A New Application of the $\bar{\partial}$ -Method

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Abstract

By constructing a new calculating rule of Lie bracket, we construct a new nonlinear Schrödinger hierarchy and its reduction equations via using the $\bar{\partial}$ -method. Furthermore, some soliton solutions of such the equation are obtained by making use of Dirac function.

Keywords Integrable hierarchy · Soliton solution · $\bar{\partial}$ -Method

1 Introduction

The inverse scattering method has been known a tool for constructing nonlinear integrable equations and solutions. An effective procedure for simultaneous construction of integrable systems and of their solutions, known presently as the dressing method, in two distinct variants in papers [1] and [2], which was used to construct solutions of nonlinear integrable equations by using infinite dimensional analogs of the Gauss' problem of factorizing a matrix into a product of two triangular ones. In paper [3], a systematic exposition of the procedure of constructing higher dimensional equations integrable by means of the Riemann problem was presented, and further the resources of the dressing method was extended which was applied to equations already known. By applying the $\bar{\partial}$ -dressing method, Dubrosky et al. [4–6] had generated some (2+1)-dimensional integrable equations, such as the Nizhik–Veselov–Novikov equation, the generalization of dispersive long wave equations, and so on, whose exact solutions, such as line solitons, rational solutions were produced, respectively. The starting point for the method presented in Refs. [4, 5] reads the linear spectral problems $L_1\psi = 0, L_2\psi = 0$, where L_1 and L_2 are nonlinear differential operators with respect to the “long derivatives”. In paper [6], with the help of the systems of L-A-B triad representation, the equivalent matrix

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commutativity representation is an efficient tool for working out two-dimensional integrable equations. In monograph [7], the detailed statement on the dressing method were introduced which extensively applied to nonlinear equations in mathematical physics. Based on the work, Fan [8] singled out some integrable equations and exact solutions by the $\bar{\partial}$ -dressing method. Luo and Fan [9] applied the $\bar{\partial}$ -method to get the coupled Gerdjikov–Ivanov equation and its solitons solutions. In the paper, we want to apply the $\bar{\partial}$ -dressing method improved by us to generate the generalization of the nonlinear Schrödinger (NLS) and its reduction equations, and further we work out some soliton solutions with the help of the Dirac functions. The novel point in the paper presents that a new Lie bracket is introduced, whose special case is just the standard classical Lie bracket. The work in the paper extends some known results in Ref. [7].

2 The Basic Ingredients of the $\bar{\partial}$ -Method

The $\bar{\partial}$ -dressing method is a very powerful method for the integrable systems and the solutions. The method was discovered by Zakharov [3]. The $\bar{\partial}$ -method allows one to construct Lax pairs, to solve initial and boundary value problems and to compute the broad classes of exact solutions [4–9]. Now we base on the known works as above, we simply recall the basic ingredients of the $\bar{\partial}$ -method.

We consider the special $\bar{\partial}$ -problem

$$\bar{\partial}\psi(x, t, z) = \psi(x, t, z)R(x, t, z), \quad (1)$$

where $z \in \mathbb{C}$ is any complex number, $\psi(x, t, z)$, $R(x, t, z)$ are 2×2 matrices. Assume that $\psi(x, t, z) \rightarrow I, z \rightarrow \infty$, then Eq. (1) has the solution

$$\psi(x, t, z) = I + \frac{1}{2\pi i} \int \int \frac{\psi(\xi)R(\xi)}{\xi - z} d\xi \wedge d\bar{\xi} =: I + \psi RC_z, \quad (2)$$

here C_z is known as left Cauchy–Green integral operator.

Define an inner product as follows

$$\langle f, g \rangle = \frac{1}{2\pi i} \int \int f(\xi)g^T(\xi)d\xi \wedge d\bar{\xi},$$

it is easy to verify that [7, 8]:

$$\begin{aligned} \langle f, g \rangle^T &= \langle f, g \rangle, \\ \langle fR, g \rangle &= \langle f, gR^T \rangle, \\ \langle fC_z, g \rangle &= -\langle f, gC_z \rangle, \\ \langle f(I - RC_z), g \rangle &= \langle f, g(I + R^T C_z) \rangle, \\ \langle h(I - RC_z)^{-1}, g \rangle &= \langle h, g(I + R^T C_z)^{-1} \rangle. \end{aligned}$$

Based on the above identities, it is easy to verify that

$$zI = \langle \psi R \rangle + z\psi(I - RC_z),$$

$$z(I - RC_z)^{-1} = \langle \psi R \rangle \psi + z\psi, \quad (3)$$

where

$$\langle \psi R \rangle =: \langle \psi R, I \rangle = \frac{1}{2\pi i} \iint \psi(\xi) R(\xi) d\xi \wedge d\bar{\xi},$$

$$iz\psi RC_z = i\langle \psi R \rangle + iz(\psi - I), \quad (4)$$

$$z^2(I - RC_z)^{-1} = (\langle \xi \psi R \rangle + \langle \psi R \rangle^2 + z\langle \psi R \rangle + z^2)\psi. \quad (5)$$

Concerning the verification on Eqs. (3)–(5), Fan has completed them in his monograph [8].

3 A New Integrable NSL Hierarchy and Some Soliton Solutions

Equation (2) can be written as

$$\psi = I \cdot (I - RC_z)^{-1}. \quad (6)$$

Thus, the derivative of ψ with respect to x reads

$$\psi_x = \psi R_x C_z (I - RC_z)^{-1}. \quad (7)$$

In Refs. [7, 8], R satisfies

$$R_x = iz[R, \sigma_3] = R\sigma_3 - \sigma_3 R, \quad (8)$$

$$R_t = [R, \Omega] = R\Omega - \Omega R, \quad (9)$$

where $\Omega(z)$ is a dispersion relation. Suppose $\Omega(z)$ comprises both a polynomial part $\Omega_p(z)$ and a singular (nonanalytic) part $\Omega_s(z)$. We put

$$\Omega = \Omega_p + \Omega_s = \alpha_n z^n \sigma_3 + \frac{1}{2\pi i} \iint \frac{d\xi \wedge d\bar{\xi}}{\xi - z} \omega(\xi) \sigma_3, \quad (10)$$

here α_n is a constant, and $\omega(z)$ is some scalar function. The NLS integrable hierarchy was obtained under the condition

$$\Omega_p = 2iz^2 \sigma_3. \quad (11)$$

In the paper, we introduce a new Lie bracket

$$\{F, G\} = FMG - GMF, \quad (12)$$

where M is a constant matrix. Obviously, when $M = I$, Eq. (12) reduces to the standard Lie bracket

$$[F, G] = FG - GF.$$

In addition, we replace iz with $iz + \lambda$ in Eq. (8), then we have

$$R_x = (iz + \lambda)\{R, \sigma_3\}, \quad (13)$$

$$R_t = \{R, \Omega\}. \quad (14)$$

Hence, applying Eq. (13) gives rise to

$$\begin{aligned} \psi_x &= (iz + \lambda)\psi\{R, \sigma_3\}C_z(I - RC_z)^{-1} \\ &= iz\psi RC_z M\sigma_3(I - RC_z)^{-1} - iz\psi\sigma_3 MRC_z(I - RC_z)^{-1} \\ &\quad + \lambda\psi RC_z(I - RC_z)^{-1} - \lambda\psi\sigma_3 MRC_z(I - RC_z)^{-1} \\ &= i\langle\psi R\rangle M\sigma_3\psi - iz\psi\sigma_3 M(I - RC_z)^{-1} + iz\psi\sigma_3 M \\ &\quad + iz\psi M\sigma_3(I - RC_z)^{-1} - izM\sigma_3(I - RC_z)^{-1} \\ &\quad + \lambda(\psi - I)M\sigma_3\psi - \lambda\psi\sigma_3 MRC_z(I - RC_z)^{-1}. \end{aligned} \quad (15)$$

Applying Eq. (3) to (15) yields

$$\begin{aligned} \psi_x &= i\{\langle\psi R\rangle, \sigma_3\}\psi + i\psi[M, \sigma_3]\langle\psi R\rangle\psi \\ &\quad + iz\psi[M, \sigma_3]\psi + iz\{\psi, \sigma_3\} + \lambda\psi[M, \sigma_3]\Psi + \lambda\{\psi, \sigma_3\}. \end{aligned} \quad (16)$$

Due to M being diagonal, Eq. (16) reduces to

$$\psi_x - iz\{\psi, \sigma\} - \lambda\{\psi, \sigma\} = Q\psi, \quad (17)$$

where

$$Q = i\{\langle\psi R\rangle, \sigma_3\}. \quad (18)$$

Thus, we arrive at a generalized Zakharov–Shabat spectral problem (17).

It is easy to see by using Eq. (14) that

$$\psi_t = \psi R_t C_z(I - RC_z)^{-1} = \psi\{R, \Omega\}C_z(I - RC_z)^{-1}. \quad (19)$$

By using Eqs. (4), (5) and (11), Eq. (19) becomes

$$\begin{aligned} \psi_t &= 2i(\langle\xi\psi R\rangle + z\langle\psi R\rangle + z^2(\psi - I))M\sigma_3(I - RC_z)^{-1} \\ &\quad - 2iz^2\psi\sigma_3 M[I - (I - RC_z)](I - RC_z)^{-1} \\ &= 2i\langle\xi\psi R\rangle M\sigma_3\psi + 2i\langle\psi R\rangle M\sigma_3(\langle\psi R\rangle\psi + z\psi) \\ &\quad - 2iM\sigma_3(\langle\xi\psi R\rangle\psi + \langle\psi R\rangle^2\psi + z\langle\psi R\rangle\psi + z^2\psi) \\ &= 2iz^2\{\psi, \sigma_3\} + 2iz\{\langle\psi R\rangle, \sigma_3\}\psi + 2i\{\langle\xi\psi R\rangle, \sigma_3\}\psi \\ &\quad + 2i\{\langle\psi R\rangle, \sigma_3\}\langle\psi R\rangle\psi. \end{aligned}$$

Hence, we find

$$\begin{aligned}\psi_t + 2iz^2\{\sigma_3, \psi\} &= 2i\{\langle \xi \psi R \rangle, \sigma_3\} + 2Q\langle \psi R \rangle\psi + 2zQ\psi \\ &= -2i(\text{tr}M)\sigma_3\langle \xi \psi R \rangle^{\text{off}}\psi + 2Q\langle \psi R \rangle\psi + 2zQ\psi.\end{aligned}\quad (20)$$

A direct calculation gives

$$\begin{aligned}(\psi R)_x &= Q\psi R + iz\{\psi, \sigma_3\}R + \lambda\{\psi, \sigma_3\}R + iz\psi RM\sigma_3 \\ &\quad - iz\psi\sigma_3 MR + \lambda\psi RM\sigma_3 - \lambda\psi\sigma_3 MR \\ &= Q\psi R + iz\{\sigma_3, \psi R\} + \lambda\{\psi R, \sigma_3\}.\end{aligned}$$

Thus, one gets

$$\begin{aligned}\langle \psi R \rangle_x &= \frac{1}{2\pi i} \iint (\psi R)_x d\xi \wedge d\bar{\xi} \\ &= Q\langle \psi R \rangle - i\{\sigma_3, \langle \psi R \rangle\} - \{\sigma_3, \lambda\langle \psi R \rangle\}, \\ \langle \psi R \rangle_x^{\text{off}} &= Q\langle \psi R \rangle^{\text{diag}} - i(\text{tr}M)\sigma_3\langle \psi R \rangle^{\text{off}} - \lambda(\text{tr}M)\sigma_3\langle \psi R \rangle^{\text{off}}.\end{aligned}$$

Therefore, one infers that

$$\begin{aligned}Q_x &= i(\text{tr}M)\sigma_3\langle \Psi R \rangle_x^{\text{off}} \\ &= -(\text{tr}M)^2\langle \psi R \rangle^{\text{off}} - i(\text{tr}M)\sigma_3Q\langle \psi R \rangle^{\text{diag}} + i\lambda(\text{tr}M)^2\langle \psi R \rangle^{\text{off}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\psi_t + 2iz^2\{\sigma_3, \psi\} &= 2i(\text{tr}M)^{-1}\sigma_3Q_x\psi - 2Q\langle \psi R \rangle^{\text{diag}}\psi \\ &\quad + 2\lambda(\text{tr}M)\sigma_3\langle \psi R \rangle^{\text{off}}\psi + 2Q\langle \psi R \rangle\psi + 2zQ\psi \\ &= 2i(\text{tr}M)^{-1}\sigma_3Q_x\psi + 2i(\text{tr}M)^{-1}Q\sigma_3Q\psi + 2i\lambda Q\psi + 2zQ\psi \\ &= (2i(\text{tr}M)^{-1}\sigma_3Q_x + 2i(\text{tr}M)^{-1}\sigma_3Q^2 + 2i\lambda Q + 2zQ)\psi,\end{aligned}\quad (21)$$

which is a generalized Zakharov–Shabat time-spectral problem.

In what follows, we want to work out a new integrable NLS hierarchy and its reduction based on Eqs. (17) and (21).

Since

$$\begin{aligned}(\psi R)_t &= \bar{\partial}\psi_t = \bar{\partial}(I \cdot (I - RC_z)_t^{-1}) \\ &= \bar{\partial}[I \cdot (I - RC_z)^{-1}R_tC_z(I - RC_z)^{-1}] = \psi R_t(I - RC_z)^{-1}, \\ \langle \psi R \rangle_t &= \frac{1}{2\pi i} \iint (\Psi R)_t d\xi \wedge d\bar{\xi} = \langle \psi R_t(I - RC_z)^{-1} \rangle.\end{aligned}$$

Hence,

$$\begin{aligned}Q_t &= -i\{\sigma_3, \langle \psi R \rangle_t\} \\ &= -i\{\sigma_3, \langle \psi R_t, I \cdot (I + R^T C_z)^{-1} \rangle\}\end{aligned}\quad (22)$$

From Eq. (1) we deduce that

$$\begin{aligned}\bar{\partial}(\psi^{-1}\psi) &= (\bar{\partial}^{-1})\psi + \psi^{-1}\bar{\partial}\psi = \bar{\partial}(I) = 0, \\ \bar{\partial}\psi^{-1} &= -\psi^{-1}(\bar{\partial}\psi)\psi^{-1} = -\psi^{-1}\psi R\psi^{-1} = -R\psi^{-1},\end{aligned}$$

which implies that

$$\bar{\partial}(\psi^{-1})^T = -(\psi^{-1})^T R^T,$$

which has the solution as follows

$$(\psi^{-1})^T = I - (\psi^{-1})^T R^T C_z,$$

that is

$$(\psi^{-1})^T = I \cdot (I + R^T C_z)^{-1}. \quad (23)$$

Substituting Eq. (23) into (22) reads that

$$\begin{aligned}Q_t &= -i\{\sigma_3, \langle \psi(RM\Omega - \Omega MR), (\psi^{-1})^T \rangle\} \\ &= -i\{\sigma_3, \langle \psi RM\Omega\psi^{-1}, I \rangle\} - i\{\sigma_3, \langle -\psi\Omega MR\psi^{-1}, I \rangle\} \\ &= -i\{\sigma_3, \langle (\bar{\partial}\psi)M\Omega\psi^{-1}, I \rangle\} - i\{\sigma_3, \langle \psi\Omega M\bar{\partial}\psi^{-1}, I \rangle\} \\ &= -i\{\sigma_3, \langle (\bar{\partial}\psi)M\Omega\psi^{-1} \rangle + \langle \psi\Omega M\bar{\partial}\psi^{-1} \rangle\} \\ &= -i\{\sigma_3, \langle \bar{\partial}(\psi M\Omega\psi^{-1}) \rangle - \langle \psi M(\bar{\partial}\Omega)\psi^{-1} \rangle\}.\end{aligned} \quad (24)$$

Taking $\Omega = \Omega_p + \Omega_s$, $\Omega_p = \alpha_n z^n \sigma_3$, $\Omega_s = \frac{1}{2\pi i} \iint \frac{\omega(\xi)\sigma_3}{\xi-z} d\xi \wedge d\bar{\xi}$, then Eq. (24) becomes

$$Q_t = i(\text{tr}M)\sigma_3\alpha_n \langle \bar{\partial}z^n \psi M\sigma_3\psi^{-1} \rangle^{\text{off}} + i(\text{tr}M)\sigma_3 \langle \omega(z)\psi M\sigma_3\psi^{-1} \rangle^{\text{off}}. \quad (25)$$

Denoting $\tilde{M}(z) = \psi M\sigma_3\psi^{-1}$, Eq. (25) can be written as

$$Q_t = -i(\text{tr}M)\sigma_3\alpha_n \langle \bar{\partial}z^n \tilde{M} \rangle^{\text{off}} + i\left\{\sigma_3, \left\langle \omega(z)\tilde{M} \right\rangle\right\}. \quad (26)$$

In terms of Eq. (17), we infer that

$$\begin{aligned}\tilde{M}_x &= \psi_x M\sigma_3\psi^{-1} - \psi M\sigma_3(\psi^{-1})_x \\ &= Q\psi M\sigma_3\psi^{-1} - iz\{\sigma_3, \psi\}M\sigma_3\psi^{-1} - \lambda\{\sigma_3, \psi\}M\sigma_3\psi^{-1} \\ &\quad - \tilde{M}Q + iz\tilde{M}\{\sigma_3, \psi\}\psi^{-1} + \lambda\tilde{M}\{\sigma_3, \psi\}\psi^{-1} \\ &= Q\tilde{M} - iz\{\sigma_3, \tilde{M}\} - \lambda\{\sigma_3, \tilde{M}\} - \tilde{M}Q + iz\tilde{M}\{\sigma_3, I\} + \lambda\tilde{M}\{\sigma_3, I\} \\ &= [Q, \tilde{M}] - iz\{\sigma_3, \tilde{M}\} - \lambda\{\sigma_3, \tilde{M}\}.\end{aligned}$$

Thus, we get $\tilde{M}$ in Eq. (24) which satisfies the following matrix equation

$$\tilde{M}_x + iz\{\sigma_3, \tilde{M}\} + \lambda\{\sigma_3, \tilde{M}\} - [Q, \tilde{M}] = 0. \quad (27)$$

Now the $\tilde{M}$ can be decomposed into $\tilde{M} = \tilde{M}^{\text{diag}} + \tilde{M}^{\text{off}}$ and substituted into Eq. (27), one infers

$$\tilde{M}_x^{\text{diag}} = [Q, \tilde{M}^{\text{off}}], \quad (28)$$

$$\tilde{M}_x^{\text{off}} + iz(\text{tr}M)\sigma_3\tilde{M}^{\text{off}} + \lambda(\text{tr}M)\sigma_3\tilde{M}^{\text{off}} = [Q, \tilde{M}^{\text{diag}}]. \quad (29)$$

Due to $\psi \rightarrow I$, when $z \rightarrow \infty$, thus, $\tilde{M} = \psi M \sigma_3 \psi^{-1} \rightarrow M \sigma_3$.

Solving Eq. (28) gives

$$\tilde{M}^{\text{diag}} = M \sigma_3 + \partial_x^{-1} [Q, \tilde{M}^{\text{off}}]. \quad (30)$$

Inserting Eq. (30) into (29) yields

$$\tilde{M}_x^{\text{off}} + iz(\text{tr}M)\sigma_3\tilde{M}^{\text{off}} + \lambda(\text{tr}M)\sigma_3\tilde{M}^{\text{off}} = -\{\sigma_3, Q\} + [Q, \partial_x^{-1} [Q, \tilde{M}^{\text{off}}]],$$

which implies that

$$-z\tilde{M}^{\text{off}} + i(\text{tr}M)^{-1}\sigma_3\tilde{M}_x^{\text{off}} + i\lambda\tilde{M}^{\text{off}} - i(\text{tr}M)^{-1}\sigma_3[Q, \partial_x^{-1} [Q, \tilde{M}^{\text{off}}]] = -iQ. \quad (31)$$

Denoting $\Lambda = i(\text{tr}M)^{-1}\sigma_3\partial_x + i\lambda - i(\text{tr}M)^{-1}\sigma_3[Q, \partial_x^{-1} [Q, \cdot]]$, then Eq. (31) can be written as

$$(\Lambda - z)\tilde{M}^{\text{off}} = -iQ, \\ \tilde{M}^{\text{off}} = -i(\Lambda - z)^{-1}Q = -i \sum_{j=1}^{\infty} z^{-j} \Lambda^{j-1} Q.$$

Hence we have

$$Q_t = -(\text{tr}M)\sigma_3 \sum_{j=1}^{\infty} \langle \bar{\partial} z^{n-j} \rangle \Lambda^{j-1} Q + i \left\{ \sigma_3, \left\langle \omega(z) \tilde{M}(z) \right\rangle \right\}. \quad (32)$$

Since

$$\bar{\partial}(z^{n-j}) = \pi \delta(z) \delta_{j,n+1}, j = 1, 2, \dots, \\ \langle \bar{\partial} z^{n-j} \rangle = \frac{\delta_{j,n+1}}{2i} \iint \delta(z) dz \wedge d\bar{z} = -\delta_{j,n+1}, \\ \sum_{j=1}^{\infty} \langle \bar{\partial}(z^{n-j}) \rangle \Lambda^{j-1} Q = -\Lambda^n Q,$$

we obtain a new generalized integrable NLS hierarchy

$$Q_t + (\text{tr}M)\sigma_3 \Lambda^n Q = i \left\{ \sigma_3, \left\langle \omega(z) \tilde{M}(z) \right\rangle \right\}, \quad (33)$$

where $\tilde{M}$ satisfies Eq. (27).

In what follows, we consider reduction of Eq. (33). Taking $n = 2$, $\omega(z) = \pi g(z_k)\delta(z_k)$, $z = z_R + iz_I$, we have

$$\begin{aligned}\Lambda^2 Q &= (\text{tr} M)^{-2} \sigma_3 Q_{xx} - 2(\text{tr} M)^{-1} \sigma_3 Q_x - \lambda^2 Q + 2(\text{tr} M)^{-2} Q^3, \\ \Lambda Q &= i(\text{tr} M)^{-1} \sigma_3 Q_x + i\lambda Q.\end{aligned}$$

Equation (33) reduces to

$$\begin{aligned}Q_t - (\text{tr} M)^{-1} Q_{xx} - 2Q_x - (\text{tr} M)\lambda^2 \sigma_3 Q + 2(\text{tr} M)^{-1} \sigma_3 Q^3 \\ = i\left\{\sigma_3, \left\langle \omega(z)\tilde{M}(z) \right\rangle\right\}.\end{aligned}\quad (34)$$

Let $Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$, $\tilde{M} = \begin{pmatrix} w(z) & p(z) \\ s(z) & v(z) \end{pmatrix}$, then Eq. (34) admits

$$\begin{cases} iq_t - (\text{tr} M)^{-1} q_{xx} - 2q_x - (\text{tr} M)\lambda^2 q + 2(\text{tr} M)^{-1} q^2 r = \langle \langle p \rangle \rangle, \\ ir_t - (\text{tr} M)^{-1} r_{xx} - 2r_x + (\text{tr} M)\lambda^2 r - 2(\text{tr} M)^{-1} qr^2 = \langle \langle s \rangle \rangle, \end{cases}\quad (35)$$

where

$$\begin{aligned}\langle \langle p \rangle \rangle &= \frac{i}{2}(\text{tr} M) \int_{-\infty}^{+\infty} g(z_R)p(z_R)dz_R, \\ \langle \langle s \rangle \rangle &= -\frac{i}{2}(\text{tr} M) \int_{-\infty}^{+\infty} g(z_R)s(z_R)dz_R.\end{aligned}$$

From Eq. (27) we have

$$\begin{cases} w_x + iz(\text{tr} M)w + \lambda(\text{tr} M)w = qs - pr, \\ p_x + iz(\text{tr} M)p + \lambda(\text{tr} M)p = qv - qw, \\ s_x - iz(\text{tr} M)s + \lambda(\text{tr} M)s = rw - rv, \\ v_x - iz(\text{tr} M)v + \lambda(\text{tr} M)v = rp - qs. \end{cases}\quad (36)$$

When $r = -\bar{q}$, $\langle \langle p \rangle \rangle = -\langle \langle s \rangle \rangle$, Eqs. (35) and (36) reduce to

$$iq_t - (\text{tr} M)^{-1} q_{xx} - 2q_x - (\text{tr} M)\lambda^2 q + 2(\text{tr} M)^{-1} |q|^2 q = \langle \langle p \rangle \rangle, \quad (37)$$

$$\begin{cases} w_x + iz(\text{tr} M)w + \lambda(\text{tr} M)w = qs + p\bar{q}, \\ p_x + iz(\text{tr} M)p + \lambda(\text{tr} M)p = q(v - w), \\ s_x - iz(\text{tr} M)s + \lambda(\text{tr} M)s = \bar{q}(v - w), \\ v_x - iz(\text{tr} M)v + \lambda(\text{tr} M)v = -\bar{q}p - qs. \end{cases}\quad (38)$$

Specially, when $w(z) = 0$, Eq. (37) reduces to

$$iq_t - (\text{tr} M)^{-1} q_{xx} - 2q_x - (\text{tr} M)\lambda^2 q - 2(\text{tr} M)^{-1} |q|^2 q = 0,$$

which is a generalized NLS equation. This is why we call (33) the generalized NLS hierarchy.

4 Soliton Solution of Eq. (37)

In the following, we shall apply the $\bar{\partial}$ -method to construct soliton solutions of Eq. (37). Let

$$R = e^{-i(i\bar{z}+\lambda)(\text{tr}M)\sigma_3x} \begin{pmatrix} 0 & -c\delta(z-z_1) \\ \bar{c}\delta(z-\bar{z}_1) & 1 \end{pmatrix} e^{(i\bar{z}+\lambda)x(\text{tr}M)\sigma_3}, \quad (39)$$

where $z, \bar{z}_1$ are two discrete points on complex plane, $c = c(t)$ dependent on t .

In terms of Eq. (18), one has

$$q = -i(\text{tr}M)\langle \psi R \rangle_{12} = -\frac{1}{2\pi}(\text{tr}M) \iint \psi_{11} R_{12}(z) dz \wedge d\bar{z}. \quad (40)$$

Substituting Eq. (39) into (40) admits

$$q(t) = -\frac{c i \text{tr} M}{\pi} e^{-2(i\bar{z}_1+\lambda)x \text{tr} M} \psi_{11}(z_1). \quad (41)$$

According to Eq. (2), we find that

$$\psi_{11}(z) = 1 + \frac{1}{2\pi i} \iint \frac{\psi_{12}(\xi) R_{21}(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}, \quad (42)$$

$$\psi_{12}(z) = \frac{1}{2\pi i} \iint \frac{\psi_{11}(\xi) R_{12}(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}. \quad (43)$$

Inserting

$$\begin{aligned} R_{12}(z) &= -ce^{-2(i\bar{z}+\lambda)x \text{tr} M} \delta(z-z_1), \\ R_{21}(z) &= \bar{c}e^{2(i\bar{z}+\lambda)x \text{tr} M} \delta(z-\bar{z}_1) \end{aligned}$$

into Eqs. (42) and (43), it is easy to have

$$\psi_{11}(z) = 1 + \frac{\bar{c}}{\pi} \frac{\psi_{12}(\bar{z}_1) e^{2(i\bar{z}_1+\lambda)x \text{tr} M}}{z - \bar{z}_1}, \quad (44)$$

$$\psi_{12}(z) = -\frac{c}{\pi} \frac{\psi_{11}(z_1) e^{-2(i\bar{z}_1+\lambda)x \text{tr} M}}{z - z_1}. \quad (45)$$

Taking $z = z_1, z = \bar{z}_1$ in Eqs. (44) and (45), respectively, one infers

$$\begin{aligned}\psi_{12}(\bar{z}) &= -\frac{c}{\pi} \frac{\psi_{11}(z_1) e^{-2(i\bar{z}_1 + \lambda)\text{tr}M}}{\bar{z}_1 - z_1}, \\ \psi_{11}(z_1) &= 1 + \frac{\bar{c}}{\pi} \frac{\psi_{12}(\bar{z}_1) e^{2(i\bar{z}_1 + \lambda)\text{tr}M}}{z_1 - \bar{z}_1} \\ &= 1 + \frac{|c|^2}{\pi^2} \frac{\psi_{11}(z_1)}{|z_1 - \bar{z}_1|^2} e^{2i(\bar{z}_1 - z_1)\text{tr}M},\end{aligned}$$

which implies that

$$\left(1 - \frac{|c|^2}{\pi^2} \frac{e^{2i(\bar{z}_1 - z_1)\text{tr}M}}{|z_1 - \bar{z}_1|^2}\right) \psi_{11}(z_1) = 1. \quad (46)$$

Denoting $z_1 = \xi + i\eta$, and taking $|c|^2 = 4\pi^2\eta^2 e^{-4\eta\alpha\text{tr}M}$, then we get

$$c(t) = \pm 2\pi\eta e^{-2\eta\alpha\text{tr}M + i\varphi},$$

where $\alpha = \alpha(t)$, $\varphi = \varphi(t)$ are to be determined.

Assume that

$$\begin{aligned}\Omega &= \left(2iz_1^2 + \frac{1}{2\pi i} \iint \frac{w(z)}{z - z_1} dz \wedge d\bar{z}\right) \sigma_3 \\ &=: (2iz_1^2 + f_1 + if_2) \sigma_3,\end{aligned} \quad (47)$$

then Eq. (14) along with Eq. (47) admits that

$$R_t = -(2iz_1^2 + f_1 + if_2)(\text{tr}M)\sigma_3 R^{\text{off}},$$

which leads to

$$c_t = -(2i(\xi^2 - \eta^2 + 2i\xi\eta) + f_1 + if_2)(\text{tr}M)c.$$

Besides, we know that

$$c_t = \pm(-2\eta(\text{tr}M)\alpha_t + i\varphi_t)c.$$

From the above, one obtains

$$\begin{cases} \alpha = -\frac{4\xi\eta + f_1}{2\eta}t, \\ \varphi = (2(\eta^2 - \xi^2) - f_2)t(\text{tr}M)t. \end{cases} \quad (48)$$

Since $\psi_{11}(z_1) = \frac{1}{1 + e^{4\eta(x-\alpha)\text{tr}M}}$, we have a single soliton equation

$$q(t, x) = \mp \frac{i\text{tr}M}{\pi} \frac{e^{(-2i\xi + 2\eta - 2\lambda)x\text{tr}M - 2\eta\alpha\text{tr}M + i\varphi}}{1 + e^{4\eta(x-\alpha)\text{tr}M}}, \quad (49)$$

where α, φ meet Eq. (48). Next, we want to construct multi-soliton solutions to Eq. (37). Suppose $z_j, \bar{z}_j, j = 1, \dots, N$ are $2N$ Discrete spectral points on the complex plane. Let

$$R(z) = e^{-i\theta(z)\sigma_3} v(z) e^{i\theta(z)\sigma_3}. \quad (50)$$

Based on Eqs. (13) and (14), we obtain

$$\theta(z) = \frac{1}{2}(-z + i\lambda)(\text{tr}M)x.$$

Set

$$v(z) = \sum_{j=1}^N \begin{pmatrix} 0 & -c_j \delta(z - z_j) \\ \bar{c}_j \delta(z - \bar{z}_j) & 0 \end{pmatrix},$$

then Eq. (50) can be written as

$$R(z) = \sum_{j=1}^N \begin{pmatrix} 0 & -c_j e^{-2i\theta(z)} \delta(z - z_j) \\ \bar{c}_j e^{2i\theta(z)} \delta(z - \bar{z}_j) & 0 \end{pmatrix}. \quad (51)$$

Assume $Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$, from Eq. (16) we have

$$\begin{cases} q = -i(\text{tr}M \langle \psi R \rangle)_{12} = -\frac{\text{tr}M}{2\pi} \iint \psi_{11}(z) R_{12}(z) dz \wedge d\bar{z}, \\ r = i(\text{tr}M \langle \psi R \rangle)_{21} = \frac{\text{tr}M}{2\pi} \iint \psi_{22}(z) R_{21}(z) dz \wedge d\bar{z}. \end{cases} \quad (52)$$

Substituting Eq. (51) into (52), one has

$$\begin{cases} q = -\frac{i\text{tr}M}{\pi} \sum_{j=1}^N c_j e^{-2i\theta(z_j)} \psi_{11}(z_j), \\ r = -\frac{i\text{tr}M}{\pi} \sum_{j=1}^N \bar{c}_j e^{2i\theta(\bar{z}_j)} \psi_{22}(\bar{z}_j). \end{cases} \quad (53)$$

In terms of Eqs. (2) and (51), we obtain that

$$\begin{aligned}\psi_{11}(z) &= 1 + \frac{1}{\pi} \sum_{j=1}^N \frac{\bar{c}_j \psi_{12}(\bar{z}_j) e^{2i\theta(\bar{z}_j)}}{z - \bar{z}_j}, \\ \psi_{12}(z) &= -\frac{1}{\pi} \sum_{k=1}^N \frac{c_k}{z - z_k} e^{-2i\theta(z_k)} \psi_{11}(z_k), \\ \psi_{22}(z) &= 1 - \frac{1}{\pi} \sum_{j=1}^N \frac{c_j \psi_{21}(z_j) e^{-2i\theta(z_j)}}{z - z_j}, \\ \psi_{21}(z) &= \frac{1}{\pi} \sum_{k=1}^N \frac{\bar{c}_k \psi_{22}(\bar{z}_k) e^{2i\theta(\bar{z}_k)}}{z - \bar{z}_k},\end{aligned}$$

Since

$$\begin{aligned}\psi_{12}(\bar{z}_j) &= -\frac{1}{\pi} \sum_{k=1}^N \frac{c_k}{\bar{z}_j - z_k} e^{-2i\theta(z_k)} \psi_{11}(z_k), \\ \psi_{11}(z_n) &= 1 + \frac{1}{\pi^2} \sum_{j,k=1}^N \frac{\bar{c}_j e^{2i\theta(\bar{z}_j)}}{z_n - \bar{z}_j} \frac{c_k}{z_k - \bar{z}_j} e^{-2i\theta(z_k)} \psi_{11}(z_k).\end{aligned}\quad (54)$$

Noting $c_j(z) = \frac{\bar{c}_j e^{2i\theta(\bar{z}_j)}}{z - \bar{z}_j}$, then Eq. (54) becomes

$$\psi_{11}(z_n) = 1 - \frac{1}{\pi^2} \sum_{l=1}^N \sum_{k=1}^N c_l(z_n) \overline{c_k(z_l)} \psi_{11}(z_k). \quad (55)$$

Denoting $A_{nk} = \frac{1}{\pi^2} \sum_{l=1}^N c_l(z_n) \overline{c_k(z_l)}$, Eq. (55) can be written again

$$\psi_{11}(z_n) + \sum_{k=1}^N A_{nk} \psi_{11}(z_k) = 1. \quad (56)$$

Similarly, we obtain that

$$\psi_{21}(z_j) = \frac{1}{\pi} \sum_{k=1}^N \frac{\bar{c}_k e^{2i\theta(\bar{z}_k)}}{z_j - \bar{z}_k} \psi_{22}(\bar{z}_k), \quad (57)$$

$$\psi_{22}(\bar{z}_m) = 1 - \frac{1}{\pi^2} \sum_{j=1}^N \sum_{k=1}^N \frac{c_j e^{-2i\theta(z_j)}}{\bar{z}_m - z_j} \cdot \frac{\bar{c}_k e^{2i\theta(\bar{z}_k)}}{z_j - \bar{z}_k} \psi_{22}(\bar{z}_k). \quad (58)$$

Set

$$D_j(z) = \frac{c_j e^{-2i\theta(z_j)}}{z - z_j},$$

it is easy to have

$$D_j(\bar{z}_m) = \frac{c_j e^{-2i\theta(z_j)}}{\bar{z}_m - z_j},$$

$$\overline{D_k(z_k)} = \frac{\bar{c}_k e^{2i\theta(\bar{z}_k)}}{\bar{z}_k - z_j}.$$

Therefore, Eq. (58) is written as

$$\psi_{22}(\bar{z}_m) = 1 - \frac{1}{\pi^2} \sum_{j=1}^N \sum_{k=1}^N D_j(\bar{z}_m) \overline{D_k(z_k)} \psi_{22}(\bar{z}_k). \quad (59)$$

If

$$B_{mk} = \frac{1}{\pi^2} \sum_{j=1}^N \sum_{k=1}^N D_j(\bar{z}_m) \overline{D_k(z_k)},$$

then Eq. (59) becomes

$$\psi_{22}(\bar{z}_m) + \sum_{m=1}^N B_{mk} \psi_{22}(\bar{z}_k) = 1. \quad (60)$$

Equation (56) can be written as

$$U(\psi_{11}(z_1), \dots, \psi_{11}(z_N))^T = (1, \dots, 1)^T, \quad (61)$$

where $U = I + (A_{nk})$.

According to the Grammer rule, Eq. (61) has the solutions

$$\psi(z_n) = \frac{\det \tilde{U}_n}{\det U}, n = 1, \dots, N, \quad (62)$$

where

$$\tilde{U}_n = (U_1, \dots, U_{n-1}, B, U_{n+1}, \dots, U_N),$$

$$B = (1, \dots, 1)^T, \psi(z) = (\psi_{11}(z), \psi_{11}(z), \dots, \psi_{11}^T).$$

Similarly, Eq. (60) is equivalent to

$$V\varphi(\bar{z}_m) =: V(\psi_{22}(\bar{z}_1), \dots, \psi_{22}(\bar{z}_N)) = (1, \dots, 1)^T, \quad (63)$$

where $V = I + (B_{mk})$.

The solutions of Eq. (63) present that

$$\varphi(\bar{z}_m) = \frac{\det \tilde{V}_m}{\det V}, m = 1, \dots, N, \quad (64)$$

here

$$\tilde{V}_m = (V_1, V_2, \dots, V_{m-1}, B, V_{m+1}, \dots, V_N).$$

Substituting Eqs. (62) and (64) into Eq. (53), we obtain

$$\begin{aligned} q &= -\frac{i\text{tr}M}{\pi} (c_1 e^{2i\theta(z_1)} \psi_{11}(z_1) + \dots + c_N e^{-2i\theta(z_N)} \psi_{11}(z_N)) \\ &=: -\frac{i\text{tr}M}{\pi} (Y_1, \dots, Y_N) \begin{pmatrix} \psi_{11}(z_1) \\ \vdots \\ \psi_{11}(z_N) \end{pmatrix} \\ &= -\frac{i\text{tr}M}{\pi \det U} Y \det \begin{pmatrix} \tilde{U}_1 \\ \vdots \\ \tilde{U}_N \end{pmatrix} \end{aligned} \quad (65)$$

where

$$\begin{aligned} Y_k &= c_k e^{2i\theta(z_k)}, k = 1, \dots, N, \\ Y &= (Y_1, \dots, Y_N). \end{aligned}$$

$$r = -\frac{i\text{tr}M}{\pi \det V} W \det \begin{pmatrix} \tilde{V}_1 \\ \vdots \\ \tilde{V}_N \end{pmatrix}, \quad (66)$$

where

$$W = (W_1, W_2, \dots, W_N), W_k = \bar{c}_k e^{2i\theta(\bar{z}_k)}, k = 1, \dots, N.$$

When $r = -\bar{q}$, Eq. (65) or Eq. (66) is the multi-soliton solution of Eq. (37).

5 Conclusion

In the paper, we introduced the new Lie bracket and applied a more general polynomial to obtain the new generalized NLS hierarchy and its reduction by using the $\bar{\partial}$ -method. Finally, with the help of Dirac function we obtain the soliton solutions of a generalized NLS equation with source. In addition, Qiao [10] and Ma [11] once introduced some Lax pairs to generate nonisospectral integrable hierarchies, Zhang et al. [12] also presented an efficient method for generating nonisospectral integrable hierarchies. There is an open problem which presents that how to generate nonisospectral integrable hierarchies by using $\bar{\partial}$ -method, which will be considered in the forthcoming days.

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Data Availability The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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