Almost Cosymplectic (k, μ) -metrics as η -Ricci Solitons

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Abstract

In this paper, we study η -Ricci solitons on almost cosymplectic (k, μ) -manifolds. As an application, it is proved that if an almost cosymplectic (k, μ) -metric with $k < 0$ represents a Ricci soliton, then the potential vector field of the Ricci soliton is a strict infinitesimal contact transformation, and the corresponding almost cosymplectic manifold is locally isometric to a Lie group whose local structure is determined completely by $k < 0$. In addition, a concrete example is constructed to illustrate the above result.

Keywords Almost cosymplectic manifold · (k, μ) -nullity distribution · η -Ricci soliton · Lie group

Mathematics Subject Classification Primary 53D15 · Secondary 53C25

1 Introduction

Let (M, g) be a Riemannian manifold, and Ric its Ricci tensor with respect to g . For the Lie differentiation $\mathcal{L}$ on M , if there exists a constant λ and a non-zero vector field V on M satisfying

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} = \lambda g, \quad (1)$$

then the metric g is said to be a Ricci soliton (see Hamilton [14, 15]), and λ and V are said to be the soliton constant and the potential vector field, respectively. In particular, if V is the gradient of certain smooth function f for a Ricci soliton, then (1) becomes

$$\nabla \nabla f + \text{Ric} = \lambda g, \quad (2)$$

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and it is said to be a gradient Ricci soliton. A Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negative, respectively. Obviously, a Ricci soliton equation is an extension of Einstein metric because it reduces to an Einstein metric if V is homothetic, that is, the Lie differentiation of g along V is a constant multiple of the metric g . Therefore, a Ricci soliton is said to be trivial if the potential vector field is Killing. Moreover, Ricci solitons are self-similar solutions to the Ricci flow

$$\frac{\partial}{\partial t} g_t + 2\text{Ric}(g_t) = 0,$$

where g_t evolves from an initial metric $g = g_0$ by a diffeomorphism and a scaling factor.

The notion of Ricci solitons is very important in modern differential geometry and has been studied extensively by a large number of geometers. The study of Ricci solitons in almost contact Riemannian geometry was initiated by Sharma [20] who proved that if the metric of a complete K -contact manifold represents a gradient Ricci soliton, then the soliton must be shrinking and the manifold is compact Einstein and Sasakian. Such a result has stimulated the great research interest of many scholars who worked in almost contact Riemannian geometry. In the last decade, there are lots of research on Ricci solitons in almost contact Riemannian geometry. Next we exhibit only those interesting results related with Ricci solitons in cosymplectic geometry.

Cho [9] started the study of Ricci solitons on cosymplectic manifolds. He proved that the metric of a three-dimensional cosymplectic manifold being a Ricci soliton with the Reeb potential vector field or a unit transversal (i.e., perpendicular to the Reeb vector field) potential vector field is locally flat. Later, such a result was generalized by Y. Wang in [24] who proved that if the metric of a three-dimensional cosymplectic manifold is a Ricci soliton, then either the manifold is locally flat or the potential vector field is an infinitesimal contact transformation. Moreover, a concrete example for the former case has been constructed in [24]. Where by an infinitesimal contact transformation defined on an almost contact metric manifold with almost contact 1-form η , the author in [24] refers to a vector field satisfying

$$\mathcal{L}_V \eta = \sigma \eta \quad (3)$$

for certain smooth function σ . In particular, if $\sigma = 0$, an infinitesimal contact transformation is said to be strictly. Very recently, the present author in [22] generalized Y. Wang's result (see [24]) for K -cosymplectic manifolds with arbitrary dimensions greater than three under η -Einstein conditions. A concrete example of three-dimensional cosymplectic manifolds admitting an infinitesimal contact transformation has been constructed. By applying the notion of Ricci solitons, Y. Wang in [23] obtained a generalization of the Goldberg's conjecture for compact almost cosymplectic manifolds. Moreover, Ricci solitons on almost cosymplectic manifolds under the η -Einstein conditions having constant coefficients were also studied by Y. Wang [26]. For some other results related with (gradient) Ricci solitons on (almost) cosymplectic manifolds, we refer the reader to [6, 7, 12, 21].

In view of those interesting results regarding Ricci solitons on (almost) cosymplectic manifolds, in the present paper, we aim to investigate η -Ricci solitons on a special type of almost cosymplectic manifolds, namely the (k, μ) -almost cosymplectic manifolds, i.e., the Reeb vector field belongs to the so called (k, μ) -nullity distribution. Where by an η -Ricci soliton defined on an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we mean the metric g of M satisfying

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} = \lambda g + \delta \eta \otimes \eta \quad (4)$$

for certain constants λ and δ . An η -Ricci soliton is said to be a gradient η -Ricci soliton if the potential vector field is the gradient of certain function. Obviously, an η -Ricci soliton reduces to the classical Ricci soliton when $\delta = 0$. The notion of η -Ricci solitons was first introduced by Cho and Kimura [10] in which an η -Ricci soliton means (4) for $V = \xi$. In general, one regards η -Ricci solitons in almost contact metric manifolds as an analogy of Ricci solitons in Riemannian manifolds.

Very recently, η -Ricci solitons on contact metric, Kenmotsu and cosymplectic manifolds have been considered in [13, 18, 22], respectively. Wang [23] proved that there exist no gradient Ricci solitons on a non-cosymplectic almost cosymplectic (k, μ) -manifold. Extending this result, in this paper we prove that if the metric of a non-cosymplectic almost cosymplectic (k, μ) -manifold is a Ricci soliton, then the potential vector field of the Ricci soliton is a strict infinitesimal contact transformation, and $\mu = 0$. Moreover, the corresponding almost cosymplectic manifold is locally isometric to a Lie group G whose local structure is determined completely by $k < 0$. Finally, for the 3-dimensional case, we also construct an almost cosymplectic k -manifold with $k < 0$ whose metric represents a non-trivial expanding Ricci soliton to illustrate our main result. Besides, some interesting corollaries are also provided.

2 Almost cosymplectic manifolds

By an almost contact manifold we mean a smooth differentiable manifold M of dimension $2n + 1$, $n \geq 1$, together with a $(1, 1)$ -type tensor field ϕ , a $(1, 0)$ -type tensor field ξ and a $(0, 1)$ -type tensor field η satisfying the following two tensorial equations

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (5)$$

where I denotes the identity transformation. If in addition there exists a Riemannian metric g on an almost contact manifold M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (6)$$

for any vector fields X, Y , then M is said to be an almost contact metric (Riemannian) manifold and g is said to be a compatible metric with respect to the almost contact structure (ϕ, ξ, η) (see Blair [2]).

An almost cosymplectic manifold is defined as an almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 0$, where the fundamental two-form Φ of the almost

contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M (see [2, 16]). On the product $M \times \mathbb{R}$ of an almost contact metric manifold M and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right),$$

where X denotes a vector field tangent to M , t is the coordinate of $\mathbb{R}$ and f is a C^∞ -function on $M \times \mathbb{R}$. If J is integrable, the almost contact metric structure on M is said to be normal. A normal almost cosymplectic manifold is said to be a cosymplectic manifold (see [3]). An almost cosymplectic manifold is a cosymplectic manifold if and only if

$$\nabla \phi = 0 \text{ (or equivalently, } \nabla \Phi = 0\text{)}. \quad (7)$$

We remark that in this paper the notion of (almost) cosymplectic manifolds follows from D. E. Blair [3], and it corresponds to the notion of (almost) coKähler manifolds in [4], but different to the notions provided in [1]. With regard to an introduction of relationships among these similar notions, we refer the reader to [4].

Let M be an almost cosymplectic manifold. There exist two important tensor fields $h = \frac{1}{2} \mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on M , where $\mathcal{L}$ denotes the Lie differentiation. The behavior of these two tensor fields are critical for geometry of almost cosymplectic manifolds (see survey on cosymplectic geometry in [4]). According to Olszak [16], both the two $(1, 1)$ -tensor fields h and h' are symmetric and they satisfy $h\xi = 0$, $\text{tr}h = 0$, $\text{tr}(h') = 0$ and $h\phi + \phi h = 0$ and

$$\nabla \xi = h', \quad (8)$$

$$\phi l \phi - l = 2h^2, \quad (9)$$

where $l := R(\cdot, \cdot)\xi$ and R is the Riemannian curvature tensor.

3 Almost cosymplectic (k, μ) -metrics as η -Ricci solitons

Let (M, ϕ, ξ, η, g) be an almost cosymplectic manifold of dimension $2n + 1$, $n \geq 1$. The so called almost cosymplectic (k, μ) -manifold is defined as an almost cosymplectic manifold such that the Reeb vector field belongs to the following (k, μ) -nullity distribution

$$N_p(k, \mu) := \{Z \in T_p M \mid R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}$$

for certain constants k and μ , where X, Y denote arbitrary vector fields on M . On an almost cosymplectic (k, μ) -manifold, by definition we have

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \quad (10)$$

In particular, when $\mu = 0$, an almost cosymplectic (k, μ) -manifold becomes an almost cosymplectic k -manifold studied by Dacko [11].

It is easily seen that (10) implies $l = -k\phi^2 + \mu h$. Substituting this into (9), we obtain

$$h^2 = k\phi^2. \quad (11)$$

Obviously, from (11) and (5) we obtain $k \leq 0$. The equality holds if and only if $h = 0$, or equivalently, with the help of (8), the Reeb vector field is Killing. We remark that $h = 0$ is the necessary condition for an almost cosymplectic manifold to be cosymplectic. In fact, as pointed out in Section one, Ricci solitons on almost cosymplectic manifolds with $h = 0$ have been considered in [9, 22, 24], therefore in this section we investigate only Ricci solitons on non-cosymplectic almost cosymplectic (k, μ) -manifolds, namely $k < 0$.

Lemma 3.1 *The Ricci operator of an almost cosymplectic (k, μ) -manifold with $k < 0$ is $Q = \mu h + 2nk\eta \otimes \xi$ and hence the scalar curvature is a constant $r = 2nk$.*

This lemma can be seen in [4, 23]. Where by a Ricci operator Q with respect to the metric g , it is defined by $\text{Ric}(X, Y) = g(QX, Y)$.

Lemma 3.2 *On an almost cosymplectic (k, μ) -manifold with $k < 0$, we have*

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X = & k\eta(Y)\phi X - k\eta(X)\phi Y \\ & - 2kg(X, \phi Y)\xi - \mu\eta(Y)h'X + \mu\eta(X)h'Y \end{aligned} \quad (12)$$

for any vector fields X, Y .

The above lemma can be seen in [5, 17]. It is known that the induced almost Kählerian structure on the contact distribution $\ker \eta$ is integrable for an almost cosymplectic (k, μ) -manifolds with $k < 0$, that is,

$$(\nabla_X \phi)Y = g(hX, Y)\xi - \eta(Y)hX \quad (13)$$

for any vector fields X, Y . Actually, comparing (7) with (13) implies a characterization for an almost cosymplectic manifold to be cosymplectic, namely an almost cosymplectic manifold is cosymplectic if and only if the Reeb vector field is Killing and the induced almost Kählerian structure on $\ker \eta$ is integrable. Together (13) with (12), on an almost cosymplectic (k, μ) -manifolds with $k < 0$, we obtain

$$(\nabla_X h')Y - (\nabla_Y h')X = k\eta(Y)X - k\eta(X)Y + \mu\eta(Y)hX - \mu\eta(X)hY \quad (14)$$

for any vector fields X, Y . In particular, considering $X = \xi$ in (12), with the help of (8) and (11) we obtain

$$\nabla_\xi h = \mu h'. \quad (15)$$

Based on the above facts, by K. Yano's Lie derivative formulas we obtain

Lemma 3.3 *If the metric of an almost cosymplectic (k, μ) -manifold with $k < 0$ is an η -Ricci soliton, then*

$$(\mathcal{L}_V \text{Ric})(X, \xi) = -2nk\mu^2\eta(X) \quad (16)$$

for any vector field X .

Proof By definition, if the metric g of an almost cosymplectic (k, μ) -manifold with $k < 0$ is an η -Ricci soliton, from (4) we have

$$\mathcal{L}_V g = 2\lambda g + 2(\delta - 2nk)\eta \otimes \eta - 2\mu g(h, \cdot), \quad (17)$$

where we have applied Lemma 3.1. Since the Riemannian metric g is parallel and k, μ are constants, taking the covariant differentiation of (17) along arbitrary vector field X , we get

$$\nabla_X \mathcal{L}_V g = 2(\delta - 2nk)((\nabla_X \eta) \otimes \eta + \eta \otimes (\nabla_X \eta)) - 2\mu g((\nabla_X h) \cdot, \cdot). \quad (18)$$

By (8), a direct calculation gives

$$(\nabla_X \eta)Y = g(h'X, Y)$$

for any vector fields X, Y . Substituting the above relation into (18) gives

$$\begin{aligned} (\nabla_X \mathcal{L}_V g)(Y, Z) = & 2(\delta - 2nk)g(h'X, Y)\eta(Z) \\ & + 2(\delta - 2nk)g(h'X, Z)\eta(Y) - 2\mu g((\nabla_X h)Y, Z) \end{aligned} \quad (19)$$

for any vector fields X, Y, Z .

From K. Yano [27, pp. 23] we have the following formula

$$(\nabla_Z \mathcal{L}_V g - \mathcal{L}_V \nabla_Z g + \nabla_{[V, Z]} g)(X, Y) = g((\mathcal{L}_V \nabla)(Z, X), Y) + g((\mathcal{L}_V \nabla)(Z, Y), X)$$

for any vector fields X, Y, Z . Note that on a Riemannian manifold (M, g) , the Riemannian metric g is parallel. Thus, it follows directly that

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = g((\mathcal{L}_V \nabla)(Z, X), Y) + g((\mathcal{L}_V \nabla)(Z, Y), X) \quad (20)$$

for any vector fields X, Y, Z . In view of the symmetry of the $(1, 2)$ -type tensor field $\mathcal{L}_V \nabla$, i.e., $(\mathcal{L}_V \nabla)(X, Y) = (\mathcal{L}_V \nabla)(Y, X)$, interchanging cyclicly the roles of X, Y, Z in (20), we obtain

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X, Y), Z) = & \frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(Z, X) \\ & - \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X, Y) \end{aligned} \quad (21)$$

for any vector fields X, Y, Z . Now, substituting (19) into (21), we obtain

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X, Y), Z) = & 2(\delta - 2nk)g(h'X, Y)\eta(Z) - \mu g((\nabla_Y h)X, Z) \\ & + \mu g((\nabla_Z h)X - (\nabla_X h)Z, Y) \end{aligned} \quad (22)$$

for any vector fields X, Y, Z . Substituting (12) into (22), we get

$$\begin{aligned}
g((\mathcal{L}_V \nabla)(X, Y), Z) &= 2(\delta - 2nk)g(h'X, Y)\eta(Z) - \mu g((\nabla_Y h)X, Z) + k\mu\eta(X)g(\phi Z, Y) \\
&\quad - k\mu\eta(Z)g(\phi X, Y) + 2k\mu g(X, \phi Z)\eta(Y) \\
&\quad - \mu^2\eta(X)g(h'Z, Y) + \mu^2\eta(Z)g(h'X, Y)
\end{aligned} \tag{23}$$

for any vector fields X, Y, Z . It follows immediately from (23) that

$$\begin{aligned}
(\mathcal{L}_V \nabla)(X, Y) &= (2\delta + \mu^2 - 4nk)g(h'X, Y)\xi - k\mu\eta(X)\phi Y \\
&\quad - k\mu g(\phi X, Y)\xi - 2k\mu\eta(Y)\phi X \\
&\quad - \mu^2\eta(X)h'Y - \mu(\nabla_Y h)X
\end{aligned} \tag{24}$$

for any vector fields X, Y .

In (24), replacing Y with ξ , with the aid of (15) we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = -2k\mu\phi X - \mu^2 h'X \tag{25}$$

for any vector field X . In view of $k, \mu \in \mathbb{R}$, taking the covariant differentiation of (25), we obtain

$$(\nabla_X \mathcal{L}_V \nabla)(Y, \xi) = -2k\mu(\nabla_X \phi)Y - \mu^2(\nabla_X h')Y$$

for any vector fields X, Y . Substituting (13) into the above relation, we obtain

$$(\nabla_X \mathcal{L}_V \nabla)(Y, \xi) = -2k\mu g(hX, Y)\xi + 2k\mu\eta(Y)hX - \mu^2(\nabla_X h')Y \tag{26}$$

for any vector fields X, Y .

Note that Yano [27, pp. 23] proved the following formula:

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$$

for any vector fields X, Y, Z . Now, substituting (26) into the above formula, we obtain

$$(\mathcal{L}_V R)(X, Y, \xi) = 2k\mu(\eta(Y)hX - \eta(X)hY) + \mu^2((\nabla_Y h')X - (\nabla_X h')Y) \tag{27}$$

for any vector fields X, Y . Substituting (14) into (27), we obtain

$$\begin{aligned}
(\mathcal{L}_V R)(X, Y, \xi) &= \mu(2k - \mu^2)(\eta(Y)hX - \eta(X)hY) \\
&\quad - k\mu^2(\eta(Y)X - \eta(X)Y)
\end{aligned} \tag{28}$$

for any vector fields X, Y . Finally, contracting X in (28), we obtain (16). $\square$

Lemma 3.4 *If the metric of an almost cosymplectic (k, μ) -manifold with $k < 0$ is an η -Ricci soliton, then either $\mu^2 + 4n^2k = 0$ or $\mathcal{L}_V \eta = -\frac{1}{2}\mu^2\eta$.*

Proof On an almost cosymplectic (k, μ) -manifold with $k < 0$, by a direct calculation we get

$$(\mathcal{L}_V \eta)X = g(\nabla_V \xi, X) + \eta(\nabla_X V) = (\mathcal{L}_V g)(X, \xi) + g(\mathcal{L}_V \xi, X) \tag{29}$$

for any vector field X . On the other hand, from (17) we get

$$(\mathcal{L}_V g)(X, \xi) = 2(\delta + \lambda - 2nk)\eta(X)$$

for any vector field X . In view of (8), combining the above relation with the previous one, we get

$$g(h'V, X) + \eta(\nabla_X V) = 2(\delta + \lambda - 2nk)\eta(X) + g(\mathcal{L}_V \xi, X) \quad (30)$$

for any vector field X . In particular, setting $X = \xi$ in (30) implies $\eta(\nabla_\xi V) = \delta + \lambda - 2nk$. Moreover, from Lemma 3.1 we obtain $\text{Ric}(X, \xi) = 2nk\eta(X)$ for any vector field X . Taking the Lie differentiation on the above relation, we have

$$(\mathcal{L}_V \text{Ric})(X, \xi) = 2nk(\mathcal{L}_V \eta)X - \text{Ric}(\mathcal{L}_V \xi, X) \quad (31)$$

for any vector field X . Substituting (29) into (31) and applying Lemma 3.1, we obtain

$$(\mathcal{L}_V \text{Ric})(X, \xi) = 6nk(\delta + \lambda - 2nk)\eta(X) + 2nkg(\mathcal{L}_V \xi, X) - \mu g(\mathcal{L}_V \xi, hX)$$

for any vector field X . Comparing this with (16), we obtain

$$2nk(3\delta + 3\lambda - 6nk + \mu^2)\eta(X) + 2nkg(\mathcal{L}_V \xi, X) - \mu g(\mathcal{L}_V \xi, hX) = 0 \quad (32)$$

for any vector field X .

In (32), with the help of $\eta(\nabla_\xi V) = \delta + \lambda - 2nk$, considering $X = \xi$ yields

$$2\delta + 2\lambda - 4nk + \mu^2 = 0, \quad (33)$$

where we have applied the assumption $k < 0$. On the other hand, it follows immediately from (32) that

$$\mu h \mathcal{L}_V \xi = 2nk(\delta + \lambda - 2nk)\xi + 2nk \mathcal{L}_V \xi, \quad (34)$$

where we have applied (33).

The action of h on (34) gives an equation which is simplified by using (11) implying

$$2nh \mathcal{L}_V \xi = -\mu(\delta + \lambda - 2nk)\xi - \mu \mathcal{L}_V \xi, \quad (35)$$

where we have applied again $k < 0$. Eliminating $\mathcal{L}_V \xi$, the subtraction of (34) multiplied by $2n$ from (35) multiplied by μ gives

$$(\mu^2 + 4n^2k)(\mathcal{L}_V \xi + (\delta + \lambda - 2nk)\xi) = 0.$$

Finally, notice that $\mathcal{L}_V \xi = -(\delta + \lambda - 2nk)\xi$ can be simplified by using (33) to $\mathcal{L}_V \xi = \frac{1}{2}\mu^2 \xi$, and this is also equivalent to $\mathcal{L}_V \eta = -\frac{1}{2}\mu^2 \eta$ because of (29), (30) and (33). This completes the proof. $\square$

Applying (3), the above lemma can also be written as the following form.

Lemma 3.5 *If the metric of an almost cosymplectic (k, μ) -manifold with $k < 0$ is an η -Ricci soliton, then either $\mu^2 + 4n^2k = 0$ or the potential vector field is an infinitesimal contact transformation.*

The potential vector field of an η -Ricci soliton is said to be transversal if it is orthogonal to the Reeb vector field ξ . By this, the first main theorem of this paper is given as the following.

Theorem 3.1 *If the metric of an almost cosymplectic (k, μ) -manifold satisfying $k < 0$ is an η -Ricci soliton whose potential vector field is transversal, then the potential vector field is an infinitesimal contact transformation.*

Proof By Lemma 3.5, it suffices to prove that the first case in this lemma does not occur. In fact, if the potential vector field of an η -Ricci soliton is transversal, by the proof of Lemma 3.5, we have $\eta(\nabla_\xi V) = \delta + \lambda - 2nk = 0$. If $\mu^2 + 4n^2k = 0$ is true, substituting this into (33) gives $\delta + \lambda - 4nk = 0$. Consequently, comparing this with the previous relation we get $k = 0$, a contradiction.

Note that (11) is still true if (k, μ) -nullity condition on an almost cosymplectic manifold was weakened to k -nullity condition. Therefore, in view of (11), from the above Lemma 3.5 we also have the following corollary.

Corollary 3.1 *If the metric of an almost cosymplectic k -manifold with $k < 0$ is an η -Ricci soliton, then the potential vector field is a strict infinitesimal contact transformation.*

Suppose that the metric g of an almost cosymplectic (k, μ) -manifold M with $k < 0$ is an η -Ricci soliton whose potential vector field is collinear with the Reeb vector field, i.e., $V = \eta(V)\xi$, with the help of (8), then we have

$$\nabla_X V = X(\eta(V))\xi + \eta(V)h'X,$$

for any vector field X , which is substituted into (17) giving

$$\begin{aligned} X(\eta(V))\eta(Y) + Y(\eta(V))\eta(X) + 2\mu g(hX, Y) + 2\eta(V)g(h'X, Y) \\ = 2\lambda g(X, Y) + 2(\delta - 2nk)\eta(X)\eta(Y) \end{aligned}$$

for any vector fields X, Y . In view of (11), in the above equation, if considering $X = Y$ two unit eigenvector fields of h' with eigenvalue $\sqrt{-k}$, we obtain $hX = \sqrt{-k}\phi X$ and hence $\lambda = \sqrt{-k}\eta(V)$. Similarly, when considering $X = Y$ two unit eigenvector fields of h' with eigenvalue $-\sqrt{-k}$, we obtain $hX = -\sqrt{-k}\phi X$ and hence $\lambda = -\sqrt{-k}\eta(V)$. Consequently, in view of $k < 0$, it follows directly that $\lambda = 0$ and $\eta(V) = 0$. This implies that the potential vector field vanishes identically and it is meaningless. In fact, when V vanishes identically, an η -Ricci soliton equation (4) becomes $\text{Ric} = \lambda g + \delta\eta \otimes \eta$, which is compared with Lemma 3.1 implying

$$\lambda g(X, Y) + (\delta - 2nk)\eta(X)\eta(Y) = \mu g(hX, Y)$$

for any vector fields X, Y . Considering the trace of the above relation by setting $X = Y$ we obtain $(2n + 1)\lambda + \delta - 2nk = 0$. On the other hand, setting $X = Y = \xi$ in the above relation we get $\lambda + \delta - 2nk = 0$, which is compared with the previous one

implying $\lambda = 0$ and $\delta = 2nk$. This gives also $\mu = 0$ and the soliton equation becomes Einstein metric. Based on these analyses, we obtain

Theorem 3.2 *There exist no almost cosymplectic (k, μ) -manifolds with $k < 0$ whose metric represents non-trivial η -Ricci solitons and the potential vector field is collinear with the Reeb vector field.*

Ricci solitons on almost cosymplectic (k, μ) -manifolds whose potential vector field is the Reeb vector field were studied by Suh and De [21].

Remark 3.1 The above Theorem 3.2 is an extension of [21, Theorem 5.1].

By Theorem 3.2, we also have the following corollary.

Corollary 3.2 *Let M be an almost cosymplectic (k, μ) -manifold satisfying $k < 0$. If the metric of M represents an η -Ricci soliton and the potential vector field is collinear with the Reeb vector field, then M is an almost cosymplectic k -manifold.*

Applying the above several lemmas, one of our main results can be given as the following.

Theorem 3.3 *If the metric of an almost cosymplectic (k, μ) -manifold M with $k < 0$ is a Ricci soliton, then the potential vector field is a strict infinitesimal contact transformation and M is locally isometric to a Lie group G whose local structure is determined completely by k .*

Proof For a Ricci soliton (1), according to R. Sharma [20, pp. 139], we have

$$\mathcal{L}_V r = \Delta r - 2\lambda r + 2\|Q\|^2, \quad (36)$$

where $\Delta f = \text{div}(\text{grad}f)$. In our case, by Lemma 3.1 the scalar curvature $r = 2nk$ is a constant. By a direct calculation, from Lemma 3.1 we have $\|Q\|^2 = \mu^2\|h\|^2 + 4n^2k^2$. Putting (11) into this relation, we obtain

$$\|Q\|^2 = 4n^2k^2 - 2nk\mu^2.$$

Now, taking into account of the above relation and $r = 2nk$, (36) becomes

$$\lambda + \mu^2 = 2nk, \quad (37)$$

where we have used again $k < 0$.

If the metric of an almost cosymplectic (k, μ) -manifold M satisfying $k < 0$ is a Ricci soliton, from Lemma 3.4, either $\mu^2 + 4n^2k = 0$ or $\mathcal{L}_V \eta = \frac{1}{2}\mu^2\eta$. However, the former case is impossible. In fact, if $\mu^2 + 4n^2k = 0$ is true, eliminating μ^2 , from (37) we obtain $\lambda = 2nk + 4n^2k$. Substituting this and $\mu^2 = -4n^2k$ into (33), we obtain $k = 0$, however, this contradicts our assumption. For the later case, comparing (37) with (33) we obtain $\mu = 0$. Using this in (37) we have

$$\lambda = 2nk. \quad (38)$$

In this situation, we also see that the potential vector field leaves the almost contact 1-form η invariant. Moreover, in view of $\mu = 0$, an almost cosymplectic (k, μ) -manifold becomes an almost cosymplectic k -manifold. We remark that P. Dacko in [11] proved that the manifold is locally isometric to a Lie group whose local structure is determined completely by $k < 0$. $\square$

According to (38) and proof of Theorem 3.3, we have the following corollary.

Corollary 3.3 *There exist neither shrinking nor steady Ricci solitons on almost cosymplectic (k, μ) -manifolds with $k < 0$.*

Remark 3.2 Chen [8] has corrected his Theorem 1.1 in [7] and proved that there do not exist Ricci solitons on an almost cosymplectic (k, μ) -manifold with $k < 0$ and $-\frac{1}{2} < \mu < 0$. Actually, such a result can be implied immediately from Theorem 3.3.

Let M be an almost cosymplectic (k, μ) -manifold with $k < 0$ whose metric is a gradient Ricci soliton, i.e., $V = Df$, where D denotes the gradient operator and f is a smooth function. According to (4) and (17) we have

$$\nabla_X Df = \lambda X + (\delta - 2nk)\eta(X)\xi - \mu hX \quad (39)$$

for any vector field X . With the help of (8), taking the covariant differentiation of the above relation, we obtain

$$\begin{aligned} \nabla_X \nabla_Y Df = & \lambda \nabla_X Y + (\delta - 2nk)(g(h'X, Y)\xi + \eta(\nabla_X Y)\xi + \eta(Y)h'X) \\ & - \mu(\nabla_X h)Y - \mu h \nabla_X Y \end{aligned}$$

for any vector fields X, Y . By the above relation, we get

$$R(X, Y)Df = (\delta - 2nk)(\eta(Y)h'X - \eta(X)h'Y) - \mu((\nabla_X h)Y - (\nabla_Y h)X)$$

for any vector fields X, Y . The application of Lemma 3.2 in the above relation gives

$$\begin{aligned} R(X, Y)Df = & (\delta - 2nk)(\eta(Y)h'X - \eta(X)h'Y) - k\mu\eta(Y)\phi X \\ & + k\mu\eta(X)\phi Y + 2k\mu g(X, \phi Y)\xi + \mu^2\eta(Y)h'X - \mu^2\eta(X)h'Y \end{aligned}$$

for any vector fields X, Y .

Taking the inner product of the above equation with ξ and substituting Y with ξ give

$$g(R(X, \xi)Df, \xi) = 0$$

for any vector field X . However, from $g(R(X, \xi)Df, \xi) + g(R(X, \xi)\xi, Df) = 0$ and (10) we obtain

$$\mu h Df = -k(Df - \xi(f)\xi).$$

With the aid of (11), the action of h on the above relation yields

$$hDf = \mu(Df - \xi(f)\xi), \quad (40)$$

where we have used the assumption $k < 0$. Substituting (40) into the previous one we obtain

$$(\mu^2 + k)(Df - \xi(f)\xi) = 0. \quad (41)$$

Therefore, it is easily seen that either $\mu^2 + k = 0$ or $Df = \xi(f)\xi$. Actually, the former case is impossible.

If $\mu^2 + k = 0$, in view of $k < 0$, according to Lemma 3.4 we obtain $\mathcal{L}_V \xi = \frac{1}{2}\mu^2 \xi$, or equivalently, $\nabla_\xi Df = h'Df - \frac{1}{2}\mu^2 \xi$, which is simplified by using (40) giving

$$\nabla_\xi Df = -\mu\phi Df - \frac{1}{2}\mu^2 \xi.$$

However, replacing X with ξ in (39) yields $\nabla_\xi Df = (\delta + \lambda - 2nk)\xi$, which is simplified by using (33) giving $\nabla_\xi Df = -\frac{1}{2}\mu^2 \xi$. Finally, comparing this with the above relation, we obtain $\mu\phi Df = 0$. If $\mu = 0$, from $\mu^2 + k = 0$ we obtain $k = 0$, a contradiction. For the case $\mu \neq 0$, $\phi Df = 0$ means that Df is collinear with the Reeb vector field. In this situation, it follows immediately from proof of Theorem 3.2 that such a case is meaningless. Therefore, Wang's result in [23, Theorem 4.1] becomes a direct corollary of the following theorem. Based on these analyses, we obtain

Theorem 3.4 *There exist no non-trivial gradient η -Ricci solitons on almost cosymplectic (k, μ) -manifolds with $k < 0$.*

4 Example

Before closing this paper, we construct a concrete example admitting a non-cosymplectic almost cosymplectic k -metric ($k < 0$) being a non-trivial expending Ricci soliton.

Example 4.1 Let M be a nonempty open subset of the Cartesian three-space $\mathbb{R}^3$ with global coordinates $\{x, y, z\}$. On M we define three vector fields as the following:

$$e_0 = \frac{\partial}{\partial z}, \quad e_1 = e^{-\sqrt{-k}z} \frac{\partial}{\partial x}, \quad e_2 = e^{\sqrt{-k}z} \frac{\partial}{\partial y}$$

for an arbitrary negative constant $k < 0$. An almost contact metric structure (ϕ, ξ, η, g) on M is defined as the following:

$$\begin{aligned} \phi &= e^{2\sqrt{-k}z} dx \otimes \frac{\partial}{\partial y} - e^{-2\sqrt{-k}z} dy \otimes \frac{\partial}{\partial x}, \\ g &= e^{2\sqrt{-k}z} dx \otimes dx + e^{-2\sqrt{-k}z} dy \otimes dy + dz \otimes dz, \\ \xi &:= e_0, \quad \eta := g(\xi, \cdot). \end{aligned}$$

One can check that $\{e_0, e_1, e_2\}$ are global orthonormal basis of M with respect to the Riemannian metric g . Moreover, we also observe $\phi e_0 = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$, i.e., $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. In addition, it is easily verified that the Riemannian metric g is compatible with respect to the $(1, 1)$ -type tensor field ϕ , i.e., $g(\phi e_i, \phi e_j) = \delta_{ij}$ for any $i, j \in \{0, 1, 2\}$. Let Φ be the fundamental two-form defined by $\Phi(\cdot, \cdot) = g(\cdot, \phi \cdot)$. Taking into account the above ϕ , we have

$$\Phi = -2dx \wedge dy.$$

Furthermore, by a simple calculation, it is easily seen that

$$d\eta = 0, \quad d\Phi = 0.$$

That is, M is an almost cosymplectic three-manifold. Moreover, by a direct calculation we obtain

$$[e_0, e_1] = -\sqrt{-k}e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_0] = -\sqrt{-k}e_2. \quad (42)$$

Actually, following the above relations, M is locally isometric to a 3-dimensional unimodular Lie group $E(2)$ of rigid motions of Euclidean 2-plane equipped with a left invariant Riemannian metric.

Furthermore, by using the well known Koszul formula, the Levi-Civita connection of g is given by

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{-k}e_1 & -\sqrt{-k}e_0 & 0 \\ -\sqrt{-k}e_2 & 0 & \sqrt{-k}e_0 \end{pmatrix} \quad (43)$$

with respect to $i, j \in \{0, 1, 2\}$. Taking into account (43), we obtain $h'e_1 = \sqrt{-k}e_1$ and $h'e_2 = -\sqrt{-k}e_2$. This means that M is a strictly almost cosymplectic manifold.

Applying (43), we observe that

$$R(e_1, e_2)e_0 = 0, \quad R(e_0, e_1)e_0 = -ke_1, \quad R(e_0, e_1)e_0 = -ke_2. \quad (44)$$

Obviously, M is a 3-dimensional almost cosymplectic k -manifold in which k can be an arbitrary negative constant.

On M , we set $V = 2ke^{\sqrt{-k}z}xe_1 + 2ke^{-\sqrt{-k}z}ye_2$. A direct calculation gives

$$\frac{1}{2}\mathcal{L}_V g = 2k(g - \eta \otimes \eta).$$

From this, one can check that the above metric g on M defines a non-trivial expanding Ricci soliton whose soliton constant $\lambda = 2k$ and the potential vector field is V .

An infinitesimal contact transformation has been constructed on a cosymplectic 3-manifold (see [22]). The above Example 4.1 first gives an infinitesimal contact transformation on a strict almost cosymplectic 3-manifold.

It is known that on any three-dimensional unimodular Lie group $E(2)$ of rigid motions of Euclidean 2-plane equipped with a left invariant Riemannian metric,

there exists a left invariant almost cosymplectic structure (see [19]). Based on the above Example 4.1, we have

Theorem 4.1 *A three-dimensional unimodular Lie group $E(2)$ of rigid motions of Euclidean 2-plane admits a left invariant almost cosymplectic metric being an expanding Ricci soliton.*

5 Conclusions

The geometry and topology of almost contact metric manifolds which admit Ricci solitons has been an active research field for a long time. In this paper, we aim to study η -Ricci solitons which is an extension of Ricci solitons on a special almost contact metric manifold, namely almost cosymplectic (k, μ) -manifold. It was proved that if a non-cosymplectic almost cosymplectic (k, μ) -metric represents a Ricci soliton, then the potential vector field of the Ricci soliton is a strict infinitesimal contact transformation, and the corresponding almost cosymplectic manifold is locally isometric to a Lie group whose local structure is determined completely by $k < 0$. This extends some earlier results in this field. In particular, when the potential vector field of the soliton is the gradient of a function, the corresponding situation was also investigated and a non-existence theorem was obtained. Finally, a concrete example was constructed to illustrate main results. This paper will motivate some further research concerning η -Ricci solitons on a more general almost cosymplectic manifold without (k, μ) -nullity condition, and even on some other almost contact metric manifolds.

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Declarations

Conflict of interest The author declares that there are no conflicts of interest.

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