A Blow-up Result for the Periodic Solutions to an Integrable Dispersive Hunter–Saxton Equation

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Abstract

We prove a finite time blow-up result for the periodic solutions to an integrable Hunter–Saxton equation with a dispersion term on the basis of its conserved quantities and blow-up criterion, as well as the Gagliardo–Nirenberg inequality.

Keywords Integrable Hunter–Saxton equation · Dispersion term · Periodic solution · Blow-up

1 Introduction

Recently Hone, Novikov and Wang derived a group of second-order integrable nonlinear partial differential equations. This group contains at least four kinds of physically important nonlinear wave models, these are, the short-pulse equation, the Vakhnenko equation, the Hunter–Saxton equation, the nonlinear Klein–Gordon equation (up to a linear coordinate transformation), which have been studied intensively (see [1–4], for example). One novel member of this group, taking the form

$$u_{tx} = u + 2uu_{xx} + u_x^2, \quad (1.1)$$

has been shown to possess a compatible Hamiltonian pair as well as a Lax pair, and be linked to the sine-Gordon equation via a reciprocal transformation [5].

It is worthwhile to mention that Eq. (1.1) can be viewed as a short-wave limit of the celebrated Camassa–Holm (CH) equation

$$v_t - v_{txx} - 2\omega^2 v_x + 3vv_x = 2v_x v_{xx} + vv_{xxx}. \quad (1.2)$$

As a matter of fact after introducing the scaling transformation

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$$t \mapsto -\frac{1}{\sqrt{2\omega\epsilon}}t, \quad x \mapsto \frac{\epsilon}{\sqrt{2\omega}}x, \quad v \mapsto 2\epsilon^2v$$

with $v(t, x) = \sum_{i=0}^{\infty} v_i(t, x)\epsilon^i$ being expanded in powers of the small positive parameter ϵ , it can be found that $u = v_{0,x}(t, x)$ satisfies Eq. (1.1). The CH equation was first included in a bi-Hamiltonian generalization of KdV equation by Fuchssteiner and Fokas [6], and later proposed as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [7]. Moreover, it was also derived independently as a model for the propagation of nonlinear waves in cylindrical compressible hyper-elastic rods by Dai [8]. Geometrically, it is a re-expression of geodesic flow on the diffeomorphism group over the unit cycle [9, 10], and this geometric illustration leads to a proof that the Least Action Principle holds. The CH equation is completely integrable in the sense of possessing the bi-Hamiltonian structure and Lax pair representation [6, 11, 12]. The two most attractive features of the CH equation may be referred as the presence of peaked soliton solutions and the occurrence of wave-breaking phenomena [7, 12, 13].

It is of interest to point out that Eq. (1.1) also can be regarded as the Hunter–Saxton equation with a dispersion term u . In fact, neglecting the dispersion term u in Eq. (1.1) and then replacing u and t by $u/2$ and $-t$, respectively, one obtains the following Hunter–Saxton (HS) equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \quad (1.3)$$

which was first established by Hunter and Saxton [3] to model the propagation of weakly nonlinear orientation waves in a massive director field of a nematic liquid crystal. The HS equation is also a completely integrable equation which admits a bi-Hamiltonian structure, and it also describes the geodesic flow on homogeneous spaces related to the Virasoro group [14–17]. For the initial value problem for the HS equation on the line, the authors of [3] demonstrated that smooth solutions exist locally and break down in finite time relying on the technique of characteristics. The occurrence of blow-up for the HS equation can be interpreted physically as an altering of the director field from its unperturbed state. The periodic Cauchy problem for the HS equation on the unit circle was discussed by Yin [18]. He proved the local existence of solutions of the periodic HS equation and showed that the finite time blow-up occurs for every non-constant initial data. More results on its inverse scattering solutions, admissible weak solutions, global dissipative and conservative weak solutions, and their stability can be found in ref. [15, 19–26].

Very recently, Li and Yin [27] considered the Cauchy problem associated to Eq. (1.1) in a periodic domain

$$\begin{cases} u_{tx} = u + 2uu_{xx} + u_x^2, & x \in \mathbb{S}, \quad t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \end{cases} \quad (1.4)$$

where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ denotes the unit circle endowed with periodic boundary conditions. For initial profile $u_0 \in H^s(\mathbb{S})$ with the exponent $s > 3/2$ satisfying certain constraint,

they proved that Eq. (1.4) is locally well-posed in $\mathbb{S}$. Furthermore, they showed that the quantities

$$\int_{\mathbb{S}} u(t, x) dx = \int_{\mathbb{S}} u_0(x) dx \quad \text{and} \quad \int_{\mathbb{S}} u_x^2(t, x) dx = \int_{\mathbb{S}} u_{0,x}^2(x) dx \quad (1.5)$$

are conserved in time with the relation

$$\int_{\mathbb{S}} u dx = \int_{\mathbb{S}} u_x^2 dx := K, \quad (1.6)$$

where $K \geq 0$ is a constant depending only on u_0 . Based on a precise blow-up criterion, they also showed that if $u_0 \in H^s(\mathbb{S})$ with $s > 3$ and $K \geq 1$, then the corresponding solution $u(t, x)$ to Eq. (1.4) blows up in finite time. Subsequently they presented its traveling wave solutions by use of the traveling wave solutions of the sinh-Gordon equation and a period stretch between these two equations. The global conservative solutions were discussed in ref. [28].

Motivated by the references cited above, we are interested in seeking new sufficient condition on the initial datum, which guarantees the finite time singularity formation for the corresponding solutions to Eq. (1.4). In the present paper, following the idea developed in ref. [29] we will give a new blow-up result for Eq. (1.4). Concretely, we will show that for $u_0 \in H^s(\mathbb{S})$, $s \geq 2$, $n \in \mathbb{N}^*$ such that $\int_{\mathbb{S}} (u_0 - u_{0,x}^2) dx = 0$ and $0 < K < 1$ if the initial datum u_0 satisfies

$$\int_{\mathbb{S}} u_{0,x}^{2n+1} dx > C,$$

with $C > 0$ being a suitable constant that depends on $\|\partial_x u_0\|_{L^2(\mathbb{S})}$ and the Gagliardo-Nirenberg constant c , then the corresponding solution $u(t, x)$ to Eq. (1.4) with the initial datum will blow up in finite time.

The remainder of paper is organized as follows. In Sect. 2, we recall the well-posedness and blow-up scenario for Eq. (1.4), and the Gagliardo-Nirenberg interpolation inequality. In Sect. 3, we state and prove a new finite time blow-up result for Eq. (1.4), which has never been reported in the literature.

2 Preliminaries

We use the following notations and definitions. For the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, the Sobolev space $H^s(\mathbb{S})$ consists of the periodic functions $f \in L^2(\mathbb{S})$ such that $\sum_{m \in \mathbb{Z}} (1 + m^2)^s |\hat{f}(m)|^2 < \infty$, where $\hat{f}$ is the Fourier transform of f . For a function $f \in H^s(\mathbb{S})$, the norm is denoted by $\|f\|_{H^s(\mathbb{S})}$. Moreover, for an open set $\Omega \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$ and $p \geq 1$, the Sobolev space $W^{k,p}(\Omega)$ is the collection of functions $f \in L^p(\Omega)$ that have partial weak derivatives $\partial^\alpha f \in L^p(\Omega)$ of all orders $|\alpha| \leq k$, equipped with the norm $\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\Omega)}$. For $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$.

We first recall the local well-posedness result for Eq. (1.4), which was established by using Kato's semigroup theory in the following paper.

Lemma 2.1 (see [27]) *Let the initial datum $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$ satisfying $\int_{\mathbb{S}}(u_0 - u_{0,x}^2) dx = 0$, then there exists a maximal $T = T(u_0) > 0$ and a unique solution $u(t, x)$, which depends continuously on the initial datum u_0 to Eq. (1.4) such that*

$$u \in C([0, T]; H^s(\mathbb{S})) \cap C^1([0, T]; H^{s-1}(\mathbb{S})).$$

Moreover, for any $(t, x) \in [0, T) \times \mathbb{S}$, there holds that

$$K - \sqrt{K} \leq u(t, x) \leq K + \sqrt{K}, \quad (2.1)$$

where K is defined as in (1.6).

The next lemma concerns a precise blow-up scenario for Eq. (1.4), which indicates that the solution blows up if and only if the first-order derivative blows up.

Lemma 2.2 (See [27]) *Suppose that $u_0(x) \in H^s(\mathbb{S})$ with $s \geq 2$. Then the corresponding solution $u(t, x)$ to Eq. (1.4) blows up in finite time $T > 0$ if and only if*

$$\limsup_{t \uparrow T} \sup_{x \in \mathbb{S}} u_x(t, x) = +\infty. \quad (2.2)$$

We also recall the following Gagliardo-Nirenberg inequality, which plays a key role in our proof of the main result.

Lemma 2.3 (See [30]) *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, smooth domain. If $f \in W^{l,q}(\Omega) \cap W^{j,r}(\Omega)$ for some $j \leq l$, and if $j \leq k \leq l$ and for some $0 < \theta \leq 1$,*

$$\frac{1}{p} - \frac{k}{d} = \theta \left(\frac{1}{q} - \frac{l}{d} \right) + (1 - \theta) \left(\frac{1}{r} - \frac{j}{d} \right),$$

then $f \in W^{k,p}(\Omega)$. Moreover,

$$\|f\|_{W^{k,p}(\Omega)} \leq c \|f\|_{W^{l,q}(\Omega)}^\theta \|f\|_{W^{j,r}(\Omega)}^{1-\theta}$$

for some constant $c > 0$ independent of f .

3 Blow up

In this section, we state and prove our main blow-up result.

Theorem 3.1 *Let $u_0 \in H^s(\mathbb{S})$ with $s \geq 2$ satisfying $\int_{\mathbb{S}}(u_0 - u_{0,x}^2) dx = 0$, $u_0 \not\equiv \lambda$ for any $\lambda \in \mathbb{R}$, and T be the maximal existence time of the solution $u(t, x)$ to Eq. (1.4) corresponding to the initial datum u_0 . If the slope of u_0 satisfies*

$$\int_{\mathbb{S}} u_{0,x}^{2n+1} dx > \left(\frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1} \right)^{1-\frac{1}{2n}}, \quad (3.1)$$

for some constant $c > 0$, $n \in \mathbb{N}^*$ and $0 < K < 1$ with

$$\mu = \sqrt{\frac{(2n)^n(2n+1)^{n+1}}{(2n-1)^n(n+1)^{n+1}}} (\sqrt{K} - K)^{\frac{n+1}{2}}, \quad (3.2)$$

then the corresponding solution to Eq. (1.4) blows up in finite time $T \in (0, +\infty)$. Moreover, for the function $m(0) = \int_{\mathbb{S}} u_{0,x}^{2n+1} dx$, the upper bound of the maximal existence time satisfies

$$T \leq \int_{m(0)}^{+\infty} \frac{dy}{\frac{2n-1}{2cK^{\frac{1}{2n-1}}} y^{\frac{2n}{2n-1}} - \mu^2}. \quad (3.3)$$

Proof Multiplying the first equation in (1.4) by $(2n+1)u_x^{2n}$ and integrating the resultant equation over $\mathbb{S}$ with respect to x subsequently, it yields that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}} u_x^{2n+1} dx &= (2n+1) \int_{\mathbb{S}} u_x^{2n} u_{xt} dx \\ &= (2n+1) \int_{\mathbb{S}} u_x^{2n} (u + 2uu_{xx} + u_x^2) dx \\ &= (2n-1) \int_{\mathbb{S}} u_x^{2n+2} dx + (2n+1) \int_{\mathbb{S}} uu_x^{2n} dx \\ &\geq (2n-1) \int_{\mathbb{S}} u_x^{2n+2} dx + (2n+1) \min_{x \in \mathbb{S}} u(t, x) \int_{\mathbb{S}} u_x^{2n} dx \\ &\geq (2n-1) \int_{\mathbb{S}} u_x^{2n+2} dx + (2n+1)(K - \sqrt{K}) \int_{\mathbb{S}} u_x^{2n} dx, \end{aligned} \quad (3.4)$$

where we have used (2.1) and the following identity

$$\int_{\mathbb{S}} uu_x^{2n} u_{xx} dx = -\frac{1}{2n+1} \int_{\mathbb{S}} u_x^{2n+2} dx.$$

If $0 < K < 1$, by Hölder's inequality and Young's inequality with ϵ , we arrive at

$$\begin{aligned} (\sqrt{K} - K) \int_{\mathbb{S}} u_x^{2n} dx &\leq (\sqrt{K} - K) \left(\int_{\mathbb{S}} u_x^{2n+2} dx \right)^{\frac{n}{n+1}} \\ &\leq \frac{\epsilon^{n+1}}{n+1} (\sqrt{K} - K)^{n+1} + \frac{n}{(n+1)\epsilon^{\frac{n+1}{n}}} \int_{\mathbb{S}} u_x^{2n+2} dx. \end{aligned} \quad (3.5)$$

Setting $\epsilon = \left(\frac{2n(2n+1)}{(2n-1)(n+1)} \right)^{\frac{n}{n+1}}$, it then follows from (3.4) and (3.5) that

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^{2n+1} dx \geq \frac{2n-1}{2} \int_{\mathbb{S}} u_x^{2n+2} dx - \frac{(2n+1)(\sqrt{K}-K)^{n+1}}{n+1} \left(\frac{2n(2n+1)}{(2n-1)(n+1)} \right)^n. \quad (3.6)$$

In the case when

$$d = j = k = l = 1, \quad p = 2n + 1, \quad q = 2, \quad r = 2n + 2, \quad \theta = \frac{1}{n(2n+1)},$$

Lemma 2.3 suggests the following special version of Gagliardo-Nirenberg inequality as

$$\|\partial_x u\|_{L^{2n+1}(\mathbb{S})} \leq c \|\partial_x u\|_{L^2(\mathbb{S})}^{\frac{1}{n(2n+1)}} \|\partial_x u\|_{L^{2n+2}(\mathbb{S})}^{1 - \frac{1}{n(2n+1)}}.$$

Using this together with (1.5) and (1.6), it can be deduced that

$$\begin{aligned} \int_{\mathbb{S}} u_x^{2n+2} dx &\geq \frac{1}{c \|\partial_x u\|_{L^2(\mathbb{S})}^{\frac{2}{2n-1}}} \left(\int_{\mathbb{S}} u_x^{2n+1} dx \right)^{\frac{2n}{2n-1}} \\ &= \frac{1}{cK^{\frac{1}{2n-1}}} \left(\int_{\mathbb{S}} u_x^{2n+1} dx \right)^{\frac{2n}{2n-1}}. \end{aligned} \quad (3.7)$$

If we denote

$$m(t) = \int_{\mathbb{S}} u_x^{2n+1} dx, \quad t \geq 0,$$

combination of (3.6) with (3.7) infers that

$$\frac{d}{dt} m(t) \geq \frac{2n-1}{2cK^{\frac{1}{2n-1}}} (m(t))^{\frac{2n}{2n-1}} - \mu^2, \quad (3.8)$$

where the positive constant μ is as in (3.2). From the assumption $m(0) > [(2c\mu^2 K^{\frac{1}{2n-1}})/(2n-1)]^{1-\frac{1}{2n}}$, it follows that $(m(0))^{\frac{2n}{2n-1}} > \frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1}$. We claim that

$$m(t) > \left(\frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1} \right)^{1-\frac{1}{2n}} \quad \text{for any } t \in [0, T). \quad (3.9)$$

If (3.9) does not hold, by the continuity of $m(t)$, we know that there exists some $t_0 \in (0, T)$ such that

$$(m(t))^{\frac{2n}{2n-1}} > \frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1} \quad \text{for } t \in [0, t_0),$$

but

$$(m(t_0))^{\frac{2n}{2n-1}} = \frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1}.$$

Then it follows from (3.8) that

$$\frac{d}{dt} m(t) > 0 \text{ on } [0, t_0]. \quad (3.10)$$

Integration of (3.10) gives the contradiction

$$m(t_0) > m(0) > \left(\frac{2c\mu^2 K^{\frac{1}{2n-1}}}{2n-1} \right)^{1-\frac{1}{2n}}.$$

Hence the claim holds true. Combining this claim with (3.10), we see that $m(t)$ is a strictly increasing function. If the solution u exists globally, then there exists some t_1 such that

$$\frac{d}{dt} m(t) \geq \frac{2n-1}{4cK^{\frac{1}{2n-1}}} (m(t))^{\frac{2n}{2n-1}} \text{ for } t \geq t_1.$$

Therefore

$$0 \leq \frac{1}{(m(t))^{\frac{1}{2n-1}}} \leq -\frac{1}{4cK^{\frac{1}{2n-1}}} (t - t_1) + \frac{1}{(m(t_1))^{\frac{1}{2n-1}}}.$$

Due to $m(t_1) > 0$, for $t \geq t_1$ large enough the above inequality will lead a contradiction. Thus there exists a finite time T such that $\lim_{t \rightarrow T} m(t) = +\infty$. We also notice that integration of (3.8) with respect to t over $[0, T]$ yields the upper bound of the maximal existence time

$$T \leq \int_0^T \frac{m'(t) dt}{\frac{2n-1}{2cK^{\frac{1}{2n-1}}} (m(t))^{\frac{2n}{2n-1}} - \mu^2} = \int_{m(0)}^{+\infty} \frac{dy}{\frac{2n-1}{2cK^{\frac{1}{2n-1}}} y^{\frac{2n}{2n-1}} - \mu^2}. \quad (3.11)$$

On the other hand,

$$\int_{\mathbb{S}} u_x^{2n+1} dx \leq (\sup_{x \in \mathbb{S}} u_x(x, t))^{2n-1} \int_{\mathbb{S}} u_x^2 dx = (\sup_{x \in \mathbb{S}} u_x(x, t))^{2n-1} K.$$

In view of Lemma 2.2, we know that the solution u must blow up in finite time. $\square$

Remark 3.2 Notice that $n \in \mathbb{N}^* = \mathbb{N}/\{0\}$ is necessary in Theorem 3.1 since $\int_{\mathbb{S}} u_x dx = \int_{\mathbb{S}} u_{0,x} dx$ is a conserved quantity of Eq.(1.4). Moreover, for the special case $n = 1$, it follows from (3.8) that

$$\begin{aligned}\frac{d}{dt} m(t) &\geq \frac{1}{2cK} (m(t))^2 - \frac{9}{2} (\sqrt{K} - K)^2 \\ &= \frac{1}{2cK} [m(t) + 3(\sqrt{K} - K)\sqrt{cK}] [m(t) - 3(\sqrt{K} - K)\sqrt{cK}].\end{aligned}$$

Solving the above inequality with $m(0) > 3\sqrt{cK}(\sqrt{K} - K)$, it yields that

$$m(t) \rightarrow +\infty \text{ as } t \rightarrow T \leq \frac{\sqrt{cK}}{3(\sqrt{K} - K)} \log \frac{m(0) + 3\sqrt{cK}(\sqrt{K} - K)}{m(0) - 3\sqrt{cK}(\sqrt{K} - K)}.$$

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Declarations

Conflict of interest The author declares that I have no conflict of interest.

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