_{H^{\mu-1}} + \|v\|_{L^\infty} \|u^{\omega,\lambda} + u_{\omega,\lambda}\|_{H^{\mu-1}}) \|v\|_{H^\mu} \\
&\lesssim (\|u_{\omega,\lambda}\|_{H^\mu} + \|u^{\omega,\lambda}\|_{H^\mu}) \|v\|_{H^\mu}^2.
\end{aligned} \tag{18}$$

By Lemma 4.1, we have

$$\begin{aligned}
& \left| \int_{\mathbb{S}} \left[\frac{3-k_1}{2} (u_x^{\omega,\lambda} + \partial_x u_{\omega,\lambda}) v_x + \frac{k_2}{2} \sigma (\rho^{\omega,\lambda} + \rho_{\omega,\lambda}) \right] \Lambda^\mu v dx \right| \\
& \lesssim \left| \int_{\mathbb{S}} [\Lambda^{\mu-2} \partial_x (u_x^{\omega,\lambda} + \partial_x u_{\omega,\lambda})] v_x \cdot \Lambda^\mu v dx + \int_{\mathbb{S}} (u_x^{\omega,\lambda} + \partial_x u_{\omega,\lambda}) \Lambda^{\mu-2} \partial_x v_x \cdot \Lambda^\mu v dx \right| \\
& \quad + \left| \int_{\mathbb{S}} [\Lambda^{\mu-2} \partial_x (\rho^{\omega,\lambda} + \rho_{\omega,\lambda})] \sigma \cdot \Lambda^\mu v dx + \int_{\mathbb{S}} (\rho^{\omega,\lambda} + \rho_{\omega,\lambda}) \Lambda^{\mu-2} \partial_x \sigma \cdot \Lambda^\mu v dx \right| \\
& \lesssim (\|u^{\omega,\lambda}\|_{H^{\gamma+1}} + \|u_{\omega,\lambda}\|_{H^{\gamma+1}}) \|v\|_{H^\mu}^2 + (\|\rho^{\omega,\lambda}\|_{H^\gamma} + \|\rho_{\omega,\lambda}\|_{H^\gamma}) (\|v\|_{H^\mu}^2 + \|\sigma\|_{H^{\mu-1}}^2),
\end{aligned} \tag{19}$$

where γ satisfies $3/2 < \gamma < s-1$. We remark there exists γ satisfying $3/2 < \gamma < s-1$ because the index s considered here is strictly larger than $5/2$.

Integrating by part and using Lemma 4.1, we have

$$\begin{aligned}
& \left| k_3 \int_{\mathbb{S}} \Lambda^{\mu-1} \partial_x (u_{\omega,\lambda} \sigma + v \rho^{\omega,\lambda}) \Lambda^{\mu-1} \sigma dx \right| \\
& \lesssim \|v \rho^{\omega,\lambda}\|_{H^\mu} \|\sigma\|_{H^{\mu-1}} + \left| \int_{\mathbb{S}} [\Lambda^{\mu-1} \partial_x u_{\omega,\lambda}] \sigma \cdot \Lambda^{\mu-1} \sigma dx \right. \\
& \quad \left. + \int_{\mathbb{S}} u_{\omega,\lambda} \partial_x \Lambda^{\mu-1} \sigma \cdot \Lambda^{\mu-1} \sigma dx \right| \\
& \lesssim \|\rho^{\omega,\lambda}\|_{H^\mu} \|v\|_{H^\mu} \|\sigma\|_{H^{\mu-1}} \\
& \quad + \|[\Lambda^{\mu-1} \partial_x u_{\omega,\lambda}] \sigma\|_{L^2} \|\sigma\|_{H^{\mu-1}} + \left| \int_{\mathbb{S}} \partial_x u_{\omega,\lambda} (\Lambda^{\mu-1} \sigma)^2 dx \right| \\
& \lesssim (\|\rho^{\omega,\lambda}\|_{H^\mu} + \|u_{\omega,\lambda}\|_{H^\gamma}) (\|v\|_{H^\mu}^2 + \|\sigma\|_{H^{\mu-1}}^2).
\end{aligned} \tag{20}$$

Substituting (18)–(20) into (16) and (17), and adding the resulting equations, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|v(t)\|_{H^\mu}^2 + \|\sigma(t)\|_{H^{\mu-1}}^2) \\
& \lesssim \|E\|_{H^\mu} \|v(t)\|_{H^\mu} + \|F\|_{H^{\mu-1}} \|\sigma(t)\|_{H^{\mu-1}} \\
& \quad + (\|\rho^{\omega,\lambda}\|_{H^\gamma} + \|\rho_{\omega,\lambda}\|_{H^\gamma} + \|u^{\omega,\lambda}\|_{H^{\gamma+1}} + \|u_{\omega,\lambda}\|_{H^{\gamma+1}}) (\|v\|_{H^\mu}^2 + \|\sigma\|_{H^{\mu-1}}^2).
\end{aligned} \tag{21}$$

Note that

$$u^{\omega,\lambda} = \omega \lambda^{-1} + \lambda^{-s} \cos(\lambda x + \omega t), \quad \rho^{\omega,\lambda} = \omega \lambda^{-1} + \lambda^{-s+1} \cos(\lambda x + k_3 \omega t),$$

we have

$$\begin{aligned}
\|u^{\omega,\lambda}\|_{H^{\gamma+1}} & \lesssim \lambda^{-1} + \lambda^{-s+\gamma+1}, \\
\|\rho^{\omega,\lambda}\|_{H^\gamma} & \lesssim \lambda^{-1} + \lambda^{-s+\gamma+1}.
\end{aligned}$$

By (9), we have

$$\|u_{\omega,\lambda}(t)\|_{H^s}, \|\rho_{\omega,\lambda}(t)\|_{H^{s-1}} \leq \|z_{\omega,\lambda}(t)\|_{H^s \times H^{s-1}} \lesssim \|z_{\omega,\lambda}(0)\|_{H^s \times H^{s-1}},$$

where $z_{\omega,\lambda}(t) = (u_{\omega,\lambda}(t), \rho_{\omega,\lambda}(t))$. Since $z_{\omega,\lambda}(0) = z^{\omega,\lambda}(0)$, we get

$$\begin{aligned}
& \|u_{\omega,\lambda}(t)\|_{H^{\gamma+1}}, \|\rho_{\omega,\lambda}(t)\|_{H^{\gamma}} \\
& \lesssim \|z^{\omega,\lambda}(0)\|_{H^{\gamma+1} \times H^{\gamma}} \\
& \lesssim \|u^{\omega,\lambda}(0)\|_{H^{\gamma+1}} + \|\rho^{\omega,\lambda}(0)\|_{H^{\gamma}} \\
& \lesssim \lambda^{-1} + \lambda^{-s+\gamma+1}.
\end{aligned}$$

Let $\tilde{z}(t, x) = (v(t, x), \sigma(t, x))$ and $\|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}}^2 = \|v(t)\|_{H^{\mu}}^2 + \|\sigma(t)\|_{H^{\mu-1}}^2$, then by (21), we obtain that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}}^2 & \lesssim (E\|_{H^{\mu}} + F\|_{H^{\mu-1}}) \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}} + \lambda^{-\rho_s} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}}^2 \\
& \lesssim \lambda^{-r_s} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}} + \lambda^{-s+\gamma+1} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}}^2, \quad \lambda \gg 1,
\end{aligned}$$

where we have used Theorem 3.1. Consequently,

$$\frac{d}{dt} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}} \lesssim \lambda^{-s+\gamma+1} \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}} + \lambda^{-r_s}, \quad \lambda \gg 1. \quad (22)$$

Since $\|\tilde{z}(0)\|_{H^{\mu} \times H^{\mu-1}} = (\|v(0)\|_{H^{\mu}}^2 + \|\sigma(0)\|_{H^{\mu-1}}^2)^{1/2} = 0$ and for $s > 5/2$, we can choose γ such that $-s + \gamma + 1 < 0$, by (22) and Gronwall's inequality, we obtain

$$\|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \quad \lambda \gg 1.$$

Note that

$$\|v(t)\|_{H^{\mu}}, \quad \|\sigma(t)\|_{H^{\mu-1}} \leq \|\tilde{z}(t)\|_{H^{\mu} \times H^{\mu-1}},$$

we see that

$$\|v(t)\|_{H^{\mu}}, \quad \|\sigma(t)\|_{H^{\mu-1}} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \quad \lambda \gg 1.$$

This completes the proof of this theorem. $\square$

5 Non-uniform Dependence

In this section, we will prove non-uniform dependence for system (5) by taking advantage of the information provided by Theorems 2.1, 2.2, 3.1 and 4.1. Our main result is the following theorem.

Theorem 5.1 *If $s > 5/2$, then the data-to-solution $z(0) \rightarrow z(t)$ for system (5) is not uniformly continuous from any bounded subset of $H^s \times H^s$ into $C([-T, T]; H^s) \times C([-T, T]; H^{s-1})$, where $z(0) = (u_0(x), \rho_0(x))$ and $z(t) = (u(t, x), \rho(t, x))$. More precisely, there exist two sequences of solutions $(u_{\lambda}(t), \rho_{\lambda}(t))$ and $(\tilde{u}_{\lambda}(t), \tilde{\rho}_{\lambda}(t))$ to the differential equations of (5) in $C([-T, T]; H^s) \times C([-T, T]; H^{s-1})$ such that*

$$\begin{aligned} \|u_\lambda(t)\|_{H^s} + \|\tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t)\|_{H^{s-1}} + \|\tilde{\rho}_\lambda(t)\|_{H^{s-1}} &\lesssim 1, \\ \lim_{\lambda \rightarrow \infty} \|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} &= \lim_{\lambda \rightarrow \infty} \|\rho_\lambda(0) - \tilde{\rho}_\lambda(0)\|_{H^{s-1}} = 0, \end{aligned} \quad (23)$$

and

$$\liminf_{\lambda \rightarrow \infty} (\|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t) - \tilde{\rho}_\lambda(t)\|_{H^{s-1}}) \gtrsim \sin t, \quad |t| < T \leq 1. \quad (24)$$

Proof Let $(u_\lambda(t), \rho_\lambda(t)) = (u_{1,\lambda}(t, x), \rho_{1,\lambda}(t, x))$ and $(\tilde{u}_\lambda(t), \tilde{\rho}_\lambda(t)) = (u_{-1,\lambda}(t, x), \rho_{-1,\lambda}(t, x))$, where $(u_{1,\lambda}(t, x), \rho_{1,\lambda}(t, x))$ and $(u_{-1,\lambda}(t, x), \rho_{-1,\lambda}(t, x))$ are the unique solution to Cauchy problem (11) with initial data $(u^{1,\lambda}(0, x), \rho^{1,\lambda}(0, x))$ and $(u^{-1,\lambda}(0, x), \rho^{-1,\lambda}(0, x))$, respectively.

It follows from Theorem 2.1 that these solutions belong in $C([0, T]; H^s) \times C([0, T]; H^{s-1})$. By (12) and the assumptions after Theorem 2.1, we see that T is independent of $\lambda \gg 1$. Letting $k = [s] + 2$ and using estimate (9), we have

$$\|u_{\pm 1,\lambda}(t)\|_{H^k}, \|\rho_{\pm 1,\lambda}(t)\|_{H^k} \lesssim \|z^{\pm 1,\lambda}(0)\|_{H^k \times H^{k-1}}, \quad (25)$$

where $z^{\pm 1,\lambda}(0) = (u^{\pm 1,\lambda}(0), \rho^{\pm 1,\lambda}(0))$ and $\|z^{\pm 1,\lambda}(0)\|_{H^k \times H^{k-1}}^2 = \|u^{\pm 1,\lambda}(0)\|_{H^k}^2 + \|\rho^{\pm 1,\lambda}(0)\|_{H^{k-1}}^2$. If λ is large enough, then from Lemma 3.1 we have

$$\begin{aligned} \|u^{\pm 1,\lambda}(t)\|_{H^k} &\leq \|\omega \lambda^{-1}\|_{H^k} + \lambda^{-s} \|\cos(\lambda x + \omega t)\|_{H^k} \\ &\lesssim \lambda^{-1} + \lambda^{k-s}, \\ \|\rho^{\pm 1,\lambda}(t)\|_{H^{k-1}} &\lesssim \lambda^{-1} + \lambda^{k-s}, \end{aligned}$$

which gives

$$\|u^{\pm 1,\lambda}(t)\|_{H^k}, \|\rho^{\pm 1,\lambda}(t)\|_{H^{k-1}} \lesssim \lambda^{k-s}. \quad (26)$$

Combining (25) with (26), we obtain

$$\|u_{\pm 1,\lambda}(t)\|_{H^k}, \|\rho_{\pm 1,\lambda}(t)\|_{H^{k-1}} \lesssim \lambda^{k-s}, \quad \lambda \gg 1. \quad (27)$$

Estimate (26) together with (27) yield

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^k}, \|\rho^{\pm 1,\lambda}(t) - \rho_{\pm 1,\lambda}(t)\|_{H^{k-1}} \lesssim \lambda^{k-s}, \quad \lambda \gg 1. \quad (28)$$

Theorem 4.1 implies that

$$\|u^{\pm 1,\lambda}(t) - u_{\pm 1,\lambda}(t)\|_{H^\mu}, \|\rho^{\pm 1,\lambda}(t) - \rho_{\pm 1,\lambda}(t)\|_{H^{\mu-1}} \lesssim \lambda^{-r_s}, \quad \lambda \gg 1. \quad (29)$$

Now, applying the interpolation inequality

$$\|\varphi\|_{H^s} \leq \|\varphi\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)} \|\varphi\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)}$$

with $s_1 = \mu$ and $s_2 = [s] + 2 = k$, and using estimates (28) and (29), we get

$$\begin{aligned}
\|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^s} &\leq \|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^\mu}^{(k-s)/(k-\mu)} \|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^k}^{(s-\mu)/(k-\mu)} \\
&\lesssim \lambda^{-r_s(k-s)/(k-\mu)} \lambda^{(k-s)(s-\mu)/(k-\mu)} \\
&\lesssim \lambda^{-(r_s-s+\mu)(k-s)/(k-\mu)}, \quad \lambda \gg 1.
\end{aligned}$$

Hence

$$\|u^{\pm 1, \lambda}(t) - u_{\pm 1, \lambda}(t)\|_{H^s} \lesssim \lambda^{-\varepsilon_s}, \quad \lambda \gg 1, \quad (30)$$

where $\varepsilon_s = 1/(s+2)$.

Now, we prove (23) and (24). We only consider $u_{\pm 1, \lambda}(t, x)$ because $\rho_{\pm 1, \lambda}(t, x)$ can be similarly proved. Firstly, we have

$$\|u_{1, \lambda}(0) - u_{-1, \lambda}(0)\|_{H^s} = 2\lambda^{-1} \|\omega\|_{H^s} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,$$

which implies that (23) holds. Next, we prove (24). It is easy to see that

$$\liminf_{\lambda \rightarrow \infty} (\|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t) - \tilde{\rho}_\lambda(t)\|_{H^{s-1}}) \geq \liminf_{\lambda \rightarrow \infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s}.$$

Thus we only prove that

$$\liminf_{\lambda \rightarrow \infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} \gtrsim \sin t, \quad |t| < T \leq 1.$$

Obviously,

$$\|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s} \geq \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s} - \|u^{1, \lambda}(t) - u_{1, \lambda}(t)\|_{H^s} - \|u^{-1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s}.$$

It follows from (30) that

$$\|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s} \geq \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s} - c\lambda^{-\varepsilon_s}, \quad \lambda \gg 1,$$

which implies that

$$\liminf_{\lambda \rightarrow \infty} \|u_{1, \lambda}(t) - u_{-1, \lambda}(t)\|_{H^s} \geq \liminf_{\lambda \rightarrow \infty} \|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s}. \quad (31)$$

Using the identity

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

gives

$$u^{1, \lambda}(t) - u^{-1, \lambda}(t) = 2\lambda^{-1} - 2\lambda^{-s} \sin(\lambda x) \sin t.$$

Thus,

$$\begin{aligned}
\|u^{1, \lambda}(t) - u^{-1, \lambda}(t)\|_{H^s} &\geq 2\lambda^{-s} \left\| \sin \lambda x \right\|_{H^s} |\sin t| - 2\lambda^{-1} \|1\|_{H^s} \\
&\gtrsim \lambda^{-s} \left\| \sin \lambda x \right\|_{H^s} |\sin t| - \lambda^{-1}, \quad \lambda \gg 1.
\end{aligned}$$

Letting $\lambda \rightarrow \infty$ in the above inequality, we have

$$\liminf_{\lambda \rightarrow \infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \gtrsim |\sin t|. \quad (32)$$

Summing inequalities (31) and (32) up, it yields inequality (24). This completes the proof of this theorem. $\square$

6 Conclusion

In this paper, we considered the initial data problem of a periodic two-component b-family system. The solution map of the Cauchy problem of the b-family system is not uniformly continuous in $H^s(\mathbb{S})$, $s > 5/2$, is obtained.

Comparing with the earlier results, we studied a more general shallow water equation. By constructing the approximating solutions, we prove the solution map is continuous but not uniformly continuous in H^s .

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