Some Inequalities for $LR-(h_1, h_2)$ -Convex Interval-Valued Functions by Means of Pseudo Order Relation

Muhammad Bilal Khan¹ · Muhammad Aslam Noor¹ · Khalida Inayat Noor¹ · Kottakkaran Sooppy Nisar² · Khadiga Ahmed Ismail³ · Ashraf Elfakhany⁴

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Abstract

In both theoretical and applied mathematics fields, integral inequalities play a critical role. Due to the behavior of the definition of convexity, both concepts convexity and integral inequality depend on each other. Therefore, the relationship between convexity and symmetry is strong. Whichever one we work on, we introduced the new class of generalized convex function is known as $LR-(h_1, h_2)$ -convex interval-valued function ($LR-(h_1, h_2)$ -IVF) by means of pseudo order relation. Then, we established its strong relationship between Hermite–Hadamard inequality (HH -inequality) and their variant forms. Besides, we derive the Hermite–Hadamard–Fejér inequality (HH –Fejér inequality) for $LR-(h_1, h_2)$ -convex interval-valued functions. Several exceptional cases are also obtained which can be viewed as its applications of this new concept of convexity. Useful examples are given that verify the validity of the theory established in this research. This paper's concepts and techniques may be the starting point for further research in this field.

Keywords Interval-valued function · Riemann integral · $LR-(h_1, h_2)$ -convex interval-valued function · Interval Hermite · Hadamard inequality · Interval Hermite · Hadamard · Fejér inequality

✉ Kottakkaran Sooppy Nisar
n.sooppy@psau.edu.sa

Muhammad Bilal Khan
bilal42742@gmail.com

Muhammad Aslam Noor
noormaslam@gmail.com

Khalida Inayat Noor
khalidanoor@hotmail.com

Khadiga Ahmed Ismail
khadigaah.aa@tu.edu.sa

Ashraf Elfakhany
a.taha@tu.edu.sa

¹ Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

² Department of Mathematics, College of Arts and Sciences, Prince Sattam Bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia

³ Department of Clinical Laboratory Sciences, College of Applied Medical Sciences, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

⁴ Mechanical Engineering Department, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

1 Introduction

The theory of convexity in pure and applied sciences has become a rich source of inspiration. This theory has not only motivated new and profound results in many branches of mathematical and engineering sciences, but also provides a consistent and general for studying a wide range of problems. For convex sets and convex functions, several new notions of convexity have been developed and investigated. Various fundamental integral inequalities for convex functions and their variant forms are being developed using novel and innovative ideas and techniques. Every function is convex function, if and only if it satisfies an integral inequality, which is known as HH -inequality. This inequality was proposed by Hermite and was independently introduced by Hadamard, see [12, 13, 16].

A set $K \subset \mathbb{R}$ is said to be convex if $\rho u + (1 - \rho)v \in K$ for all $u, v \in K$ and $\rho \in [0, 1]$. We call a real-valued function $f : K \rightarrow \mathbb{R}$ is called convex if

$$f(\rho x + (1 - \rho)y) \leq \rho f(x) + (1 - \rho)f(y), \quad (1)$$

for all $x, y \in K, \rho \in [0, 1]$. It is generally known that, if $f : [u, \vartheta] \rightarrow \mathbb{R}$ is convex, then

$$f\left(\frac{u + \vartheta}{2}\right) \leq \frac{1}{\vartheta - u} \int_u^\vartheta f(x) dx \leq \frac{f(u) + f(\vartheta)}{2}. \quad (2)$$

Fejér considered the major generalizations of *HH*-inequality in [11] which is known as *HH*-Fejér inequality. It can be expressed as follows:

Let $f : [u, \vartheta] \rightarrow \mathbb{R}$ be a convex function. Then,

$$\begin{aligned} f\left(\frac{u + \vartheta}{2}\right) &\leq \frac{1}{\int_u^\vartheta W(x) dx} \int_u^\vartheta f(x) W(x) dx \\ &\leq \frac{f(u) + f(\vartheta)}{2} \end{aligned} \quad (3)$$

If $W(x) = 1$, then we obtain (2) from (3). With the assistance of inequality (3), many classical inequalities can be obtained through special convex function. In addition, these inequalities have a very significant role for convex functions in both pure and applied mathematics. We urge the readers for further analysis of literature on the applications and properties of generalized convex functions and *HH*-integral inequalities, see [2, 3, 10, 12–15, 24] and the references therein.

From the other side, because of the absence of implementations of the theory of interval analysis in other sciences, this theory fell into oblivion for a long time. Moore [18] suggested and investigated the idea of interval analysis. In numerical analysis, it is first used to evaluate the error bounds of a finite state machine's numerical solutions. We refer to the readers for basic information and applications [22, 23, 25] and the references therein. Since its inspection five decades ago, the theory of fuzzy sets and system has advanced in variety of ways, see [26]. Therefore, it plays an important role in the study of a broad-based class problems in pure mathematics and applied sciences including operational analysis, computer science, management sciences, artificial intelligence, control engineering and decision-makings. The convex analysis has played an important and fundamental part in development of various fields of applied and pure science. Similarly, the notions of convexity and non-convexity play a vital role in optimization under fuzzy domain because during characterization of the optimality condition of convexity, we obtain fuzzy variational inequalities, so variational inequality theory and fuzzy complementary problem theory established powerful mechanism of the mathematical problems and they have a friendly relationship. Recently, several classical discrete and integral inequalities have been generalized not only to the environment of the IVF and fuzzy-IVFs by Costa [7], Costa and Roman-Flores [8], Roman-Flores et al. [20, 21], and Chalco-Cano et al. [5, 6], but also to more general set-valued

maps by Nikodem et al. [19], and Matkowski and Nikodem [17]. In particular, Zhang et al. [27] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized form of Costa Jensen's inequalities [7]. Motivated by the above literature, Zhao et al. [28] introduced *h*-convex interval-valued functions (*h*-convex-IVFs, in short) and demonstrated that the *HH*-type inequalities and Jensen *HH*-type inequalities for *h*-convex-IVFs. Besides, Yanrong An et al. [1] defined the class of (h_1, h_2) -convex-IVFs and established following interval *HH*-inequality for (h_1, h_2) -convex-IVFs:

Let $f : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$ be an (h_1, h_2) -convex-IVF given by $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, \vartheta]$, with $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ with $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$, where $f_*(x)$ and $f^*(x)$ both are (h_1, h_2) -concave function. If f is Riemann integrable (in sort, *IR*-integrable), then

$$\begin{aligned} \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{u + \vartheta}{2}\right) &\supseteq \frac{1}{\vartheta - u} (IR) \int_u^\vartheta f(x) dx \\ &\supseteq [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1 - \rho) d\rho. \end{aligned} \quad (4)$$

For further review of the literature on the applications and properties of generalized convex functions and Hermite–Hadamard inequalities, see [4, 5, 9, 20] and the references therein. Inspired by Costa and Roman-Flores [8], and Zhang et al. [27], we present interval inequality and interval *HH*-Fejér inequality for LR- (h_1, h_2) -convex-IVFs by means of pseudo order relation.

A one step forward, Khan et al. introduced new classes of convex and generalized convex FIVF, and derived new fractional *H*-*H* type and *H*-*H* type inequalities for convex FIVF [29], *h*-convex FIVF [30], (h_1, h_2) -convex FIVF [31], (h_1, h_2) -preinvex FIVF [32], log-*h*-convex FIVFs [33], log-s-convex FIVFs in the second sense [34], harmonically convex FIVFs [35], coordinated convex FIVFs [36], LR-log-*h*-convex IVFs [37], and the references therein. We refer to the readers for further analysis of literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex fuzzy mappings, see [38–47] and the references therein.

The main purpose of this paper is to introduce a new class of LR- (h_1, h_2) -convex-IVF and to develop a close relationship between Hermite–Hadamard type and Hermite–Hadamard–Fejér type inequalities for LR- (h_1, h_2) -convex-IVF.

2 Preliminaries

Interval analysis is generally defined as an analysis of functions valued at intervals. It is an extension of numerical analysis that uses intervals as its operating element instead

of real number. In this section, we collate some basic notions and terms, and essentials of theory of interval analysis. Let $\mathbb{R}_I$ be the collection of all closed and bounded intervals of $\mathbb{R}$ that is $\mathbb{R}_I = \{[\eta_*, \eta^*] : \eta_*, \eta^* \in \mathbb{R} \text{ and } \eta_* \leq \eta^*\}$. If $\eta_* \geq 0$, then $[\eta_*, \eta^*]$ is called non-negative interval. The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[\eta_*, \eta^*] : \eta_*, \eta^* \in \mathbb{R}_I \text{ and } \eta_* \geq 0\}$.

We now discuss some properties of intervals under the arithmetic operations addition, multiplication and scalar multiplication. If $[\varpi_*, \varpi^*], [\eta_*, \eta^*] \in \mathbb{R}_I$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$\begin{aligned} [\varpi_*, \varpi^*] + [\eta_*, \eta^*] &= [\varpi_* + \eta_*, \varpi^* + \eta^*], \\ [\varpi_*, \varpi^*] \times [\eta_*, \eta^*] &= \left[\min \left\{ \frac{\varpi_* \eta_*, \varpi^* \eta_*}{\varpi_* \eta^*, \varpi^* \eta^*} \right\}, \max \left\{ \frac{\varpi_* \eta_*, \varpi^* \eta_*}{\varpi_* \eta^*, \varpi^* \eta^*} \right\} \right], \\ \rho \cdot [\varpi_*, \varpi^*] &= \begin{cases} [\rho \varpi_*, \rho \varpi^*] & \text{if } \rho > 0, \\ \{0\} & \text{if } \rho = 0, \\ [\rho \varpi^*, \rho \varpi_*] & \text{if } \rho < 0. \end{cases} \end{aligned}$$

For $[\varpi_*, \varpi^*], [\eta_*, \eta^*] \in \mathbb{R}_I$, the inclusion “ $\subseteq$ ” is defined by

$$[\varpi_*, \varpi^*] \subseteq [\eta_*, \eta^*], \text{ if and only if } \eta_* \leq \varpi_*, \varpi^* \leq \eta^*.$$

Remark 2.1. [27] (i) The relation $\leq_p$ defined on $\mathbb{R}_I$ by $[\varpi_*, \varpi^*] \leq_p [\eta_*, \eta^*]$ if and only if $\varpi_* \leq \eta_*, \varpi^* \leq \eta^*$, for all $[\varpi_*, \varpi^*], [\eta_*, \eta^*] \in \mathbb{R}_I$, it is an pseudo order relation. For given $[\varpi_*, \varpi^*], [\eta_*, \eta^*] \in \mathbb{R}_I$, we say that $[\varpi_*, \varpi^*] \leq_p [\eta_*, \eta^*]$ if and only if $\varpi_* \leq \eta_*, \varpi^* \leq \eta^*$ or $\varpi_* \leq \eta_*, \varpi^* < \eta^*$. The relation $[\varpi_*, \varpi^*] \leq_p [\eta_*, \eta^*]$ coincident to $[\varpi_*, \varpi^*] \leq [\eta_*, \eta^*]$ on $\mathbb{R}_I$.

(ii) It can be easily seen that $\leq_p$ looks like “left and right” on the real line $\mathbb{R}$, so we call $\leq_p$ is “left and right” (or “LR” order, in short).

The concept of Riemann integral for IVF first introduced by Moore [29] is defined as follows:

Theorem 2.2. [18] If $f : [u, \vartheta] \subset \mathbb{R} \rightarrow \mathbb{R}_I$ is an IVF on such that $f(x) = [f_*(x), f^*(x)]$, then f is Riemann integrable over $[u, \vartheta]$ if and only if, f_* and f^* both are Riemann integrable over $[u, \vartheta]$ such that

$$\begin{aligned} (IR) \int_u^\vartheta f(x) dx &= (R) \int_u^\vartheta [f_*(x), f^*(x)] dx \\ &= \left[(R) \int_u^\vartheta f_*(x) dx, (R) \int_u^\vartheta f^*(x) dx \right] \end{aligned}$$

The collection of all Riemann integrable real-valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[u, \vartheta]}$ and $\mathcal{IR}_{[u, \vartheta]}$, respectively.

Definition 2.3. A function $f : [u, \vartheta] \rightarrow \mathbb{R}^+$ is said to be convex function if

$$\begin{aligned} f(\rho x + (1 - \rho)y) \\ \leq \rho f(x) + (1 - \rho)f(y), \forall x, y \in [u, \vartheta], \rho \in [0, 1]. \end{aligned} \quad (5)$$

If (5) is reversed, then f is called concave.

Definition 2.4. [10] A function $f : [u, \vartheta] \rightarrow \mathbb{R}^+$ is said to be P -convex function if

$$\begin{aligned} f(\rho x + (1 - \rho)y) \\ \leq f(x) + f(y), \forall x, y \in [u, \vartheta], \rho \in [0, 1] \end{aligned} \quad (6)$$

If (6) is reversed, then f is called P -concave.

Definition 2.5. [3] A function $f : K \rightarrow \mathbb{R}^+$ is said to be s -convex function in the second sense if.

$$\begin{aligned} f(\rho x + (1 - \rho)y) \\ \leq \rho^s f(x) + (1 - \rho)^s f(y), \forall x, y \in [u, \vartheta], \rho \in [0, 1], \end{aligned} \quad (7)$$

where $s \in (0, 1)$. If (7) is reversed, then f is called s -concave in the second sense.

Definition 2.6. [24] A function $f : [u, \vartheta] \rightarrow \mathbb{R}^+$ is said to be h -convex function if for all $x, y \in [u, \vartheta], \rho \in [0, 1]$, we have.

$$f(\rho x + (1 - \rho)y) \leq h(\rho)f(x) + h(1 - \rho)f(y), \quad (8)$$

where $h : [0, 1] \rightarrow \mathbb{R}^+$. If (8) is reversed, then f is called h -concave.

Definition 2.7. [2] Let non-negative real-valued function $f : [u, \vartheta] \rightarrow \mathbb{R}^+$ is said to be (h_1, h_2) -convex function if for all $x, y \in [u, \vartheta]$ and $\rho \in [0, 1]$ we have.

$$\begin{aligned} f(\rho x + (1 - \rho)y) \\ \leq h_1(\rho)h_2(1 - \rho)f(x) + h_1(1 - \rho)h_2(\rho)f(y), \end{aligned} \quad (9)$$

where $h_1, h_2 : [0, 1] \subseteq [u, \vartheta] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. If inequality (9) is reversed, then f is said to be (h_1, h_2) -concave on $[u, \vartheta]$. f is (h_1, h_2) -affine if and only if it is both (h_1, h_2) -convex and (h_1, h_2) -concave. The set of all (h_1, h_2)

-convex $((h_1, h_2)$ -concave, (h_1, h_2) -affine) functions is denoted by.

$$SX([u, \vartheta], \mathbb{R}^+, (h_1, h_2))SV([u, \vartheta], \mathbb{R}^+, (h_1, h_2)) \\ SA([u, \vartheta], \mathbb{R}^+, (h_1, h_2)).$$

Definition 2.8. The IVF $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ is said to be LR- (h_1, h_2) -convex-IVF if for all $x, y \in [u, \vartheta]$ and $\rho \in [0, 1]$ we have,

$$f(\rho x + (1 - \rho)y) \\ \leq_p h_1(\rho)h_2(1 - \rho)f(x) + h_1(1 - \rho)h_2(\rho)f(y), \quad (10)$$

where $h_1, h_2 : [0, 1] \subseteq [u, \vartheta] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. If inequality (10) is reversed, then f is said to be LR- (h_1, h_2) -concave on $[u, \vartheta]$. f is LR- (h_1, h_2) -affine if and only if it is both LR- (h_1, h_2) -convex and LR- (h_1, h_2) -concave. The set of all LR- (h_1, h_2) -convex (LR- (h_1, h_2) -concave, LR- (h_1, h_2) -affine) IVFs is denoted by

$$LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2)) LRSV([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2)), \\ LRSA([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2)).$$

Now we discuss some special cases of LR- (h_1, h_2) -convex-IVFs:

Remark 2.9. If $h_2(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF becomes LR- h_1 -convex-IVF, that is $f(\rho x + (1 - \rho)y) \leq_p h_1(\rho)f(x) + h_1(1 - \rho)f(y)$, $\forall x, y \in K, \rho \in [0, 1]$.

If $h_1(\rho) = \rho^s$, $h_2(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF becomes LR- s -convex-IVF in the second sense, that is

$$f(\rho x + (1 - \rho)y) \\ \leq_p \rho^s f(x) + (1 - \rho)^s f(y), \\ \forall x, y \in K, \rho \in [0, 1].$$

If $h_1(\rho) = \rho$, $h_2(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF becomes LR-convex-IVF, that is

$$f(\rho x + (1 - \rho)y) \\ \leq_p \rho f(x) + (1 - \rho)f(y), \\ \forall x, y \in K, \rho \in [0, 1].$$

If $h_1(\rho) = h_2(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF becomes LR- P -convex-IVF, that is

$$f(\rho x + (1 - \rho)y) \\ \leq_p f(x) + f(y), \\ \forall x, y \in K, \rho \in [0, 1].$$

Theorem 2.10. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$ and $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF defined by $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u, \vartheta]$.

Then $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ if and only if $f_*, f^* \in SX([u, \vartheta], \mathbb{R}^+, (h_1, h_2))$.

Proof. Assume that $f_*, f^* \in SX([u, \vartheta], \mathbb{R}^+, (h_1, h_2))$. Then, for all $x, y \in [u, \vartheta]$, $\rho \in [0, 1]$, we have

$$f_*(\rho x + (1 - \rho)y) \\ \leq h_1(\rho)h_2(1 - \rho)f_*(x) + h_1(1 - \rho)h_2(\rho)f_*(y) \\ \text{and}$$

$$f^*(\rho x + (1 - \rho)y) \\ \leq h_1(\rho)h_2(1 - \rho)f^*(x) + h_1(1 - \rho)h_2(\rho)f^*(y).$$

From Definition 2.7 and order relation $\leq_p$, we have

$$[f_*(\rho x + (1 - \rho)y), f^*(\rho x + (1 - \rho)y)] \\ \leq_p \left[\begin{array}{l} h_1(\rho)h_2(1 - \rho)f_*(x) + h_1(1 - \rho)h_2(\rho)f_*(y), \\ h_1(\rho)h_2(1 - \rho)f^*(x) + h_1(1 - \rho)h_2(\rho)f^*(y) \end{array} \right] \\ = h_1(\rho)h_2(1 - \rho)[f_*(x), f^*(x)] \\ + h_1(1 - \rho)h_2(\rho)[f_*(y), f^*(y)]$$

that is

$$f(\rho x + (1 - \rho)y) \\ \leq_p h_1(\rho)h_2(1 - \rho)f(x) + h_1(1 - \rho)h_2(\rho)f(y), \\ \forall x, y \in [u, \vartheta], \rho \in [0, 1]$$

Hence, $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$.

Conversely, let $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$. Then for all $x, y \in [u, \vartheta]$ and $\rho \in [0, 1]$, we have

$$f(\rho x + (1 - \rho)y) \\ \leq_p h_1(\rho)h_2(1 - \rho)f(x) + h_1(1 - \rho)h_2(\rho)f(y)$$

That is

$$[f_*(\rho x + (1 - \rho)y), f^*(\rho x + (1 - \rho)y)] \\ \leq_p h_1(\rho)h_2(1 - \rho)[f_*(x), f^*(x)] \\ + h_1(1 - \rho)h_2(\rho)[f_*(y), f^*(y)], = \\ \left[\begin{array}{l} h_1(\rho)h_2(1 - \rho)f_*(x) + h_1(1 - \rho)h_2(\rho)f_*(y), \\ h_1(\rho)h_2(1 - \rho)f^*(x) + h_1(1 - \rho)h_2(\rho)f^*(y) \end{array} \right]$$

It follows that

$$f_*(\rho x + (1 - \rho)y) \\ \leq h_1(\rho)h_2(1 - \rho)f_*(x) + h_1(1 - \rho)h_2(\rho)f_*(y)$$

and

$$f^*(\rho x + (1 - \rho)y) \\ \leq h_1(\rho)h_2(1 - \rho)f^*(x) + h_1(1 - \rho)h_2(\rho)f^*(y)$$

hence, the result follows.

Example 2.11. We consider $h_1(\rho) = h_2(\rho) \equiv \geq m(m \geq 1)$, for $\rho \in [0, 1]$ and the IVF $f : [1, 4] \rightarrow \mathbb{R}_I^+$ defined by, $f(x) = [e^x, e^{x^2}]$. Since end point functions $f_*(x)$ and $f^*(x)$ both are (h_1, h_2) -convex functions then, by Theorem 2.10 $f(x)$ is LR- (h_1, h_2) -convex-IVF.

Remark 2.12. If $f_*(x) = f^*(x)$, then LR- (h_1, h_2) -convex-IVF becomes (h_1, h_2) -convex function, see [2].

If $f_*(x) = f^*(x)$ and $h_2(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF becomes h_1 -convex function, see [24].

If $f_*(x) = f^*(x)$ with $h_2(\rho) \equiv 1$ and $h_1(\rho) = \rho^s$ with $s \in (0, 1)$, then LR- (h_1, h_2) -convex-IVF becomes s -convex function in the second sense, see [3].

If $f_*(x) = f^*(x)$ with $h_2(\rho) \equiv 1$ and $h_1(\rho) \equiv 1$, then LR- (h_1, h_2) -convex-IVF reduces to the P -convex function, see [30].

If $f_*(x) = f^*(x)$ with $h_2(\rho) \equiv 1$ and $h_1(\rho) = \rho$, then LR- (h_1, h_2) -convex-IVF becomes classical convex function.

Theorem 2.13. Let $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF defined by $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u, \vartheta]$ and $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. Then $f \in LRSV([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ if and only if, $f_*, f^* \in SV([u, \vartheta], \mathbb{R}^+, (h_1, h_2))$.

Proof. Proof is similar to the proof of Theorem 2.10.

Example 2.14. We consider the IVF $f : [u, \vartheta] = [0, 4] \rightarrow \mathbb{R}_I^+$ defined by,

$$f(x) = [-x^2, -x].$$

where $h_1(\rho) = \rho$ and $h_2(\rho) \equiv 1$, $\rho \in [0, 1]$. Since end point functions $f_*(x) = -x^2$, $f^*(x) = -x$ are (h_1, h_2) -concave functions, by Theorem 2.13 $f(x)$ is LR- (h_1, h_2) -concave-IVF.

3 The New Hermite–Hadamard Inequalities

In view of the HH -inequality in Eq. (2) and HH -inequality in Eq. (3), we can deduce the following version of the HH -inequalities for LR- (h_1, h_2) -concave-IVFs.

Theorem 3.1. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$ and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$, and let $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, \vartheta]$. If $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ and $f \in \mathcal{IR}_{([u, \vartheta])}$, then

$$\begin{aligned} \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{u+\vartheta}{2}\right) &\leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x) dx \\ &\leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho) d\rho, \end{aligned} \quad (11)$$

If $f \in LRSV([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$, then

$$\begin{aligned} \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{u+\vartheta}{2}\right) &\geq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x) dx \\ &\geq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho) d\rho, \end{aligned} \quad (12)$$

Proof. Let $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$. Then, by hypothesis, we have

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{u+\vartheta}{2}\right) \\ \leq_p f(\rho u + (1-\rho)\vartheta) + f((1-\rho)u + \rho\vartheta), \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f^*\left(\frac{u+\vartheta}{2}\right) \\ \leq f_*(\rho u + (1-\rho)\vartheta) + f_*((1-\rho)u + \rho\vartheta), \\ \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f^*\left(\frac{u+\vartheta}{2}\right) \\ \leq f^*(\rho u + (1-\rho)\vartheta) + f^*((1-\rho)u + \rho\vartheta). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} \int_0^1 f^*\left(\frac{u+\vartheta}{2}\right) d\rho \\ \leq \int_0^1 f_*(\rho u + (1-\rho)\vartheta) d\rho + \int_0^1 f_*((1-\rho)u + \rho\vartheta) d\rho, \end{aligned}$$

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} \int_0^1 f^*\left(\frac{u+\vartheta}{2}\right) d\rho \\ \leq \int_0^1 f^*(\rho u + (1-\rho)\vartheta) d\rho + \int_0^1 f^*((1-\rho)u + \rho\vartheta) d\rho, \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f^*\left(\frac{u+\vartheta}{2}\right) &\leq \frac{2}{\vartheta-u} \int_u^\vartheta f_*(x) dx, \\ \frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})} f^*\left(\frac{u+\vartheta}{2}\right) &\leq \frac{2}{\vartheta-u} \int_u^\vartheta f^*(x) dx. \end{aligned}$$

That is

$$\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\left[f_*\left(\frac{u+\vartheta}{2}\right), f^*\left(\frac{u+\vartheta}{2}\right)\right] \\ \leq_p \frac{2}{\vartheta-u} \left[\int_u^\vartheta f_*(x)dx, \int_u^\vartheta f^*(x)dx \right],$$

Thus,

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx, \quad (13)$$

In a similar way as above, we have

$$\frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \\ \leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho, \quad (14)$$

Combining (13) and (14), we have

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \\ \leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho,$$

Hence, the required result.

Remark 3.2. If $h_2(\rho) \equiv 1$, then Theorem 7 reduces to the result for LR- h_1 -convex-IVF:

$$\frac{1}{2h_1\left(\frac{1}{2}\right)}f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \\ \leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)d\rho,$$

If $h_1(\rho) = \rho^s$, $s \in (0, 1)$ and $h_2(\rho) \equiv 1$, then Theorem 3.1 reduces to the result for LR- s -convex-IVF in the second sense:

$$2^{s-1}f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \\ \leq_p \frac{1}{s+1} [f(u) + f(\vartheta)],$$

If $h_1(\rho) = \rho$ and $h_2(\rho) \equiv 1$, then Theorem 3.1 reduces to the result for LR-convex-IVF:

$$f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \leq_p \frac{f(u) + f(\vartheta)}{2}.$$

If $h_1(\rho) = h_2(\rho) \equiv 1$, then Theorem 3.1 reduces to the result for LR- P -convex-IVF:

$$\frac{1}{2}f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)dx \leq_p f(u) + f(\vartheta),$$

If $f_*(x) = f^*(x)$, then Theorem 3.1 reduces to the result for (h_1, h_2) -convex function, see [2]:

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u} (R) \int_u^\vartheta f(x)dx \\ \leq [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho.$$

Example 3.3: We consider $h_1(\rho) = \rho$, $h_2(\rho) \equiv 1$, for $\rho \in [0, 1]$, and the IVF $f : [u, \vartheta] = [0, 2] \rightarrow \mathbb{R}_I^+$ defined by $f(x) = [2x^2, 4x^2]$. Since end point functions $f_*(x) = 2x^2$, $f^*(x) = 4x^2$, both are (h_1, h_2) -convex functions. Hence $f(x)$ is LR- (h_1, h_2) -convex-IVF. We computing the following

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f_*\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u} \int_u^\vartheta f_*(x)dx \\ \leq [f_*(u) + f_*(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho \\ \frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f^*\left(\frac{u+\vartheta}{2}\right) = f_*(1) = 2, \\ \frac{1}{\vartheta-u} \int_u^\vartheta f_*(x)dx = \frac{1}{2} \int_0^2 2x^2dx = \frac{8}{3}, \\ [f_*(u) + f_*(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho = 4,$$

That means

$$2 \leq \frac{8}{3} \leq 4.$$

Similarly, it can be easily show that

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f^*\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u} \int_u^\vartheta f^*(x)dx \\ \leq_p [f^*(u) + f^*(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho,$$

Such that

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f^*\left(\frac{u+\vartheta}{2}\right) = f^*(1) = 4, \\ \frac{1}{\vartheta-u} \int_u^\vartheta f^*(x)dx = \frac{1}{2} \int_0^2 4x^2dx = \frac{16}{3}, \\ [f^*(u) + f^*(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)d\rho = 8.$$

From which, it follows that

$$4 \leq \frac{16}{3} \leq 8$$

that is

$$[2, 4] \leq_p \left[\frac{8}{3}, \frac{16}{3} \right] \leq_p [4, 8]$$

Theorem 3.4. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$, and let $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, \vartheta]$. If $f \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ and $f \in \mathcal{IR}_{([u, \vartheta])}$, then.

$$\begin{aligned} & \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f\left(\frac{u+\vartheta}{2}\right) \leq_p \triangleright_2 \\ & \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x) dx \leq_p \triangleright_1 \\ & \leq_p \left[\frac{f(u)+}{f(\vartheta)} \right] \left[h_1\left(\frac{1}{2}\right)^+ h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho)h_2(1-\varrho)d\varrho, \end{aligned}$$

Where,

$$\triangleright_1 = \left[\frac{f(u)+f(\vartheta)}{2} + f\left(\frac{u+\vartheta}{2}\right) \right] \int_0^1 h_1(\varrho)h_2(1-\varrho)d\varrho,$$

$$\triangleright_2 = \frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \left[f\left(\frac{3u+\vartheta}{4}\right) + f\left(\frac{u+3\vartheta}{4}\right) \right],$$

$$\text{and } \triangleright_1 = [\triangleright_1, \triangleright_1 *], \triangleright_2 = [\triangleright_2, \triangleright_2 *].$$

Proof. Take $\left[u, \frac{u+\vartheta}{2} \right]$, we have

$$\begin{aligned} & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{\varrho u + (1-\varrho)\frac{u+\vartheta}{2}}{2} + \frac{\varrho u + (1-\varrho)\frac{u+\vartheta}{2}}{2} \right) \\ & \leq_p f\left(\varrho u + (1-\varrho)\frac{u+\vartheta}{2} \right) + f\left((1-\varrho)u + \varrho\frac{u+\vartheta}{2} \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f^*\left(\frac{\varrho u + (1-\varrho)\frac{u+\vartheta}{2}}{2} + \frac{(1-\varrho)u + \varrho\frac{u+\vartheta}{2}}{2} \right) \\ & \leq f^*\left(\varrho u + (1-\varrho)\frac{u+\vartheta}{2} \right) + f^*\left((1-\varrho)u + \varrho\frac{u+\vartheta}{2} \right), \\ & \frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f^*\left(\frac{\varrho u + (1-\varrho)\frac{u+\vartheta}{2}}{2} + \frac{(1-\varrho)u + \varrho\frac{u+\vartheta}{2}}{2} \right) \\ & \leq f^*\left(\varrho u + (1-\varrho)\frac{u+\vartheta}{2} \right) + f^*\left((1-\varrho)u + \varrho\frac{u+\vartheta}{2} \right). \end{aligned}$$

In consequence, we obtain

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f^*\left(\frac{3u+\vartheta}{4}\right) \leq \frac{1}{\vartheta-u} \int_u^{\frac{u+\vartheta}{2}} f_*(x) dx,$$

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f^*\left(\frac{3u+\vartheta}{4}\right) \leq \frac{1}{\vartheta-u} \int_u^{\frac{u+\vartheta}{2}} f^*(x) dx.$$

That is

$$\begin{aligned} & \frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \left[f^*\left(\frac{3u+\vartheta}{4}\right), f^*\left(\frac{3u+\vartheta}{4}\right) \right] \\ & \leq_p \frac{1}{\vartheta-u} \left[\int_u^{\frac{u+\vartheta}{2}} f_*(x) dx, \int_u^{\frac{u+\vartheta}{2}} f^*(x) dx \right]. \end{aligned}$$

It follows that

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{3u+\vartheta}{4}\right) \leq_p \frac{1}{\vartheta-u} \int_u^{\frac{u+\vartheta}{2}} f(x) dx. \quad (15)$$

In a similar way as above, we have

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{u+3\vartheta}{4}\right) \leq_p \frac{1}{\vartheta-u} \int_{\frac{u+\vartheta}{2}}^\vartheta f(x) dx. \quad (16)$$

Combining (15) and (16), we have

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} \left[f\left(\frac{3u+\vartheta}{4}\right) + f\left(\frac{u+3\vartheta}{4}\right) \right] \leq_p \frac{1}{\vartheta-u} \int_u^\vartheta f(x) dx.$$

By using Theorem 3.1, we have

$$\begin{aligned} & \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f\left(\frac{u+\vartheta}{2}\right) = \\ & \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f\left(\frac{1}{2} \cdot \frac{3u+\vartheta}{4} + \frac{1}{2} \cdot \frac{u+3\vartheta}{4}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f^*\left(\frac{u+\vartheta}{2}\right) \\ & = \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f^*\left(\frac{1}{2} \cdot \frac{3u+\vartheta}{4} + \frac{1}{2} \cdot \frac{u+3\vartheta}{4}\right), \\ & \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f^*\left(\frac{u+\vartheta}{2}\right) \\ & = \frac{1}{4 \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \right]^2} f^*\left(\frac{1}{2} \cdot \frac{3u+\vartheta}{4} + \frac{1}{2} \cdot \frac{u+3\vartheta}{4}\right) \end{aligned}$$

$$\leq \frac{1}{4 \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right]^2} \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f_*\left(\frac{3u+\vartheta}{4}\right) + \right. \\ \left. h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f_*\left(\frac{u+3\vartheta}{4}\right) \right],$$

$$\leq \frac{1}{4 \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right]^2} \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f^*\left(\frac{3u+\vartheta}{4}\right) + \right. \\ \left. h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f^*\left(\frac{u+3\vartheta}{4}\right) \right],$$

$$= \triangleright_{2_*},$$

$$= \triangleright_2^*,$$

$$\leq \frac{1}{\vartheta-u} \int_u^\vartheta f_*(x) dx,$$

$$\leq \frac{1}{\vartheta-u} \int_u^\vartheta f^*(x) dx,$$

$$\leq \left[\frac{f_*(u)+f_*(\vartheta)}{2} + f_*\left(\frac{u+\vartheta}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho,$$

$$\leq \left[\frac{f^*(u)+f^*(\vartheta)}{2} + f^*\left(\frac{u+\vartheta}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho,$$

$$= \triangleright_{1_*},$$

$$= \triangleright_1^*,$$

$$\leq \left[\frac{f_*(u)+f_*(\vartheta)}{2} + h_1(\varrho) h_2(1-\varrho) (f_*(u) + f_*(\vartheta)) \right] \\ \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho, \\ \leq \left[\frac{f^*(u)+f^*(\vartheta)}{2} + h_1(\varrho) h_2(1-\varrho) (f^*(u) + f^*(\vartheta)) \right] \\ \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho,$$

$$= [f_*(u) + f_*(\vartheta)] \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho, \\ = [f^*(u) + f^*(\vartheta)] \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho,$$

that is

$$\frac{1}{4 \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right]^2} f\left(\frac{u+\vartheta}{2}\right) \leq_p \triangleright_2 \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x) dx \leq_p \triangleright_1 \\ \leq_p \left[\frac{f(u)+}{f(\vartheta)} \right] \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho$$

hence, the result follows.

Example 3.5. We consider $h_1(\varrho) = \varrho, h_2(\varrho) \equiv 1$, for $\varrho \in [0, 1]$, and the IVF $f : [u, \vartheta] = [0, 2] \rightarrow \mathbb{R}_I^+$ defined by, $f(x) = [2x^2, 4x^2]$, as in Example 3.3, then $f(x)$ is LR- (h_1, h_2) -convex-IVF and satisfying inequality (11). We have $f_*(x) = 2x^2$ and $f^*(x) = 4x^2$. We now computing the following

$$\left[\begin{matrix} f_*(u)+ \\ f_*(\vartheta) \end{matrix} \right] \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho = 4, \\ \left[\begin{matrix} f^*(u)+ \\ f^*(\vartheta) \end{matrix} \right] \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho = 8,$$

$$\triangleright_{1*} = \left[\frac{f_*(u)+f_*(\vartheta)}{2} + f_*\left(\frac{u+\vartheta}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho = 3, \\ \triangleright_1^* = \left[\frac{f^*(u)+f^*(\vartheta)}{2} + f^*\left(\frac{u+\vartheta}{2}\right) \right] \int_0^1 h_1(\varrho) h_2(1-\varrho) d\varrho = 6$$

$$\triangleright_{2*} = \frac{1}{4 \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right]^2} \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f_*\left(\frac{3u+\vartheta}{4}\right) + \right. \\ \left. h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f_*\left(\frac{u+3\vartheta}{4}\right) \right] = \frac{5}{2}, \\ \triangleright_2^* = \frac{1}{4 \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \right]^2} \left[h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f^*\left(\frac{3u+\vartheta}{4}\right) + \right. \\ \left. h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) f^*\left(\frac{u+3\vartheta}{4}\right) \right] = 5$$

Then we obtain that

$$2 \leq \frac{5}{2} \leq \frac{8}{3} \leq 3 \leq 4, \\ 4 \leq 5 \leq \frac{16}{3} \leq 6 \leq 8.$$

Hence, Theorem 3.4 is demonstrated.

Theorem 3.6. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$, and let $f, g : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ and $g(x) = [g_*(x), g^*(x)]$ for all $x \in [u, \vartheta]$. If $f, g \in \text{LRSX}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ and $fg \in \mathcal{IR}_{([u, \vartheta])}$, then

$$\frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x) g(x) dx \\ \leq_p M(u, \vartheta) \int_0^1 [h_1(\varrho) h_2(1-\varrho)]^2 d\varrho \\ + (u, \vartheta) \int_0^1 h_1(\varrho) h_2(\varrho) h_1(1-\varrho) h_2(1-\varrho) d\varrho,$$

where $M(u, \vartheta) = f(u)g(u) + f(\vartheta)g(\vartheta)$, $N(u, \vartheta) = f(u)g(\vartheta) + f(\vartheta)g(u)$,
 $a \text{ n } d \quad M(u, \vartheta) = [M_*((u, \vartheta)), M^*((u, \vartheta))]$ $a \text{ n } d$
 $N(u, \vartheta) = [N_*((u, \vartheta)), N^*((u, \vartheta))].$

Proof. Since f and g both are LR- (h_1, h_2) -convex-IVFs on $[u, \vartheta]$, then, we have

$$g_*(\varrho u + (1-\varrho)\vartheta) \\ \leq h_1(\varrho) h_2(1-\varrho) g_*(u) + h_1(1-\varrho) h_2(\varrho) g_*(\vartheta), \\ g^*(\varrho u + (1-\varrho)\vartheta) \\ \leq h_1(\varrho) h_2(1-\varrho) g^*(u) + h_1(1-\varrho) h_2(\varrho) g^*(\vartheta)$$

And

$$\begin{aligned} & g_*(\rho u + (1 - \rho)\vartheta) \\ & \leq h_1(\rho)h_2(1 - \rho)g_*(u) + h_1(1 - \rho)h_2(\rho)g_*(\vartheta), \\ & g^*(\rho u + (1 - \rho)\vartheta) \\ & \leq h_1(\rho)h_2(1 - \rho)g^*(u) + h_1(1 - \rho)h_2(\rho)g^*(\vartheta) \end{aligned}$$

From the definition of LR- (h_1, h_2) -convex-IVFs, it follows that $f(x) \geq_p 0$ and $g(x) \geq_p 0$, so

$$\begin{aligned} & f_*(\rho u + (1 - \rho)\vartheta)g_*(\rho u + (1 - \rho)\vartheta) \\ & \leq \begin{pmatrix} h_1(\rho)h_2(1 - \rho)f_*(u) \\ +h_1(1 - \rho)h_2(\rho)f_*(\vartheta) \end{pmatrix} \begin{pmatrix} h_1(\rho)h_2(1 - \rho)g_*(u) \\ +h_1(1 - \rho)h_2(\rho)g_*(\vartheta) \end{pmatrix} \\ & = f_*(u)g_*(u)[h_1(\rho)h_2(1 - \rho)]^2 \\ & + f_*(\vartheta)g_*(\vartheta)[h_1(\rho)h_2(1 - \rho)]^2 \\ & + f_*(u)g_*(\vartheta)h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho) \\ & + f_*(\vartheta)g_*(u)h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho), \\ & f^*(\rho u + (1 - \rho)\vartheta)g^*(\rho u + (1 - \rho)\vartheta) \\ & \leq \begin{pmatrix} h_1(\rho)h_2(1 - \rho)f^*(u) \\ +h_1(1 - \rho)h_2(\rho)f^*(\vartheta) \end{pmatrix} \begin{pmatrix} h_1(\rho)h_2(1 - \rho)g^*(u) \\ +h_1(1 - \rho)h_2(\rho)g^*(\vartheta) \end{pmatrix} \\ & = f^*(u)g^*(u)[h_1(\rho)h_2(1 - \rho)]^2 \\ & + f^*(\vartheta)g^*(\vartheta)[h_1(\rho)h_2(1 - \rho)]^2 \\ & + f^*(u)g^*(\vartheta)h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho) \\ & + f^*(\vartheta)g^*(u)h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho). \end{aligned}$$

Integrating both sides of above inequality over $[0, 1]$ we get

$$\begin{aligned} & \int_0^1 f_*(\rho u + (1 - \rho)\vartheta)g_*(\rho u + (1 - \rho)\vartheta) \\ & = \frac{1}{\vartheta - u} \int_u^\vartheta f_*(x)g_*(x)dx \\ & \leq (f_*(u)g_*(u) + f_*(\vartheta)g_*(\vartheta)) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + \left(\frac{f_*(u)g_*(\vartheta) + f_*(\vartheta)g_*(u)}{f_*(\vartheta)g_*(u)} \right) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho, \\ & \int_0^1 f^*(\rho u + (1 - \rho)\vartheta)g^*(\rho u + (1 - \rho)\vartheta) \\ & = \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x)g^*(x)dx \\ & \leq (f^*(u)g^*(u) + f^*(\vartheta)g^*(\vartheta)) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + \left(\frac{f^*(u)g^*(\vartheta) + f^*(\vartheta)g^*(u)}{f^*(\vartheta)g^*(u)} \right) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho. \end{aligned}$$

It follows that,

$$\begin{aligned} & \frac{1}{\vartheta - u} \int_u^\vartheta f_*(x)g_*(x)dx \\ & \leq M_*((u, \vartheta)) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + N_*((u, \vartheta)) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho, \\ & \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x)g^*(x)dx \\ & \leq M^*((u, \vartheta)) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + N^*((u, \vartheta)) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho, \end{aligned}$$

that is

$$\begin{aligned} & \frac{1}{\vartheta - u} \left[\int_u^\vartheta f_*(x)g_*(x)dx, \int_u^\vartheta f^*(x)g^*(x)dx \right] \\ & \leq_p [M_*((u, \vartheta)), M^*((u, \vartheta))] \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + [N_*((u, \vartheta)), N^*((u, \vartheta))] \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{1}{\vartheta - u} (FR) \int_u^\vartheta f(x) + g(x)dx \leq_p M(u, \vartheta) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + N(u, \vartheta) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho, \end{aligned}$$

and the theorem has been established.

Theorem 3.7. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$ and $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$, and let $f, g : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ and $g(x) = [g_*(x), g^*(x)]$ for all $x \in [u, \vartheta]$. If $f, g \in LRSX([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ and $fg \in \mathcal{IR}_{([u, \vartheta])}$ then

$$\begin{aligned} & \frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2} f\left(\frac{u + \vartheta}{2}\right)g\left(\frac{u + \vartheta}{2}\right) \\ & \leq_p \frac{1}{\vartheta - u} (IR) \int_u^\vartheta f(x)g(x)dx \\ & + N(u, \vartheta) \int_0^1 [h_1(\rho)h_2(1 - \rho)]^2 d\rho \\ & + M(u, \vartheta) \int_0^1 h_1(\rho)h_2(\rho)h_1(1 - \rho)h_2(1 - \rho)d\rho \end{aligned}$$

$$\begin{aligned} & w \quad h \quad e \quad r \quad e \quad M(u, \vartheta) = f(u)g(u) + f(\vartheta)g(\vartheta), \\ & N(u, \vartheta) = f(u)g(\vartheta) + f(\vartheta)g(u), \quad a \quad n \quad d \\ & M(u, \vartheta) = [M_*((u, \vartheta)), M^*((u, \vartheta))] \quad a \quad n \quad d \\ & N(u, \vartheta) = [N_*((u, \vartheta)), N^*((u, \vartheta))] \end{aligned}$$

Proof. By hypothesis, we have

$$\begin{aligned} & f_*\left(\frac{u+\vartheta}{2}\right)g_*\left(\frac{u+\vartheta}{2}\right) \\ & f^*\left(\frac{u+\vartheta}{2}\right)g^*\left(\frac{u+\vartheta}{2}\right) \end{aligned}$$

$$\begin{aligned} & \leq \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & f_*(\rho u + (1-\rho)\vartheta)g_*\left(\frac{\rho u + (1-\rho)\vartheta}{(1-\rho)\vartheta}\right) \\ & + f_*(\rho u + (1-\rho)\vartheta)g_*\left(\frac{(1-\rho)u}{+\rho\vartheta}\right) \end{aligned} \right] \\ & + \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & f_*((1-\rho)u + \rho\vartheta)g_*\left(\frac{\rho u + (1-\rho)\vartheta}{(1-\rho)\vartheta}\right) \\ & + f_*((1-\rho)u + \rho\vartheta)g_*\left(\frac{(1-\rho)u}{+\rho\vartheta}\right) \end{aligned} \right], \\ & \leq \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & f^*(\rho u + (1-\rho)\vartheta)g^*\left(\frac{\rho u + (1-\rho)\vartheta}{(1-\rho)\vartheta}\right) \\ & + f^*(\rho u + (1-\rho)\vartheta)g^*\left(\frac{(1-\rho)u}{+\rho\vartheta}\right) \end{aligned} \right] \\ & + \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & f^*((1-\rho)u + \rho\vartheta)g^*\left(\frac{\rho u + (1-\rho)\vartheta}{(1-\rho)\vartheta}\right) \\ & + f^*((1-\rho)u + \rho\vartheta)g^*\left(\frac{(1-\rho)u}{+\rho\vartheta}\right) \end{aligned} \right], \end{aligned}$$

-convex functions. Hence, f and g both are $\text{LR-}(h_1, h_2)$ -convex-IVF. We now compute the following

By simple modification, we have

$$\begin{aligned} & f\left(\frac{u+\vartheta}{2}\right)g\left(\frac{u+\vartheta}{2}\right) \\ & \leq_p \left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & f(\rho u + (1-\rho)\vartheta)g\left(\frac{\rho u + (1-\rho)\vartheta}{(1-\rho)\vartheta}\right) \\ & + f((1-\rho)u + \rho\vartheta)g\left(\frac{(1-\rho)u}{+\rho\vartheta}\right) \end{aligned} \right] \\ & + 2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2 \left[\begin{aligned} & h_1(\rho)h_2(\rho)h_1(1-\rho) \\ & h_2(1-\rho)M(u, \vartheta) + \\ & [h_1(\rho)h_2(1-\rho)]^2 N(u, \vartheta) \end{aligned} \right] \end{aligned}$$

Taking IR -Integrand over $[0, 1]$, we have

$$\begin{aligned} & \frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2} f\left(\frac{u+\vartheta}{2}\right)g\left(\frac{u+\vartheta}{2}\right) \\ & \leq_p \frac{1}{\vartheta-u} (IR) \int_u^\vartheta f(x)g(x)dx \\ & + N(u, \vartheta) \int_0^1 [h_1(\rho)h_2(1-\rho)]^2 d\rho \\ & + M(u, \vartheta) \int_0^1 h_1(\rho)h_2(\rho)h_1(1-\rho)h_2(1-\rho)d\rho \end{aligned}$$

hence, the result is proof.

Example 3.8. We consider $h_1(\rho) = \rho, h_2(\rho) \equiv 1$, for $\rho \in [0, 1]$, and the IVFs $f, g : [u, \vartheta] = [0, 1] \rightarrow \mathbb{R}_+^+$ defined by, $f(x) = [2x^2, 4x^2]$ and $g(x) = [x, 2x]$. Since end point functions $f_*(x) = 2x^2, f^*(x) = 4x^2$ and $g_*(x) = x, g^*(x) = 2x, (h_1, h_2)$

$$\begin{aligned} & \frac{1}{\vartheta-u} \int_u^\vartheta f_*(x)g_*(x)dx = \int_0^1 (2x^2)(x)dx = \frac{1}{2} \\ & \frac{1}{\vartheta-u} \int_u^\vartheta f^*(x)g^*(x)dx = \int_0^1 (4x^2)(2x)dx = 2, \end{aligned}$$

$$\begin{aligned} & M_*((u, \vartheta)) \int_0^1 [h_1(\rho)h_2(1-\rho)]^2 d\rho = \frac{2}{3} \\ & M^*((u, \vartheta)) \int_0^1 [h_1(\rho)h_2(1-\rho)]^2 d\rho = \frac{8}{3}, \end{aligned}$$

$$\begin{aligned} & N_*((u, \vartheta)) \int_0^1 h_1(\rho)h_2(\rho)h_1(1-\rho)h_2(1-\rho)d\rho = 0 \\ & N^*((u, \vartheta)) \int_0^1 h_1(\rho)h_2(\rho)h_1(1-\rho)h_2(1-\rho)d\rho = 0, \end{aligned}$$

that means

$$\begin{aligned} & \frac{1}{2} \leq \frac{2}{3} + 0 = \frac{2}{3} \\ & 2 \leq \frac{8}{3} + 0 = \frac{8}{3}, \end{aligned}$$

hence, Theorem 3.6 is demonstrated.

For Theorem 3.7, we have

$$\begin{aligned} & \frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2} f\left(\frac{u+\vartheta}{2}\right)g\left(\frac{u+\vartheta}{2}\right) = \frac{1}{2} \\ & \frac{1}{2\left[h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\right]^2} f^*\left(\frac{u+\vartheta}{2}\right)g^*\left(\frac{u+\vartheta}{2}\right) = 2 \end{aligned}$$

$$\begin{aligned} & \frac{1}{\vartheta-u} \int_u^\vartheta f_*(x)g_*(x)dx = \frac{1}{2}, \\ & \frac{1}{\vartheta-u} \int_u^\vartheta f^*(x)g^*(x)dx = 2, \end{aligned}$$

$$N_*(u, \vartheta) \int_0^1 [h_1(\rho)h_2(1-\rho)]^2 d\rho = 0,$$

$$N^*(u, \vartheta) \int_0^1 [h_1(\rho)h_2(1-\rho)]^2 d\rho = 0,$$

$$M_*(u, \vartheta) \int_0^1 h_1(\rho)h_2(\rho)h_1(1-\rho)h_2(1-\rho)d\rho = \frac{1}{3},$$

$$M^*(u, \vartheta) \int_0^1 h_1(\rho)h_2(\rho)h_1(1-\rho)h_2(1-\rho)d\rho = \frac{4}{3},$$

$$\text{that means } \frac{1}{2} \leq \frac{1}{2} + 0 + \frac{1}{3} = \frac{5}{6},$$

$$2 \leq 2 + 0 + \frac{4}{3} = \frac{10}{3},$$

hence, the result follows.

Theorem 3.9. (Second *HH*-Fejér inequality for LR- (h_1, h_2) -convex-IVF) Let $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF with $u < \vartheta$, such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, \vartheta]$ and $f \in \mathcal{LR}_{([u, \vartheta])}$. If $f \in \mathcal{LRSX}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$, then $\mathcal{W} : [u, \vartheta] \rightarrow \mathbb{R}, \mathcal{W}(x) \geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, then.

$$\frac{1}{\vartheta - u} (IR) \int_u^\vartheta f(x) \mathcal{W}(x) dx$$

$$\leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho)h_2(1-\rho)W((1-\rho)u + \rho\vartheta)d\rho, \quad (17)$$

If $f \in \mathcal{LRSV}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$, then inequality (17) is reversed.

Proof. Let $f \in \mathcal{LRSX}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$. Then, we have.

$$f_*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta)$$

$$\leq \begin{pmatrix} h_1(\rho)h_2(1-\rho)f_*(u) \\ h_1(1-\rho)h_2(\rho)f_*(\vartheta) \end{pmatrix} \mathcal{W}(\rho u + (1-\rho)\vartheta),$$

$$f^*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta) \quad (18)$$

$$\leq \begin{pmatrix} h_1(\rho)h_2(1-\rho)f^*(u) \\ h_1(1-\rho)h_2(\rho)f^*(\vartheta) \end{pmatrix} \mathcal{W}(\rho u + (1-\rho)\vartheta).$$

And

$$f_*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta)$$

$$\leq \begin{pmatrix} h_1(1-\rho)h_2(\rho)f_*(u) \\ +h_1(\rho)h_2(1-\rho)f_*(\vartheta) \end{pmatrix} \mathcal{W}((1-\rho)u + \rho\vartheta),$$

$$f^*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta) \quad (19)$$

$$\leq \begin{pmatrix} h_1(1-\rho)h_2(\rho)f^*(u) \\ +h_1(\rho)h_2(1-\rho)f^*(\vartheta) \end{pmatrix} \mathcal{W}((1-\rho)u + \rho\vartheta).$$

After adding (18) and (19), and integrating over $[0, 1]$, we get

$$\int_0^1 f_*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta) d\rho$$

$$+ \int_0^1 f_*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta) d\rho$$

$$\leq \int_0^1 \left[\begin{matrix} f_*(u) \left\{ \begin{matrix} h_1(\rho)h_2(1-\rho)W(\rho u + (1-\rho)\vartheta) \\ +h_1(1-\rho)h_2(\rho)W((1-\rho)u + \rho\vartheta) \end{matrix} \right\} \\ +f_*(\vartheta) \left\{ \begin{matrix} h_1(1-\rho)h_2(\rho)W(\rho u + (1-\rho)\vartheta) \\ +h_1(\rho)h_2(1-\rho)W((1-\rho)u + \rho\vartheta) \end{matrix} \right\} \end{matrix} \right] dt,$$

$$\int_0^1 f^*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta) d\rho$$

$$+ \int_0^1 f^*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta) d\rho$$

$$\leq \int_0^1 \left[\begin{matrix} f^*(u) \left\{ \begin{matrix} h_1(\rho)h_2(1-\rho)W(\rho u + (1-\rho)\vartheta) \\ +h_1(1-\rho)h_2(\rho)W((1-\rho)u + \rho\vartheta) \end{matrix} \right\} \\ +f^*(\vartheta) \left\{ \begin{matrix} h_1(1-\rho)h_2(\rho)W(\rho u + (1-\rho)\vartheta) \\ +h_1(\rho)h_2(1-\rho)W((1-\rho)u + \rho\vartheta) \end{matrix} \right\} \end{matrix} \right] dt.$$

$$= 2f_*(u) \int_0^1 h_1(\rho)h_2(1-\rho) \mathcal{W}(\rho u + (1-\rho)\vartheta) dt$$

$$+ 2f_*(\vartheta) \int_0^1 h_1(\rho)h_2(1-\rho) \mathcal{W}((1-\rho)u + \rho\vartheta) dt,$$

$$= 2f^*(u) \int_0^1 h_1(\rho)h_2(1-\rho) \mathcal{W}(\rho u + (1-\rho)\vartheta) dt$$

$$+ 2f^*(\vartheta) \int_0^1 h_1(\rho)h_2(1-\rho) \mathcal{W}((1-\rho)u + \rho\vartheta) dt.$$

Since $\mathcal{W}$ is symmetric then,

$$= 2 \left[\begin{matrix} f_*(u) \\ +f_*(\vartheta) \end{matrix} \right] \int_0^1 h_1(\rho)h_2(1-\rho)W \begin{pmatrix} (1-\rho)u \\ +\rho\vartheta \end{pmatrix} dt,$$

$$= 2 \left[\begin{matrix} f^*(u) \\ +f^*(\vartheta) \end{matrix} \right] \int_0^1 h_1(\rho)h_2(1-\rho)W \begin{pmatrix} (1-\rho)u \\ +\rho\vartheta \end{pmatrix} dt. \quad (20)$$

Since

$$\int_0^1 f_*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta) d\rho$$

$$= \int_0^1 f_*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta) d\rho$$

$$= \frac{1}{\vartheta - u} \int_u^\vartheta f_*(x) \mathcal{W}(x) dx,$$

$$\int_0^1 f^*((1-\rho)u + \rho\vartheta) \mathcal{W}((1-\rho)u + \rho\vartheta) d\rho \quad (21)$$

$$= \int_0^1 f^*(\rho u + (1-\rho)\vartheta) \mathcal{W}(\rho u + (1-\rho)\vartheta) d\rho$$

$$= \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x) \mathcal{W}(x) dx.$$

From (20) and (21), we have

$$\begin{aligned} & \frac{1}{\vartheta - u} \int_u^\vartheta f_*(x) \mathcal{W}(x) dx \\ & \leq \left[\frac{f_*(u)}{+f_*(\vartheta)} \right] \int_0^1 h_1(\rho) h_2(1 - \rho) W \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) dt, \\ & \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x) \mathcal{W}(x) dx \\ & \leq \left[\frac{f^*(u)}{+f^*(\vartheta)} \right] \int_0^1 h_1(\rho) h_2(1 - \rho) W \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) dt, \end{aligned}$$

that is

$$\begin{aligned} & \left[\frac{1}{\vartheta - u} \int_u^\vartheta f_*(x) \mathcal{W}(x) dx, \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x) \mathcal{W}(x) dx \right], \\ & \leq_p [f_*(u) + f_*(\vartheta), f^*(u) + f^*(\vartheta)] \int_0^1 h_1(\rho) h_2(1 - \rho) W \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) dt. \end{aligned}$$

Hence,

$$\frac{1}{\vartheta - u} (IR) \int_u^\vartheta f(x) \mathcal{W}(x) dx \leq_p [f(u) + f(\vartheta)] \int_0^1 h_1(\rho) h_2(1 - \rho) \mathcal{W} \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) d\rho,$$

then, we complete the proof.

Theorem 3.10. (First *HH*-Fejér inequality for LR- (h_1, h_2) -convex-IVF) Let $f : [u, \vartheta] \rightarrow \mathbb{R}_I^+$ be an IVF with $u < \vartheta$, such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u, \vartheta]$ and $f \in \mathcal{LR}_{([u, \vartheta])}$. If $f \in \mathcal{LRSX}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ and $\mathcal{W} : [u, \vartheta] \rightarrow \mathbb{R}, \mathcal{W}(x) \geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, and $\int_u^\vartheta \mathcal{W}(x) dx > 0$, then

$$f \left(\frac{u + \vartheta}{2} \right) \leq_p \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x) dx} (IR) \int_u^\vartheta f(x) \mathcal{W}(x) dx \quad (22)$$

If $f \in \mathcal{LRSV}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ then, inequality (22) is reversed.

Proof. Since $f \in \mathcal{LRSX}([u, \vartheta], \mathbb{R}_I^+, (h_1, h_2))$ then, we have.

$$\begin{aligned} f_* \left(\frac{u + \vartheta}{2} \right) & \leq h_1 \left(\frac{1}{2} \right) h_2 \left(\frac{1}{2} \right) \left(\frac{f_*(\rho u + (1 - \rho)\vartheta)}{+f_*((1 - \rho)u + \rho\vartheta)} \right), \\ f^* \left(\frac{u + \vartheta}{2} \right) & \leq h_1 \left(\frac{1}{2} \right) h_2 \left(\frac{1}{2} \right) \left(\frac{f^*(\rho u + (1 - \rho)\vartheta)}{+f^*((1 - \rho)u + \rho\vartheta)} \right). \end{aligned} \quad (23)$$

By multiplying (23) by $\mathcal{W}(\rho u + (1 - \rho)\vartheta) = \mathcal{W}((1 - \rho)u + \rho\vartheta)$ and integrate it by ρ over $[0, 1]$, we obtain

$$\begin{aligned} & f_* \left(\frac{u + \vartheta}{2} \right) \int_0^1 \mathcal{W}((1 - \rho)u + \rho\vartheta) d\rho \\ & \leq h_1 \left(\frac{1}{2} \right) h_2 \left(\frac{1}{2} \right) \left(\int_0^1 \frac{f_* \left(\frac{\rho u + (1 - \rho)\vartheta}{(1 - \rho)\vartheta} \right) \mathcal{W} \left(\frac{\rho u + (1 - \rho)\vartheta}{(1 - \rho)\vartheta} \right) d\rho}{+ \int_0^1 \frac{f_* \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) \mathcal{W} \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) d\rho} \right), \\ & f^* \left(\frac{u + \vartheta}{2} \right) \int_0^1 \mathcal{W}((1 - \rho)u + \rho\vartheta) d\rho \\ & \leq h_1 \left(\frac{1}{2} \right) h_2 \left(\frac{1}{2} \right) \left(\int_0^1 \frac{f^* \left(\frac{\rho u + (1 - \rho)\vartheta}{(1 - \rho)\vartheta} \right) \mathcal{W} \left(\frac{\rho u + (1 - \rho)\vartheta}{(1 - \rho)\vartheta} \right) d\rho}{+ \int_0^1 \frac{f^* \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) \mathcal{W} \left(\frac{(1 - \rho)u}{+\rho\vartheta} \right) d\rho} \right). \end{aligned} \quad (24)$$

Since

$$\begin{aligned} & \int_0^1 f_*(\rho u + (1 - \rho)\vartheta) \mathcal{W}(\rho u + (1 - \rho)\vartheta) d\rho \\ & = \int_0^1 f_*((1 - \rho)u + \rho\vartheta) \mathcal{W}((1 - \rho)u + \rho\vartheta) d\rho \\ & = \frac{1}{\vartheta - u} \int_u^\vartheta f_*(x) \mathcal{W}(x) dx \\ & \int_0^1 f^*((1 - \rho)u + \rho\vartheta) \mathcal{W}((1 - \rho)u + \rho\vartheta) d\rho \\ & = \int_0^1 f^*(\rho u + (1 - \rho)\vartheta) \mathcal{W}(\rho u + (1 - \rho)\vartheta) d\rho \\ & = \frac{1}{\vartheta - u} \int_u^\vartheta f^*(x) \mathcal{W}(x) dx. \end{aligned} \quad (25)$$

From (24) and (25), we have

$$\begin{aligned} f_* \left(\frac{u + \vartheta}{2} \right) & \leq \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x) dx} \int_u^\vartheta f_*(x) \mathcal{W}(x) dx, \\ f^* \left(\frac{u + \vartheta}{2} \right) & \leq \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x) dx} \int_u^\vartheta f^*(x) \mathcal{W}(x) dx. \end{aligned}$$

From which, we have

$$\left[f_*\left(\frac{u+\vartheta}{2}\right), f^*\left(\frac{u+\vartheta}{2}\right) \right] \\ \leq_p \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} \left[\int_u^\vartheta f_*(x)\mathcal{W}(x)dx, \int_u^\vartheta f^*(x)\mathcal{W}(x)dx \right],$$

that is

$$f\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} (IR) \int_u^\vartheta f(x)\mathcal{W}(x)dx,$$

this completes the proof.

Remark 3.11. If $h_2(\varrho) \equiv 1$, $\varrho \in [0, 1]$ then, inequalities in Theorem 3.9 and 3.10 reduce for LR- h_1 -convex-IVFs which are also new one.

If in the Theorems 3.9 and 3.10 $h_2(\varrho) \equiv 1$ then, we obtain the appropriate theorems for LR- h_1 -convex-IVFs which are also new one.

$$\begin{aligned} [f_*(u) + f_*(\vartheta)] \int_0^1 h_1(\varrho)h_2(1-\varrho)W\left(\frac{(1-\varrho)u}{+\varrho\vartheta}\right) d\varrho &= [e + e^4] \left[\int_0^{\frac{1}{2}} 3\varrho^2 dx + \int_{\frac{1}{2}}^1 \varrho(3-3\varrho)d\varrho \right] \approx \frac{43}{2}, \\ [f^*(u) + f^*(\vartheta)] \int_0^1 h_1(\varrho)h_2(1-\varrho)W\left(\frac{(1-\varrho)u}{+\varrho\vartheta}\right) d\varrho &= [e + e^4] \left[\int_0^{\frac{1}{2}} 3\varrho^2 dx + \int_{\frac{1}{2}}^1 \varrho(3-3\varrho)d\varrho \right] \approx 86. \end{aligned} \quad (26)$$

If $h_1(\varrho) = \varrho^s$ with $s \in (0, 1)$ and $h_2(\varrho) \equiv 1$, $\varrho \in [0, 1]$ then, inequalities in Theorems 3.9 and 3.10 reduce for LR- s -convex-IVFs in the second sense which are also new one.

If in the Theorems 3.9 and 3.10 $h_1(\varrho) = \varrho$ and $h_2(\varrho) \equiv 1$ then, we obtain the appropriate theorems for LR-convex-IVFs which are also new one.

If $f_*(x) = f^*(x)$ with $h_2(\varrho) \equiv 1$ then, Theorems 3.9 and 3.10 reduce to classical first and second HH-Fejér inequality for h -convex function, see [4].

If $\mathcal{W}(x) = 1$ then, combining Theorems 3.9 and 3.10, we get Theorem 3.1.

Example 3.12. We consider $h_1(\varrho) = \varrho$, $h_2(\varrho) = 1$ for $\varrho \in [0, 1]$ and the IVF $f : [1, 4] \rightarrow \mathbb{R}_I^+$ defined by, $f(x) = [e^x, 2e^x]$. Since end point functions $f_*(x)$ and $f^*(x)$ are (h_1, h_2) -convex functions, then $f(x)$ is LR- (h_1, h_2) -convex-IVF. If

$$\mathcal{W}(x) = \begin{cases} x-1, \sigma \in \left[1, \frac{5}{2}\right], \\ 4-x, \sigma \in \left(\frac{5}{2}, 4\right], \end{cases}$$

then, we have

$$\begin{aligned} &\frac{1}{3} \int_1^4 f_*(x)\mathcal{W}(x)dx \\ &= \frac{1}{3} \int_1^{\frac{5}{2}} f_*(x)\mathcal{W}(x)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 f_*(x)\mathcal{W}(x)dx, \\ &\frac{1}{3} \int_1^4 f^*(x)\mathcal{W}(x)dx \\ &= \frac{1}{3} \int_1^{\frac{5}{2}} f^*(x)\mathcal{W}(x)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 f^*(x)\mathcal{W}(x)dx, \\ &= \frac{1}{3} \int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 11, \\ &= \frac{4}{3} \int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{4}{3} \int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 42, \end{aligned} \quad (26)$$

And

$$\begin{aligned} &\frac{1}{3} \int_1^4 f_*(x)\mathcal{W}(x)dx \\ &= \frac{1}{3} \int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{1}{3} \int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 11, \\ &= \frac{4}{3} \int_1^{\frac{5}{2}} e^x(x-1)dx + \frac{4}{3} \int_{\frac{5}{2}}^4 e^x(4-x)dx \approx 42, \end{aligned} \quad (27)$$

From (26) and (27), we have $[11, 42] \leq_p \left[\frac{43}{2}, 86\right]$.

Hence, Theorem 3.9 is verified.

For Theorem 3.10, we have

$$\begin{aligned} f_*\left(\frac{u+\vartheta}{2}\right) &\approx \frac{64}{5}, \\ f^*\left(\frac{u+\vartheta}{2}\right) &\approx \frac{244}{5}, \end{aligned} \quad (28)$$

$$\int_u^\vartheta \mathcal{W}(x)dx = \int_1^{\frac{5}{2}} (x-1)dx + \int_{\frac{5}{2}}^4 (4-x)dx = \frac{9}{4},$$

$$\begin{aligned} \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} \int_u^\vartheta f_*(x)\mathcal{W}(x)dx &\approx \frac{73}{5} \\ \frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} \int_u^\vartheta f^*(x)\mathcal{W}(x)dx &\approx \frac{293}{5}. \end{aligned} \quad (29)$$

From (28) and (29), we have

$$\left[\frac{64}{5}, \frac{244}{5}\right] \leq_p \left[\frac{73}{5}, \frac{293}{5}\right],$$

Hence, Theorem 3.10 is verified.

4 Conclusion

In this study, we introduced the concept of $LR(h_1, h_2)$ -convex-IVFs for the IVFs and established several novel HH -type and HH -Fejér type inequalities for $LR(h_1, h_2)$ -convex-IVFs. Our results provided the interval-valued counterparts of the inequalities presented in [2, 3, 10, 24], and our ideas may lead to a lot of follow-up research. In future, we will try to explore this for fuzzy-interval-valued functions using fuzzy fractional operators.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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