Equivalence Conditions and Invariants for the General Form of Burgers' Equations

Mostafa Hesamiarshad¹

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Abstract

Equivalence of differential equations is one of the most important concepts in the theory of differential equations. In this paper, the moving coframe method is applied to solve the local equivalence problem for the general form of Burgers' equation, which has two independent variables under action of a pseudo-group of contact transformations. Using this method, we found the structure equations and invariants of these equations, as a result some conditions for equivalence of them will be given.

Keywords Equivalence problem · General form of Burgers' equation · Invariants · moving coframe

1 Introduction

In the beginning of twentieth century, Elie Cartan developed a uniform method, called the “Cartan equivalence method”, for analyzing the differential invariants of many geometric structures, which is a systematic procedure that allows one to decide whether two sub manifolds can be mapped on each other by a transformation taken in a given pseudo-group [1]. Later, Erhesmann and Chern introduced two important concepts, jets spaces and G-structures to the method of equivalence [2, 3]. In recent years, with the help of mathematical software, many authors have successfully applied the method of equivalence to many interesting problems: classifications of differential equations, see [4–6] and references therein, holonomy groups [7], inverse variational problems [8, 9], general relativity [10, 11], and image processing [12]. (We also mention for completeness that in [13, 14] the authors classify differential equations using methods from classical theory of surfaces).

In this paper we consider a local equivalence problem for the general form of Burgers' equation:

✉ Mostafa Hesamiarshad
mostaf.hesami@gmail.com

¹ Department of Mathematics, Tuyserkan Branch, Islamic Azad University, Tuyserkan, Iran

$$u_{xx} = u_t + Q(u)u_x, \quad (1)$$

under the contact transformation of a pseudo-group. We use Elie Cartan's method of equivalence [1], in its original form, developed by Fels and Olver [15–17], and as stated by morozov [18], to compute the Maurer–Cartan forms, structure equations, basic invariants, and the invariant derivatives for symmetry groups of equations. The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence. Note that, the symmetry groups of two equations are necessarily isomorphic if these equations are equivalent, while the converse statement is not true in general. For the symmetry analysis of (1) the reader is referred to [19–21].

2 Pseudo-group of Contact Transformations and Symmetries of Differential Equations

In this section we describe the local equivalence problem for differential equations under the action of a pseudo-group of contact transformations. Two equations are said to be equivalent, if there exists a contact transformation that maps the equations to each other. We apply Cartan's structure theory of Lie pseudo-groups to obtain some necessary and sufficient conditions under which equivalence mappings can be found. This theory describes a Lie pseudo-group in terms of a set of invariant differential 1-forms, called Maurer–Cartan forms, which contains all information about the pseudo-group. In particular, they allow us to solve equivalence problem for submanifolds under the action of the pseudo-group. Recall that expansions of exterior differentials of Maurer–Cartan forms in terms of the forms themselves, leads to the Cartan's structure equation for the prescribed pseudo-group. Suppose $\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a trivial bundle with the local base coordinates $(x_1, \dots, x_n)$ and the local fibre coordinates $(u_1, \dots, u_m)$. Then $J^1(\pi)$ is the bundle of the first-order jets of sections of π , with local coordinates $(x_i, u_\alpha, p_i^\alpha)$, $i \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, m\}$, where $p_i^\alpha = \frac{\partial u_\alpha}{\partial x_i}$. For every local section $(x_i, f_\alpha(x))$ of π , the corresponding 1-jet $\left(x_i, f_\alpha(x), \frac{\partial f_\alpha(x)}{\partial x_i}\right)$ is denoted by $j_1(f)$. A differential 1-form ν on $J^1(\pi)$ is called a contact form, if it is annihilated by all 1-jets of local sections i.e., $j_1(f)^*\nu = 0$. Note that, in local coordinates every contact 1-form is a linear combination of the forms $\nu^\alpha = du_\alpha - p_i^\alpha dx_i$, $\alpha \in \{1, \dots, m\}$. Recall that a local diffeomorphism

$$\Delta : J^1(\pi) \longrightarrow J^1(\pi), \quad \Delta : (x_i, u_\alpha, p_i^\alpha) \longmapsto (\bar{x}_i, \bar{u}_\alpha, \bar{p}_i^\alpha),$$

is called a contact transformation, if for every contact 1-form ν , the form $\Delta^*\nu$ is also a contact. Let $\mathcal{R}$ be a first-order differential equation with m dependent and n independent variables. We consider $\mathcal{R}$ as a sub-bundle in $J^1(\pi)$. Suppose $\text{Cont}(\mathcal{R})$ is the group of contact symmetries for $\mathcal{R}$, that consists of all contact transformations on $J^1(\pi)$ mapping $\mathcal{R}$ to itself. To obtain the collection of invariant 1-forms of the pseudo-group of contact transformations on $J^1(\pi)$ we apply the methods described in [18]. The following differential 1-forms,

$$\begin{aligned}
\Theta^\alpha &= a_\beta^\alpha (du_\beta - p_j^\beta dx_j), \\
\Xi^i &= b_j^i dx_j + c_\beta^i \Theta^\beta, \\
\Sigma_i^\alpha &= a_\beta^\alpha B_j^i dp_j^\beta + f_{i\beta}^\alpha \Theta^\beta + g_{ij}^\alpha \Xi^j.
\end{aligned} \tag{2}$$

are the lifted coframe of $Cont(J^1(\pi))$. They are defined on $J^1(\pi) \times \mathcal{H}$, where $\mathcal{H} = \left\{ (a_\beta^\alpha, b_j^i, c_\beta^i, f_{i\beta}^\alpha, g_{ij}^\alpha) \mid \alpha, \beta \in \{1, \dots, m\}, i, j \in \{1, \dots, n\}, \det(a_\beta^\alpha) \cdot \det(b_j^i) \neq 0, g_{ij}^\alpha = g_{ji}^\alpha \right\}$ and (B_j^i) is the inverse matrix for (b_j^i) . They satisfy the following structure equations

$$\begin{aligned}
d\Theta^\alpha &= \Phi_\beta^\alpha \wedge \Theta^\beta + \Xi^k \wedge \Sigma_k^\alpha, \\
d\Xi^i &= \Psi_k^i \wedge \Xi^k + \Pi_\gamma^i \wedge \Theta^\gamma, \\
d\Sigma_i^\alpha &= \Phi_\gamma^\alpha \wedge \Sigma_i^\gamma - \Psi_i^k \wedge \Sigma_k^\alpha + \Lambda_{i\beta}^\alpha \wedge \Theta^\beta + \Omega_{ij}^\alpha \wedge \Xi^j,
\end{aligned} \tag{3}$$

where the forms $\Phi_\beta^\alpha, \Psi_j^i, \Pi_\beta^i, \Lambda_{i\beta}^\alpha$ and Ω_{ij}^α are the modified Maurer–Cartan forms depend on differentials of the coordinates of $\mathcal{H}$.

These differential equations define a submanifold $\mathcal{R} \subset J^1(\pi)$, and the Maurer–Cartan forms for its symmetry pseudo-group $Cont(\mathcal{R})$ can be found via the restrictions $\theta^\alpha = \iota^* \Theta^\alpha$, $\xi^i = \iota^* \Xi^i$ and $\sigma_i^\alpha = \iota^* \Sigma_i^\alpha$, where $\iota = \iota_0 \times id : \mathcal{R} \times \mathcal{H} \longrightarrow J^1(\pi) \times \mathcal{H}$ and $\iota_0 : \mathcal{R} \longrightarrow J^1(\pi)$ are defined by our differential equations. In order to compute the Maurer–Cartan forms, for the symmetry pseudo-group, we apply Cartan’s equivalence method in the following steps:

Step-1: First of all, the forms $\theta^\alpha, \xi^i, \sigma_i^\alpha$ are linearly dependent, i.e. there exists a nontrivial set of functions $U_\alpha, V_i, W_\alpha^i$ on $\mathcal{R} \times \mathcal{H}$, such that $U_\alpha \theta^\alpha + V_i \xi^i + W_\alpha^i \sigma_i^\alpha \equiv 0$. Setting these functions equal to some appropriate constants allows us to introduce a part of the coordinates of $\mathcal{H}$ as functions of the other coordinates of $\mathcal{R} \times \mathcal{H}$.

Step-2: We substitute the obtained values in step-1, into the forms $\phi_\beta^\alpha = \iota^* \Phi_\beta^\alpha$ and $\psi_k^i = \iota^* \Psi_k^i$. Then the coefficients of the semi-basic forms ϕ_β^α at σ_j^γ, ξ^j , and the coefficients of the semi-basic forms ψ_j^i at σ_j^γ are lifted invariants of $Cont(\mathcal{R})$. We set them equal to an appropriate constants by considering, $\det(a_\beta^\alpha) \cdot \det(b_j^i) \neq 0$, and express them as functions of the other coordinates of $\mathcal{R} \times \mathcal{H}$.

Step-3: Now, consider the following reduced structure equations

$$\begin{aligned}
d\theta^\alpha &= \phi_\beta^\alpha \wedge \theta^\beta + \xi^k \wedge \sigma_k^\alpha, \\
d\xi^i &= \psi_k^i \wedge \xi^k + \pi_\gamma^i \wedge \theta^\gamma, \\
d\sigma_i^\alpha &= \phi_\gamma^\alpha \wedge \sigma_i^\gamma - \psi_i^k \wedge \sigma_k^\alpha + \lambda_{i\beta}^\alpha \wedge \theta^\beta + \omega_{ij}^\alpha \wedge \xi^j.
\end{aligned} \tag{4}$$

If the essential torsion coefficients are dependent to the group parameters, then we may normalize them to some constants to compute some new parts of the group parameters. Substituting these new computed parameters into the reduced modified Maurer–Cartan forms, allows us to repeat the procedure of normalization. This process has two results. First, when the reduced lifted coframe appears to be involutive as explained in [15], this coframe is the set of invariant 1-forms which characterize

the pseudo-group $Cont(\mathcal{R})$. Second, when the coframe is not involutive we should apply the procedure of prolongation as described in [15].

3 Structure Equations and Invariants of Symmetry Groups for the General Form of Burgers' Equations

Consider the following system equivalent to (1) of first order:

$$u_x = v, \quad u_t = v_x + Q(u)v. \quad (5)$$

We apply the method described in the previous section to the class of equations (5). Denoting, $x = x_1, t = x_2, v = u_1, u = u_2, v_x = p_1^1, v_t = p_2^1, u_t = p_2^2, u_x = p_1^2$. We consider this system as a sub-bundle of the bundle $J^1(\varepsilon), \varepsilon = \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, with local coordinates $\{x_1, x_2, u_1, u_2, p_1^1, p_1^2, p_2^1, p_2^2\}$, where the embedding ι is defined by the equalities:

$$p_1^2 = u_1, \quad p_1^1 = p_2^2 - Q(u_2)u_1. \quad (6)$$

The forms $\theta^\alpha = i^* \Theta^\alpha, \alpha \in \{1, 2\}, \xi^i = i^* \Xi^i, i \in \{1, 2\}$, are linearly dependent. The group parameters a_β^α, b_j^i must satisfy the conditions $\det(a_\beta^\alpha) \neq 0, \det(b_j^i) \neq 0$. linear dependence between the forms σ_i^α are

$$\sigma_1^2 = 0, \quad \sigma_1^1 = \sigma_2^2. \quad (7)$$

Computing the linear dependence conditions (7) gives the following group parameters as a functions of other group parameters and the local coordinates $\{x_1, x_2, u_1, u_2, p_1^1, p_2^2\}$ of $\mathcal{R}$. In particular,

$$\begin{aligned} a_1^2 &= 0, & b_1^2 &= 0, & b_2^2 &= \frac{b_1^1 a_2^2}{a_1^1}, \\ g_{12}^2 &= -\frac{a_1^1 (Q b_2^1 u_1 + b_1^1 p_2^1 - b_2^1 p_2^2)}{(b_1^1)^3}, & g_{11}^2 &= \frac{a_2^2 (Q u_1 - p_2^2)}{(b_1^1)^2}, \\ g_{11}^1 &= \frac{\frac{dQ}{du_2} u_1^2 a_1^1 - Q^2 a_1^1 u_1 + Q a_1^1 p_2^2 + Q a_2^1 u_1 - a_1^1 p_2^1 - a_2^1 p_2^2}{(b_1^1)^2}, \\ f_{12}^2 &= \frac{Q(a_1^1)^2 b_2^1 b_1^1 c_1^2 u_1 - Q a_1^1 a_2^2 b_1^1 c_1^1 u_1 + (a_1^1)^2 b_1^1 c_2^1 p_2^1}{(b_1^1)^3 a_1^1} \\ &\quad + \frac{-(a_1^1)^2 b_2^1 c_1^2 p_2^2 + a_1^1 a_2^2 b_1^1 c_1^1 p_2^2 - a_2^2 (b_1^1)^2}{(b_1^1)^3 a_1^1}, \\ + f_{11}^2 &= \frac{Q(a_1^1)^2 b_2^1 b_1^1 c_2^2 u_1 - Q a_1^1 a_2^2 b_1^1 c_2^1 u_1 + (a_1^1)^2 b_1^1 c_2^2 p_2^1}{(b_1^1)^3 a_1^1} \\ &\quad - \frac{(a_1^1)^2 b_2^1 c_2^2 p_2^2 + a_1^1 a_2^2 b_1^1 c_2^1 p_2^2 - a_2^2 (b_1^1)^2}{(b_1^1)^3 a_1^1}. \end{aligned} \quad (8)$$

The expression for $f_{21}^2, f_{22}^2, g_{22}^2$ is too long to be written out in full here. The analysis of the semi-basic modified Maurer–Cartan forms $\phi_\beta^a, \psi_k^i, \pi_\gamma^i$ at the obtained values of the group parameters gives the following normalizations:

$$a_2^2 = a_1^1 b_1^1, \quad c_1^2 = 0, \quad c_2^2 = c_1^1. \quad (9)$$

The expression for f_{11}^1 is too long to be written out in full here. The analysis of the structure equations gives the following normalizations:

$$\begin{aligned} c_2^1 &= 0, \quad c_1^1 = 0, \quad a_2^1 = \frac{1}{2} \frac{a_1^1 (Qb_1^1 - b_2^1)}{b_1^1}, \\ f_{12}^1 &= \frac{1}{12} \frac{4(b_1^1)^4 f_{21}^1 + 8\left(\frac{dQ}{du_2}\right)(b_1^1)^2 u_1 - (Qb_1^1)^2 + 2Qb_1^1 b_2^1 - (b_2^1)^2}{(b_1^1)^4}, \\ f_{22}^1 &= \frac{1}{2} \frac{\left(\frac{dQ}{du_2}\right)(2Qb_1^1 u_1 - b_1^1 p_2^2 - b_2^1 u_1) - \left(\frac{d^2 Q}{du_2^2}\right)b_1^1 u_1^2}{(b_1^1)^4}. \end{aligned} \quad (10)$$

Regarding the appearance of different derivatives of $Q(u)$ in the essential torsion coefficients and with respect to vanishing or non-vanishing of these derivatives and their effects on normalizations process, we have to impose some restrictions on the function $Q(u_2)$. As a result of these restrictions, the following cases arise.

Case-1 After normalization (10), if Q is a constant then we have the following structure equations

$$\begin{aligned} d\theta^1 &= \phi_1^1 \wedge \theta^1 - \frac{1}{2} \psi_2^1 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\ d\theta^2 &= \phi_1^1 \wedge \theta^2 + \psi_1^1 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2, \\ d\xi^1 &= \psi_1^1 \wedge \xi^1 + \psi_2^1 \wedge \xi^2, \\ d\xi^2 &= 2\psi_1^1 \wedge \xi^2, \\ d\sigma_2^1 &= \phi_1^1 \wedge \sigma_2^1 - 2\psi_1^1 \wedge \sigma_2^1 - \frac{3}{2} \psi_2^1 \wedge \sigma_2^2 + \lambda_{21}^1 \wedge \theta^1 \\ &\quad + \omega_{12}^1 \wedge \xi^1 + \omega_{22}^1 \wedge \xi^2, \\ d\sigma_2^2 &= \phi_1^1 \wedge \sigma_2^2 - \psi_1^1 \wedge \sigma_2^2 - \psi_2^1 \wedge \theta^1 + \frac{1}{3} \lambda_{21}^1 \wedge \theta^2 \\ &\quad + \omega_{12}^1 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1. \end{aligned} \quad (11)$$

The structure equations (11) do not contain any torsion coefficient depending on the group parameters. The first reduced character is $s'_1 = 5$, and the degree of indeterminacy is 2. The Cartan involutivity test is not satisfied. Therefore we should use the procedure of prolongation, which gives us the following structure equations.

$$\begin{aligned}
d\theta^1 &= \eta^1 \wedge \theta^1 - \frac{1}{2}\eta^3 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\
d\theta^2 &= \eta^1 \wedge \theta^2 + \eta^2 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2, \\
d\xi^1 &= \eta^2 \wedge \xi^1 + \eta^3 \wedge \xi^2, \\
d\xi^2 &= 2\eta^2 \wedge \xi^2, \\
d\sigma_2^1 &= \eta^1 \wedge \sigma_2^1 - 2\eta^2 \wedge \sigma_2^1 - \frac{3}{2}\eta^3 \wedge \sigma_2^2 + \eta^4 \wedge \theta^1 \\
&\quad + \eta^5 \wedge \xi^1 + \eta^6 \wedge \xi^2, \\
d\sigma_2^2 &= \eta^1 \wedge \sigma_2^2 - \eta^2 \wedge \sigma_2^2 - \eta^3 \wedge \theta^1 + \frac{1}{3}\eta^4 \wedge \theta^2 \\
&\quad + \eta^5 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1. \\
d\eta_1 &= \frac{1}{2}\eta^3 \wedge \xi^1 + \eta^4 \wedge \xi^2, \\
d\eta_2 &= \frac{2}{3}\eta^4 \wedge \xi^2, \\
d\eta_3 &= \frac{2}{3}\eta^4 \wedge \xi^1 - \eta^2 \wedge \eta^3, \\
d\eta_4 &= -2\eta^2 \wedge \eta^4, \\
d\eta_5 &= -\pi_1 \wedge \xi^2 - \eta^6 \wedge \xi^1 + 2\eta^3 \wedge \sigma_2^1 - 2\eta^4 \wedge \sigma_2^2 + \eta^1 \wedge \eta^5 - 3\eta^2 \wedge \eta^5, \\
d\eta_6 &= -\pi_1 \wedge \xi^1 - \pi_2 \wedge \xi^2 - \frac{10}{3}\eta^4 \wedge \sigma_2^1 + \eta^1 \wedge \eta^6 - 4\eta^2 \wedge \eta^6 - \frac{5}{2}\eta^3 \wedge \eta^5.
\end{aligned} \tag{12}$$

The forms $\eta_1, \dots, \eta_6$ depend on differentials of the parameters of $\mathcal{H}$, while the forms π_1, π_2 depend on differentials of the prolongation variables.

In structure equations (12), the degree of indeterminacy is 2 and the reduced characters of the coframe are $s'_1 = 2, s'_2 = \dots = s'_{12} = 0$. Since the Cartan involutivity test for the lifted coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma_2^1, \sigma_2^2, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ is satisfied, then the coframe is involutive. Also all the essential torsion coefficients in the structure equations (12) are constants, then from the Theorem 11.8 of [15], we have:

Theorem 1 *The equation $u_t = \kappa u_x + u_{xx}$ is equivalent to the $u_t = u_{xx}$ under a contact transformation.*

Case-2 Suppose $Q = \kappa u_2 + \lambda$, is a linear function ($\kappa \neq 0$).

In this case the analysis of the structure equations gives the following extra normalizations to (10).

$$\begin{aligned}
a_1^1 &= 4\kappa(4\kappa^2 u_1 u_2 + 4\kappa \lambda u_1 - 4\kappa p_2^2)^{-\frac{2}{3}}, \\
b_1^1 &= \frac{1}{2}\kappa(4\kappa^2 u_1 u_2 + 4\kappa \lambda u_1 - 4\kappa p_2^2)^{\frac{1}{3}}, \\
b_2^1 &= \frac{1}{2}(\kappa u_2 + \lambda)(4\kappa^2 u_1 u_2 + 4\kappa \lambda u_1 - 4\kappa p_2^2)^{\frac{1}{3}}, \\
f_{21}^1 &= -8\kappa u_1(4\kappa^2 u_1 u_2 + 4\kappa \lambda u_1 - 4\kappa p_2^2)^{-\frac{2}{3}}, \\
g_{12}^1 &= 6\sqrt[3]{2\kappa u_1(\kappa^2 u_1 u_2 + \kappa \lambda u_1 - \kappa p_2^2)^{-\frac{2}{3}}}.
\end{aligned} \tag{13}$$

The expression for g_{22}^1 are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinates $\{x_1, x_2, u_1, u_2, p_2^1, p_2^2\}$. After normalization (13) the structure equations of coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma_2^1, \sigma_2^2\}$, is

$$\begin{aligned}
d\theta^1 &= -\frac{2I}{3}\theta^1 \wedge \xi^1 - \frac{1}{3}\theta^1 \wedge \sigma_2^2 + \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\
d\theta^2 &= -\theta^1 \wedge \xi^1 - \frac{I}{3}\theta^2 \wedge \xi^1 - \frac{1}{6}\theta^2 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^2, \\
d\xi^1 &= \theta^2 \wedge \xi^2 + \frac{1}{6}\xi^1 \wedge \sigma_2^2, \\
d\xi^2 &= -\frac{2I}{3}\xi^1 \wedge \xi^2 + \frac{1}{3}\xi^2 \wedge \sigma_2^2, \\
d\sigma_2^1 &= \theta^1 \wedge \xi^1 - 8I\theta^1 \wedge \xi^2 - I\theta^2 \wedge \xi^2 - \frac{1}{2}\theta^2 \wedge \sigma_2^2 + 40I\xi^1 \wedge \xi^2 \\
&\quad + \frac{4I}{3}\xi^1 \wedge \sigma_2^1 - 13\xi^2 \wedge \sigma_2^2 - \frac{2}{3}\sigma_2^1 \wedge \sigma_2^2, \\
d\sigma_2^2 &= 6\theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^1 - 2\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1 + I\xi^1 \wedge \sigma_2^2,
\end{aligned} \tag{14}$$

where

$$I = \frac{\sqrt[3]{2(\kappa^2 u^2 u_x + 2\kappa \lambda u u_x - \kappa u u_t - \kappa(u_x)^2 + \lambda^2 u_x - \lambda u_t + u_{tx})\kappa}}{\sqrt[3]{(\kappa^2 u u_x + \kappa \lambda u_x - \kappa u_t)^4}}$$

is the only invariant of the symmetry group of equations of the form Case-2.

Note that, the exterior differential of I is

$$dI = \frac{1}{2}\theta^2 + \frac{4I^2}{3}\xi^1 + 6\xi^2 + \frac{1}{2}\sigma_2^1 + \frac{2I}{3}\sigma_2^2.$$

All derived invariants of the group are expressed as functions of I . Therefore, the rank of the coframe, is 1 and our manifold is 6-dimensional and by theorem 8.22 from [15], we deduce the following theorem.

Theorem 2 *The equation $u_t = (\kappa u + \lambda)u_x + u_{xx}$, ($\kappa \neq 0$) admits a contact transformation symmetry group of dimension 5.*

If Q is not a linear function, the analysis of the structure equations gives the following normalizations in addition to (10).

$$\begin{aligned} a_1^1 &= \frac{4\left(\frac{d^2 Q}{du^2}\right)^2}{\left(\frac{dQ}{du}\right)^3}, & b_1^1 &= -\frac{1}{2} \frac{4\left(\frac{dQ}{du}\right)^2}{\left(\frac{d^2 Q}{du^2}\right)}, \\ b_2^1 &= -\frac{1}{2} \frac{\left(\frac{dQ}{du}\right)\left(-2\left(\frac{d^2 Q}{du^2}u_x\right) + \left(\frac{dQ}{du}\right)Q\right)}{\left(\frac{d^2 Q}{du^2}\right)}. \end{aligned} \quad (15)$$

The expression for $f_{21}^1, g_{22}^1, g_{12}^1$ are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinate.

In case that, $\frac{d^3 Q}{du^3}$ is nonzero, then an invariant of the form:

$$2\left(\frac{d^2 Q}{du^2}\right)^2 - \left(\frac{d^3 Q}{du^3}\right)\left(\frac{dQ}{du}\right) \quad (16)$$

could be computed at which vanishing or non-vanishing of this invariant leads into different results. In what follows, we will investigate these issues. Note that, according to our computations, vanishing of (16) or being $Q(u)$ of quadratic type leads, on the type and number of these invariants, into the same results.

Case-3 $Q(u)$ is a quadratic polynomial or

$$2\left(\frac{d^2 Q}{du^2}\right)^2 - \left(\frac{d^3 Q}{du^3}\right)\left(\frac{dQ}{du}\right) = 0, \quad \frac{d^3 Q}{du^3} \neq 0. \quad (17)$$

If (17) satisfied then the structure equations of the coframe are

$$\begin{aligned} d\theta^1 &= J_3\theta^1 \wedge \xi^2 - 8J_1\theta^2 \wedge \xi^1 + J_2\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\ d\theta^2 &= -\theta^1 \wedge \xi^1 + \frac{J_3}{3}\theta^2 \wedge \xi^2 + \xi^2 \wedge \sigma_2^2, \\ d\xi^1 &= \theta^1 \wedge \xi^2 + \theta^2 \wedge \xi^2 - (16J_1 + \frac{J_3}{2})\xi^1 \wedge \xi^2, \\ d\xi^2 &= 0, \\ d\sigma_2^2 &= -16J_1\theta^1 \wedge \xi^1 + 2(20J_1 - J_2)\theta^1 \wedge \xi^2 - J_2\theta^2 \wedge \xi^1 - \theta^2 \wedge \sigma_2^2 \\ &\quad - (64J_1^2 + 4J_1J_3 - 8J_1 + J_2)\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1 - \xi^2 \wedge \sigma_2^1 \\ &\quad - \frac{J_3}{2}\xi^2 \wedge \sigma_2^2. \end{aligned} \quad (18)$$

The expression for $d\sigma_2^2$ is too long to be written out in full here.

If $Q(u)$ is a quadratic polynomial then the structure equations of coframe is different from (18), and expressed only by $\{J_1, J_2, J_3\}$, where

$$\begin{aligned}
J_1 &= \frac{\left(\frac{d^2 Q}{du^2}\right)^3 \left(\left(\frac{d^2 Q}{du^2}\right)u_x^2 - Q\left(\frac{dQ}{du}\right)u_x + \left(\frac{dQ}{du}\right)u_t\right)}{\left(\frac{dQ}{du}\right)^6}, \\
J_2 &= -\frac{4\left(\frac{d^2 Q}{du^2}\right)^3 \left(\left(\frac{dQ}{du}\right)^4 Qu_x + \left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right)^3 u_x^2 \right.}{\left(\frac{dQ}{du}\right)^9} \\
&\quad - 2\left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Q^2 u_x + 6\left(\frac{d^2 Q}{du^2}\right)^2 \left(\frac{dQ}{du}\right) Qu_x^2 \\
&\quad - 4\left(\frac{d^2 Q}{du^2}\right)^3 u_x^3 + 2\left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Qu_t - 6\left(\frac{d^2 Q}{du^2}\right)^2 \left(\frac{dQ}{du}\right)u_x u_t \\
&\quad \left. - 2\left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 u_{tx} - \left(\frac{dQ}{du}\right)^4 u_t\right), \\
J_3 &= -\frac{8\left(\frac{d^2 Q}{du^2}\right)^2 \left(\left(\frac{dQ}{du}\right)^3 u_x + 2\left(\frac{d^2 Q}{du^2}\right)^3 u_x^2 - 2\left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right) Qu_x + 2\left(\frac{d^2 Q}{du^2}\right)\left(\frac{dQ}{du}\right)u_t\right)}{\left(\frac{dQ}{du}\right)^6},
\end{aligned} \tag{19}$$

are invariants of the symmetry group of an equation from Case-3.

All derived invariants of the group are expressed as functions of $\{J_1, J_2, J_3\}$. Therefore the rank of the coframe, is 3. Again by theorem 8.22 from [15], we have

Theorem 3 *If $Q(u)$ is a quadratic polynomial or satisfy $2\left(\frac{d^2 Q}{du^2}\right)^2 - \left(\frac{d^3 Q}{du^3}\right)\left(\frac{dQ}{du}\right) = 0$, $\left(\frac{d^3 Q}{du^3} \neq 0\right)$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 3.*

Case-4 We will make the following assumption for (1):

$$2\left(\frac{d^2 Q}{du^2}\right)^2 - \left(\frac{d^3 Q}{du^3}\right)\left(\frac{dQ}{du}\right) \neq 0, \quad \frac{d^3 Q}{du^3} \neq 0. \tag{20}$$

The structure equations of the coframe, in this case, is

$$\begin{aligned}
d\theta^1 &= -J_4\theta^1 \wedge \theta^2 + (4J_1J_3J_4 + J_3 + 32J_1^2J_4 + \frac{1}{8}J_3^2J_4 + 8J_1J_4)\theta^1 \wedge \xi^2 \\
&\quad - (16J_1^2J_4 + 2J_1J_3J_4 + \frac{1}{16}J_3^2J_4 - 8J_1)\theta^2 \wedge \xi^1 + \frac{J_4(16J_1 + J_3)}{2}\theta^1 \wedge \xi^1 \\
&\quad + (64J_1^3J_4 + \frac{1}{64}J_3^3J_4 + 16J_1^2J_4 + 12J_1^2J_3J_4 + \frac{3}{4}J_1J_3^2J_4 + J_1J_3J_4 \\
&\quad + J_2)\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1, \\
d\theta^2 &= -\theta^1 \wedge \xi^1 + \frac{J_4(16J_1 + J_3)}{4}\theta^2 \wedge \xi^1 + (2J_1J_3J_4 + \frac{1}{2}J_3 + 16J_1^2J_4 \\
&\quad + \frac{1}{16}J_3^2J_4 + 4J_1J_4)\theta^2 \wedge \xi^2 + \xi^2 \wedge \sigma_2^2, \\
d\xi^1 &= \theta^1 \wedge \xi^2 - \frac{J_4}{2}\theta^2 \wedge \xi^1 + (1 - 4J_1J_4 - \frac{1}{4}J_3J_4)\theta^2 \wedge \xi^2 \\
&\quad + (2J_1J_3J_4 - 16J_1 - \frac{1}{2}J_3 + 16J_1^2J_4 + \frac{1}{16}J_3^2J_4 - 4J_1J_4)\xi^1 \wedge \xi^2, \\
d\xi^2 &= -J_4\theta^2 \wedge \xi^2 + \frac{J_4(16J_1 + J_3)}{2}\xi^1 \wedge \xi^2.
\end{aligned} \tag{21}$$

The expression for $d\sigma_2^1, d\sigma_2^2$ are too long to be written out in full here.

There is an invariant extra to (19), for the symmetry group of equations from **case-4**, which is

$$J_4 = \frac{2\left(\frac{d^2Q}{du^2}\right)^2 - \left(\frac{d^3Q}{du^3}\right)\left(\frac{dQ}{du}\right)}{\left(\frac{d^2Q}{du^2}\right)^2}.$$

All derived invariants of the group are functionally expressed as functions of $\{J_1, J_2, J_3, J_4\}$. The rank of the coframe, is 4, therefore we have:

Theorem 4 If $Q(u)$ satisfy, $2\left(\frac{d^2Q}{du^2}\right)^2 - \left(\frac{d^3Q}{du^3}\right)\left(\frac{dQ}{du}\right) \neq 0$ and $\left(\frac{d^3Q}{du^3} \neq 0\right)$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 2.

4 Conclusion

In this paper, the moving coframe method of [18] is applied to the local equivalence problem for a class of systems of the general form of Burgers' equations under the action of a pseudo-group of contact transformations. We have found four subclasses and showed that every type of the general form of Burgers' equations belongs to a system from one of these subclasses. The equivalence condition of first subclass, structure equations and invariants for all subclasses are found.

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Declarations

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