Riemann–Hilbert Problem Associated with the Fourth-Order Dispersive Nonlinear Schrödinger Equation in Optics and Magnetic Mechanics

Beibei Hu^{1,2} · Ling Zhang¹ · Qinghong Li¹ · Ning Zhang³

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Abstract

In this paper, by using Fokas method, we study the initial-boundary value problems (IBVPs) of the fourth-order dispersive nonlinear Schrödinger (FODNLS) equation on the half-line, which can simulate the nonlinear transmission and interaction of ultrashort pulses in the high-speed optical fiber transmission system, and can also describe the nonlinear spin excitation phenomenon of one-dimensional Heisenberg ferromagnetic chain with eight poles and dipole interaction. By discussing the eigenfunctions of Lax pair of FODNLS equation and analyzing symmetry of the scattering matrix, we get a matrix Riemann–Hilbert (RH) problem from for the IBVPs of FODNLS equation. Moreover, we get the potential function solution $u(x, t)$ of the FODNLS equation by solving this matrix RH problem. In addition, we also obtain that some spectral functions satisfy an important global relation.

Keywords Lax pair · Riemann–Hilbert problem · Fourth-order dispersive nonlinear Schrödinger equation · Initial-boundary value problems · Fokas method

Mathematics Subject Classification 35G31 · 35Q15 · 37K10 · 45D05

✉ Beibei Hu
hubsquare@chzu.edu.cn

✉ Ling Zhang
originzhang@chzu.edu.cn

¹ School of Mathematics and Finance, Chuzhou University, Chuzhou 239000, Anhui, People's Republic of China

² Department of Physics, Zhejiang Normal University, Jinhua 321004, People's Republic of China

³ Department of Basical Courses, Shandong University of Science and Technology, Taian 271019, People's Republic of China

1 Introduction

For a long time, finding the solutions of integrable equations has been a very important research topic in theory and application. The expression of exact solutions and some special solutions about integrable equations can provide important guidelines for the analysis of their various properties. However, there is no unified method to solve all integrable equations. With the in-depth study of integrable systems by scholars, a series of methods to solve the classic integrable equations have emerged, such as, inverse scattering transform (IST) method [1], Hirota method [2], Bäcklund transform [3], Darboux transform (DT) [4] and so on [5]. Among them, the IST method is the main analytical method for the exact solution of nonlinear integrable systems. However, due to the IST method is suitable for the limitations of the initial value conditions at infinity, it is almost only used to study the pure initial value problem of integrable equations. About many real-world phenomena and some studies in the fluctuation process, not only the initial value conditions need to be considered, but the boundary value conditions also need to be considered. Naturally, people need to replace the initial value problems with the initial-boundary value problems (IBVPs) in the research process.

In 1997, Fokas proposed a unified transformation method from the initial value problem to the IBVPs based on the IST method idea. This method can be used to investigate IBVPs of partial differential equation [6]. In the past 24 years, IBVPs of some classical integrable equations have been discussed via the Fokas method. For example, the modified Korteweg–de Vries (MKdV) equation [7], the nonlinear Schrödinger (NLS) equation [8], the Kaup–Newell equation [9], the stationary axisymmetric Einstein equations [10], the Ablowitz–Ladik system [11], the Kundu–Eckhaus equation [12], the Hirota equation [13, 14] and other equations [15–18]. In 2012, Lenells extended the Fokas method to the integrable equation with higher-order matrix spectrum, he proposed a more general unified transformation approach to solve IBVPs of integrable models [19] and he used the unified transformation approach to analyze IBVPs of Degasperis–Procesi equation [20]. After that, more and more researchers began to study the IBVPs of integrable model with higher-order matrix spectrum [21–28]. In particular, the long-time asymptotic behavior of the solution of the integrable systems were discussed by the nonlinear steepest descent method proposed by Deift and Zhou [29], see for instance the cases of the three-component coupled mKdV system [30] and the three-component coupled NLS system [31].

In this paper, our work is related to the fourth-order dispersive NLS (FODNLS) equation [32, 33]

$$\begin{aligned}
 iu_t + \alpha_1 u_{xx} + \alpha_2 u|u|^2 + \frac{\varepsilon^2}{12} \left(\alpha_3 u_{xxxx} + \alpha_4 |u|^2 u_{xx} \right. \\
 \left. + \alpha_5 u^2 \bar{u}_{xx} + \alpha_6 u_x^2 \bar{u} + \alpha_7 u|u_x|^2 + \alpha_8 |u|^4 u \right) = 0,
 \end{aligned}
 \tag{1.1}$$

where u represents the amplitude of the slowly varying envelope of the wave, x and t are the normalized space and time variables, ε^2 is an infinitesimal dimensionless parameter representing the high-order linear and nonlinear strength, and $\alpha_j (j = 1, 2, \dots, 8)$ is the actual parameter. The Eq. (1.1) is mainly derived from fiber optics and magnetism. On the one hand, in optics, Eq. (1.1) can simulate the nonlinear propagation and interaction of ultrashort pulses in high-speed fiber-optic transmission systems [34]. On the other hand, in magnetic mechanics, Eq. (1.1) can be used to describe the nonlinear spin excitation of a one-dimensional Heisenberg ferromagnetic chain with octuple and dipole interactions [35]. In particular, when the parameter value is $\alpha_1 = \alpha_3 = 1, \alpha_2 = \alpha_5 = 2, \alpha_4 = 8, \alpha_6 = \alpha_8 = 6, \alpha_7 = 4$, and $\gamma = \frac{\varepsilon^2}{12}$, Eq. (1.1) becomes to

$$iu_t + u_{xx} + 2u|u|^2 + \gamma(u_{xxx} + 8|u|^2u_{xx} + 2u^2\bar{u}_{xx} + 6u_x^2\bar{u} + 4u|u_x|^2 + 6|u|^4u) = 0, \quad (1.2)$$

which is an integrable model, and many properties have been widely studied, such as, the Lax pair, the infinite conservation laws [36], the breather solution, and the higher-order rogue wave solution based on the DT method [37–39], the multi-soliton solutions through Riemann–Hilbert (RH) approach [40], the dark and bright solitary waves and rogue wave solution through phase plane analysis method [41], the bilinear form and the N -soliton solution via the Hirota approach [42, 43]. However, as far as we know, the FODNLS equation (1.2) on the half-line has not been studied. In this paper, we utilize the Fokas method to discuss the IBVPs of the FODNLS equation (1.2) on the half-line domain $\Omega = \{(x, t) : 0 < x < \infty, 0 < t < T\}$.

The paper is organized as follows. In Sect. 2, the eigenfunction for spectral analysis of the Lax pair are introduced. In Sect. 3, the essential functions $y(\zeta), z(\zeta), Y(\zeta), Z(\zeta)$ are further discussed. In Sect. 4, an important theorem is proposed. And the last section is devoted to conclusions.

2 The spectral analysis

Based on Ablowitz–Kaup–Newell–Segur scheme, the Lax pair of Eq. (1.2) is expressed as [36–40]

$$\Psi_x = (-i\zeta\Lambda + P)\Psi, \quad (2.1a)$$

$$\Psi_t = [(8i\gamma\zeta^4 - 2i\zeta^2)\Lambda - 8\gamma\zeta^3P - 4i\gamma\zeta^2A_1 - 2\zeta A_2 + iA_3]\Psi, \quad (2.1b)$$

where ζ is a complex spectral parameter, $\Psi = (\Psi_1, \Psi_2)^T$ is the vector eigenfunction, the 2×2 matrices $\Lambda = \text{diag}\{1, -1\}$, and P, A_1, A_2 and A_3 are defined by

$$\begin{aligned}
P &= \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, A_1 = \begin{pmatrix} |u|^2 & u_x \\ \bar{u}_x & -|u|^2 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} \gamma(u\bar{u}_x - \bar{u}u_x) & -\gamma u_{xx} - (2\gamma|u|^2 + 1)u \\ -\gamma\bar{u}_{xx} - (2\gamma|u|^2 + 1)\bar{u} & -\gamma(u\bar{u}_x - \bar{u}u_x) \end{pmatrix}, \\
A_3 &= \begin{pmatrix} A_3^{(11)} & \gamma u_{xxx} + (6\gamma|u|^2 + 1)u_x \\ \gamma\bar{u}_{xxx} + (6\gamma|u|^2 + 1)\bar{u}_x & -A_3^{(11)} \end{pmatrix},
\end{aligned} \quad (2.2)$$

with $A_3^{(11)} = \gamma(3|u|^4 - |u_x|^2 + \bar{u}u_{xx} + u\bar{u}_{xx}) + |u|^2$.

2.1 The exact one-form

The Lax pair Eqs. (2.1a)–(2.1b) are rewritten as

$$\Psi_x + i\zeta \Lambda \Psi = P(x, t, \zeta) \Psi, \quad (2.3a)$$

$$\Psi_t - (8i\gamma\zeta^4 - 2i\zeta^2)\Lambda \Psi = R(x, t, \zeta) \Psi, \quad (2.3b)$$

where

$$\begin{aligned}
R(x, t, \zeta) &= -8\gamma\zeta^3 P - 4i\gamma\zeta^2 A_1 - 2\zeta A_2 + iA_3 \\
&= -8\gamma\zeta^3 P + 4i\gamma\zeta^2 (P^2 + P_x) \Lambda + 2\gamma\zeta (PP_x - P_x P - P_{xx} + 2P^3) \Lambda \\
&\quad - 2\zeta P \Lambda + i\gamma(3P^4 + P_x^2 - P_{xx} P - PP_{xx} - P_{xxx} + 6P^2 P_x - P_x) \Lambda - iP^2.
\end{aligned}$$

Introducing the following function transformation

$$\Psi(x, t, \zeta) = G(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x] \Lambda}, \quad 0 < x < \infty, 0 < t < T, \quad (2.4)$$

we get

$$G_x + i\zeta[\Lambda, G] = PG, \quad (2.5a)$$

$$G_t - i(8\gamma\zeta^4 - 2\zeta^2)[\Lambda, G] = RG, \quad (2.5b)$$

which can be expressed as the following full differential form

$$d(e^{i[\zeta x - (8\gamma\zeta^4 - 2\zeta^2)t] \Lambda} G(x, t, \zeta)) = F(x, t, \zeta), \quad (2.6)$$

where exact one-form $F(x, t, \zeta)$ is

$$F(x, t, \zeta) = e^{i[\zeta x - (8\gamma\zeta^4 - 2\zeta^2)t] \Lambda} (P(x, t, \zeta) dx + R(x, t, \zeta) dt) G(x, t, \zeta), \quad (2.7)$$

and $\hat{\Lambda}$ represents a matrix operator acting on a second order matrix Λ , i.e. $\hat{\Lambda}P = [\Lambda, P]$ and $e^{\hat{\Lambda}}P = e^{\Lambda}Pe^{-\Lambda}$.

2.2 The Analytic and Bounded Eigenfunctions

We assume that $u(x, t) \in \mathcal{S}$ with $(x, t) \in \Omega = \{(x, t) : 0 < x < \infty, 0 < t < T\}$, and define eigenfunctions $\{G_j(x, t, \zeta)\}_1^3$ of Eqs. (2.5a)–(2.5b) as follows

$$G_j(x, t, \zeta) = \mathbf{I} + \int_{(x_j, t_j)}^{(x, t)} e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x]\hat{\Lambda}} F(\xi, \tau, \zeta), \quad (2.8)$$

where the integration path is $(x_j, t_j) \rightarrow (x, t)$, which is a directed smooth curves. It follows from the closed of the exact one-form that the integral of Eq. (1.2) is independent of the integration path. Therefore, one can choose three integral curve parallel to the axis shown in Fig. 1.

We might as well take $(x_1, t_1) = (0, 0)$, $(x_2, t_2) = (0, T)$, and $(x_3, t_3) = (\infty, t)$, then we have

$$\begin{aligned} G_1(x, t, \zeta) = \mathbf{I} + \int_0^x e^{-i\zeta(x-\xi)\hat{\Lambda}} (PG_1)(\xi, t, \zeta) d\xi \\ + e^{-i\zeta x\hat{\Lambda}} \int_0^t e^{i(8\gamma\zeta^4 - 2\zeta^2)(t-\tau)\hat{\Lambda}} (R_1 G_1)(0, \tau, \Lambda) d\tau, \end{aligned} \quad (2.9a)$$

$$\begin{aligned} G_2(x, t, \zeta) = \mathbf{I} + \int_0^x e^{-i\zeta(x-\xi)\hat{\Lambda}} (PG_2)(\xi, t, \zeta) d\xi \\ - e^{-i\zeta x\hat{\Lambda}} \int_t^T e^{i(8\gamma\zeta^4 - 2\zeta^2)(t-\tau)\hat{\Lambda}} (R_1 G_2)(0, \tau, \Lambda) d\tau, \end{aligned} \quad (2.9b)$$

$$G_3(x, t, \zeta) = \mathbf{I} - \int_x^\infty e^{-i\zeta(x-\xi)\hat{\Lambda}} (PG_3)(\xi, t, \zeta) d\xi. \quad (2.9c)$$

On the one hand, any point (x, t) on the integral curves $\{\gamma_j\}_1^3$ satisfies the following inequalities

$$\gamma_1 : x - \xi \geq 0, t - \tau \geq 0, \quad (2.10a)$$

$$\gamma_2 : x - \xi \geq 0, t - \tau \leq 0, \quad (2.10b)$$

Fig. 1 The three contours $\gamma_1, \gamma_2, \gamma_3$ in the (x, t) -domain

$$\gamma_3 : x - \xi \leq 0. \quad (2.10c)$$

On the other hand, it follows from the Eq. (2.8) that the first column of $G_j(x, t, \zeta)$ contains $e^{2i\zeta(x-\xi)-2i(8\gamma\zeta^4-2\zeta^2)(t-\tau)}$. Thus, for $\zeta \in \mathbb{C}$, we can calculate the bounded analytic region of $[G_j(x, t, \zeta)]_1$, where ζ must satisfy

$$[G_1]_1(x, t, \zeta) : \{\operatorname{Im}\zeta \geq 0\} \cap \{\operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \geq 0\}, \quad (2.11a)$$

$$[G_2]_1(x, t, \zeta) : \{\operatorname{Im}\zeta \geq 0\} \cap \{\operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \leq 0\}, \quad (2.11b)$$

$$[G_3]_1(x, t, \zeta) : \{\operatorname{Im}\zeta \leq 0\}. \quad (2.11c)$$

Similarly, the second column of $G_j(x, t, \zeta)$ contains $e^{-2i\zeta(x-\xi)+2i(8\gamma\zeta^4-2\zeta^2)(t-\tau)}$. Then, for $\zeta \in \mathbb{C}$, we can also calculate the bounded analytic region of the eigenfunctions $[G_j(x, t, \zeta)]_2$, where ζ must satisfy

$$[G_1]_2(x, t, \zeta) : \{\operatorname{Im}\zeta \leq 0\} \cap \{\operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \leq 0\}, \quad (2.12a)$$

$$[G_2]_2(x, t, \zeta) : \{\operatorname{Im}\zeta \leq 0\} \cap \{\operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \geq 0\}, \quad (2.12b)$$

$$[G_3]_2(x, t, \zeta) : \{\operatorname{Im}\zeta \geq 0\}. \quad (2.12c)$$

Here the $[G_j]_k(x, t, \zeta)$ denotes the k -columns of $G_j(x, t, \zeta)$. By calculation, we get the bounded analytic region of $G_j(x, t, \zeta)$ as follows

$$G_1(x, t, \zeta) = ([G_1]_1^{L_1 \cup L_3}(x, t, \zeta), [G_1]_2^{L_6 \cup L_8}(x, t, \zeta)), \quad (2.13a)$$

$$G_2(x, t, \zeta) = ([G_2]_1^{L_2 \cup L_4}(x, t, \zeta), [G_2]_2^{L_5 \cup L_7}(x, t, \zeta)), \quad (2.13b)$$

$$G_3(x, t, \zeta) = ([G_3]_1^{L_3 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta)), \quad (2.13c)$$

where $G_j^{L_i}(x, t, \zeta)$ represents that the bounded analytic region of $\{G_j(x, t, \zeta)\}_1^3$ is $\zeta \in L_i$, $i = 1, 2, \dots, 8$, and L_i , $i = 1, 2, \dots, 8$ are shown in Fig. 2.

To establish the RH problem of the FODNLS Eq. (1.2), we define two special functions $\psi(\zeta)$ and $\phi(\zeta)$ with the eigenfunction $\{G_j(x, t, \zeta)\}_1^3$ as follows

$$G_3(x, t, \zeta) = G_1(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x] \hat{\Lambda}} \psi(\zeta), \quad (2.14a)$$

$$G_2(x, t, \zeta) = G_1(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x] \hat{\Lambda}} \phi(\zeta). \quad (2.14b)$$

Let $(x, t) = (0, 0)$ in Eq. (2.14a), and let $(x, t) = (0, T)$ in Eq. (2.14b), we obtain the following relationship

Fig. 2 The areas $L_i, i = 1, \dots, 8$ division on the complex ζ plane

$$\psi(\zeta) = G_3(0, 0, \zeta), \quad \phi(\zeta) = G_2(0, 0, \zeta) = [e^{i(8\gamma\zeta^4 - 2\zeta^2)T\hat{\Lambda}} G_1(0, T, \zeta)]^{-1}, \quad (2.15)$$

then, we get

$$G_3(x, t, \zeta) = G_1(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x]\hat{\Lambda}} G_3(0, 0, \zeta), \quad (2.16)$$

and

$$G_2(x, t, \zeta) = G_1(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x]\hat{\Lambda}} [e^{-i(8\gamma\zeta^4 - 2\zeta^2)T\hat{\Lambda}} G_1(0, T, \zeta)]^{-1}, \quad (2.17)$$

it follows from the Eqs. (2.16)–(2.17) that

$$G_2(x, t, \zeta) = G_3(x, t, \zeta) e^{i[(8\gamma\zeta^4 - 2\zeta^2)t - \zeta x]\hat{\Lambda}} (\psi(\zeta))^{-1} \phi(\zeta). \quad (2.18)$$

Particularly, taking $x = 0$ in the eigenfunction $G_j(x, t, \zeta), j = 1, 2$, we have

$$\begin{aligned} G_1(0, t, \zeta) &= ([G_1]_1^{L_1 \cup L_3 \cup L_5 \cup L_7}(0, t, \zeta), [G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta)) \\ &= I + \int_0^t e^{i(8\gamma\zeta^4 - 2\zeta^2)(t-\tau)\hat{\Lambda}} (RG_1)(0, \tau, \zeta) d\tau, \end{aligned} \quad (2.19a)$$

$$\begin{aligned}
G_2(0, t, \zeta) &= ([G_2]_1^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta), [G_2]_2^{L_1 \cup L_3 \cup L_5 \cup L_7}(0, t, \zeta)) \\
&= \mathbf{I} - \int_t^T e^{i(8\gamma\zeta^4 - 2\zeta^2)(t-\tau)\hat{\Lambda}} (RG_2)(0, \tau, \zeta) d\tau,
\end{aligned} \quad (2.19b)$$

and when $t = 0$ in the eigenfunction $G_1(x, t, \zeta)$, $G_3(x, t, \zeta)$, we have

$$\begin{aligned}
G_1(x, 0; \zeta) &= ([G_1]_1^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta), [G_1]_2^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, 0, \zeta)) \\
&= \mathbf{I} + \int_0^x e^{-i\zeta(x-\xi)\hat{\Lambda}} (PG_1)(\xi, 0, \zeta) d\xi,
\end{aligned} \quad (2.20a)$$

$$\begin{aligned}
G_3(x, 0; \zeta) &= ([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(z, 0, \zeta), [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(z, 0, \zeta)) \\
&= \mathbf{I} - \int_x^\infty e^{-i\zeta(x-\xi)\hat{\Lambda}} (PG_3)(\xi, 0, \zeta) d\xi.
\end{aligned} \quad (2.20b)$$

Assume that $u_0(x) = u(x, t = 0)$ is an initial data of the functions $u(x, t)$, and $v_0(t) = u(x = 0, t)$, $v_1(t) = u_x(x = 0, t)$, $v_2(t) = u_{xx}(x = 0, t)$, $v_3(t) = u_{xxx}(x = 0, t)$ be boundary data of the functions $u_x(x, t)$, $u_{xx}(x, t)$, $u_{xxx}(x, t)$. Then, the matrices $P(x, 0, \zeta)$ and $R(0, t, \zeta)$ have the following form

$$P(x, 0, \zeta) = \begin{pmatrix} 0 & u_0 \\ -\bar{u}_0 & 0 \end{pmatrix}, \quad R(0, t, \zeta) = \begin{pmatrix} R_{11}(0, t, \zeta) & R_{12}(0, t, \zeta) \\ R_{21}(0, t, \zeta) & -R_{11}(0, t, \zeta) \end{pmatrix}, \quad (2.21)$$

with

$$\begin{aligned}
R_{11}(0, t, \zeta) &= -4i\gamma\zeta^2 v_0^2 - 2\gamma\zeta(v_0 \bar{v}_1 - v_1 \bar{v}_0) + i\gamma(3|v_0|^4 - |v_1|^2 + \bar{v}_0 v_2 + v_0 \bar{v}_2) + i|v_0|^2, \\
R_{12}(0, t, \zeta) &= -8\gamma\zeta^3 v_0 - 4i\gamma\zeta^2 v_1 + 2\gamma\zeta v_2 + 2\zeta(2\gamma|v_0|^2 + 1)v_0 + i\gamma v_3 + i(6\gamma|v_0|^2 + 1)v_1, \\
R_{21}(0, t, \zeta) &= -8\gamma\zeta^3 \bar{v}_0 - 4i\gamma\zeta^2 \bar{v}_1 + 2\gamma\zeta \bar{v}_2 + 2\zeta(2\gamma|v_0|^2 + 1)\bar{v}_0 + i\gamma \bar{v}_3 + i(6\gamma|v_0|^2 + 1)\bar{v}_1.
\end{aligned}$$

2.3 The Other Properties of the Eigenfunctions

Proposition 2.1 *The matrix-valued functions $G_j(x, t, \zeta) = ([G_j]_1(x, t, \zeta), [G_j]_2(x, t, \zeta))$ ($j = 1, 2, 3$) given in Eq. (2.8) possess the following analytical properties:*

- $\det G_j(x, t, \zeta) = 1$;
- The $[G_1]_1(x, t, \zeta)$ is an analytic function for $\zeta \in L_1 \cup L_3$, and the $[G_1]_2(x, t, \zeta)$ is also an analytic function for $\zeta \in L_6 \cup L_8$;
- The $[G_2]_1(x, t, \zeta)$ is an analytic function for $\zeta \in L_2 \cup L_4$, and the $[G_2]_2(x, t, \zeta)$ is also an analytic function for $\zeta \in L_5 \cup L_7$;
- The $[G_3]_1(x, t, \zeta)$ is an analytic function for $\zeta \in L_5 \cup L_6 \cup L_7 \cup L_8$, and the $[G_3]_2(x, t, \zeta)$ is also an analytic function for $\zeta \in L_1 \cup L_2 \cup L_3 \cup L_4$;
- The $[G_j]_1(x, t, \zeta) \rightarrow (1, 0)^T$ and $[G_j]_2(x, t, \zeta) \rightarrow (0, 1)^T$, as $\zeta \rightarrow \infty$.

Proposition 2.2 *Indeed, $\psi(\zeta)$, $\phi(\zeta)$ defined in Eqs. (2.14a)–(2.14b) or Eq. (2.15) are expressed as*

$$\psi(\zeta) = I - \int_0^\infty e^{i\zeta\xi\hat{\Lambda}}(PG_3)(\xi, 0, \zeta)d\xi, \quad (2.22a)$$

$$\phi^{-1}(\zeta) = I + \int_0^T e^{i(8\gamma\zeta^4 - 2\zeta^2)\tau\hat{\Lambda}}(RG_1)(0, \tau, \zeta)d\tau. \quad (2.22b)$$

It follows from the symmetry properties of $P(x, t, \zeta)$ and $R(x, t, \zeta)$ that

$$(G_j(x, t, \zeta))_{11} = \overline{(G_j(x, t, \bar{\zeta}))_{11}}, \quad (G_j(x, t, \zeta))_{21} = -\overline{(G_j(x, t, \bar{\zeta}))_{12}},$$

then we have

$$\begin{aligned} \psi_{11}(\zeta) &= \overline{\psi_{11}(\bar{\zeta})}, & \psi_{21}(\zeta) &= -\overline{\psi_{12}(\bar{\zeta})}, \\ \phi_{11}(\zeta) &= \overline{\phi_{11}(\bar{\zeta})}, & \phi_{21}(\zeta) &= -\overline{\phi_{12}(\bar{\zeta})}. \end{aligned}$$

Assume that the $\psi(\zeta)$ and $\phi(\zeta)$ admit the matrix form as follows

$$\psi(\zeta) = \begin{pmatrix} \overline{y(\bar{\zeta})} & z(\zeta) \\ -\overline{z(\bar{\zeta})} & y(\zeta) \end{pmatrix}, \quad \phi(\zeta) = \begin{pmatrix} \overline{Y(\bar{\zeta})} & Z(\zeta) \\ -\overline{Z(\bar{\zeta})} & Y(\zeta) \end{pmatrix}. \quad (2.23)$$

In terms of the Eq. (2.15) and Eqs. (2.22a)–(2.22b), we know that the following properties are true.

$$\bullet \quad \begin{pmatrix} z(\zeta) \\ y(\zeta) \end{pmatrix} = [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(0, 0, \zeta) = \begin{pmatrix} (G_3)_{12}^{L_1 \cup L_2 \cup L_3 \cup L_4}(0, 0, \zeta) \\ (G_3)_{22}^{L_1 \cup L_2 \cup L_3 \cup L_4}(0, 0, \zeta) \end{pmatrix},$$

where the vector function $[G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta)$ satisfies the ordinary differential equation as follows

$$\begin{aligned} \partial_x [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta) + 2i\zeta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta) \\ = P(x, 0, \zeta) [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta), \quad 0 < x < \infty, \end{aligned} \quad (2.24)$$

here $P(x, 0, \zeta)$ is given in Eq. (2.21) and

$$\lim_{x \rightarrow \infty} [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, 0, \zeta) = (0, 1)^T.$$

$$\bullet \quad \begin{pmatrix} -e^{-2i(8\gamma\zeta^4-2\zeta^2)T}Z(\zeta) \\ \overline{Y(\bar{\zeta})} \end{pmatrix} = [G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, T, \zeta) \\ = \begin{pmatrix} (G_1)_{22}^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, T, \zeta) \\ (G_1)_{21}^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, T, \zeta) \end{pmatrix},$$

where the vector function $[G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta)$ satisfies the ordinary differential equation as follows

$$\partial_t [G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta) - 2i(8\gamma\zeta^4 - 2\zeta^2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta) \\ = R(0, t, \zeta) [G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, t, \zeta), \quad 0 < t < T, \quad (2.25)$$

here $R(0, t, \zeta)$ is given in Eq. (2.21) and

$$[G_1]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(0, 0, \zeta) = (0, 1)^T.$$

$$\bullet \quad y(-\zeta) = y(\zeta), \quad z(-\zeta) = -z(\zeta), \quad Y(-\zeta) = Y(\zeta), \quad Z(-\zeta) = -Z(\zeta).$$

$$\text{For } \zeta \in \mathbb{R}, \quad \det \psi(\zeta) = |y(\zeta)|^2 + |z(\zeta)|^2 = 1.$$

$$\bullet \quad \text{For } \zeta \in \mathbb{C}, \quad \det \phi(\zeta) = Y(\zeta) \overline{Y(\bar{\zeta})} + Z(\zeta) \overline{Z(\bar{\zeta})} = 1, \quad (\zeta \in \gamma_m, \text{ if } T = \infty),$$

where curve γ_m , $m = 1, 2, 3, 4$ are given in Eq. (2.30).

$$\bullet \quad \begin{aligned} y(\zeta) &= 1 + O\left(\frac{1}{\zeta}\right), \quad z(\zeta) = O\left(\frac{1}{\zeta}\right), \text{ as } \zeta \rightarrow \infty, \\ Y(\zeta) &= 1 + O\left(\frac{1}{\zeta}\right) + O\left(\frac{e^{-2i(8\gamma\zeta^4-2\zeta^2)T}}{\zeta}\right), \\ Z(\zeta) &= O\left(\frac{1}{\zeta}\right) + O\left(\frac{e^{-2i(8\gamma\zeta^4-2\zeta^2)T}}{\zeta}\right), \text{ as } \zeta \rightarrow \infty. \end{aligned}$$

2.4 The Basic Riemann–Hilbert Problem

In order to facilitate calculation, we introduce the symbolic assumptions as follows

$$\omega(x, t, \zeta) = \zeta x - (8\gamma\zeta^4 - 2\zeta^2)t, \quad (2.26a)$$

$$\rho(\zeta) = y(\zeta) \overline{Y(\bar{\zeta})} + z(\zeta) \overline{Z(\bar{\zeta})}, \quad (2.26b)$$

$$\kappa(\zeta) = \overline{y(\bar{\zeta})Z(\bar{\zeta})} - \overline{z(\bar{\zeta})Y(\bar{\zeta})}, \quad (2.26c)$$

$$\delta(\zeta) = \frac{z(\zeta)}{y(\bar{\zeta})}, \Delta(\zeta) = -\frac{\overline{Z(\bar{\zeta})}}{y(\zeta)\rho(\zeta)}, \quad (2.26d)$$

then, we have

$$\begin{aligned} \overline{Z(\bar{\zeta})} &= y(\zeta)\kappa(\zeta) + \overline{z(\bar{\zeta})}\rho(\zeta), \\ \rho(\zeta)\overline{\rho(\bar{\zeta})} - \kappa(\zeta)\overline{\kappa(\bar{\zeta})} &= 1, \\ \rho(\zeta) &= 1 + O\left(\frac{1}{\zeta}\right), \kappa(\zeta) = O\left(\frac{1}{\zeta}\right) \text{ as } \zeta \rightarrow \infty, \\ \rho(-\zeta) &= \rho(\zeta), \kappa(-\zeta) = -\kappa(\zeta), \end{aligned}$$

and the matrix function $D(x, t, \zeta)$ is defined by

$$D_+(x, t, \zeta) = \left(\frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right), \zeta \in L_1 \cup L_3, \quad (2.27a)$$

$$D_-(x, t, \zeta) = \left(\frac{[G_2]_1^{L_2 \cup L_4}(x, t, \zeta)}{\rho(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right), \zeta \in L_2 \cup L_4, \quad (2.27b)$$

$$D_+(x, t, \zeta) = \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_2]_2^{L_5 \cup L_7}(x, t, \zeta)}{\rho(\bar{\zeta})} \right), \zeta \in L_5 \cup L_7, \quad (2.27c)$$

$$D_-(x, t, \zeta) = \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_1]_2^{L_6 \cup L_8}(x, t, \zeta)}{y(\bar{\zeta})} \right), \zeta \in L_6 \cup L_8. \quad (2.27d)$$

Obviously, the above definitions indicate that

$$\det D(x, t, \zeta) = 1, \quad D(x, t, \zeta) \rightarrow I, \text{ as } \zeta \rightarrow \infty. \quad (2.28)$$

Theorem 2.3 *The matrix function $D(x, t, \zeta)$ defined by Eqs. (2.27a)–(2.27d) admits the jump relation on the curves $\Gamma_m, m = 1, \dots, 4$ as follows*

$$D_+(x, t, \zeta) = D_-(x, t, \zeta)Q(x, t, \zeta), \zeta \in \Gamma_m, m = 1, \dots, 4, \quad (2.29)$$

where

$$Q(x, t, \zeta) = \begin{cases} Q_1(x, t, \zeta), \zeta \in \Gamma_1 \doteq \{\bar{L}_1 \cup \bar{L}_3\} \cap \{\bar{L}_2 \cup \bar{L}_4\}, \\ Q_2(x, t, \zeta) = Q_3 Q_4^{-1} Q_1, \zeta \in \Gamma_2 \doteq \{\bar{L}_2 \cup \bar{L}_4\} \cap \{\bar{L}_5 \cup \bar{L}_7\}, \\ Q_3(x, t, \zeta), \zeta \in \Gamma_3 \doteq \{\bar{L}_5 \cup \bar{L}_7\} \cap \{\bar{L}_6 \cup \bar{L}_8\}, \\ Q_4(x, t, \zeta), \zeta \in \Gamma_4 \doteq \{\bar{L}_6 \cup \bar{L}_8\} \cap \{\bar{L}_1 \cup \bar{L}_3\}, \end{cases} \quad (2.30)$$

and

$$\begin{aligned} Q_1(x, t, \zeta) &= \begin{pmatrix} 1 & 0 \\ \Delta(\zeta)e^{2i\omega(\zeta)} & 1 \end{pmatrix}, \\ Q_2(x, t, \zeta) &= \begin{pmatrix} 1 - (\delta(\zeta) + \overline{\Delta(\bar{\zeta})})(\Delta(\zeta) + \overline{\delta(\bar{\zeta})}) & (\delta(\zeta) + \overline{\Delta(\bar{\zeta})})e^{-2i\omega(\zeta)} \\ (\Delta(\zeta) + \overline{\delta(\bar{\zeta})})e^{2i\omega(\zeta)} & 1 \end{pmatrix}, \\ Q_3(x, t, \zeta) &= \begin{pmatrix} 1 & \overline{\Delta(\bar{\zeta})}e^{-2i\omega(\zeta)} \\ 0 & 1 \end{pmatrix}, \\ Q_4(x, t, \zeta) &= \begin{pmatrix} 1 & -\delta(\zeta)e^{-2i\omega(\zeta)} \\ -\overline{\delta(\bar{\zeta})}e^{2i\omega(\zeta)} & 1 + |\delta(\zeta)|^2 \end{pmatrix}. \end{aligned}$$

Proof In terms of the Eqs. (2.14a)–(2.14b) and Eq. (2.23), we have

$$\overline{y(\bar{\zeta})}[G_1]_1^{L_1 \cup L_3}(x, t, \zeta) - \overline{z(\bar{\zeta})}e^{2i\omega(\zeta)}[G_1]_2^{L_6 \cup L_8}(x, t, \zeta) = [G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \quad (2.31a)$$

$$z(\zeta)e^{-2i\omega(\zeta)}[G_1]_1^{L_1 \cup L_3}(x, t, \zeta) + y(\zeta)[G_1]_2^{L_6 \cup L_8}(x, t, \zeta) = [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta), \quad (2.31b)$$

and

$$\overline{Y(\bar{\zeta})}[G_1]_1^{L_1 \cup L_3}(x, t, \zeta) - \overline{Z(\bar{\zeta})}e^{2i\omega(\zeta)}[G_1]_2^{L_6 \cup L_8}(x, t, \zeta) = [G_2]_1^{L_2 \cup L_4}(x, t, \zeta), \quad (2.32a)$$

$$Z(\zeta)e^{-2i\omega(\zeta)}[G_1]_1^{L_1 \cup L_3}(x, t, \zeta) + Y(\zeta)[G_1]_2^{L_6 \cup L_8}(x, t, \zeta) = [G_2]_2^{L_5 \cup L_7}(x, t, \zeta), \quad (2.32b)$$

according to the Eqs. (2.31a)–(2.32b) and Eqs. (2.26a)–(2.26d), we get

$$\rho(\zeta)[G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta) - \kappa(\zeta)e^{2i\omega(\zeta)}[G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) = [G_2]_1^{L_2 \cup L_4}(x, t, \zeta), \quad (2.33a)$$

$$\overline{\kappa(\bar{\zeta})}e^{-2i\omega(\zeta)}[G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta) + \overline{\rho(\bar{\zeta})}[G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) = [G_2]_2^{L_5 \cup L_7}(x, t, \zeta). \quad (2.33b)$$

By the Eqs. (2.27a)–(2.27d) and Eq. (2.29), we have

$$\begin{aligned} &\left(\frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right) \\ &= \left(\frac{[G_2]_1^{L_2 \cup L_4}(x, t, \zeta)}{\rho(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right) Q_1(x, t, \zeta), \end{aligned} \quad (2.34a)$$

$$\begin{aligned} & \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_2]_2^{L_5 \cup L_7}(x, t, \zeta)}{\overline{\rho(\zeta)}} \right) \\ &= \left(\frac{[G_2]_1^{L_2 \cup L_4}(x, t, \zeta)}{\rho(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right) Q_2(x, t, \zeta), \end{aligned} \quad (2.34b)$$

$$\begin{aligned} & \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_2]_2^{L_5 \cup L_7}(x, t, \zeta)}{\overline{\rho(\zeta)}} \right) \\ &= \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_1]_2^{L_6 \cup L_8}(x, t, \zeta)}{\overline{y(\zeta)}} \right) Q_3(x, t, \zeta), \end{aligned} \quad (2.34c)$$

$$\begin{aligned} & \left(\frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \zeta) \right) \\ &= \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, t, \zeta), \frac{[G_1]_2^{L_6 \cup L_8}(x, t, \zeta)}{\overline{y(\zeta)}} \right) Q_4(x, t, \zeta). \end{aligned} \quad (2.34d)$$

Therefore, from the Eqs. (2.34a)–(2.34d), we can get that the jump matrices $\{Q_i(x, t, \zeta)\}_1^4$ satisfy the Eq. (2.30). $\square$

Assumption 2.4 Assume that the zeros of $\rho(\zeta)$ and $y(\zeta)$ satisfy the assumptions as follows

- The spectral function $y(\zeta)$ has $2n$ simple zeros $\{\zeta_j\}_{j=1}^{2n}$, $2n = 2n_1 + 2n_2$, if $\{\zeta_j\}_1^{2n_1} \in L_1 \cup L_3$, then $\{\bar{\zeta}_j\}_1^{2n_2} \in L_8 \cup L_6$.
- The spectral function $\rho(\zeta)$ has $2N$ simple zeros $\{\eta_j\}_{j=1}^{2N}$, $2N = 2N_1 + 2N_2$, if $\{\eta_j\}_1^{2N_1} \in L_5 \cup L_7$, then $\{\bar{\eta}_j\}_1^{2N_2} \in L_4 \cup L_2$.
- The spectral function $y(\zeta)$ and $\rho(\zeta)$ have no common zeros.

Proposition 2.5 The matrix function $D(x, t, \zeta)$ defined by Eqs. (2.27a)–(2.27d) meets the following residue conditions:

$$\text{Res}\{[D(x, t, \zeta)]_1, \zeta_j\} = \frac{1}{z(\zeta_j)\dot{y}(\zeta_j)} e^{2i\omega(\zeta_j)} [D(x, t, \zeta_j)]_2, j = 1, \dots, 2n_1, \quad (2.35a)$$

$$\text{Res}\{[D(x, t, \zeta)]_2, \bar{\zeta}_j\} = -\frac{1}{z(\zeta_j)\dot{y}(\zeta_j)} e^{-2i\omega(\zeta_j)} [D(x, t, \bar{\zeta}_j)]_1, j = 1, \dots, 2n_2, \quad (2.35b)$$

$$\text{Res}\{[D(x, t, \zeta)]_1, \eta_j\} = -\frac{\overline{Z(\bar{\eta}_j)}}{y(\eta_j)\dot{\rho}(\eta_j)} e^{2i\omega(\eta_j)} [D(x, t, \eta_j)]_1, j = 1, \dots, 2N_1, \quad (2.35c)$$

$$\text{Res}\{[D(x, t, \zeta)]_2, \bar{\eta}_j\} = \frac{Z(\bar{\eta}_j)}{y(\eta_j)\dot{\rho}(\eta_j)} e^{-2i\omega(\bar{\eta}_j)} [D(x, t, \bar{\eta}_j)]_2, j = 1, \dots, 2N_2, \quad (2.35d)$$

where $\dot{\rho}(\zeta) = \frac{d\rho}{d\zeta}$.

Proof We only verify the residue relationship Eq. (2.35a) as follows:

Due to $D(x, t, \zeta) = (\frac{[G_1]_1^{L_1 \cup L_3}}{y(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4})$, which means that the zeros $\{\varsigma_j\}_1^{2n_1}$ of $y(\zeta)$ are the poles of $\frac{[G_1]_1^{L_1 \cup L_3}}{y(\zeta)}$. Then, we have

$$\text{Res}\left\{\frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)}, \varsigma_j\right\} = \lim_{\zeta \rightarrow \varsigma_j} (\zeta - \varsigma_j) \frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)} = \frac{[G_1]_1^{L_1 \cup L_3}(x, t, \varsigma_j)}{\dot{y}(\varsigma_j)}. \quad (2.36)$$

Taking $\zeta = \varsigma_j$ into the second equation of Eqs. (2.33a)–(2.33b) yields

$$[G_1]_1^{L_1 \cup L_3}(x, t, \varsigma_j) = \frac{1}{y(\varsigma_j)} e^{2i\omega(\varsigma_j)} [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \varsigma_j). \quad (2.37)$$

According to the Eqs. (2.36) and (2.37), we get

$$\text{Res}\left\{\frac{[G_1]_1^{L_1 \cup L_3}(x, t, \zeta)}{y(\zeta)}, \varsigma_j\right\} = \frac{1}{z(\varsigma_j)\dot{y}(\varsigma_j)} e^{2i\omega(\varsigma_j)} [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, t, \varsigma_j). \quad (2.38)$$

Therefore, the Eq. (2.38) leads to the Eq. (2.35a), and the remaining three residue relationships Eqs. (2.35b)–(2.35d) can be similarly proved. $\square$

2.5 The Global Relation

In this subsection, we give the spectral functions which are not independent but satisfy a nice global relation. In fact, at the boundary of the region $(\xi, \tau) : 0 < \xi < \infty, 0 < \tau < t$, the integral of the one-form $F(x, t, \zeta)$ given by Eq.(2.7) vanishes. Let $G(x, t, \zeta) = G_3(x, t, \zeta)$, we get

$$\begin{aligned}
& \int_{-\infty}^0 e^{i\zeta\xi\hat{\Lambda}}(PG_3)(\xi, 0, \zeta)d\xi + \int_0^t e^{-i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}(RG_3)(0, \tau, \zeta)d\tau \\
& + e^{-i(8\gamma\zeta^4-2\zeta^2)t\hat{\Lambda}} \times \int_0^{\infty} e^{i\zeta\xi\hat{\Lambda}}(PG_3)(\xi, t, \zeta)d\xi \\
& = \lim_{x \rightarrow \infty} e^{i\zeta x\hat{\Lambda}} \int_0^t e^{-i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}(RG_3)(x, \tau, \zeta)d\tau.
\end{aligned} \quad (2.39)$$

On the one hand, according to the definition of $\psi(\zeta)$ in Eq. (2.15), together with the Eq. (2.20b), we can get the first term of the Eq. (2.39)

$$\psi(\zeta) - I.$$

Let $x = 0$ in the Eq. (2.16), we get

$$G_3(0, \tau, \zeta) = G_1(0, \tau, \zeta)e^{i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}\psi(\zeta), \quad (2.40)$$

therefore

$$e^{-i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}(RG_3)(0, \tau, \zeta) = [e^{-i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}(RG_1)(0, \tau, \zeta)]\psi(\zeta). \quad (2.41)$$

On the other hand, the Eqs. (2.41) and (2.19a) mean that the second term of the Eq. (2.39) is

$$\int_0^t e^{-i(8\gamma\zeta^4-2\zeta^2)\tau\hat{\Lambda}}(RG_3)(0, \tau, \zeta)d\tau = [e^{-i(8\gamma\zeta^4-2\zeta^2)t\hat{\Lambda}}RG_1(0, t, \zeta) - I]\psi(\zeta).$$

For $x \rightarrow \infty$, $u(x, t) \in \mathcal{S}$, the Eq. (2.39) is equivalent to

$$\phi^{-1}(t, \zeta)\psi(\zeta) + e^{-i(8\gamma\zeta^4-2\zeta^2)t\hat{\Lambda}} \times \int_0^{\infty} e^{i\zeta\xi\hat{\Lambda}}(PG_3)(\xi, t, \zeta)d\xi = I, \quad (2.42)$$

where the first column of the Eq. (2.42) is valid for $\zeta \in L_5 \cup L_6 \cup L_7 \cup L_8$ and the second column of the Eq. (2.42) is valid for $\zeta \in L_1 \cup L_2 \cup L_3 \cup L_4$, and $\phi(t, \zeta)$ is given by

$$\phi^{-1}(t, \zeta) = e^{-i(8\gamma\zeta^4-2\zeta^2)t\hat{\Lambda}}G_1(0, t, \zeta).$$

Owing to $\phi(\zeta) = \phi(T, \zeta)$ and let $t = T$, then, the Eq. (2.42) is equivalent to

$$\phi^{-1}(\zeta)\psi(\zeta) + e^{-i(8\gamma\zeta^4-2\zeta^2)T\hat{\Lambda}} \times \int_0^{\infty} e^{i\zeta\xi\hat{\Lambda}}(PG_3)(\xi, T, \zeta)d\xi = I. \quad (2.43)$$

Hence, the 1,2 component of the Eq. (2.43) equals

$$y(\zeta)Z(\zeta) - Y(\zeta)z(\zeta) = e^{-2i(8\gamma\zeta^4-2\zeta^2)T}E(\zeta), \quad (2.44)$$

where $E(\zeta)$ is expressed as

$$E(\zeta) = \int_0^\infty e^{i\zeta\xi} (PG_3)_{12}(\xi, T, \zeta) d\xi, \quad (2.45)$$

which is the so-called global relation.

3 The Spectral Functions

Definition 3.1 (Related to $y(\zeta)$, $z(\zeta)$) Let $u_0(x) = u(x, 0) \in \mathcal{S}$, the mapping

$$H_1 : \{u_0(x)\} \rightarrow \{y(\zeta), z(\zeta)\},$$

is defined by

$$\begin{pmatrix} z(\zeta) \\ y(\zeta) \end{pmatrix} = [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(0; \zeta) = \begin{pmatrix} (G_3)_{12}^{L_1 \cup L_2 \cup L_3 \cup L_4}(0; \zeta) \\ (G_3)_{22}^{L_1 \cup L_2 \cup L_3 \cup L_4}(0; \zeta) \end{pmatrix}, \operatorname{Im} \zeta \geq 0,$$

where $G_3(x, \zeta)$ and $P(x, \zeta)$ are given by Eqs. (2.20b) and (2.21), respectively.

Proposition 3.2 The $y(\zeta)$ and $z(\zeta)$ satisfy the properties as follows

- (i) For $\operatorname{Im} \zeta < 0$, $y(\zeta)$ and $z(\zeta)$ are analytic functions,
- (ii) $y(\zeta) = 1 + O(\frac{1}{\zeta})$, $z(\zeta) = O(\frac{1}{\zeta})$, as $\zeta \rightarrow \infty$,
- (iii) For $\zeta \in \mathbb{R}$, $\det \psi(\zeta) = |y(\zeta)|^2 + |z(\zeta)|^2 = 1$,
- (iv) $S_1 = H_1^{-1} : \{y(\zeta), z(\zeta)\} \rightarrow \{u_0(x)\}$ has the following form

$$u_0(x) = 2i \lim_{\zeta \rightarrow \infty} (\zeta D^{(x)}(x, \zeta))_{12},$$

where $D^{(x)}(x, \zeta)$ meets the following RH problem.

Remark 3.3 Assume that

$$D_+^{(x)}(x, \zeta) = \left(\frac{[G_1]_1^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, \zeta)}{y(\zeta)}, [G_3]_2^{L_1 \cup L_2 \cup L_3 \cup L_4}(x, \zeta) \right), \operatorname{Im} \zeta \geq 0, \quad (3.1a)$$

$$D_-^{(x)}(x, \zeta) = \left([G_3]_1^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, \zeta), \frac{[G_1]_2^{L_5 \cup L_6 \cup L_7 \cup L_8}(x, \zeta)}{\overline{y(\bar{\zeta})}} \right), \operatorname{Im} \zeta \leq 0, \quad (3.1b)$$

hence, $D^{(x)}(x, \zeta)$ admits the RH problem

- $D^{(x)}(x, \zeta) = \begin{cases} D_+^{(x)}(x, \zeta), & \zeta \in L_1 \cup L_2 \cup L_3 \cup L_4 \\ D_-^{(x)}(x, \zeta), & \zeta \in L_5 \cup L_6 \cup L_7 \cup L_8 \end{cases}$ is a piecewise analytic function.
- $D_+^{(x)}(x, \zeta) = D_-^{(x)}(x, \zeta)(Q^{(x)}(x, \zeta))^{-1}$, $\zeta \in \mathbb{R}$, and

$$Q^{(x)}(x, \zeta) = \begin{pmatrix} \frac{1}{y(\zeta)y(\bar{\zeta})} \frac{z(\zeta)}{y(\zeta)} e^{-2i\zeta x} \\ -\frac{z(\bar{\zeta})}{y(\zeta)} e^{2i\bar{\zeta}x} & 1 \end{pmatrix}. \quad (3.2)$$

- $D^{(x)}(x, \zeta) \rightarrow I, \zeta \rightarrow \infty$.
- $y(\zeta)$ possesses $2n$ simple zeros $\{\zeta_j\}_1^{2n}$, $2n = 2n_1 + 2n_2$, assume that $\{\zeta_j\}_1^{2n_1}$ are parts of $L_1 \cup L_2 \cup L_3 \cup L_4$, then, $\{\bar{\zeta}_j\}_1^{2n_2}$ are parts of $L_5 \cup L_6 \cup L_7 \cup L_8$.
- $[D^{(x)}]_1(x, \zeta)$ enjoys simple poles for $\zeta = \{\zeta_j\}_1^{2n_1}$ and the $[D^{(x)}]_2(x, \zeta)$ has simple poles for $\zeta = \{\bar{\zeta}_j\}_1^{2n_2}$. In this case, the residue relations are define by

$$\text{Res}\{[D^{(x)}(x, \zeta)]_1, \zeta_j\} = \frac{e^{2i\zeta_j x}}{z(\zeta_j)\dot{y}(\zeta_j)} [D^{(x)}(x, \zeta_j)]_2, \quad (3.3a)$$

$$\text{Res}\{[D^{(x)}(x, \zeta)]_2, \bar{\zeta}_j\} = \frac{e^{-2i\bar{\zeta}_j x}}{z(\bar{\zeta}_j)\dot{y}(\bar{\zeta}_j)} [D^{(x)}(x, \bar{\zeta}_j)]_1. \quad (3.3b)$$

Definition 3.4 (Related to $Y(\zeta), Z(\zeta)$). Let $v_0(t), v_1(t), v_2(t), v_3(t) \in \mathcal{S}$, the mapping

$$H_2 : \{v_0(t), v_1(t), v_2(t), v_3(t)\} \rightarrow \{Y(\zeta), Z(\zeta)\},$$

is defined by

$$\begin{pmatrix} Y(\zeta) \\ Z(\zeta) \end{pmatrix} = [G_2]_2^{L_2 \cup L_4 \cup L_6 \cup L_8}(t, \zeta) = \begin{pmatrix} (G_2)_{12}^{L_2 \cup L_4 \cup L_6 \cup L_8}(t, \zeta) \\ (G_2)_{22}^{L_2 \cup L_4 \cup L_6 \cup L_8}(t, \zeta) \end{pmatrix},$$

where $G_2(t, \zeta)$ and $R(t, \zeta)$ are given by Eqs. (2.19b) and (2.21), respectively.

Proposition 3.5 The $Y(\zeta)$ and $Z(\zeta)$ satisfy the properties as follows

- For $\text{Im}(8\gamma\zeta^4 - 2\zeta^2) \leq 0$, $Y(\zeta), Z(\zeta)$ are analytic functions,
- $Y(\zeta) = 1 + O(\frac{1}{\zeta}) + O(\frac{e^{-2i(8\gamma\zeta^4 - 2\zeta^2)T}}{\zeta})$, $Z(\zeta) = O(\frac{1}{\zeta}) + O(\frac{e^{-2i(8\gamma\zeta^4 - 2\zeta^2)T}}{\zeta})$, as $\zeta \rightarrow \infty$,
- For $\zeta \in \mathbb{C}$, $\det\phi(\zeta) = Y(\zeta)\bar{Y}(\bar{\zeta}) + Z(\zeta)\bar{Z}(\bar{\zeta}) = 1$, $((8\gamma\zeta^4 - 2\zeta^2) \in \mathbb{R}$, if $T = \infty$),
- $S_2 = H_2^{-1} : \{Y(\zeta), Z(\zeta)\} \rightarrow \{v_0(t), v_1(t), v_2(t), v_3(t)\}$

$$v_0(t) = 2i \lim_{\zeta \rightarrow \infty} (\zeta D^{(t)}(t, \zeta))_{12}, \quad (3.4a)$$

$$v_1(t) = \lim_{\zeta \rightarrow \infty} [4(\zeta^2 D^{(t)}(t, \zeta))_{12} + 2iv_0(t)(\zeta D^{(t)}(t, \zeta))_{22}], \quad (3.4b)$$

$$\begin{aligned} \gamma v_2(t) = & \lim_{\zeta \rightarrow \infty} [-8i\gamma(\zeta^3 D^{(t)}(t, \zeta))_{12} + 4\gamma v_0(t)(\zeta^2 D^{(t)}(t, \zeta))_{22} \\ & + 2i\gamma(v_0^2(t)(\zeta D^{(t)}(t, \zeta))_{12} + v_1(t)(\zeta D^{(t)}(t, \zeta))_{22}) \\ & + 2i(\zeta D^{(t)}(t, \zeta))_{12} - (2\gamma v_0^2(t) + 1)v_0(t)], \end{aligned} \quad (3.4c)$$

$$\begin{aligned} \gamma v_3(t) = & \lim_{\zeta \rightarrow \infty} \{-16\gamma(\zeta^4 D^{(t)}(t, \zeta))_{12} + 4(\zeta^2 D^{(t)}(t, \zeta))_{12} \\ & - 8i\gamma v_0(t)(\zeta^3 D^{(t)}(t, \zeta))_{22} + 4\gamma(v_0^2(t)(\zeta^2 D^{(t)}(t, \zeta))_{12} + v_1(t)(\zeta^2 D^{(t)}(t, \zeta))_{22}) \\ & + i(6\gamma v_0^2(t) + 1)v_1(t) - 2i[\gamma(v_0(t)v_1(t) - \bar{v}_0(t)v_1(t))(\zeta D^{(t)}(t, \zeta))_{12} \\ & - (\gamma v_2(t) + (2\gamma v_0^2(t) + 1)v_0(t))(\zeta D^{(t)}(t, \zeta))_{22}]\}, \end{aligned} \quad (3.4d)$$

where $D^{(1)}(t), D^{(2)}(t), D^{(3)}(t), D^{(4)}(t)$ meet the following asymptotic expansion

$$\begin{aligned} D^{(t)}(t, \zeta) = & \mathbf{I} + \frac{D^{(1)}(t, \zeta)}{\zeta} + \frac{D^{(2)}(t, \zeta)}{\zeta^2} + \frac{D^{(3)}(t, \zeta)}{\zeta^3} + \frac{D^{(4)}(t, \zeta)}{\zeta^4} \\ & + O\left(\frac{1}{\zeta^5}\right), \text{ as } \zeta \rightarrow \infty, \end{aligned}$$

and $D^{(t)}(t, \zeta)$ meets the following RH problem.

Remark 3.6 Assume that

$$D_+^{(t)}(t, \zeta) = \left(\frac{[G_1]_1^{L_1 \cup L_3 \cup L_5 \cup L_7}(t, \zeta)}{Y(\zeta)}, [G_3]_2^{L_1 \cup L_3}(t, \zeta) \right), \operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \geq 0, \quad (3.5a)$$

$$D_-^{(t)}(t, \zeta) = \left([G_3]_1^{L_2 \cup L_4}(t, \zeta), \frac{[G_1]_2^{L_2 \cup L_4}(t, \zeta)}{Y(\bar{\zeta})} \right), \operatorname{Im}(8\gamma\zeta^4 - 2\zeta^2) \leq 0, \quad (3.5b)$$

hence, $D^{(t)}(t, \zeta)$ admits the RH problem as follows.

- $D^{(t)}(t, \zeta) = \begin{cases} D_+^{(t)}(t, \zeta), & \zeta \in L_1 \cup L_4 \cup L_5 \cup L_8 \\ D_-^{(t)}(t, \zeta), & \zeta \in L_2 \cup L_3 \cup L_6 \cup L_7 \end{cases}$ is a piecewise analytic function.
- $D_+^{(t)}(t, \zeta) = D_-^{(t)}(t, \zeta)Q^{(t)}(t, \zeta)$, $\zeta \in \Gamma_m$, $m = 1, 2, 3, 4$, and

$$Q^{(t)}(t, \zeta) = \begin{pmatrix} 1 & -\frac{Z(\zeta)}{Y(\bar{\zeta})}e^{2i(8\gamma\zeta^4 - 2\zeta^2)t} \\ -\frac{\bar{Z}(\bar{\zeta})}{Y(\zeta)}e^{-2i(8\gamma\zeta^4 - 2\zeta^2)t} & \frac{1}{Y(\zeta)Y(\bar{\zeta})} \end{pmatrix}. \quad (3.6)$$

- $D^{(t)}(t, \zeta) \rightarrow \mathbf{I}, \zeta \rightarrow \infty$.
- $Y(\zeta)$ possesses $2k$ simple zeros $\{v_j\}_1^{2k}$, $2k = 2k_1 + 2k_2$, assume that $\{v_j\}_1^{2k_1}$ are parts of $L_1 \cup L_3$, then, $\{\bar{v}_j\}_1^{2k_2}$ are parts of $L_2 \cup L_4$.
- $[D_+^{(t)}]_1(t, \zeta)$ enjoys simple poles for $\zeta = \{v_j\}_1^{2k_1}$ and the $[D_-^{(t)}]_2(t, \Lambda)$ has simple poles for $\zeta = \{\bar{v}_j\}_1^{2k_2}$. In this case, the residue relations are defined by

$$\text{Res}\{[D^{(t)}(t, \zeta)]_1, v_j\} = \frac{e^{-2(8\gamma v_j^4 - 2v_j^2)t}}{Z(v_j)\dot{Y}(v_j)} [D^{(t)}(t, v_j)]_2, \quad (3.7a)$$

$$\text{Res}\{[D^{(t)}(t, \zeta)]_2, \bar{v}_j\} = \frac{e^{2(8\gamma \bar{v}_j^4 - 2\bar{v}_j^2)t}}{Z(\bar{v}_j)\dot{Y}(\bar{v}_j)} [D^{(t)}(t, v_j)]_1. \quad (3.7b)$$

4 The Riemann–Hilbert Problem

In this part, we give two important results in theorem form.

Theorem 4.1 *Let $u_0(x) \in \mathcal{S}(\mathbb{R}^+)$, and $\Psi(\zeta), \phi(\zeta)$ defined in terms of $y(\zeta), z(\zeta), Y(\zeta), Z(\zeta)$ are showed in Eq. (2.23). And the $y(\zeta), z(\zeta), Y(\zeta), Z(\zeta)$ denoted by functions $u_0(x), v_j(t), j = 0, \dots, 3$ are showed in Definitions 3.1 and 3.2. Assume that the function $y(\zeta)$ possesses the possible simple zeros $\{\zeta_j\}_{j=1}^{2n}$, and the function $\rho(\zeta)$ possesses the possible simple zeros $\{\eta_j\}_{j=1}^{2N}$. Therefore, the solution of the FODNLS equation (1.2) is*

$$u(x, t) = 2i \lim_{\zeta \rightarrow \infty} (\zeta D(x, t, \zeta))_{12}, \quad (4.1)$$

where $D(x, t, \zeta)$ is the solution of the RH problems as follows:

- $D(x, t, \zeta)$ is a piecewise analytic function for $\zeta \in \mathbb{C} \setminus \Gamma_m$ ($m = 1, \dots, 4$).
- $D(x, t, \zeta)$ jump appears on the curves Γ_m , which meets the jump conditions as

$$D_+(x, t, \zeta) = D_-(x, t, \zeta)Q(x, t, \zeta), \quad \zeta \in \Gamma_m, m = 1, \dots, 4. \quad (4.2)$$

- $D(x, t, \zeta) = I + O(\frac{1}{\zeta}), \zeta \rightarrow \infty$.
- $D(x, t, \zeta)$ possesses residue relations showed in Proposition 2.3.

Thus, the matrix function $D(x, t, \zeta)$ exists and is unique. Furthermore

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u(0, t) = v_0(t), \quad u_x(0, t) = v_1(t), \\ u_{xx}(0, t) &= v_2(t), \quad u_{xxx}(0, t) = v_3(t). \end{aligned}$$

Proof In fact, if we assume that $y(\zeta)$ and $\rho(\zeta)$ have no zeros, then the matrix-valued function $D(x, t, \zeta)$ satisfies a non-singular RH problem, and then we prove that this non-singular RH problem has a unique global solution using the symmetric property of the jump matrix $Q(x, t, \zeta)$ and the scattering matrix. If $y(\zeta)$ and $\rho(\zeta)$ have only a finite number of zeros, we can map this situation to a situation without zeros, which can transform into a system of algebraic equations, this algebraic equations can always be solved uniquely.

Similar Ref. [12], we give the following vanishing theorem without proof. $\square$

Theorem 4.2 (*The vanishing theorem*) *If the matrix function $D(x, t, \zeta) \rightarrow 0$ ($\zeta \rightarrow \infty$), then, the RH problem in Theorem 4.1 possesses only the zero solution.*

5 Conclusions

In this paper, we use the Fokas method to construct the RH problem of the FODNLS equation on the half-line. When the parameter $\gamma = 0$, it can be reduced to the RH problem of the classical nonlinear Schrödinger equation on the half-line. We introduced important functions for spectral analysis of the Lax pair, established the basic RH problem, and gave the global relationship between spectral functions. Furthermore, we can analyze the integrable FODNLS equation (1.2) on a finite interval, and also discuss the asymptotic behavior for the solution of the integrable FODNLS equation (1.2). These two questions will be studied in our future investigation.

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Declarations

Conflict of interest The authors declare they have no conflicts of interest.

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