A Note on Function Space and Boundedness of the General Fractional Integral in Continuous Time Random Walk

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Abstract

The general fractional calculus becomes popular in continuous time random walk recently. However, the boundedness condition of the general fractional integral is one of the fundamental problems. It wasn't given yet. In this short communication, the classical norm space is used, and a general boundedness theorem is presented. Finally, various long-tailed waiting time probability density functions are suggested in continuous time random walk since the general fractional integral is well defined.

Keywords General fractional integral · Minkowski inequality · Boundedness

Mathematics Subject Classification 26A33 · 34A08 · 35R11

1 Introduction

A fractional integral was proposed with general kernels in [1–3]

$$I_{a+}^{\alpha, g} f(t) := \int_a^t \frac{1}{\Gamma(\alpha)} (g(t) - g(s))^{\alpha-1} g'(s) f(s) ds,$$

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where α is a positive real number, $g(t)$ is strictly increasing function on $[a, b]$ and $g(a) \geq 0$. Due to the new features in comparison with the standard fractional derivatives, much attention has been paid to theoretical research and applications, for example, fractional calculus of variations [4], the Laplace transform [5, 6], exact solution [7, 8] and numerical methods [9].

Motivated by the continuous time random walk understood by means of the standard fractional calculus [10, 11], suppose a long-tailed waiting time probability density function [12]

$$\phi(t) \sim \frac{\alpha A_\alpha}{\Gamma(1-\alpha)(g(t)-g(a))^{\alpha+1}} \quad (1)$$

which is more general than the power law function

$$\phi(t) \sim \frac{\alpha A_\alpha}{\Gamma(1-\alpha)(t-a)^{\alpha+1}} \quad (2)$$

Then, one comes across a general time-fractional Fokker–Planck equation with the general fractional integral

$$P(x, t) - P(x, 0) = I_{a^+}^{\alpha, g} \left(K_\alpha \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{F(x)}{k_b T} \right) P(x, t) \right), 0 < \alpha \leq 1. \quad (3)$$

where T is the temperature, $P(x, t)$ is the probability density function, K_α is the diffusion coefficient, $F(x)$ represents external force field and k_b represents the Boltzmann constant. So the physical meaning of the general fractional derivative or the kernel function $g(t)$ was provided. More details were shown in [12]. However, one of the fundamental problems is still not addressed yet. That is, what is the function space of the general fractional integral and what are the constraint conditions with respect to $g(t)$?

A norm of the general fractional integral was recently given (see [13]). However, the condition $c = \frac{1}{p}$ and $g(t) = t$ is too strong there. For a general function $g(t) \neq t$, it cannot be reduced to the Lebesgue space $L_p[a, b]$ ($1 \leq p$). In order to address this problem and improve the existing results, we reconsider the classical norm used by Kilbas et al for the Riemann–Liouville integral [3]. We present the boundedness of the general fractional integral operator in $X_c^p(a, b)$ and we give the constraint conditions of $g(t)$.

2 General Fractional Integral

Let us revisit the general fractional integral in $X_c^p(a, b)$ with the norm.

Definition 2.1 [3] Let $c \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < a < b < \infty$. The space $X_c^p(a, b)$ is defined to consist of those complex-valued Lebesgue measurable functions on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, with

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} \quad (1 \leq p < \infty, c \in \mathbb{R}) \quad (4)$$

and

$$\|f\|_{X_c^\infty} = \operatorname{ess\,sup}_{a \leq t \leq b} [t^c |f(t)|] \quad (p = \infty). \quad (5)$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(a, b)$ coincides with the space $L_p[a, b]$ with

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \quad (1 \leq p < \infty, c \in \mathbb{R})$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{a \leq t \leq b} |f(t)| \quad (p = \infty).$$

The following proves that the norm defined in $X_c^p(a, b)$ satisfies three axioms.

Lemma 2.2 For arbitrary $f(t)$ and $h(t) \in X_c^p(a, b)$, $k \in \mathbb{R}$, (4) and (5) can be a norm of $X_c^p(a, b)$.

Proof For $1 \leq p < \infty$, obviously, $\|f\|_{X_c^p} \geq 0$ and $\|f\|_{X_c^p} = 0$ if and only if $f = 0$.

$$\|kf\|_{X_c^p} = \left(\int_a^b |t^c kf(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} = |k| \|f\|_{X_c^p} \quad (k \in \mathbb{R}).$$

The trigonometric inequality is true. In fact, from Minkowski inequality $\|f + h\|_{L_p} \leq \|f\|_{L_p} + \|h\|_{L_p}$, we have

$$\begin{aligned} \|f + h\|_{X_c^p} &= \left(\int_a^b |t^c (f(t) + h(t))|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &= \left(\int_a^b |t^{c-\frac{1}{p}} (f(t) + h(t))|^p dt \right)^{\frac{1}{p}} \\ &\leq \left(\int_a^b |t^{c-\frac{1}{p}} f(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_a^b |t^{c-\frac{1}{p}} h(t)|^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} + \left(\int_a^b |t^c h(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &= \|f\|_{X_c^p} + \|h\|_{X_c^p}. \end{aligned} \quad (6)$$

Moreover, since $f(t)$ and $h(t) \in X_c^p(a, b)$ we know that $\|f\|_{X_c^p} < \infty$ and $\|h\|_{X_c^p} < \infty$. Therefore, from the above formula, we derive that $t^{c-\frac{1}{p}} f(t)$ and $t^{c-\frac{1}{p}} h(t)$ are L_p

integrable. Thus, the inequality (6) holds and $X_c^p(a, b)$ ($1 \leq p \leq \infty$) is a normed linear space.

For $p = \infty$, the norm (5) also can be easily verified. $\square$

The general fractional integral can be derived by the n -fold integral method from [3]. Suppose $n \in \mathbb{N} = \{1, 2, \dots\}$ and $g(t) \in C^1[a, b]$ is a strictly increasing function and $g(a) \geq 0$ for $t \in [a, b]$, the n -fold integral reads

$$\begin{aligned} I_{a^+}^{n,g} f(t) &:= \int_a^t g'(t_1) dt_1 \int_a^{t_1} g'(t_2) dt_2 \cdots \int_a^{t_{n-1}} g'(s) f(s) ds \\ &= \int_a^t g'(s) f(s) ds \int_s^t g'(t_1) dt_1 \int_s^{t_1} g'(t_2) dt_2 \cdots \int_s^{t_{n-2}} g'(t_{n-1}) dt_{n-1} \quad (7) \\ &= \frac{1}{\Gamma(n)} \int_a^t (g(t) - g(s))^{n-1} g'(s) f(s) ds. \end{aligned}$$

As a result, for any positive real number α , we give the definition general fractional integral in $X_c^p(a, b)$ space as follows.

Definition 2.3 (General fractional integral) Suppose $f(t) \in X_c^p(a, b)$ and $g(t) \in C^1[a, b]$ is a strictly increasing function with $g(a) \geq 0$. The general fractional integral of order α ($\alpha > 0$) is defined by

$$I_{a^+}^{\alpha,g} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} g'(s) f(s) ds. \quad (8)$$

For some specific functions, we obtain some well known definitions of fractional integrals, for example, the Riemann–Liouville integral [3] when $g(t) = t$, the Hadamard integral when $g(t) = \ln t$ [3] and the generalized integral [14] when $g(t) = \frac{t^\sigma}{\sigma+1}$, the fractional integral with exponential memory [15, 16] when $g(t) = e^{\lambda t}$.

The general fractional integral operator $I_{a^+}^{\alpha,g}$ is bounded in the space $X_c^p(a, b)$ ($1 \leq p \leq \infty$). We assume $a > 0$ and $g(a) > 0$ for simplicity in the sequel.

Theorem 2.4 (Boundedness theorem) Let $f \in X_c^p(a, b)$ and $u = \frac{g(t)}{g(s)}$ where $a \leq s \leq t$, $1 \leq p \leq \infty$, $c \geq \frac{1}{p}$. If there exists a function $F(u) \in C[1, \frac{g(b)}{g(a)}]$ such that $g^{-1}(ug(s)) \leq F(u)s$ and $\frac{d(g^{-1}(ug(s)))}{ds} \leq F(u)$, respectively, then

$$\|I_{a^+}^{\alpha,g} f(t)\|_{X_c^p} \leq K \|f\|_{X_c^p}$$

where

$$K := \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} F(u)^c du. \quad (9)$$

Proof For $1 \leq p < \infty$, since $g(a) > 0$, from (4) and (8), we have

$$\begin{aligned}\|I_{a^+}^{\alpha,g}f(t)\|_{X_c^p} &= \left(\int_a^b \left| t^c \int_a^t \frac{1}{\Gamma(\alpha)} (g(t) - g(s))^{\alpha-1} g'(s) f(s) ds \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &= \left(\int_a^b dt \left| \int_1^{\frac{g(t)}{g(a)}} \frac{t^{c-\frac{1}{p}}}{\Gamma(\alpha)} \left(\frac{g(t)}{u} \right)^{\alpha-1} (u-1)^{\alpha-1} f(g^{-1}(\frac{g(t)}{u})) \frac{g(t)}{u^2} du \right|^p \right)^{\frac{1}{p}}\end{aligned}$$

Since $f(t) \in X_c^p(a, b)$ and $g(t) \in C^1[a, b]$ is a strictly increasing function, then $t^{c-\frac{1}{p}} g(t)^\alpha f(g^{-1}(\frac{g(t)}{u})) \in L_p[a, b]$. Hence, we can apply the general Minkowski inequality [2] and give

$$\begin{aligned}\|I_{a^+}^{\alpha,g}f(t)\|_{X_c^p} &\leq \int_1^{\frac{g(b)}{g(a)}} \frac{1}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} du \left(\int_{g^{-1}(ug(a))}^b \left| t^{c-\frac{1}{p}} g(t)^\alpha f(g^{-1}(\frac{g(t)}{u})) \right|^p dt \right)^{\frac{1}{p}} \\ &\leq \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} du \left(\int_{g^{-1}(ug(a))}^b \left| t^{c-\frac{1}{p}} f(g^{-1}(\frac{g(t)}{u})) \right|^p dt \right)^{\frac{1}{p}} \\ &= \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} du \\ &\quad \left(\int_a^{g^{-1}(\frac{g(b)}{u})} \left| (g^{-1}(ug(s)))^{c-\frac{1}{p}} f(s) \right|^p (g^{-1}(ug(s)))' ds \right)^{\frac{1}{p}}.\end{aligned}$$

It follows from that $g^{-1}(ug(s)) \leq F(u)s$, $\frac{d(g^{-1}(ug(s)))}{ds} \leq F(u)$, $u \geq 1$ and $c \geq \frac{1}{p}$ that

$$\begin{aligned}&\left(\int_a^{g^{-1}(\frac{g(b)}{u})} \left| (g^{-1}(ug(s)))^{c-\frac{1}{p}} f(s) \right|^p (g^{-1}(ug(s)))' ds \right)^{\frac{1}{p}} \\ &\leq F(u)^c \left(\int_a^b \left| s^{c-\frac{1}{p}} f(s) \right|^p ds \right)^{\frac{1}{p}}\end{aligned}$$

and

$$\|I_{a^+}^{\alpha,g}f(t)\|_{X_c^p} \leq \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} F(u)^c du \cdot \|f\|_{X_c^p}.$$

Due to $F(u) \in C[1, \frac{g(b)}{g(a)}]$, we note that $K = \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} F(u)^c du < \infty$.

For $p = \infty$, $\|I_{a^+}^{\alpha,g}f(t)\|_{X_c^\infty} = \text{ess sup}_{a \leq t \leq b} [t^c |I_{a^+}^{\alpha,g}f(t)|]$.

$$\begin{aligned}
\left| t^c (I_{a^+}^{\alpha, g} f(t)) \right| &= \left| t^c \int_a^t \frac{1}{\Gamma(\alpha)} (g(t) - g(s))^{\alpha-1} f(s) g'(s) ds \right| \\
&= \left| \int_1^{\frac{g(t)}{g(a)}} \frac{t^c}{\Gamma(\alpha)} \left(\frac{g(t)}{u} \right)^{\alpha-1} (u-1)^{\alpha-1} f(g^{-1}(\frac{g(t)}{u})) \frac{g(t)}{u^2} du \right| \\
&\leq g(b)^\alpha \left| \int_1^{\frac{g(b)}{g(a)}} \frac{t^c}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} f(g^{-1}(\frac{g(t)}{u})) du \right| \\
&= g(b)^\alpha \left| \int_1^{\frac{g(b)}{g(a)}} \frac{(g^{-1}(ug(s)))^c}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} f(s) du \right| \\
&\leq \left| \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} F(u)^c s^c f(s) du \right| \\
&\leq \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)} \frac{(u-1)^{\alpha-1}}{u^{\alpha+1}} F(u)^c du \cdot \|f\|_{X_c^\infty}.
\end{aligned}$$

As a result, this completes the proof. $\square$

For $c = \frac{1}{p}$ in Definition 2.1, we return to the space $L_p[a, b]$ which is a special case of $X_c^p(a, b)$. The general fractional integral $I_{a^+}^{\alpha, g}$ is bounded in $L_p[a, b]$ ($1 \leq p \leq \infty$, $0 < a < b < \infty$). It can be derived from Theorem 2.4.

Corollary 2.5 *Let $c = \frac{1}{p}$ in $X_c^p(a, b)$ and $f(t) \in L_p[a, b]$, then the general fractional integral is bounded, that is*

$$\|I_{a^+}^{\alpha, g} f(t)\|_p \leq K \|f(t)\|_p$$

where the constant K is defined by (9).

3 Conclusion

We can verify that Theorem 2.4 can hold for several known fractional integrals:

- when $g(s) = s$ (The Riemann–Liouville integral [3]), we can find a function $F(u) = u$ such that $g^{-1}(ug(s)) \leq F(u)s$ and $\frac{d(g^{-1}(ug(s)))}{ds} \leq F(u)$, hence K is defined by (9) where $F(u) = u$.
- when $g(s) = \ln s$ ($s > 1$) (The Hadamard integral [3]), we also can choose $F(u) = \frac{\ln b}{\ln a} b^{u-1}$ such that those two conditions hold, hence K is given where $F(u) = \frac{\ln b}{\ln a} b^{u-1}$.

- when $g(s) = \frac{s^{\beta+1}}{\beta+1}$ ($\beta > -1$) (The generalized fractional integral [14]), we can use $F(u) = u^{\frac{1}{\beta+1}}$ and two constraint conditions with respect to $F(u)$ and $g(s)$ hold.
- when $g(s) = \frac{e^{\lambda s}}{\lambda}$ ($\lambda \geq \frac{1}{a}$) (The fractional integral with exponential memory [16]), we can take $F(u) = u$ to satisfy the conditions of Theorem 2.4.

As a result, we can conclude that the function space X_c^p and the general fractional integral are well-defined when we use the classical norm (Definition 2.1). The above $g(t)$ with constraints conditions can be employed as long-tailed waiting time probability density function. On the other hand, they are well-defined in the general fractional calculus.

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