Applications of Approximate Optimal Control to Nonlinear Systems of Tracked Vehicle Suspensions

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Received: 3 July 2020 / Accepted: 17 September 2021
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Abstract

Technique of approximate optimal vibration control and simulation for vehicle active suspension systems are developed. Considered the nonlinear damping of springs, mechanical model and a nonlinear dynamic system for a class of tracked vehicle suspension vibration control are established and the corresponding system of state space form is described. To prolong the working life of suspension system and improve ride comfort, based on the active suspension vibration control devices and using optimal control approach, an approximate optimal vibration controller is designed, and an algorithm is presented for the vibration controller. Numerical simulation results illustrate the effectiveness of the proposed technique.

Keywords Optimal control · Vibration control · Suspension system · Nonlinear system

1 Introduction

The vibration caused by severe pavement conditions not only affects the normal use of vehicle-mounted instruments, but also damages the instruments and reduces ride ability, even threatens the safety of passengers and vehicles. Therefore, vibration control of the vehicle suspension is necessary.

Since people put forward higher requirements on the ride comfort and the operational stability, the passive suspension cannot meet the current needs. However, the active suspension has outstanding advantage in the riding comfort

and operational stability. Therefore, the active suspension vibration control technique has attracted many scholars' research interests. To improve suspension performance on driver ride comfort, integrated seat and suspension models, and driver body models were investigated [1, 2], in which the problems of vibration control was studied. A nonlinear mathematical model of the dynamic suspension system with two degrees of freedom was developed and Proportional Integral Derivative (PID) controller was designed for a kind of air suspension system in [3]. An active suspension system utilizing a low-cost high-performance linear switched reluctance actuator with proportional-derivative control was presented in [4]. To study the vibration control problems of the vehicle suspension, fuzzy control strategy was employed [5–9], in which, three fuzzy controllers, including nonlinear fuzzy logic controller, fuzzy finite-frequency output feedback controller and fuzzy PID controller, are obtained; H_∞ control theory was applied [10–14], in which robust H_∞ control and linear matrix inequality optimization method, non-fragile H_∞ control method and adaptive neuro-fuzzy inference system inverse magnetorheological damper model were used, respectively. Sliding mode control strategy such as self-organizing fuzzy sliding-mode control and adaptive sliding-mode control methods were utilized in [15, 17–21] to obtain vibration controllers for vehicle suspension systems.

In most of the previous studies on vehicle suspension control, linear control theory has been used though there are nonlinearities such as a hardening spring, the hysteretic

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stiffness and the 'tyre lift-off' phenomenon, caused by the temporary loss of ground adhesion ability in a real suspension system. Nonlinearity is one of the typical properties of complexity in nature; compared with linearity, nonlinearity is closer to the nature of objective things, and it is one of the important methods to quantitatively study and understand complex knowledge. To simplify the research object, nonlinear factors are often ignored or linearized, resulting in inaccurate description of the problem. Thus, it is very difficult to achieve desired performance using linear control techniques. Therefore, the nonlinear dynamics characteristic of vehicle suspension has attracted much attention recently in [2, 5, 6, 9, 16]. Recently, with the development of the modern control method, many researchers concern the intelligent control, that way, the plants can be very complex and can consider many uncertainties and no need to be precise. However, intelligent control mostly needs more complex control laws and needs more compute power. In this paper, we plan to use traditional optimal control to obtain suboptimal control law with a simple version and demonstrate the efficiency of the vibration control for a class of tracked vehicle suspension. With the consideration of the nonlinear damping of springs, a nonlinear vibration control system for a class of tracked vehicle suspension is established. Using optimal control approach, optimal vibration control for tracked vehicle suspension systems is studied. An optimal vibration controller is designed, and the control effect is demonstrated by numerical simulations.

The remainder of this paper is structured as follows. In Sect. 2, we establish active suspension systems and disturbance model. In Sect. 3, optimal vibration problem of the suspension systems is presented. In Sect. 4, two lemmas are introduced. In Sect. 5, approximate optimal vibration controller and algorithm are designed for the suspension systems. In Sect. 6, numerical experiments are presented. Finally, in Sect. 7 some conclusions are drawn.

2 Suspension Systems and Disturbance Model

2.1 Mechanical Model of Suspension

Vehicle suspensions are complex dynamic systems, and the responses of suspensions are affected by many factors. The suspension systems of four-wheel drive and four-wheel steering vehicles, strictly speaking, are eight degrees of freedom vibration systems. Based on practical problem and principle of being sufficient to analyze and study the vibration control for vehicle suspensions, the mechanical model of quarter-car suspension system with two degrees of freedom is investigated in this paper, as shown in Fig. 1, in which m_s is the sprung mass representing the tracked car chassis; m_u is

Fig. 1 Quarter-car model with active suspension

the unsprung mass representing the wheel assembly; k_s and b_s are stiffness and damping of the uncontrolled suspension, respectively; k_t and b_t stand for stiffness and damping of the wheel, respectively; $x_s(t)$ and $x_u(t)$ are the displacements of the sprung and unsprung masses, respectively; $x_r(t)$ is the road displacement input; $u(t)$ denotes the actuator control force, which is normally generated by an actuator between the two masses.

2.2 Dynamic Model of Suspension

Consider the quarter-car suspension system with two degrees of freedom as shown in Fig. 1, nonlinear stiffness k_s of the uncontrolled suspension is represented by linear part k_{1s} and nonlinear part k_{2s} . The dynamic equations of m_s and m_u are given by

$$\begin{aligned} u(t) &= m_s \ddot{x}_s(t) + b_s [\dot{x}_s(t) - \dot{x}_u(t)] + k_{1s} [x_s(t) - x_u(t)] \\ &\quad + k_{2s} [x_s(t) - x_u(t)]^3, \\ u(t) &= -m_u \ddot{x}_u(t) + b_s [\dot{x}_s(t) - \dot{x}_u(t)] \\ &\quad + k_{1s} [x_s(t) - x_u(t)] + k_{2s} [x_s(t) - x_u(t)]^3 \\ &\quad - k_t [x_u(t) - x_r(t)] - b_t [\dot{x}_u(t) - \dot{x}_r(t)], \end{aligned} \quad (1)$$

where $\ddot{x}_s(t)$ and $\dot{x}_s(t)$, respectively, are acceleration and velocity of the sprung masses; $\ddot{x}_u(t)$ and $\dot{x}_u(t)$, respectively, are acceleration and velocity of the unsprung masses; $\dot{x}_r(t)$ is the road velocity.

Define the set of state variables for Eq. (1):

$$\begin{aligned}
 x_1(t) &= x_s(t) - x_u(t), \\
 x_2(t) &= x_u(t) - x_r(t), \\
 x_3(t) &= \dot{x}_s(t), \\
 x_4(t) &= \dot{x}_u(t),
 \end{aligned} \quad (2)$$

where $x_1(t)$ is the suspension deflection, $x_2(t)$ is the wheel deflection, $x_3(t)$ is the sprung mass velocity, $x_4(t)$ is the unsprung mass velocity. Then, the state vector is in the following form:

$$x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T. \quad (3)$$

The principal variables for the active suspension design and evaluation are sprung mass acceleration $\ddot{x}_s(t)$, which determines ride comfort, the suspension deflection $x_s(t) - x_u(t)$, which indicates the limit of the vehicle body motion, and the wheel deflection $x_u(t) - x_r(t)$, which ensures the road holding ability. To satisfy the performance requirements, the controlled output vector is chosen as

$$y_c(t) = \begin{bmatrix} \ddot{x}_s(t) \\ x_s(t) - x_u(t) \\ x_u(t) - x_r(t) \end{bmatrix} = \begin{bmatrix} \ddot{x}_s(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}. \quad (4)$$

Then, from motion equation (1), state variables (2), state vector (3) and output vector (4), the dynamic system model of vehicle suspension is rewritten in the state-space representation:

$$\begin{aligned}
 \dot{x}(t) &= Ax(t) + Bu(t) + f(x) + Dp(t), \\
 y_c(t) &= \bar{C}x(t) + \bar{B}u(t) + \bar{f}(x), \\
 x(0) &= x_0,
 \end{aligned} \quad (5)$$

where $p(t) = \dot{x}_r(t)$ is external input disturbance, and

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \frac{-k_{1s}}{m_s} & 0 & \frac{-b_s}{m_s} & \frac{b_s}{m_s} \\ \frac{k_{1s}}{m_u} & \frac{-k_t}{m_u} & \frac{b_s}{m_u} & \frac{-(b_t+b_s)}{m_u} \end{bmatrix}, \\
 \bar{f}(x) &= \begin{bmatrix} \frac{-k_{2s}}{m_s} x_1^3 \\ 0 \\ 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} \frac{1}{m_s} \\ 0 \\ 0 \end{bmatrix}, \\
 B &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ \frac{-1}{m_u} \end{bmatrix}, \bar{C} = \begin{bmatrix} \frac{-k_{1s}}{m_s} & 0 & \frac{-b_s}{m_s} & \frac{b_s}{m_s} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
 D &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{b_t}{m_u} \end{bmatrix}, f(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{-k_{2s}}{m_s} x_1^3 \\ \frac{k_{2s}}{m_u} x_1^3 \end{bmatrix}.
 \end{aligned} \quad (6)$$

2.3 Disturbance Model

To evaluate the suspension characteristics, the road profiles variability is taken into account. According to ISO 2631 standards in [22], the road displacement Power Spectral Density (PSD) is approximately represented in the formulation of

$$S(W_1) = C_s W_1^{-2} = 4^k \times 10^{-7} W_1^{-2}, \quad (7)$$

where W_1 is the spatial frequency, C_s is the road roughness constant and k denotes the sorts of road grade as shown in Table 1.

To deduce the calculation formula for the road displacement input $x_r(t)$, assume that the vehicle travels at a constant horizontal velocity v_0 and the given road segment length is l . The road disturbances are approximately considered as periodic vibrations. Since vehicle wheels and the active suspension system have the low-pass filter characteristic, the road displacement $x_r(t)$ can be approximately simulated by a finite series sum

$$x_r(t) = \sum_{j=1}^i \xi_j(t) \Delta \sum_{j=1}^i \phi_j \sin(j\omega_0 t + \theta_j) \quad (8)$$

with amplitudes ϕ_j and initial phases θ_j which follows a uniform distribution in $[0, 2\pi]$. According to random process theory, the j th average power satisfies

$$S(j\Delta W_1) \times \Delta W_1 = \phi_j^2 / 2. \quad (9)$$

Hence, we get

$$\phi_j = \sqrt{2S(j\Delta W_1)\Delta W_1} = \frac{2^k}{10^{3j}} \sqrt{\frac{l}{10\pi}}. \quad (10)$$

Choose the spatial frequency interval $\Delta W_1 = 2\pi/l$, then the time frequency interval $\omega_0 = 2\pi v_0/l$. Positive integer i limits the considered frequency band and in this simplified model, it is generally lower than 20Hz.

Defining the disturbance state vector

$$\begin{aligned}
 w(t) &= [w_1(t), w_2(t), \dots, w_{2i}(t)]^T \\
 &= [\xi_1(t), \dots, \xi_i(t), \dot{\xi}_1(t), \dots, \dot{\xi}_i(t)]^T.
 \end{aligned} \quad (11)$$

The road velocity disturbance $p(t) = \dot{x}_r(t)$ is described by the following exosystem:

Table 1 Road grades and PSDs

Road grade	A	B	C	D	E
$C_s (10^{-7} \text{ m}^3/\text{rad})$	1	4	16	64	256
Road sort k	0	1	2	3	4

$$\begin{aligned}\dot{w}(t) &= Gw(t), \\ p(t) &= Fw(t),\end{aligned}\quad (12)$$

where

$$\begin{aligned}G &= \begin{bmatrix} 0 & I_i \\ -\tilde{\Omega}^2 & 0 \end{bmatrix} \in \mathbb{R}^{2i \times 2i}, \\ F &= [\underbrace{0, \dots, 0}_i, \underbrace{1, \dots, 1}_i] \in \mathbb{R}^{1 \times 2i},\end{aligned}\quad (13)$$

in which $\tilde{\Omega} = \text{diag}\{\omega_0, 2\omega_0, \dots, i\omega_0\} \in \mathbb{R}^{i \times i}$. Evidently, the pair (F, G) is observable completely.

3 Optimal Vibration Problem of the Suspension Systems

To study optimal vibration control problem for tracked vehicle, we choose an average performance index for system (5), as follows:

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x^T(t)Qx(t) + Ru^2(t)]dt, \quad (14)$$

where $Q = \bar{C}^T \bar{C} \in \mathbb{R}^{4 \times 4}$ is a positive semi-definite matrix and $R \in \mathbb{R}$ is a positive definite matrix.

The objective of this paper is to find a control law $u^*(t)$ for system (5) and make the value of the performance index (14) a minimum.

Apply the maximum principle to the optimal control problem in (5) and (14), and the optimal control law is described by

$$u^*(t) = -R^{-1}B^T \lambda(t), \quad (15)$$

where $\lambda(t)$ is the solution to the following nonlinear Two-point boundary value (TPBV) problem:

$$\begin{aligned}-\dot{\lambda}(t) &= Qx(t) + A^T \lambda(t) + f_x^T(x) \lambda(t), \\ \dot{x}(t) &= Ax(t) - S \lambda(t) + f(x) + Dp(t), \\ x(0) &= x_0, \\ \lambda(\infty) &= 0,\end{aligned}\quad (16)$$

where $S = BR^{-1}B^T$ and $f_x^T(x) = \partial f^T(x)/\partial x$ is the Jacobian matrix of $f(x)$ with respect to vector x .

Unfortunately, for the nonlinear TPBV problem in (16), with the exception of the simplest cases, there is no analytic solution. Therefore, many researchers tend to find the numerical solution to such problem. The main purpose of this article is to develop the Successive Approximation Approach (SAA) to find the optimal control law for the system described by (5) with average performance index (14).

4 Preliminaries

Consider the nonlinear system

$$\begin{aligned}\dot{z}(t) &= G(t)z(t) + h(z(t), t) + Fv(t), \\ z(t_1) &= \eta,\end{aligned}\quad (17)$$

where $z \in \mathbb{R}^n$ is the state vector, $v \in \mathbb{R}^m$ is the input vector, $h : C^1(\mathbb{R}^n \times \mathbb{R}_+) \rightarrow U$, $h(0, t) \equiv 0$, $\tilde{G} : C^1(\mathbb{R}_+) \rightarrow \mathbb{R}^{n \times n}$, $F \in \mathbb{R}^{n \times m}$, η is the initial state vector (for $t_1 = t_0$) or the terminal state vector (for $t_1 = t_f$). Assume that h satisfies the Lipschitz conditions on $\mathbb{R}^n \times \mathbb{R}_+$. Lipschitz conditions are also called Lipschitz continuity. Lipschitz conditions limit the change speed of the function. The slope of the function that meets the Lipschitz condition must be smaller than a real number called the Lipschitz constant, which is determined by the function itself.

To use the SAA, we introduce two lemmas [23].

Lemma 1 Define the vector function sequence $\{z^{(k)}(t)\}$ as

$$\begin{aligned}z^{(0)}(t) &= \Phi(t, t_0)\eta + \int_{t_0}^t \Phi(t, r)Fv(r)dr, \\ z^{(k)}(t) &= \Phi(t, t_0)\eta + \int_{t_0}^t \Phi(t, r)[h(z^{(k-1)}(r), r) \\ &\quad + Fv(r)]dr, \\ k &= 1, 2, \dots, \quad t \in \mathbb{R}_T,\end{aligned}\quad (18)$$

where $\Phi(t, t_0)$ is the state transition matrix corresponding to $G(t)$; then the sequence $\{z^{(k)}(t)\}$ uniformly converges to the solution of (17) for $t_1 = t_0$.

Lemma 2 Define the vector function sequence $\{z^{(k)}(t)\}$ as

$$\begin{aligned}z^{(0)}(t) &= \Phi(t, t_f)\eta + \int_{t_f}^t \Phi(t, r)Fv(r)dr, \\ z^{(k)}(t) &= \Phi(t, t_f)\eta + \int_{t_f}^t \Phi(t, r)[h(z^{(k-1)}(r), r) \\ &\quad + Fv(r)]dr, \\ k &= 1, 2, \dots, \quad t \in \mathbb{R}_T,\end{aligned}\quad (19)$$

then the sequence $\{z^{(k)}(t)\}$ converges uniformly to the solution of system (17) as $t_1 = t_f$.

5 Approximate Optimal Controller and Algorithm Design

5.1 Approximate Optimal Controller Design

Then, we design vibration controller for system (5). The optimal control law can be presented in the following theorem.

Theorem 1 Consider the optimal control problem described by system (5) with performance index (14), the optimal control law $u^*(t)$ exists and is unique. Its form is as follows:

$$u^*(t) = -R^{-1}B^T [P_1x(t) + P_2p(t) + P_3p_\omega(t) + \lim_{k \rightarrow \infty} g^{(k)}(t)], \quad (20)$$

where P_1 is the unique positive definite solution of the following Riccati matrix equation:

$$A^TP_1 + P_1A - P_1SP_1 + Q = 0, \quad (21)$$

P_2 and P_3 are the unique solutions of the following Sylvester matrix equations:

$$\begin{aligned} (A^T - P_1S)^2P_2F + P_2F\Omega^2 &= -(A^T - P_1S)P_1DF, \\ (A^T - P_1S)^2P_3F + P_3F\Omega^2 &= P_1DF, \end{aligned} \quad (22)$$

the k th adjoint vector is found from the following adjoint vector sequences:

$$\begin{aligned} g^{(0)}(t) &= \int_t^\infty \Phi^T(r-t)f_x^T(0)[P_2p(r) + P_3p_\omega(r)]dr, \\ g^{(k)}(t) &= \int_t^\infty \Phi^T(r-t) \{ P_1f(x^{(k-1)}(r)) \\ &\quad + f_x^T(x^{(k-1)}(r))[P_1x^{(k-1)}(r) \\ &\quad + P_2p(r) + P_3p_\omega(r) + g^{(k-1)}(r)] \} dr, \\ k &= 1, 2, \dots, \end{aligned} \quad (23)$$

the k th state vectors $x^{(k)}(t)$ satisfies the following state vector sequences:

$$\begin{aligned} x^{(0)}(t) &= \Phi(t)x_0 + \int_0^t \Phi(r-t)[(D - SP_2)p(r) \\ &\quad - SP_3p_\omega(r) - Sg^{(0)}(r)]dr, \\ x^{(k)}(t) &= \Phi(t)x_0 + \int_0^t \Phi(r-t)[(D - SP_2)p(r) \\ &\quad - SP_3p_\omega(r) - Sg^{(k)}(r) + f(x^{(k-1)}(r))]dr, \\ k &= 1, 2, \dots, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \Phi(t) &= e^{(A-SP_1)t}, \\ \Omega &= \text{block-diag}\{\bar{\Omega}, \bar{\Omega}\}, \\ p_\omega(t) &= \dot{p}(t) = FGw(t). \end{aligned} \quad (25)$$

Proof To use SAA to solve the nonlinear TPBV problem (16), we separate the linear part in the nonlinear TPBV problem (16), and let

$$\lambda(t) = P_1x(t) + P_2p(t) + P_3\dot{p}(t) + g(t), \quad (26)$$

where $g(t)$ is an adjoint vector introduced to compensate for the effect of the nonlinear term in system (5).

Substituting the first equation of (5) and (15) into the derivative of (26), we get

$$\begin{aligned} \dot{\lambda}(t) &= P_1\dot{x}(t) + P_2\dot{p}(t) + P_3\ddot{p}(t) + \dot{g}(t) \\ &= (P_1A - P_1SP_1)x(t) + (P_1D - P_1SP_2)p(t) \\ &\quad + (P_2 - P_1SP_3)\dot{p}(t) + P_3\ddot{p}(t) - P_1Sg(t) \\ &\quad + P_1f(x) + \dot{g}(t). \end{aligned} \quad (27)$$

Noting that

$$\begin{aligned} p_\omega(t) &= \dot{p}(t), \\ \ddot{p}(t) &= FG^2w(t) = -F\Omega^2w(t). \end{aligned} \quad (28)$$

From Eqs. (16) and (26), we obtain

$$\begin{aligned} \dot{\lambda}(t) &= - \left(Q + A^TP_1 \right) x(t) - A^TP_2p(t) \\ &\quad - A^TP_3p_\omega(t) - A^Tg(t) - f_x^T(x)\lambda(t). \end{aligned} \quad (29)$$

Substituting (28) into (27), and comparing the coefficients of (27) and (29), we obtain the matrix equations:

$$\begin{aligned} A^TP_1 + P_1A - P_1SP_1 + Q &= 0, \\ A^TP_2F + P_1DF - P_3F\Omega^2 - P_1SP_2F &= 0, \\ A^TP_3 + P_2 - P_1SP_3 &= 0, \end{aligned} \quad (30)$$

and the adjoint vector differential equations:

$$\begin{aligned} \dot{g}(t) &= -(A - SP_1)^Tg(t) - P_1f(x) - f_x^T(x)\lambda(t), \\ g(\infty) &= 0. \end{aligned} \quad (31)$$

From the matrix equations in (30), we obtain

$$\begin{aligned} A^TP_1 + P_1A - P_1SP_1 + Q &= 0, \\ (A^T - P_1S)^2P_2F + P_2F\Omega^2 &= -(A^T - P_1S)P_1DF, \\ (A^T - P_1S)^2P_3F + P_3F\Omega^2 &= P_1DF. \end{aligned} \quad (32)$$

Substituting (26) into the second equation of TPBV problem (16), we have

$$\begin{aligned} \dot{x}(t) &= (A - SP_1)x(t) + (D - SP_2)p(t) \\ &\quad - SP_3p_\omega(t) - Sg(t) + f(x), \\ x(t_0) &= x_0. \end{aligned} \quad (33)$$

To obtain the solution of TPBV problem (16) from SAA, according to (33) and (31), we construct the following state equation sequences:

$$\begin{aligned}
\dot{x}^{(0)}(t) &= (A - SP_1)x^{(0)}(t) + (D - SP_2)p(t) \\
&\quad - SP_3p_\omega(t) - Sg^{(0)}(t), \\
x^{(0)}(t_0) &= x_0, \\
\dot{x}^{(k)}(t) &= (A - SP_1)x^{(k)}(t) + (D - SP_2)p(t) \\
&\quad - SP_3p_\omega(t) - Sg^{(k)}(t) + f(x^{(k-1)}), \\
x^{(k)}(t_0) &= x_0, \\
k &= 1, 2, \dots,
\end{aligned} \tag{34}$$

and the adjoint vector differential equation sequences:

$$\begin{aligned}
\dot{g}^{(0)}(t) &= -(A - SP_1)^T g^{(0)}(t) - f_x^T(0)[P_2p(t) \\
&\quad + P_3p_\omega(t)], \\
\lim_{T \rightarrow \infty} g^{(0)}(T) &= 0, \\
\dot{g}^{(k)}(t) &= -(A - SP_1)^T g^{(k)}(t) - P_1f(x^{(k-1)}(t)) \\
&\quad - f_x^T(x^{(k-1)}(t))\lambda^{(k-1)}(t), \\
\lim_{T \rightarrow \infty} g^{(k)}(T) &= 0, \\
k &= 1, 2, \dots,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\lambda^{(k)}(t) &= P_1x^{(k)}(t) + P_2p(t) + P_3p_\omega(t) + g^{(k)}(t), \\
k &= 1, 2, \dots
\end{aligned} \tag{36}$$

According to Lemmas 1 and 2, the adjoint vector sequence in (23) and the state vector sequences in (24) uniformly converge to the solution of (31) and (33), respectively.

Correspondingly, the control sequence is given in the form:

$$\begin{aligned}
u^{(k)}(t) &= -R^{-1}B^T\lambda^{(k)}(t) \\
&= -R^{-1}B^T[P_1x^{(k)}(t) + P_2p(t) + P_3p_\omega(t) \\
&\quad + g^{(k)}(t)].
\end{aligned} \tag{37}$$

when $k \rightarrow \infty$, we obtain the optimal control law $u^*(t)$ in (20). $\square$

Remark 1 In fact, it is impossible to calculate optimal control law in (20). We can find the approximate optimal control law by replacing ∞ with a certain number N in (20)

$$\begin{aligned}
u_N(t) &= -R^{-1}B^T[P_1x(t) + P_2p(t) + P_3p_\omega(t) \\
&\quad + g^{(N)}(t)].
\end{aligned} \tag{38}$$

5.2 Algorithm Design

To implement the control law described in theorem 1, we design an algorithm in the following.

Algorithm 1.

Step 1. Calculate the matrices P_1 , P_2 and P_3 from the matrix equations (21) and (22);

Step 2. Establish the wave force $p(t)$ by the model (12) and relevant data from the Eq. (28);

Step 3. Given allowable error σ ;

Step 4. Calculate $g^{(0)}(t)$ and $x^{(0)}(t)$ by the first equations of (23) and (24), respectively, and let $i = 1$;

Step 5. Calculate $g^{(i)}(t)$ and $x^{(i)}(t)$ by the second equations of (23) and (24), respectively;

Step 6. Calculate the i th approximate optimal controller $u^{(i)}(t)$ by the equations of (37);

Step 7. Calculate $J^{(i)}$

$$J^{(i)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [x^{(i)T}(t)Qx^{(i)}(t) + Ru^{(i)2}(t)]dt, \tag{39}$$

select j from $(i, i-1, i-2, i-3)$

if $(|J^{(j)} - J^{(i-1)}|/J^{(j)}) \leq \sigma$ and $i \geq 4$

then stop calculate

else

let $i = i + 1$, and go to Step 5.

The number N can be chosen from $[i-3, i]$, and the N th order approximate optimal controller $u_N(t)$ is obtained by (38).

6 Numerical Simulations

In this section, we apply the proposed optimal vibration controller to a tracked vehicle. The model parameters have the values listed in Table 2, which have been used in some references [24, 25]. The associated matrices of system (5) are as follows:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -43.9412 & 0 & -1.1263 & 1.1263 \\ 376.0528 & -890.0968 & 9.6391 & -9.7676 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ -0.0088 \end{bmatrix}, f(x) = \begin{bmatrix} 0 \\ 0 \\ -136.2177x_1^3 \\ 1165.7637x_1^3 \end{bmatrix}, \\
\bar{C} &= \begin{bmatrix} -43.9412 & 0 & -1.1263 & 1.1263 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
Q &= \begin{bmatrix} 1931.8 & 0 & 49.5 & -49.5 \\ 0 & 1 & 0 & 0 \\ 49.5 & 0 & 1.3 & -1.3 \\ -49.5 & 0 & -1.3 & 1.3 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1285 \end{bmatrix}, \\
\bar{B} &= \begin{bmatrix} 0.001 \\ 0 \\ 0 \end{bmatrix}, R = 1.
\end{aligned} \tag{40}$$

Fig. 2 Road displacement curve

Table 2 Parameters of quarter-car model

Parameter	Variable	Value	Unit
Sprung mass	m_s	972.2	kg
Unsprung mass	m_u	113.6	kg
Suspension stiffness	k_{1s}	42,719.6	N/m
Suspension stiffness	k_{2s}	132,430.8	N/m
Wheel stiffness	k_t	101,115	N/m
Suspension damping	b_s	1,095	N s/m
Wheel damping	b_t	14.6	N s/m

To generate D grade road profile, we select $C_s = 64 \times 10^{-7} \text{ m}^3/\text{rad}$ and $k = 3$ in Table 1. Set $v_0 = 20 \text{ m/s}$, $l = 400 \text{ m}$ and $i = 200$ in (8) and (10), and take the frequency band from 0.05 to 10 Hz.

Using Matlab software, numerical experiment is carried out for the proposed optimal vibration controller. Figure 2 shows the road displacement curve, which indicates the external disturbance the control system subjected to.

The main purpose of vibration control of vehicle suspensions is to reduce sprung mass acceleration to enhance ride comfort, to reduce the suspension deflection which indicates the limit of the vehicle body motion and to reduce the wheel deflection to ensure road holding ability. So, to evaluate effectiveness of the proposed control strategy, sprung mass acceleration, suspension deflection and wheel deflection are considered. Then, the corresponding curves of open loop, traditional feedback control and controlled by the proposed optimal vibration controller are compared, and shown in Figs. 3, 4, 5 and 6. In the traditional feedback control method, the deviation, which is obtained by the comparison of the output information with the input information, is used to control the system. Actually, it is using the past to guide the present and the future. Therefore, there may be time delay from finding the deviation to taking corresponding measures. When correcting, the actual situation may have changed, and the loss may have been caused.

The curves of sprung mass acceleration $\ddot{x}_s(t)$ are shown in Fig. 3, suspension deflections $x_s(t) - x_u(t)$ are in Fig. 4,

Fig. 3 Sprung mass acceleration curves

Fig. 4 Suspension deflection curves

and wheel deflections $x_u(t) - x_r(t)$ are in Fig. 5, in which, red lines represent the open loop results of the suspension systems, blue lines denote the effect of traditional feedback control, and green lines describe the results of the suspension systems controlled by the proposed control strategy. From Figs. 3, 4 and 5, compared with the open loop results of the suspension systems and the traditional feedback control results, it can be seen that sprung mass acceleration, suspension deflection and wheel deflections controlled by the proposed control strategy are all reduced greatly. These numerical results demonstrate that the proposed optimal controller is efficient in reducing sprung mass acceleration and suspension deflection, thereby it can enhance ride comfort and ensure safety of passengers and vehicles. The blue line and red line demonstrate the control forces of the traditional feedback control and the proposed optimal control in Fig. 6, and it is obvious that the proposed optimal control strategy needs less control energy.

Fig. 5 Wheel deflection curves

Fig. 6 Control force curves

7 Conclusions

Suspension control plays an important role in the modern vehicle, and it is one of the very important components to provide ride comfort, in particular, to reduce driver fatigue due to long hours driving. To enhance ride comfort and ensure safety of passengers and vehicles, nonlinear dynamic system for a class of tracked vehicle suspension vibration control is established, a kind of approximate optimal controller and the algorithm are designed. Numerical simulations demonstrate that the proposed strategy is efficient and real time. The SAA also can be used to design the approximate optimal control laws for the nonlinear systems and delay systems. In the next work, we will focus on the vibration control for the suspension systems with the consideration of time delay, and the application of the modern control method, in particularly, intelligent control method.

Acknowledgements This research was supported by Science & Technology Development Plan of Henan Province (No.182102210205), Science and Technology Project of Guizhou Province (No.[2020]1Y277,

No.[2020]1Y263 and No.[2019]1159) and Guizhou Minzu University Fund Project (No.GZMU[2019]YB01, No.GZMU[2019]YB02, No.GZMU[2019]YB03 and No.GZMU[2019]YB04), and Doctoral Cultivation Fund Project of Henan University of Engineering (DKJ2019026).

Declarations

Conflict of interest This study was funded by Science & Technology Development Plan of Henan Province (No.182102210205), Science and Technology Project of Guizhou Province (No. [2020]1Y277, No. [2020]1Y263 and No. [2019]1159) and Guizhou Minzu University Fund Project (No. GZMU[2019]YB01, No. GZMU[2019]YB02, No. GZMU[2019]YB03 and No. GZMU[2019]YB04), and Doctoral Cultivation Fund Project of Henan University of Engineering (DKJ2019026). All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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