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RESEARCH ARTICLE

Power Aggregation Operators of Interval-Valued Atanassov-Intuitionistic Fuzzy Sets Based on Aczel–Alsina t-Norm and t-Conorm and Their Applications in Decision Making

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Abstract

Aczel–Alsina t-norm and t-conorm are important t-norm and t-conorm, and they are extended from algebraic t-norm and t-conorm. Obviously, Aczel–Alsina t-norm and t-conorm are more general than some existing t-norm and t-conorm. Furthermore, the power aggregation (PA) operator is also a very famous and valuable operator which can consider the power relation between any two input parameters. In addition, Interval-valued Atanassov-intuitionistic fuzzy set (IVA-IFS) can easily express uncertain information. In order to fully use their advantages, in this analysis, we extend the PA operators based on Aczel–Alsina t-norm and t-conorm to IVA-IFS and propose the interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power averaging (IVA-IFAAPA), interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power ordered averaging (IVA-IFAAPOA), interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power geometric (IVA-IFAAPG) and interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power ordered geometric (IVA-IFAAPOG) operators. Moreover, we discuss the properties of the presented operators such as idempotency, monotonicity, and boundedness. In addition, a multi-attribute decision-making (MADM) procedure is proposed to process the IVA-IF information. Finally, a practical example is used to show the effectiveness and superiority of the proposed method by comparing it with some existing operators.

Keywords Interval-valued Atanassov-intuitionistic fuzzy sets · Aczel–Alsina t-norm and t-conorm · Power aggregation operators · Decision-making techniques

Abbreviations

PA	Power aggregation
A-IFS	Atanassov-intuitionistic fuzzy set
IVA-IFS	Interval-valued Atanassov-intuitionistic fuzzy set
IVA-IFAAPA	Interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power averaging
IVA-IFAAPOA	Interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power ordered averaging
IVA-IFAAPG	Interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power geometric
IVA-IFAAPOG	Interval-valued Atanassov-intuitionistic fuzzy Aczel–Alsina power ordered geometric
MADM	Multi-attribute decision-making
FS	Fuzzy sets
IFS	Intuitionistic fuzzy sets
IVIFS	Interval-valued intuitionistic fuzzy sets
AOs	Aggregation operators

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1 Introduction

The MADM is to select the best one or to give a ranking from the collection of a different and finite number of alternatives. In the traditional decision-making procedure, the attribute value can be only expressed by the crisp number, however, in real-life, many MADM problems cannot be described by real numbers because the attribute information is with ambiguity or uncertain. In order to process the ambiguity and complexity of the attributes, the FS [1] is presented which is an effective tool to express the uncertain information. Now, the FS theory has a lot of applications in various fields, for example, employment selection (doctors and nurses) during the COVID-19 pandemic [2], designing different clothes with the help of fuzzy superior mandelbrot set [3], business analysis and management research [4], data-driven modeling in manifolds based on the FS theory [5], environmental uncertainty and digital transformation for fuzzy information [6], N-soft set under the consideration of multi-agent preferences and their operators [7], the algebraic structure of multi-fuzzy soft information [8], multi q-fuzzy soft expert information [9], strategic decision-making analysis based on fuzzy N-soft expert information [10], and treatment decision of lung cancer with the help of fuzzy information [11].

Fuzzy set, soft set, and N-soft set are three different extensions, and the FS has received a lot of attention from various individuals, however, because there is only a membership grade in FS, in some cases, it is difficult to deal with some uncertain information. Some scholars pointed out that the negative or falsity or non-membership grade also played a very essential or critical role in some real-life problems, then, Atanassov [12] proposed a new structure called IFS which can deal with truth grade "$\overline{\overline{\zeta}}_{\mathfrak{W}_{\mathfrak{Q}}}(\widetilde{\mu_{\varrho}})$" and as well as falsity grade "$\overline{\overline{\eta}}_{\mathfrak{W}_{\mathfrak{Q}}}(\widetilde{\mu_{\varrho}})$" with a condition: $0 \leq \overline{\overline{\zeta}}_{\mathfrak{W}_{\mathfrak{Q}}}(\widetilde{\mu_{\varrho}}) + \overline{\overline{\eta}}_{\mathfrak{W}_{\mathfrak{Q}}}(\widetilde{\mu_{\varrho}}) \leq 1$. Noticed that the FS is one of the valuable and famous cases when falsity grade $\overline{\overline{\eta}}_{\mathfrak{W}_{\mathfrak{Q}}}(\widetilde{\mu_{\varrho}}) = 0$ in the IFS. Furthermore, the IFS has been used in the fields of decision-making, artificial intelligence, machine learning, and clustering analysis, for example, IVIFS [13], stock prediction under the consideration of A-IF inferences [14], MADM technique for IFSs [15], distance measures for IFS and their application in various triangle centers [16], two novel types of measures in the IFSs [17], software bug triaging under the availability of IFSs and their application [18], the enhancement of color images under the uses of IFS [19], and positron emission tomography image segmentation for IFSs and their applications in clustering analysis [20].

Aggregating the collection of information into one is a very challenging task. Some scholars have proposed different types of operators such as arithmetic and geometric AOs. Further, Yager [21] developed the PA operator which is the generalized form of the above two different operators. Now the averaging, geometric, and PA operators are computed based on algebraic t-norm and t-conorm. Further, Aczel and Alsina [22] proposed Aczel–Alsina t-norm and t-conorm which are the modified and generalized forms of algebraic norms. The Aczel–Alsina t-norm and t-conorm are the generalization of the algebraic t-norm and t-conorm, so it is clear that the aggregation operators based on algebraic t-norm and t-conorm are the special cases of the proposed operators. Then they received a lot of attention from different scholars because of their remarkable features. For instance, Xu [23] proposed the averaging AOs for IFS, Xu and Yager [24] proposed the geometric AOs for IFS, Jiang et al. [25] and Xu [26] proposed the PA operators for IFS, Senapati et al. [27] and Senapati et al. [28] proposed the Aczel–Alsina averaging AOs for IFS and the geometric Aczel–Alsina AOs for IFS. Senapati et al. [29] proposed the weighted PA operators based on Aczel–Alsina operational laws. Furthermore, geometric AOs for IVIFS was discovered by Wei and Wang [30]. Wang et al. [31] also examined the theory of averaging AOs for IVIFSs. Aczel–Alsina AOs for IVIFS was invented by Senapati et al. [32]. Where the power AOs for IVIFS was derived by He et al. [33]. Therefore, we concluded that the proposed operators from Xu [23], Xu and Yager [24], Senapati et al. [32] and He et al. [33] are the special cases of proposed studies. Furthermore, we discuss the advantages of the proposed works, such as

1. If we have only used the theory of PA operators based on algebraic laws, then the derived theory will be changed for the proposed theory of He et al. [33].
2. If we have only used the theory of simple averaging and geometric aggregation operators based on Aczel–Alsina operational laws, then the derived theory will be changed for the proposed theory of Senapati et al. [32].
3. If we have only used the theory of simple averaging and geometric aggregation operators based on algebraic operational laws, then the derived theory will be changed for the proposed theory of Wei and Wang [30] and Wang et al. [31].

From the above analysis, it is clear that some scholars studied the power aggregation operators based on different types of norms, Senapati et al. [29] proposed the weighted PA operators based on Aczel–Alsina operational laws and they thought that the monotonicity is also kept for the proposed work, but in actuality, they are not held. In this manuscript, we aim to propose the PA operators based on Aczel–Alsina operational laws for IVA-IFSs. Since the PA operators based on Aczel–Alsina t-norm and t-conorm are more general than some existing averaging and geometric AOs, and IVA-IFS can easily express the uncertain information, it is necessary to develop some new AOs for IVA-IFS based on Aczel–Alsina t-norm and t-conorm. In this scenario, our main target or contributions are shown as follows.

1. To propose the IVA-IFAAPA, IVA-IFAAPOA, IVA-IFAAPG, and IVA-IFAAPOG operators.
2. To discuss the properties of the presented operators.
3. To develop a MADM method with IVA-IF information based on the proposed operators.
4. To demonstrate the advantages of the proposed method by comparing it with various prevailing operators by some examples.

This article is summarized as follows: in Sect. 2, we review the PA operators, Aczel–Alsina operational laws for IVA-IFSs, and the comparison method for two IVA-IFSs. In Sect. 3, we develop the IVA-IFAAPA, IVA-IFAAPOA, IVA-IFAAPG, and IVA-IFAAPOG operators, and discuss the properties of the presented operators such as idempotency, monotonicity, and boundedness. In Sect. 4, a procedure for decision-making is developed for the MADM problem with A-IF information. In Sect. 5, a practical example is used to show the advantages of the proposed technique by comparing it with various prevailing operators. Some valuable conclusions and future research are stated in Sect. 6.

2 Preliminaries

In this section, we reviewed the PA operators, IVA-IFSs, and Aczel–Alsina operational laws.

Definition 1 [21] For the finite positive numbers $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \tau = 1, 2, \ldots, Z$, then the PA operator is derived by

$$PA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_Z}\right) = \frac{\sum_{\tau=1}^{Z}\left(1+\overline{\overline{\mathfrak{T}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right)\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}{\sum_{\tau=1}^{Z}\left(1+\overline{\overline{\mathfrak{T}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right)} \tag{1}$$

Noticed that $\overline{\overline{\mathfrak{T}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right) = \sum_{s=1, s\neq\tau}^{Z} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right)$, which expresses the relationship between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}$ with various characteristics:

1. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) \in [0,1]$.
2. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) = Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)$.
3. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) \geq Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{*}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}^{*}\right)$ when $\left|\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} - \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right| \leq \left|\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{*} - \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}^{*}\right|$.

Definition 2 [13] For the universal set $\widetilde{\mathbb{X}_U}$, the IVA-IF set was given such that

$$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}} = \left\{\left(\overline{\overline{\zeta}}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}), \overline{\overline{\eta}}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right) : \widetilde{\mu_\varrho} \in \widetilde{\mathbb{X}_U}\right\} \tag{2}$$

Here, $\overline{\overline{\zeta}}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) = \left[\overline{\overline{\zeta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}), \overline{\overline{\zeta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right]$ and $\overline{\overline{\eta}}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) = \left[\overline{\overline{\eta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}), \overline{\overline{\eta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right]$ represents the values of truth and falsity and meets $0 \leq \overline{\overline{\zeta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) + \overline{\overline{\eta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) \leq 1$, then a neutral grade is expressed by $\overline{\overline{\mathfrak{R}}}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) = \left[\overline{\overline{\mathfrak{R}}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}), \overline{\overline{\mathfrak{R}}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right] = \left[\left(1-\left(\overline{\overline{\zeta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) + \overline{\overline{\eta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right)\right), \left(1-\left(\overline{\overline{\zeta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) + \overline{\overline{\eta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho})\right)\right)\right]$. Furthermore, the simple form of IVA-IFN is expressed by $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$.

Definition 3 For two IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2$, then the Aczel–Alsina operational laws are shown as

$$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} \oplus \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2} = \left(\begin{array}{c} \left[1-e^{-\left(\left(-\log\left(1-\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(1-\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1-e^{-\left(\left(-\log\left(1-\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(1-\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[e^{-\left(\left(-\log\left(\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\left(-\log\left(\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right) \tag{3}$$

$$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} \otimes \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2} = \left(\begin{array}{c} \left[e^{-\left(\left(-\log\left(\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\left(-\log\left(\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[1-e^{-\left(\left(-\log\left(1-\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(1-\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1-e^{-\left(\left(-\log\left(1-\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}+\left(-\log\left(1-\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right) \tag{4}$$

$$\overline{\overline{\theta}}_S \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} = \left(\begin{array}{c} \left[1-e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(1-\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1-e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(1-\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right) \tag{5}$$

$$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}^{\overline{\overline{\theta}}_S} = \left(\begin{array}{c} \left[e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(\zeta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(\zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[1-e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(1-\eta^{l}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1-e^{-\left(\overline{\overline{\theta}}_S\left(-\log\left(1-\eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right) \tag{6}$$

Definition 4 [13] For two IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$, $\tau = 1, 2$, then,

$$\overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) = \frac{1}{2}\left(\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} + \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right) \in [-1, 1] \tag{7}$$

$$\overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) = \frac{1}{2}\left(\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} + \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} + \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} + \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right) \in [0, 1] \tag{8}$$

$\overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)$ and $\overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)$ represented the score function and accuracy function with some rules:

(1) If $\overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) > \overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right) \Rightarrow \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$;

(2) If $\overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) < \overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right) \Rightarrow \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} < \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$;

(3) If $\overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) = \overline{\overline{\Theta_{SV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right)$ then

If $\overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) > \overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right) \Rightarrow \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$;

If $\overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) < \overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right) \Rightarrow \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} < \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$;

If $\overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}\right) = \overline{\overline{\Theta_{AV}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}\right) \Rightarrow \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$.

3 Aczel–Alsina PA Operators for IVA-IFSs

The Aczel–Alsina t-norm and t-conorm and the PA operator are very famous and valuable tools for getting the finest preferences from the collection of preferences. In this section, we propose the IVA-IFAAPA, IVA-IFAAPOA, IVA-IFAAPG, and IVA-IFAAPOG operators. Moreover, we discuss some of the properties of the presented operators such as idempotency, monotonicity, and boundedness.

Definition 5 For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$, $\tau = 1, 2, \ldots, \mathrm{Z}$, then an IVA-IFAAPA operator is described as

$$IVA-IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\mathrm{Z}}\right) = \overline{\overline{\Xi}}_1\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} \oplus \overline{\overline{\Xi}}_2\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2} \oplus \ldots \oplus \overline{\overline{\Xi}}_\mathrm{Z}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\mathrm{Z}} = \oplus_{\tau=1}^{\mathrm{Z}}\left(\overline{\overline{\Xi}}_\tau\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) \tag{9}$$

Noticed that

$$\overline{\overline{\Xi}}_\tau = \frac{\left(1 + \overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)}{\sum_{\tau=1}^{\mathrm{Z}}\left(1 + \overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)} \tag{10}$$

where $\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) = \sum_{\substack{s=1,\\ s \neq \tau}}^{\mathrm{Z}} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$ which expresses the relationship between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}$ with various characteristics:

1. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) \in [0, 1]$.
2. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) = Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)$.
3. $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) \geq Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}^{*}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}^{*}_{\mathfrak{L}_k}\right)$ when $\left|\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} - \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right| \leq \left|\overline{\overline{\breve{\mathfrak{W}}}}^{*}_{\mathfrak{L}_\tau} - \overline{\overline{\breve{\mathfrak{W}}}}^{*}_{\mathfrak{L}_k}\right|$

where $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) = 1 - Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$ and

$$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) = \frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|\right) \tag{11}$$

where Eq. (11) represents the distance measures between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}$.

Theorem 1 *For a group of IVA-IFNs* $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$, $\tau = 1, 2, \ldots, \mathrm{Z}$, *the aggregation result from Eq.* (9) *is still an IVA-IFN, such as*

$$IVA-IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2},\ldots,\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right) = \left(\begin{array}{c}\left[1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right]\end{array}\right) \tag{12}$$

Proof (Using Mathematical induction).

For $Z=2$, we have

$$\overline{\overline{\Xi}}_{1}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1} = \left(\begin{array}{c}\left[1-e^{-\left(\overline{\overline{\Xi}}_{1}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},1-e^{-\left(\overline{\overline{\Xi}}_{1}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\overline{\overline{\Xi}}_{1}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},e^{-\left(\overline{\overline{\Xi}}_{1}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_1}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right]\end{array}\right)$$

$$\overline{\overline{\Xi}}_{2}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2} = \left(\begin{array}{c}\left[1-e^{-\left(\overline{\overline{\Xi}}_{2}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},1-e^{-\left(\overline{\overline{\Xi}}_{2}\left(-\log\left(1-\overline{\overline{\breve{\zeta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\overline{\overline{\Xi}}_{2}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{l}_{\mathfrak{W}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}},e^{-\left(\overline{\overline{\Xi}}_{2}\left(-\log\left(\overline{\overline{\breve{\eta}}}^{u}_{\mathfrak{W}_{\mathfrak{L}_2}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right]\end{array}\right)$$

Thus, we have

$$IVA-IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}},\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}\right)=\overline{\overline{\Xi_1}}\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}\oplus\overline{\overline{\Xi_2}}\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}$$

$$=\left(\begin{array}{c}\left[1-e^{-\left(\overline{\overline{\Xi_1}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},1-e^{-\left(\overline{\overline{\Xi_1}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\overline{\overline{\Xi_1}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},e^{-\left(\overline{\overline{\Xi_1}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right]\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\left[1-e^{-\left(\overline{\overline{\Xi_2}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},1-e^{-\left(\overline{\overline{\Xi_2}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\overline{\overline{\Xi_2}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},e^{-\left(\overline{\overline{\Xi_2}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[1-e^{-\left(\sum_{\tau=1}^{2}\overline{\overline{\Xi_\tau}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},1-e^{-\left(\sum_{\tau=1}^{2}\overline{\overline{\Xi_\tau}}\left(-\log\left(1-\overline{\overline{\zeta_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right],\\ \left[e^{-\left(\sum_{\tau=1}^{2}\overline{\overline{\Xi_\tau}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}}}^{l}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}},e^{-\left(\sum_{\tau=1}^{2}\overline{\overline{\Xi_\tau}}\left(-\log\left(\overline{\overline{\eta_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}}}^{u}\right)\right)\right)^{\Lambda\frac{1}{\Lambda}}}\right]\end{array}\right)$$

For $Z = 2$, (12) is kept.for $Z = k$, we assume that (12) is right, such as

$$IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$$
$$= \left(\begin{array}{c} \left[1 - e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right)$$

Then, for $Z = k + 1$, we aim to derive our required result such as

$$IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{k+1}}\right) = \oplus_{\tau=1}^{k+1}\left(\overline{\overline{\Xi}}_{\tau}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)$$
$$= \oplus_{\tau=1}^{k}\left(\overline{\overline{\Xi}}_{\tau}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) \oplus \overline{\overline{\Xi}}_{k+1}\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{k+1}}$$
$$= \left(\begin{array}{c} \left[1 - e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{k} \overline{\overline{\Xi}}_{\tau}\left(-\log\left(\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right)$$
$$\oplus \left(\begin{array}{c} \left[1 - e^{-\left(\overline{\overline{\Xi}}_{k+1}\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{k+1}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\overline{\overline{\Xi}}_{k+1}\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{k+1}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \\ \left[e^{-\left(\overline{\overline{\Xi}}_{k+1}\left(-\log\left(\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{k+1}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\overline{\overline{\Xi}}_{k+1}\left(-\log\left(\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{k+1}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \end{array} \right)$$

$$= \left(\left[1 - e^{-\left(\sum_{\tau=1}^{k+1} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\sum_{\tau=1}^{k+1} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \left[e^{-\left(\sum_{\tau=1}^{k+1} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{k+1} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \right)$$

Therefore, when $\mathrm{Z} = k+1$, (12) is also kept.

In a word, (12) is right.

Property 1 (Idempotency) For a group of IVA-IFNs $\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right]\right)$, $\tau = 1, 2, \ldots, \mathrm{Z}$, If $\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_\tau} = \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right]\right)$, then

$$IVA-IFAAPA\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_1}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_{\mathrm{Z}}}\right) = \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}} \tag{13}$$

Proof For a group of IVA-IFNs $\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right]\right)$, $\tau = 1, 2, \ldots, \mathrm{Z}$,

If $\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_\tau} = \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right]\right)$, then

$$IVA-IFAAPA\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_1}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{L}_{\mathrm{Z}}}\right)$$

$$= \left(\left[1 - e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \left[e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \right)$$

$$= \left(\left[1 - e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \left[e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau \left(-\log\left(\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \right)$$

$$= \left(\left[1 - e^{-\left(\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1 - e^{-\left(\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right], \left[e^{-\left(\left(-\log\left(\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\left(-\log\left(\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{L}}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}} \right] \right), \sum_{\tau=1}^{\mathrm{Z}} \overline{\overline{\Xi}}_\tau = 1$$

$$= \left(\left[1-e^{\log\left(1-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right)}, 1-e^{\log\left(1-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right)}\right], \left[e^{\log\left(\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right)}, e^{\log\left(\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right)}\right]\right)$$

$$= \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right]\right) = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}.$$

Property 2 (Non-monotonicity) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right)$, $\tau = 1, 2, \ldots, \mathrm{Z}$, If $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} \leq \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}'^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}'^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}'^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}'^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right)$, then

$$IVA-IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\mathrm{Z}}\right) \nleq IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_\mathrm{Z}}\right) \tag{14}$$

Here, we give only some counterexamples to show that Eq. (14) is held correctly. For this, we use a few IVA-IFNs such as $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} = ([0.2, 0.21], [0.2, 0.21]), \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} = ([0.4, 0.41], [0.4, 0.41]), \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} = ([0.1, 0.11], [0.88, 0.89])$ and $\overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_1} = ([0.2, 0.21], [0.2, 0.21]), \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_2} = ([0.4, 0.41], [0.4, 0.41]), \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_3} = ([0.11, 0.12], [0.11, 0.12])$ with $\Lambda = 2$. We get the following results:

$$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}\right) = 0.2$$

$$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 0.39; Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 0.39$$

$$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}\right) = 1 - Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}\right) = 1 - 0.2 = 0.8$$

$$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 0.61; Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 0.61$$

Thus,

$$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}\right) = \sum_{\substack{k=1,\\ k \neq 1}}^{\mathrm{Z}} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) = Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}\right) + Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 1.41$$

$$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}\right) = 1.41; \overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = 1.22$$

$$\sum_{\tau=1}^{\mathrm{Z}}\left(1 + \overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right) = 7.04$$

Thus,

$$\overline{\overline{\Xi}}_1 = \frac{\left(1 + \overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}\right)\right)}{\sum_{\tau=1}^{3}\left(1 + \overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right)} = \frac{1 + 1.41}{7.04} = 0.34233$$

$$\overline{\overline{\Xi}}_2 = 0.34233; \overline{\overline{\Xi}}_3 = 0.31534.$$

Hence, by Eq. (12), we have

$$IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}\right) = ([0.13408, 0.13892], [0.62399, 0.6328])$$

In the same way, we get

$$IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_2}, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_3}\right) = ([0.13144, 0.13628], [0.48398, 0.49653])$$

From the above two information, we get

$$IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\mathrm{Z}}\right) \nleq IVA - IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}'_{\mathfrak{Q}_\mathrm{Z}}\right)$$

Noticing that the monotonicity has been neglected. Further, we give the boundedness.

Property 3 (Boundedness) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, \mathrm{Z}$, If $\overline{\overline{\breve{\mathfrak{W}}}}^{-}_{\mathfrak{Q}_\tau} = \left(\left[\min_\tau \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \min_\tau \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\max_\tau \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}},\right.\right.$

$\max_{\tau}\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}])$ and $\overline{\overline{\breve{\mathfrak{W}}}}^{+}_{\mathfrak{L}_\tau} = \left(\left[\max_{\tau}\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \max_{\tau}\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\min_{\tau}\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \min_{\tau}\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$, then

$$\overline{\overline{\breve{\mathfrak{W}}}}^{-}_{\mathfrak{L}_\tau} \leq IVA-IFAAPA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right) \leq \overline{\overline{\breve{\mathfrak{W}}}}^{+}_{\mathfrak{L}_\tau}$$

Definition 6 For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, then an IVA-IFAAPOA operator is described as

$$\begin{aligned} &IVA-IFAAPOA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right) \\ &= \overline{\overline{\Xi}}_1\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(1)}} \oplus \overline{\overline{\Xi}}_2\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(2)}} \oplus \ldots \oplus \overline{\overline{\Xi}}_Z\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(Z)}} \\ &= \oplus_{\tau=1}^{Z}\left(\overline{\overline{\Xi}}_\tau\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(\tau)}}\right) \end{aligned} \tag{15}$$

where $0(\tau) \leq 0(\tau-1)$ represents the permutations and $\overline{\overline{\Xi}}_\tau = \frac{\left(1+\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)}{\sum_{\tau=1}^{Z}\left(1+\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)}$. Further, the $\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right) = \sum_{\substack{s=1,\\ s\neq\tau}}^{Z} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$, which expresses the relationship between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}$, where $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) = 1 - Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$ and $Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right) = \frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right| + \left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|\right)$, which is the distance information between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} and \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}$.

Theorem 2 *For a group of IVA-IFNs* $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, *the aggregation result from Eq. (15) is still an IVA-IFN, such as*

$$\begin{aligned} &IVA-IFAAPOA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right) \\ &= \left(\begin{array}{c} \left[1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, 1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right], \\ \left[e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right] \end{array}\right) \end{aligned} \tag{16}$$

Property 4 (Idempotency) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, If $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau} = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}}}\right]\right)$, then

$$IVA-IFAAPOA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right) = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}} \tag{17}$$

Further, the monotonicity is not kept. Further, we give the boundedness.

Property 5 (Boundedness) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, If $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{-} = \left(\left[\min_\tau \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \min_\tau \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\max_\tau \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \max_\tau \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right)$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{+} = \left(\left[\max_\tau \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \max_\tau \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\min_\tau \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \min_\tau \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right)$, then

$$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{-} \le IVA-IFAAPOA\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_Z}\right) \le \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}^{+}$$

Definition 7 For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, then an IVA-IFAAPG operator is described as

$$IVA-IFAAPG\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_Z}\right) = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}^{\overline{\overline{\Xi}}_1} \otimes \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}^{\overline{\overline{\Xi}}_2} \otimes \ldots \otimes \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_Z}^{\overline{\overline{\Xi}}_Z} = \otimes_{\tau=1}^{Z} \left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)^{\overline{\overline{\Xi}}_\tau} \tag{18}$$

Noticed that $\overline{\overline{\Xi}}_\tau = \frac{\left(1+\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right)}{\sum_{\tau=1}^{Z}\left(1+\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right)\right)}$. Further, $\overline{\overline{\Im}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}\right) = \sum_{s=1,\ s\neq\tau}^{Z} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right)$, which expresses the relationship between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}$, where $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) = 1 - Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right)$ and $Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_k}\right) = \frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}} - \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_k}}\right| + \left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_k}}\right| + \left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}} - \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_k}}\right| + \left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_k}}\right|\right)$.

Theorem 3 *For a group of IVA-IFNs* $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, *the aggregation result from Eq.* (18) *is still an IVA-IFN, such as*

$$IVA-IFAAPG\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_Z}\right) = \left(\begin{array}{c} \left[e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right], \\ \left[1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}, 1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right)\right)^{\Lambda}\right)^{\frac{1}{\Lambda}}}\right] \end{array}\right) \tag{19}$$

Property 6 (Idempotency) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_\tau}}\right]\right), \tau = 1, 2, \ldots, Z$, If $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau} = \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}, \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right], \left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}, \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}}}\right]\right)$, then

$$IVA-IFAAPG\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2},\ldots,\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right)=\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}} \tag{20}$$

Further, the monotonicity is not kept.

Property 7 (Boundedness) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}=\left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right),\tau=1,2,\ldots,Z,$ If $\overline{\overline{\breve{\mathfrak{W}}}}^{-}_{\mathfrak{L}_\tau}=\left(\left[\min_{\tau}\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\min_{\tau}\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\max_{\tau}\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\max_{\tau}\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$ and $\overline{\overline{\breve{\mathfrak{W}}}}^{+}_{\mathfrak{L}_\tau}=\left(\left[\max_{\tau}\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\max_{\tau}\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\min_{\tau}\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\min_{\tau}\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right)$, then

$$\overline{\overline{\breve{\mathfrak{W}}}}^{-}_{\mathfrak{L}_\tau}\le IVA-IFAAPG\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2},\ldots,\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right)\le\overline{\overline{\breve{\mathfrak{W}}}}^{+}_{\mathfrak{L}_\tau}$$

Definition 8 For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}=\left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right),\tau=1,2,\ldots,Z$, then an IVA-IFAAPOG operator is described as

$$\begin{aligned}IVA-IFAAPOG&\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2},\ldots,\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right)\\&=\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(1)}}^{\overline{\overline{\Xi}}_1}\otimes\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(2)}}^{\overline{\overline{\Xi}}_2}\otimes\ldots\otimes\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(Z)}}^{\overline{\overline{\Xi}}_Z}\\&=\otimes_{\tau=1}^{Z}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{0(\tau)}}^{\overline{\overline{\Xi}}_\tau}\right)\end{aligned} \tag{21}$$

where $0(\tau)\le 0(\tau-1)$ represents the permutations and $\overline{\overline{\Xi}}_\tau=\frac{\left(1+\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)}{\sum_{\tau=1}^{Z}\left(1+\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)\right)}$. Further, $\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}\right)=\sum_{s=1,\,s\neq\tau}^{Z}Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$, which expresses the relationship between $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}$ and $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}$, where $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)=1-Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)$ and $Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_k}\right)=\frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}-\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|+\left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}-\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|+\left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}-\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|+\left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}-\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_k}}\right|\right)$, stated the value of distance information.

Theorem 4 *For a group of IVA-IFNs* $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}=\left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right),\tau=1,2,\ldots,Z,$ *the aggregation result from Eq. (21) is still an IVA-IFN, such as*

$$\begin{aligned}&IVA-IFAAPOG\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_1},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2},\ldots,\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_Z}\right)\\&=\left(\begin{array}{c}\left[e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}},e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right],\\\left[1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}},1-e^{-\left(\sum_{\tau=1}^{Z}\overline{\overline{\Xi}}_\tau\left(-\log\left(1-\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_{0(\tau)}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right]\end{array}\right)\end{aligned} \tag{22}$$

Property 8 (Idempotency) For a group of IVA-IFNs $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_\tau}=\left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right],\left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}},\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{L}_\tau}}\right]\right),\tau=1,2,\ldots,Z,$

If $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau} = \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}}}\right]\right)$, then

$$IVA-IFAAPOG\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_1}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_Z}\right) = \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}} \tag{23}$$

Further, the monotonicity is not kept. Further, we give the boundedness.

Property 9 (Boundedness) For a group of IVA-IFNs $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau} = \left(\overline{\overline{\zeta}}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}, \overline{\overline{\eta}}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}\right), \tau = 1, 2, \ldots, Z.$ If $\overline{\overline{\mathfrak{W}}}^{-}_{\mathfrak{Q}_\tau} = \left(\left[\min_\tau \overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}, \min_\tau \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}\right], \left[\max_\tau \overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}, \max_\tau \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}\right]\right)$ and $\overline{\overline{\mathfrak{W}}}^{+}_{\mathfrak{Q}_\tau} = \left(\left[\max_\tau \overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}, \max_\tau \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}\right], \left[\min_\tau \overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}, \min_\tau \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_\tau}}\right]\right)$, then

$$\overline{\overline{\mathfrak{W}}}^{-}_{\mathfrak{Q}_\tau} \leq IVA-IFAAPOG\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_1}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_Z}\right) \leq \overline{\overline{\mathfrak{W}}}^{+}_{\mathfrak{Q}_\tau}$$

4 A Decision-Making Procedure Based on the Proposed Operators

The MADM is the subpart of the decision-making process, which is used to determine the best optimal form of the collection of preferences. Based on the derived operators, we develop the MADM technique with the IVA-IF information.

For a MADM problem, there are a finite number of alternatives $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_1}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_Z}$ and every alternative is evaluated by a collection of a finite number of attributes $\overline{\overline{\mathfrak{W}}}^{/}_{\mathfrak{Q}_1}, \overline{\overline{\mathfrak{W}}}^{/}_{\mathfrak{Q}_2}, \ldots, \overline{\overline{\mathfrak{W}}}^{/}_{\mathfrak{Q}_m}$. To proceed with the above problem, we need to construct a decision matrix $N = \left[\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau j}}\right]_{Z\times m}$ where $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau j}}$ is an evaluation value for the alternative $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau}$ under the attribute $\overline{\overline{\mathfrak{W}}}^{/}_{\mathfrak{Q}_j}$, which is expressed by an IVA-IF number such as $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau j}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right]\right)$, $\tau = 1, 2, \ldots, Z; j = 1, 2, \ldots, m$, where $\overline{\overline{\zeta}}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}$ and $\overline{\overline{\eta}}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}$ represent the true value and falsity value, and meet $0 \leq \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}} + \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}} \leq 1$. Furthermore, to solve the above problem, we develop the procedure of decision-making based on the proposed operators.

Step 1: Normalize the decision matrix.

Since the evaluation value $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau j}}$ for the alternative $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau}$ under the attribute $\overline{\overline{\mathfrak{W}}}^{/}_{\mathfrak{Q}_j}$ maybe the cost or benefit type should be converted to the same type. If it is cost-type information, then we need to normalize it by

$$N = \begin{cases} \left(\left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right], \left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right]\right) & \textit{for benefit types} \\ \left(\left[\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right], \left[\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}, \overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right]\right) & \textit{for cost types} \end{cases}$$

Step 2: Aggregate all attribute values of each alternative by the IVA-IFAAPA operator and IVA-IFAAPG operator:

$$\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau} = IVA-IFAAPA\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau 1}}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau 2}}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau m}}\right) = \left(\begin{array}{c} \left[1-e^{-\left(\sum_{j=1}^{m}\overline{\overline{\Xi}}_j\left(-\log\left(1-\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, 1-e^{-\left(\sum_{j=1}^{m}\overline{\overline{\Xi}}_j\left(-\log\left(1-\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right], \\ \left[e^{-\left(\sum_{j=1}^{m}\overline{\overline{\Xi}}_j\left(-\log\left(\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, e^{-\left(\sum_{j=1}^{m}\overline{\overline{\Xi}}_j\left(-\log\left(\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right] \end{array}\right)$$

$$\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_\tau} = IVA-IFAAPG\left(\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau 1}}, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau 2}}, \ldots, \overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_{\tau m}}\right) = \left(\begin{array}{c} \left[e^{-\left(\sum_{j=1}^{Z}\overline{\overline{\Xi}}_j\left(-\log\left(\overline{\overline{\zeta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, e^{-\left(\sum_{j=1}^{Z}\overline{\overline{\Xi}}_j\left(-\log\left(\overline{\overline{\zeta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right], \\ \left[1-e^{-\left(\sum_{j=1}^{Z}\overline{\overline{\Xi}}_j\left(-\log\left(1-\overline{\overline{\eta}}^{l}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}, 1-e^{-\left(\sum_{j=1}^{Z}\overline{\overline{\Xi}}_j\left(-\log\left(1-\overline{\overline{\eta}}^{u}_{\mathfrak{W}_{\mathfrak{Q}_{\tau j}}}\right)\right)^{\wedge}\right)^{\frac{1}{\wedge}}}\right] \end{array}\right)$$

Step 3: Calculate the score and accuracy functions based on Eqs. (7) and (8).

Step 4: Obtain the ranking results based on the Score value information and also get the best preference from the collection of alternatives.

In addition, we give a counterexample and try to verify the stability and effectiveness of the proposed approaches under the IVA-IF information.

4.1 Illustrative Example

An enterprise wants to invest its money in some valuable business. For this, five different businesses for investment in 2022 to 2030 are explained as $\overline{\overline{\mathfrak{W}}}_{\mathfrak{Q}_1}$: Computer business,

Table 1 Initial IVA-IF decision matrix

	$\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_1}$	$\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_2}$	$\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_3}$	$\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_4}$
$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1}$	$([0.3,0.4],[0.1,0.2])$	$([0.31,0.41],[0.11,0.21])$	$([0.32,0.42],[0.12,0.22])$	$([0.33,0.43],[0.13,0.23])$
$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}$	$([0.6,0.7],[0.1,0.2])$	$([0.61,0.71],[0.11,0.21])$	$([0.62,0.72],[0.12,0.22])$	$([0.63,0.73],[0.13,0.23])$
$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}$	$([0.4,0.5],[0.3,0.4])$	$([0.41,0.51],[0.31,0.41])$	$([0.42,0.52],[0.32,0.42])$	$([0.43,0.53],[0.33,0.43])$
$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4}$	$([0.5,0.6],[0.2,0.3])$	$([0.51,0.61],[0.21,0.31])$	$([0.52,0.62],[0.22,0.32])$	$([0.53,0.63],[0.23,0.33])$
$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$	$([0.2,0.4],[0.3,0.4])$	$([0.21,0.41],[0.31,0.41])$	$([0.22,0.42],[0.32,0.42])$	$([0.23,0.43],[0.33,0.43])$

Table 2 The distance measures

$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}}\right)=0.01$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}}\right)=0.01$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}}\right)=0.01$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}}\right)=0.01$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}}\right)=0.01$
$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.04$
$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.06$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.06$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.06$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.06$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{11}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.06$
$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}}\right)=0.02$
$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.04$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{12}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.04$
$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.02$	$Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{13}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{14}}\right)=0.02$

$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2}$: Laptop business, $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3}$: Mobile business, $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4}$: Software business, and$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$: Hardware business. To select the best one from the above five businesses, four different features are used to evaluate these alternatives, which are explained as $\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_1}$: the price, $\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_2}$: warranty, $\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_3}$: quality, and $\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_4}$: social and political impact. To solve this problem, we construct a decision matrix $N = [\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}}]_{Z\times m}(shownintable1)$ where $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}}$ is an evaluation value for the business $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_\tau}$ under the attribute $\overline{\overline{\breve{\mathfrak{W}}}}^{/}_{\mathfrak{Q}_j}$, which is expressed by an IVA-IF number such as $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}} = \left(\left[\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}},\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}}\right],\left[\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}},\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}}\right]\right), \tau = 1,2,\ldots,Z; j = 1,2,\ldots,m$, based on the proposed method, we can solve this problem by the following steps.

Step 1: Normalize the decision matrix.

Since all evaluation values are benefit types of information, it is not needed to normalize them.

Further, based on the data in Table 1, we get their distance measures by $Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau k}}\right) = \frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right|\right)$, and shown in Table 2.

Further, we calculate their support degrees using the data in Table 2 based on $Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau k}}\right) = 1 - Dis\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau j}},\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_{\tau k}}\right) = 1 - \frac{1}{4}\left(\left|\overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\zeta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\eta}}^{l}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\zeta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right| + \left|\overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau j}}} - \overline{\overline{\eta}}^{u}_{\breve{\mathfrak{W}}_{\mathfrak{Q}_{\tau k}}}\right|\right)$, and the results are shown in Table 3.

Table 3 The support degrees

$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}\right) = 0.99$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}\right) = 0.99$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}\right) = 0.99$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}\right) = 0.99$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}\right) = 0.99$
$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.96$
$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.94$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.94$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.94$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.94$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{11}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.94$
$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}\right) = 0.98$
$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.96$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{12}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.96$
$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.98$	$Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{13}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{14}}\right) = 0.98$

To evaluate the values of $\overline{\overline{\Xi_{\tau j}}}$, we need to compute the values of $\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau j}}\right)$ by $\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau j}}\right) = \sum_{\substack{k=1,\\ k \neq j}}^{m} Sup\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau j}}, \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau k}}\right)$, and the results as shown in Table 4.

Then, we can calculate the values of $\overline{\overline{\Xi_{\tau j}}}$ by $\overline{\overline{\Xi_{\tau j}}} = \frac{\left(1+\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau j}}\right)\right)}{\sum_{j=1}^{m}\left(1+\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_{\tau j}}\right)\right)}$ (from Eq. (10), and the results are shown in Table 5.

Step 2: Aggregate all attribute values for each alternative based on the IVA-IFAAPA operator and IVA-IFAAPG operator, and the results are shown in Table 6.

Step 3: Calculate the Score values based on Eq. (7), and the results are shown in Table 7.

Step 4: Obtain the ranking results based on the Score values and the results are shown in Table 8.

From Table 8, we can get that the best choice is $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$ based on the IVA-IFAAPA operator and IVA-IFAAPG operator, i.e., there are the same best choice from the two methods. However, there are the different ranking results because of some variations of parameters involved in the proposed operators. Furthermore, we check the supremacy and worth of the proposed methods by comparative analysis.

5 Comparative Analysis

In Sect. 4, we have given the decision-making results, to show the effectiveness and superiority of the derived method, we compare the proposed operators with various prevailing operators, such as averaging AOs for IFS by Xu [23], geometric AOs for IFS by Xu and Yager [24], PA operators for IFS by Jiang et al. [25], PA operators for IFS by Xu [26], Aczel–Alsina averaging AOs for IFS by Senapati et al. [27], the geometric Aczel–Alsina AOs for IFS by Senapati et al. [28], geometric AOs for IVIFS by Wei and Wang [30], the averaging AOs for IVIFSs by Wang et al. [31], Aczel–Alsina AOs for IVIFS by Senapati et al. [32], the power AOs for IVIFS by He et al. [33]. Then, the comparison results are shown in Table 9.

From Table 9, we can obtain that the finest optimal is $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{L}_2}$ for all methods.

1. Wang et al. [31] proposed the averaging AOs for IVA-IFS, then we can get the ranking result

Table 4 The values of $\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{\tau j}}}}\right)$

$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$
$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{11}}}}\right) = 2.89$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{11}}}}\right) = 2.89$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{11}}}}\right) = 2.89$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{11}}}}\right) = 2.89$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{11}}}}\right) = 2.89$
$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{12}}}}\right) = 2.93$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{12}}}}\right) = 2.93$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{12}}}}\right) = 2.93$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{12}}}}\right) = 2.93$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{12}}}}\right) = 2.93$
$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{13}}}}\right) = 2.92$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{13}}}}\right) = 2.92$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{13}}}}\right) = 2.92$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{13}}}}\right) = 2.92$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{13}}}}\right) = 2.92$
$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{14}}}}\right) = 2.88$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{14}}}}\right) = 2.88$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{14}}}}\right) = 2.88$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{14}}}}\right) = 2.88$	$\overline{\overline{\mathfrak{I}}}\left(\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_{14}}}}\right) = 2.88$

Table 5 The values of $\overline{\overline{\Xi_{\tau j}}}$

$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}}$	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$
$\overline{\overline{\Xi_{11}}} = 0.249$	$\overline{\overline{\Xi_{11}}} = 0.249$	$\overline{\overline{\Xi_{11}}} = 0.249$	$\overline{\overline{\Xi_{11}}} = 0.249$	$\overline{\overline{\Xi_{11}}} = 0.249$
$\overline{\overline{\Xi_{12}}} = 0.2516$	$\overline{\overline{\Xi_{12}}} = 0.2516$	$\overline{\overline{\Xi_{12}}} = 0.2516$	$\overline{\overline{\Xi_{12}}} = 0.2516$	$\overline{\overline{\Xi_{12}}} = 0.2516$
$\overline{\overline{\Xi_{13}}} = 0.251$	$\overline{\overline{\Xi_{13}}} = 0.251$	$\overline{\overline{\Xi_{13}}} = 0.251$	$\overline{\overline{\Xi_{13}}} = 0.251$	$\overline{\overline{\Xi_{13}}} = 0.251$
$\overline{\overline{\Xi_{14}}} = 0.2484$	$\overline{\overline{\Xi_{14}}} = 0.2484$	$\overline{\overline{\Xi_{14}}} = 0.2484$	$\overline{\overline{\Xi_{14}}} = 0.2484$	$\overline{\overline{\Xi_{14}}} = 0.2484$

Table 6 The aggregated results for each alternative

Alternatives	IVA-IFAAPA Operator	IVA-IFAAPG Operator
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}$	([0.1517,0.2079],[0.3897,0.4635])	([0.6052,0.6823],[0.052,0.1])
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}$	([0.3396,0.4206],[0.3897,0.4635])	([0.8095,0.8642],[0.052,0.1])
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}}$	([0.2079,0.2699],[0.6052,0.5564])	([0.6823,0.7494],[0.1517,0.2079])
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}}$	([0.2699,0.3396],[0.5124,0.5182])	([0.7494,0.8095],[0.1,0.1517])
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$	([0.1,0.2079],[0.6052,0.5564])	([0.5124,0.6823],[0.1517,0.2079])

Table 7 The score values

Alternatives	IVA-IFAAPA Operator	IVA-IFAAPG Operator
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}}$	−0.2468	0.5678
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}}$	−0.0465	0.7609
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}}$	−0.3419	0.5361
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}}$	−0.2106	0.6536
$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$	−0.4269	0.4176

Table 8 The ranking results

Methods	RankingValues
IVA-IFAAPA Operator	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$
IVA-IFAAPG Operator	$\overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_2}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_4}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_1}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_3}}} > \overline{\overline{\breve{\mathfrak{W}}_{\mathfrak{L}_5}}}$

Table 9 Comparative results for the different operators

Methods	Score values	Ranking values
Xu [23]	*Cannot process this problem*	*Cannot process this problem*
Xu and Yager [24]	*Cannot process this problem*	*Cannot process this problem*
Jiang et al. [25]	*Cannot process this problem*	*Cannot process this problem*
Xu [26]	*Cannot process this problem*	*Cannot process this problem*
Senapati et al. [27]	*Cannot process this problem*	*Cannot process this problem*
Senapati et al. [28]	*Cannot process this problem*	*Cannot process this problem*
Wei and Wang [30]	$0.1998, 0.4999, 0.09998, 0.2998, -0.0503$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$
Wang et al. [31]	$0.2004, 0.5005, 0.1002, 0.3003, -0.0498$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$
Senapati et al. [32]	$-0.2749, -0.0734, -0.4057, -0.2554, -0.491$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$
He et al. [33]	$0.2005, 0.5006, 0.1003, 0.3004, -0.0497$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$
IVA-IFAAPA Operator	$-0.2468, -0.0465, -0.3419, -0.2106, -0.4269$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$
IVA-IFAAPG Operator	$0.5678, 0.7609, 0.5361, 0.6536, 0.4176$	$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$

Table 10 Theoretical analysis of the proposed work

Methods	Truth grade	Falsity grade	Condition	Interval-valued information
Fuzzy sets	√	×	$0 \le \zeta_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) \le 1$	×
Intuitionistic fuzzy sets	√	√	$0 \le \zeta_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) + \eta_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) \le 1$	×
Interval-valued intuitionistic fuzzy set	√	√	$0 \le \zeta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) + \eta^{u}_{\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}}}(\widetilde{\mu_\varrho}) \le 1$	√

$\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$. Although the best choice is the same as our method, the ranking result is slightly different. The reason is that our method considers the power relation between two attributes. Wang et al. method [31] used only the Arithmetic averaging operator and the operational laws based on the algebraic t-norm and t-conorm, our method is more generalized than it.

2. Wei and Wang [30] proposed the geometric AOs for IVA-IFS, then we can get the ranking result $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$. Similarly, the best choice is the same as our method, the ranking result is slightly different. The reason is that our method considers the power relation between two attributes. Wei and Wang method [30] used only the geometric operator and the operational laws based on the algebraic t-norm and t-conorm, our method is more generalized than it.
3. He et al. [33] proposed the PA operators for IVA-IFS, then we can get the ranking result $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$. This ranking result is the same as our method based on IVA-IFAAPA Operator. He et al.' method [33] used only the operational laws based on the algebraic t-norm and t-conorm, our method is more generalized than it.
4. Senapati et al. [32] proposed the Aczel–Alsina averaging AOs for IVA-IFS, then we can get the ranking result $\overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_2} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_4} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_1} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_3} > \overline{\overline{\breve{\mathfrak{W}}}}_{\mathfrak{Q}_5}$. Although the best choice is the same as our method, the ranking result is different from the other methods. The reason is that our method considers the power relation between any two attributes. Since the ranking result is different from the other methods, its effectiveness is

questionable. Senapati et al.' method [32] used only the operational laws based on the Aczel–Alsina t-norm and t-conorm, our method is more generalized than it.

From the above analysis, we can see that all methods produced the same best choice for this example, however, the ranking results are different, because our method adopted the PA operators and the operational laws based on the Aczel–Alsina t-norm and t-conorm, it is more general than the others. In addition, we explained the novelty and supremacy of the proposed methods by comparative analysis with some existing methods shown in Table 10.

In a word, because the proposed methods are developed based on Aczel–Alsina operators with power aggregation operators for IVA-IFSs, they are the generalized form of some existing works such as the averaging AOs for IFS from Xu [23], the geometric AOs for IFS from Xu and Yager [24], the PA operators for IFS from Jiang et al. [25] and Xu [26], the Aczel–Alsina averaging AOs for IFS and the geometric Aczel–Alsina AOs for IFS from Senapati et al. [27] and Senapati et al. [28], Geometric AOs for IVIFS by Wei and Wang [30], the averaging AOs for IVIFSs by Wang et al. [31], Aczel–Alsina AOs for IVIFS by Senapati et al. [32], and the power AOs for IVIFS by He et al. [33]. We can explain them as follows:

1. If we only used the PA operators based on algebraic laws, then our proposed operators are reduced to ones by He et al. [33].
2. If we only used the simple averaging and geometric aggregation operators based on Aczel–Alsina operational laws, then our proposed operators are reduced to ones by Senapati et al. [32].
3. If we only used the simple averaging and geometric aggregation operators based on algebraic operational laws, then our proposed operators are reduced to ones by Wei and Wang [30] and Wang et al. [31].

Of course, because the proposed methods can only deal only with one-dimension information, they cannot process the phase term, obviously, there is shortcoming in this study, and it is further study on the Aczel–Alsina power aggregation operators based on complex IVA-IFSs.

6 Conclusion

The operational laws based on the Aczel–Alsina t-norm and t-conorm are more general than some existing operations, the PA operators can consider the power relation between any two attributes, and IVA-IFS can better express the uncertain information, in this paper, we fully consider their advantages, and proposed some PA operators based on the operational laws from the Aczel–Alsina t-norm and t-conorm. The main contributions are shown as follows.

1. Proposed the IVA-IFAAPA, IVA-IFAAPOA, IVA-IFAAPG, and IVA-IFAAPOG operators.
2. Discussed the properties of the presented operators such as idempotency, monotonicity, and boundedness.
3. Developed a MADM method with IVA-IF information.
4. Demonstrated the effectiveness and superiority of the derived method by comparing it with some existing operators.

In the future, we will modify the Aczel–Alsina PA operators based on the IVA-IF set. Furthermore, we also utilize the proposed operators and method in the fields of decision-making analysis [34], artificial intelligence, neural networks, pattern recognition, clustering analysis, and medical diagnosis.

Author Contributions XS: conceptualization, model construction, validation. ZA: conceptualization, model construction, methodology, writing—original draft. TM: conceptualization, methodology, and data curation. PL: methodology, investigated the data, compared the proposed method with some existing methods, funding acquisition, writing—review and editing.

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Declarations

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