Generalized Fuzzy Ideals of BCI-Algebras Based on Interval Valued m -Polar Fuzzy Structures

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Abstract

In this paper, first we introduce the concepts of interval valued m -polar fuzzy (briefly, $IVmPF$) p -ideals, $IVmPF$ q -ideals and $IVmPF$ a -ideals in BCI-algebras. Then we show that $IVmPF$ $p(q$ and $a)$ -ideals are $IVmPF$ ideals but the converse statements are not valid and an example is given in each case. We provide conditions under which an $IVmPF$ ideal becomes an $IVmPF$ $p(q$ and $a)$ -ideal. Further, the associated properties of $IVmPF$ p -ideals, $IVmPF$ q -ideals and $IVmPF$ a -ideals are considered. Moreover, we characterize $IVmPF$ $p(q$ and $a)$ -ideals in terms of fuzzy $p(q$ and $a)$ -ideals of BCI-algebras. Also, correspondences among $IVmPF$ $p(q$ and $a)$ -ideals of BCI-algebras and $p(q$ and $a)$ -ideals of BCI-algebras are investigated.

Keywords BCI-algebras · Interval valued m -polar fuzzy sets · Interval valued m -polar fuzzy subalgebras · Interval valued m -polar fuzzy $p(q$ and $a)$ -ideals

Mathematics Subject Classification 03G25 · 06F35 · 08A72 · 16D25 · 94D05

1 Introduction

In decision making, Zadeh was the first to initiate the concept known as fuzzy set [51] in which the set is characterized by assigning each element in the set to a membership grade that ranges between 0 and 1. Fuzzy set concept was extended to intuitionistic fuzzy set [9], interval valued fuzzy set [52], fuzzy multiset [50], bipolar fuzzy set [53] and m -polar fuzzy set [11] (see [29, 47, 48] for other related notions). Moreover, a novel technique based on m -polar fuzzy set was introduced in [4].

The theory of m -polar fuzzy set and interval valued fuzzy set was proposed to be used in cognitive modeling, particularly in multi-agent decision making, i.e. if there are multi-attributes or more than one attribution with further

bifurcated. The elements in both m -polar fuzzy set and interval valued fuzzy set consist of m components but in the first each component is a real number in $[0, 1]$ and in the latter each component is a subset interval of $[0, 1]$.

The m -polar fuzzy set and interval valued fuzzy set theories were studied on different algebraic structures as lie subalgebras, groups, lie ideals of lie algebras, subgroups and rings, ideals and bi-ideals of semigroup, quasi-ideals in semigroups and matroids (see for example [2, 3, 12, 13, 20, 46, 49]) and from a different point of view see [1, 27].

Moreover, the algebraic structure of BCK/BCI-algebras [14, 15] were extensively investigated and a lot of theories has been studied on the theory of BCK/BCI-algebras such as ideal theory (see the references [17–19, 21, 23, 28]). The results of BCK/BCI-algebras were developed to m -polar fuzzy set and interval valued fuzzy set frames (see [16, 25, 26] and more recent, [5–8, 10, 30]). Also, in [22, 24, 30–34], some more general ideas on bipolar fuzzy were studied. In BCK/BCI-algebras and other related algebraic structures, different kinds of related concepts were investigated in various ways (see for example, [35–45]).

In this paper, results are organized sectionwise as follows: in Sect. 3, first we define the notion of $IVmPF$ p -ideals in BCI-algebra. Then the inter-relations between $IVmPF$ p -ideals and $IVmPF$ ideals are considered. In Sect. 4, $IVmPF$

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q -ideals in BCI-algebra is defined and characterized and also the inter-relations between IVmPF q -ideals and IVmPF ideals are investigated. In Sect. 5, we define the concept of IVmPF a -ideals in BCI-algebra and relations between IVmPF a -ideals with IVmPF ideals, IVmPF subalgebras and IVmPF $p(q)$ -ideals are investigated. In Sect. 6, correspondence between (fuzzy) $p(q)$ and a -ideals of BCI-algebra and IVmPF $p(q)$ and a -ideals of BCI-algebra is given.

2 Preliminaries

In this section, we recall some basic definitions and notions that will be used throughout this paper.

An algebra $Z = (Z; *, 0)$ of type $(2, 0)$ is a BCI-algebra if for all $v, \kappa, \tau \in Z$,

- (K₁) $((v * \kappa) * (v * \tau)) * (\tau * \kappa) = 0$,
- (K₂) $(v * (v * \kappa)) * \kappa = 0$,
- (K₃) $v * v = 0$,
- (K₄) $v * \kappa = 0$ and $\kappa * v = 0 \Rightarrow v = \kappa$.

Any BCI-algebra Z satisfies:

- (P₁) $v * 0 = v$,
- (P₂) $(v * \kappa) * \tau = (v * \tau) * \kappa$,
- (P₃) $v \leq \kappa \Rightarrow v * \tau \leq \kappa * \tau$ and $\tau * \kappa \leq \tau * v$,
- (P₄) $0 * (v * \kappa) = (0 * v) * (0 * \kappa)$,
- (P₅) $0 * (0 * (v * \kappa)) = 0 * (\kappa * v)$,
- (P₆) $(v * \tau) * (\kappa * \tau) \leq (v * \kappa)$,
- (P₇) $v * (v * (v * \kappa)) = v * \kappa$,
- (P₈) $0 * (0 * ((v * \tau) * (\kappa * \tau))) = (0 * \kappa) * (0 * v)$,
- (P₉) $0 * (0 * (v * \kappa)) = (0 * \kappa) * (0 * v)$,

where $v \leq \kappa \Leftrightarrow v * \kappa = 0$. Note that $(Z, \leq)$ is a partially ordered set.

A subset $(\emptyset \neq) A$ of Z is called a subalgebra if for all $v, \kappa \in Z$, $v * \kappa \in A$ and is called an *ideal* of Z if $0 \in A$ and for all $v, \kappa \in Z$, $v * \kappa \in A$, $\kappa \in A$ implies $v \in A$. Further, A is called p -ideal (resp. q -ideal and a -ideal) of Z if $0 \in A$ and for all $v, \kappa, \tau \in Z$, $((v * \tau) * (\kappa * \tau)) \in A$, $\kappa \in A$ implies $v \in A$ (resp. $v * (\kappa * \tau) \in A$, $\kappa \in A$ implies $v * \tau \in A$ and $((v * \tau) * (0 * \kappa)) \in A$, $\tau \in A$ implies $\kappa * v \in A$).

A mapping $\mu : Z \rightarrow [0, 1]$ is called a fuzzy set of Z . If $\mu(v * \kappa) \geq \mu(v) \wedge \mu(\kappa)$ for all $v, \kappa \in Z$ then μ is called a fuzzy subalgebra. If $\mu(0) \geq \mu(v)$ and $\mu(v) \geq \mu(v * \kappa) \wedge \mu(\kappa)$ for all $v, \kappa \in Z$ then μ is called a fuzzy ideal. Moreover, if $\mu(0) \geq \mu(v)$ and $\mu(v) \geq \mu((v * \tau) * (\kappa * \tau)) \wedge \mu(\kappa)$ (resp. $\mu(v * \tau) \geq \mu(v * (\kappa * \tau)) \wedge \mu(\kappa)$ and $\mu(\kappa * v) \geq \mu((v * \tau) * (0 * \kappa)) \wedge \mu(\tau)$) for all $v, \kappa, \tau \in Z$ then μ is called a fuzzy p -ideal (resp. fuzzy q -ideal and fuzzy a -ideal) of Z .

The interval number $\hat{t}$ is the interval $[t^-, t^+]$, where $0 \leq t^- \leq t^+ \leq 1$, $D[0, 1]$ is the set of all interval numbers. For the interval numbers $\hat{t}_i = [t_i^-, t_i^+]$, $\hat{d}_i = [d_i^-, d_i^+] \in D[0, 1]$, $i \in I$, we describe:

- (a) $\hat{t}_i \wedge \hat{d}_i = [t_i^- \wedge d_i^-, t_i^+ \wedge d_i^+]$;
- (b) $\hat{t}_1 \leq \hat{t}_2 \Leftrightarrow t_1^- \leq t_2^-$ and $t_1^+ \leq t_2^+$;
- (c) $\hat{t}_1 = \hat{t}_2 \Leftrightarrow t_1^- = t_2^-$ and $t_1^+ = t_2^+$.

A mapping $\hat{\mathcal{G}} : Z \rightarrow D[0, 1]$ is called an interval valued fuzzy set of Z , where $\hat{\mathcal{G}}(v) = [\mathcal{G}^-(v), \mathcal{G}^+(v)]$ for all $v \in Z$, $\mathcal{G}^-$ and $\mathcal{G}^+$ are fuzzy sets of Z with $\mathcal{G}^-(v) \leq \mathcal{G}^+(v)$ for all $v \in Z$.

Definition 2.1 [30] A mapping $\hat{\mathcal{G}} : Z \rightarrow D[0, 1]^m$ is called an interval valued m -polar fuzzy set (briefly, IVmPF set) of Z and is defined as:

$$\hat{\mathcal{G}}(v) = (\pi_1 \circ \hat{\mathcal{G}}(v), \pi_2 \circ \hat{\mathcal{G}}(v), \dots, \pi_m \circ \hat{\mathcal{G}}(v))$$

where $\pi_i : D[0, 1]^m \rightarrow D[0, 1]$ is the i th projection mapping for $i \in \{1, 2, \dots, m\}$.

That is,

$$\hat{\mathcal{G}}(v) = ([\mathcal{G}_1^-(v), \mathcal{G}_1^+(v)], [\mathcal{G}_2^-(v), \mathcal{G}_2^+(v)], \dots, [\mathcal{G}_m^-(v), \mathcal{G}_m^+(v)])$$

for all $v \in Z$, $\mathcal{G}_i^-$ and $\mathcal{G}_i^+$ are fuzzy sets of Z with $\mathcal{G}_i^-(v) \leq \mathcal{G}_i^+(v)$ for all $v \in Z$ and $i \in \{1, 2, \dots, m\}$.

The i th projection map π_i is order preserving and vice versa i.e.,

$$v \leq \kappa \Leftrightarrow \pi_i(v) \leq \pi_i(\kappa) \quad \forall i \in \{1, 2, \dots, m\}.$$

Definition 2.2 [30] An IVmPF set $\hat{\mathcal{G}}$ is said to be an IVmPF subalgebra if:

$$(\forall v, \kappa \in Z) \hat{\mathcal{G}}(v * \kappa) \geq \hat{\mathcal{G}}(v) \wedge \hat{\mathcal{G}}(\kappa),$$

that is,

$$(\forall v, \kappa \in Z, i \in \{1, 2, \dots, m\}) \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa).$$

Definition 2.3 [30] An IVmPF set $\hat{\mathcal{G}}$ is said to be an IVmPF ideal if:

- (1) $(\forall v \in Z) \hat{\mathcal{G}}(0) \geq \hat{\mathcal{G}}(v)$,
- (2) $(\forall v, \kappa \in Z) \hat{\mathcal{G}}(v) \geq \hat{\mathcal{G}}(v * \kappa) \wedge \hat{\mathcal{G}}(\kappa)$,

that is,

- (1) $\hat{\pi}_i \circ \hat{\mathcal{G}}(0) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v)$,
- (2) $\hat{\pi}_i \circ \hat{\mathcal{G}}(v) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa)$,

for all $v, \kappa \in Z$ and $i \in \{1, 2, \dots, m\}$.

Lemma 2.4 [30] Let $\widehat{\mathcal{G}}$ be an *IVmPF* ideal of Z and $v, \kappa \in Z$ such that $v \leq \kappa$. Then,

$$\widehat{\mathcal{G}}(v) \geq \widehat{\mathcal{G}}(\kappa).$$

Lemma 2.5 [30] Let $\widehat{\mathcal{G}}$ be an *IVmPF* ideal of Z and $v, \kappa, \tau \in Z$ such that $v * \kappa \leq \tau$. Then,

$$\widehat{\mathcal{G}}(v) \geq \widehat{\mathcal{G}}(\kappa) \wedge \widehat{\mathcal{G}}(\tau).$$

Throughout following sections, the BCI-algebra will be denoted by Z .

3 IVmPF p-ideal

The notion *IVmPF* p -ideals of BCI-algebras is described in this section, and relationships are provided between the *IVmPF* ideals and *IVmPF* p -ideals.

Definition 3.1 An *IVmPF* set $\widehat{\mathcal{G}}$ of Z is called an *IVmPF* p -ideal of Z if:

- (1) $(\forall v \in Z) \widehat{\mathcal{G}}(0) \geq \widehat{\mathcal{G}}(v)$,
- (2) $(\forall v, \kappa, \tau \in Z) \widehat{\mathcal{G}}(v) \geq \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\mathcal{G}}(\kappa)$,

that is,

- (1) $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v)$,
- (2) $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$,

for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$.

Example 1 Consider a BCI-algebra $Z = \{0, \iota, J, \ell\}$ with operation $(*)$ which is defined in Table 1.

Define an *IV3PF* set $\widehat{\mathcal{G}}$ on Z as:

$$\widehat{\mathcal{G}}(x) = \begin{cases} ([0.6, 0.7], [0.5, 0.8], [0.2, 0.4]) & \text{if } x = 0, \\ ([0.5, 0.6], [0.5, 0.5], [0.1, 0.4]) & \text{if } x = \iota, \\ ([0.3, 0.4], [0.2, 0.4], [0.1, 0.2]) & \text{if } x \in \{J, \ell\}. \end{cases}$$

It is straightforward to show that $\widehat{\mathcal{G}}$ is an *IV3PF* p -ideal of Z .

Table 1 Cayley table of the binary operation*

*	0	ι	J	ℓ
0	0	ι	J	ℓ
ι	ι	0	ℓ	J
J	J	ℓ	0	ι
ℓ	ℓ	J	ι	0

Theorem 3.2 Any *IVmPF* p -ideal of Z is an *IVmPF* ideal.

Proof Suppose that $\widehat{\mathcal{G}}$ is an *IVmPF* p -ideal of Z . Then for each $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa).$$

Substitute τ by 0 in above inequality, to get

$$\begin{aligned} \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) &\geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * 0) * (\kappa * 0)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa) \\ &= \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa). \end{aligned}$$

Hence, $\widehat{\mathcal{G}}$ is an *IVmPF* ideal. $\square$

Generally, the converse of Theorem 3.2 need not be true. This is illustrated by the example below.

Example 2 Consider $Z = \{0, \iota, J, \kappa, \ell\}$ a BCI-algebra under the operation $(*)$ which is defined by Table 2.

Define an *IV4PF* set $\widehat{\mathcal{G}}$ on Z as:

$$\widehat{\mathcal{G}}(x) = \begin{cases} ([0.6, 0.7], [0.5, 0.8], [0.4, 0.6], [0.7, 0.9]) & \text{if } x = 0, \\ ([0.1, 0.2], [0.2, 0.3], [0.1, 0.2], [0.1, 0.2]) & \text{if } x \in \{\iota, \ell\}, \\ ([0.3, 0.6], [0.3, 0.7], [0.2, 0.4], [0.4, 0.5]) & \text{if } x = J, \\ ([0.2, 0.3], [0.3, 0.4], [0.1, 0.3], [0.2, 0.4]) & \text{if } x = \kappa. \end{cases}$$

It is straightforward to check that $\widehat{\mathcal{G}}$ is an *IV4PF* ideal of Z but not an *IV4PF* p -ideal because $([0.3, 0.6], [0.3, 0.7], [0.2, 0.4], [0.4, 0.5]) = \widehat{\mathcal{G}}(J) \not\geq \widehat{\mathcal{G}}((J * \ell) * (0 * \ell))$
 $\wedge \widehat{\mathcal{G}}(0) = \widehat{\mathcal{G}}(0) = ([0.6, 0.7], [0.5, 0.8], [0.4, 0.6], [0.7, 0.9])$.

Theorem 3.3 If $\widehat{\mathcal{G}}$ is an *IVmPF* p -ideal of Z , then

- (a) $\widehat{\mathcal{G}}(v) \geq \widehat{\mathcal{G}}(0 * (0 * v)) \forall v \in Z$.
- (b) $\widehat{\mathcal{G}}(v) = \widehat{\mathcal{G}}(0 * (0 * v)) \forall v \in Z$.

Proof (a) . Let $\widehat{\mathcal{G}}$ be an *IVmPF* p -ideal of Z . Then,

$$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$$

for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$. Substitute v for τ and 0 for κ to get

Table 2 Cayley table of the binary operation*

*	0	ι	J	κ	ℓ
0	0	0	0	κ	κ
ι	ι	0	ι	ℓ	κ
J	J	J	0	κ	κ
κ	κ	κ	κ	0	0
ℓ	ℓ	κ	ℓ	ι	0

$$\begin{aligned}
\hat{\pi}_i \circ \hat{\mathcal{G}}(v) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((v * v) * (0 * v)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(0) \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * v)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(0) \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * v)).
\end{aligned}$$

(b). Since $0 * (0 * v) \leq v$ for any $v \in Z$. Therefore, by Lemma 2.4 we have

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * v)) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \quad (\forall i \in \{1, 2, \dots, m\}).$$

From (a), $\hat{\pi}_i \circ \hat{\mathcal{G}}(v) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * v))$. Thus,

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v) = \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * v)).$$

□

A condition for IVmPF ideal to be IVmPF p -ideal is provided in the following two results.

Theorem 3.4 Let $\hat{\mathcal{G}}$ be an IVmPF ideal of Z satisfying the condition:

$$(\forall v, \kappa, \tau \in Z) \hat{\mathcal{G}}(v * \kappa) \geq \hat{\mathcal{G}}((v * \tau) * (\kappa * \tau)). \quad (1)$$

Then, $\hat{\mathcal{G}}$ is an IVmPF p -ideal of Z .

Proof Suppose that $\hat{\mathcal{G}}$ is an IVmPF ideal of Z satisfying (1). Then,

$$\begin{aligned}
\hat{\pi}_i \circ \hat{\mathcal{G}}(v) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa) \\
&\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa),
\end{aligned}$$

$\forall v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$. Hence, $\hat{\mathcal{G}}$ is an IVmPF p -ideal of Z . □

Theorem 3.5 Let $\hat{\mathcal{G}}$ be an IVmPF ideal of Z satisfying the identity:

$$(\forall v \in Z) \hat{\mathcal{G}}(v) \geq \hat{\mathcal{G}}(0 * (0 * v)). \quad (2)$$

Then, $\hat{\mathcal{G}}$ is an IVmPF p -ideal of Z .

Proof Let $v, \kappa, \tau \in Z$. Then for all $i \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned}
&\hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \\
&\geq \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * (v * \kappa))) \quad \text{by (2)} \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}((0 * \kappa) * (0 * v)) \quad \text{by } (P_9) \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * ((v * \tau) * (\kappa * \tau)))) \quad \text{by } (P_8) \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}((v * \tau) * (\kappa * \tau)). \quad \text{by Theorem 3.3}
\end{aligned}$$

Hence from Theorem 3.4, $\hat{\mathcal{G}}$ is an IVmPF p -ideal of Z . □

4 IVmPF q -ideal

In this section, we define IVmPF q -ideals of BCI-algebras and associated properties of IVmPF ideals and IVmPF q -ideals are considered.

Definition 4.1 An IVmPF set $\hat{\mathcal{G}}$ of Z is called an IVmPF q -ideal of Z if:

- (1) $(\forall v \in Z) \hat{\mathcal{G}}(0) \geq \hat{\mathcal{G}}(v)$,
- (2) $(\forall v, \kappa, \tau \in Z) \hat{\mathcal{G}}(v * \tau) \geq \hat{\mathcal{G}}(v * (\kappa * \tau)) \wedge \hat{\mathcal{G}}(\kappa)$,

that is,

- (1) $\hat{\pi}_i \circ \hat{\mathcal{G}}(0) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v)$,
- (2) $\hat{\pi}_i \circ \hat{\mathcal{G}}(v * \tau) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * (\kappa * \tau)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa)$,

for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$.

Example 3 Consider the BCI-algebra Z and the IV4PF set $\hat{\mathcal{G}}$ on Z defined in Example 2. It is straightforward to check that $\hat{\mathcal{G}}$ is an IV4PF q -ideal of Z .

Theorem 4.2 Every IVmPF q -ideal of Z is an IVmPF ideal.

Proof Let $\hat{\mathcal{G}}$ be an IVmPF q -ideal of Z . Then for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v * \tau) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * (\kappa * \tau)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa).$$

Substitute τ by 0, to have

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v * 0) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * (\kappa * 0)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa).$$

It follows that $\hat{\pi}_i \circ \hat{\mathcal{G}}(v) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa)$, as required. □

In general, the converse of Theorem 4.2 is not valid, as illustrated in the following example.

Example 4 Consider a BCI-algebra $Z = \{0, \iota, j, \kappa, \ell\}$ with Table 3.

Define an IV3PF set $\hat{\mathcal{G}}$ on Z as:

Table 3 Cayley table of the binary operation*

*	0	ι	j	κ	ℓ
0	0	0	ℓ	κ	j
ι	ι	0	ℓ	κ	j
j	j	j	0	ℓ	κ
κ	κ	κ	j	0	ℓ
ℓ	ℓ	ℓ	κ	j	0

$$\widehat{G}(x) = \begin{cases} ([0.6, 0.7], [0.6, 0.9], [0.3, 0.5]) & \text{if } x = 0, \\ ([0.4, 0.5], [0.5, 0.7], [0.3, 0.4]) & \text{if } x = \iota, \\ ([0.2, 0.3], [0.3, 0.4], [0.2, 0.3]) & \text{if } x \in \{j, \kappa, \ell\}. \end{cases}$$

It is straightforward to check that $\widehat{G}$ is an *IV3PF* ideal of Z but not an *IV3PF* q -ideal because $([0.2, 0.3], [0.3, 0.4], [0.2, 0.3]) = \widehat{G}(\ell * j) \not\geq \widehat{G}(\ell * (0 * j)) \wedge \widehat{G}(0) = \widehat{G}(0) = ([0.6, 0.7], [0.6, 0.9], [0.3, 0.5])$.

Theorem 4.3 *If $\widehat{G}$ is an IVmPF ideal of Z , then the following statements are equivalent:*

- (1) $\widehat{G}$ is an IVmPF q -ideal of Z .
- (2) $\widehat{G}(v * \kappa) \geq \widehat{G}(v * (0 * \kappa)) \forall v, \kappa \in Z$.
- (3) $\widehat{G}((v * \kappa) * \tau) \geq \widehat{G}(v * (\kappa * \tau)) \forall v, \kappa, \tau \in Z$.

Proof (1) $\Rightarrow$ (2). Let $\widehat{G}$ be an IVmPF q -ideal of Z . Then for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\widehat{\pi}_i \circ \widehat{G}(v * \tau) \geq \widehat{\pi}_i \circ \widehat{G}(v * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa).$$

Replacing τ by κ and κ by τ , we get

$$\widehat{\pi}_i \circ \widehat{G}(v * \kappa) \geq \widehat{\pi}_i \circ \widehat{G}(v * (\tau * \kappa)) \wedge \widehat{\pi}_i \circ \widehat{G}(\tau).$$

Substitute τ by 0,

$$\begin{aligned} \widehat{\pi}_i \circ \widehat{G}(v * \kappa) &\geq \widehat{\pi}_i \circ \widehat{G}(v * (0 * \kappa)) \wedge \widehat{\pi}_i \circ \widehat{G}(0) \\ &= \widehat{\pi}_i \circ \widehat{G}(v * (0 * \kappa)). \end{aligned}$$

(2) $\Rightarrow$ (3). Let $v, \kappa, \tau \in Z$. Then,

$$\widehat{\pi}_i \circ \widehat{G}((v * \kappa) * \tau) \geq \widehat{\pi}_i \circ \widehat{G}((v * \kappa) * (0 * \tau)) \quad (\forall i \in \{1, 2, \dots, m\}). \quad (3)$$

Now we have

$$\begin{aligned} ((v * \kappa) * (0 * \tau)) * (v * (\kappa * \tau)) &= ((v * \kappa) * (v * (\kappa * \tau))) * (0 * \tau) \\ &\leq ((\kappa * \tau) * \kappa) * (0 * \tau) \\ &= ((\kappa * \kappa) * \tau) * (0 * \tau) \\ &= (0 * \tau) * (0 * \tau) \\ &= 0. \end{aligned}$$

By Lemma 2.5, we have

$$\widehat{\pi}_i \circ \widehat{G}((v * \kappa) * (0 * \tau)) \geq \widehat{\pi}_i \circ \widehat{G}(v * (\kappa * \tau)). \quad (4)$$

It follows from (3) and (4) that $\widehat{G}((v * \kappa) * \tau) \geq \widehat{G}(v * (\kappa * \tau))$.

(3) $\Rightarrow$ (1). As $\widehat{G}$ is an IVmPF ideal of Z , so for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} \widehat{\pi}_i \circ \widehat{G}(v * \tau) &\geq \widehat{\pi}_i \circ \widehat{G}((v * \tau) * \kappa) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa) \\ &\geq \widehat{\pi}_i \circ \widehat{G}(v * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa). \end{aligned}$$

□

Lemma 4.4 *Let $\widehat{G}$ be an IVmPF q -ideal of Z . Then $\widehat{G}(0 * \tau) \geq \widehat{G}(\tau)$ for all $\tau \in Z$.*

Proof Suppose that $\widehat{G}$ is an IVmPF q -ideal of Z . Then for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\widehat{\pi}_i \circ \widehat{G}(v * \tau) \geq \widehat{\pi}_i \circ \widehat{G}(v * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa).$$

Substitute v by 0 and κ by τ , to have

$$\widehat{\pi}_i \circ \widehat{G}(0 * \tau) \geq \widehat{\pi}_i \circ \widehat{G}(0 * (\tau * \tau)) \wedge \widehat{\pi}_i \circ \widehat{G}(\tau) = \widehat{\pi}_i \circ \widehat{G}(\tau).$$

□

5 IVmPF a -ideal

In this section, the notion of IVmPF a -ideals of BCI-algebra is defined and associated properties are investigated. Inter related properties of IVmPF a -ideals with IVmPF q -ideals and IVmPF p -ideals are discussed.

Definition 5.1 An IVmPF set $\widehat{G}$ of Z is called an IVmPF a -ideal of Z if:

- (1) $\widehat{G}(0) \geq \widehat{G}(v)$,
- (2) $\widehat{G}(v * \tau) \geq \widehat{G}((\tau * \kappa) * (0 * v)) \wedge \widehat{G}(\kappa)$,

that is,

- (1) $\widehat{\pi}_i \circ \widehat{G}(0) \geq \widehat{\pi}_i \circ \widehat{G}(v)$,
- (2) $\widehat{\pi}_i \circ \widehat{G}(v * \tau) \geq \widehat{\pi}_i \circ \widehat{G}((\tau * \kappa) * (0 * v)) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa)$,

for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$.

Example 5 Consider the BCI-algebra Z and the IV3PF set $\widehat{G}$ on Z defined in Example 1. It is straightforward to verify that $\widehat{G}$ is an IV3PF a -ideal.

Theorem 5.2 *Every IVmPF a -ideal of Z is both an IVmPF ideal and an IVmPF subalgebra.*

Proof Let $\widehat{G}$ be an IVmPF a -ideal of Z . Then for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\widehat{\pi}_i \circ \widehat{G}(v * \tau) \geq \widehat{\pi}_i \circ \widehat{G}((\tau * \kappa) * (0 * v)) \wedge \widehat{\pi}_i \circ \widehat{G}(\kappa). \quad (5)$$

Substitute κ and v by 0, to get

$$\widehat{\pi}_i \circ \widehat{G}(0 * \tau) \geq \widehat{\pi}_i \circ \widehat{G}((\tau * 0) * (0 * 0)) \wedge \widehat{\pi}_i \circ \widehat{G}(0) = \widehat{\pi}_i \circ \widehat{G}(\tau). \quad (6)$$

Again in (5), substitute τ and κ by 0 and use (6), to get

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v * 0) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}((0 * 0) * (0 * v)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(0) \geq \hat{\mathcal{G}}(0 * v).$$

As $v * 0 = v$, Therefore,

$$\begin{aligned} \hat{\pi}_i \circ \hat{\mathcal{G}}(v) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * v) \\ &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((v * \tau) * (0 * 0)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\tau) \\ &= \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \tau) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\tau). \end{aligned}$$

Hence, $\hat{\mathcal{G}}$ is an IVmPF ideal.

Next, take any $v, \tau \in Z$. By above inequality and (6), we have

$$\begin{aligned} \hat{\pi}_i \circ \hat{\mathcal{G}}(v * \tau) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((v * \tau) * v) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \\ &= \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * \tau) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \\ &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\tau). \end{aligned}$$

Hence, $\hat{\mathcal{G}}$ is an IVmPF subalgebra. $\square$

The following example shows that the converse of Theorem 5.2 is not valid.

Example 6 Consider a BCI-algebra $Z = \{0, \iota, j\}$ under the operation $(*)$ defined in Table 4.

Define an IV4PF set $\hat{\mathcal{G}}$ on Z as:

$$\hat{\mathcal{G}}(x) = \begin{cases} ([0.6, 0.7], [0.5, 0.8], [0.4, 0.6], [0.7, 0.9]) & \text{if } x = 0, \\ ([0.2, 0.3], [0.4, 0.7], [0.1, 0.2], [0.1, 0.2]) & \text{if } x \in \{\iota, j\}. \end{cases}$$

It is straightforward to check that $\hat{\mathcal{G}}$ is both an IV4PF ideal and an IV4PF subalgebra but not an IV4PF a -ideal because $([0.2, 0.3], [0.4, 0.7], [0.1, 0.2], [0.1, 0.2]) = \hat{\mathcal{G}}(\iota * 0) \not\geq \hat{\mathcal{G}}(0 * 0) * (0 * \iota) \wedge \hat{\mathcal{G}}(0) = \hat{\mathcal{G}}(0) = ([0.6, 0.7], [0.5, 0.8], [0.4, 0.6], [0.7, 0.9])$.

The following result provides a condition under which every IVmPF ideal of Z is an IVmPF a -ideal.

Theorem 5.3 Let $\hat{\mathcal{G}}$ be an IVmPF ideal of Z satisfying:

$$(\forall \tau, \kappa \in Z) \hat{\mathcal{G}}(\kappa * \tau) \geq \hat{\mathcal{G}}(\tau * (0 * \kappa)).$$

Then, $\hat{\mathcal{G}}$ is an IVmPF a -ideal.

Proof As $\hat{\mathcal{G}}$ is an IVmPF ideal of Z , so for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

Table 4 Cayley table of the binary operation*

*	0	ι	j
0	0	0	0
ι	ι	0	ι
j	j	j	0

$$\begin{aligned} \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa * \tau) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}(\tau * (0 * \kappa)) \\ &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((\tau * (0 * \kappa)) * v) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(v) \\ &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((\tau * v) * (0 * \kappa)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(v). \end{aligned}$$

Hence, $\hat{\mathcal{G}}$ is an IVmPF a -ideal of Z . $\square$

Theorem 5.4 IVmPF a -ideals are IVmPF p -ideals in Z , conversely IVmPF p -ideals need not be IVmPF a -ideals.

Proof Let $\hat{\mathcal{G}}$ be an IVmPF a -ideal of Z . Then for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$, we have

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v * \tau) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}((\tau * \kappa) * (0 * v)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa).$$

Substitute κ and τ by 0, we get

$$\begin{aligned} \hat{\pi}_i \circ \hat{\mathcal{G}}(v * 0) &\geq \hat{\pi}_i \circ \hat{\mathcal{G}}((0 * 0) * (0 * v)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(0) \\ &= \hat{\mathcal{G}}(0 * (0 * v)). \end{aligned} \quad (7)$$

As $v * 0 = v$, so from (7)

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v) \geq \hat{\mathcal{G}}(0 * (0 * v)). \quad (8)$$

From (8) and Theorem 3.5, $\hat{\mathcal{G}}$ is an IVmPF p -ideal.

To show the converse, consider the BCI-algebra of Example 4 defined in Table 3. Choose $[\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m])$ and $[\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m]) \in D[0, 1]^m$ such that $[\theta, \lambda] \geq [\rho, \sigma]$. Define an IVmPF set $\hat{\mathcal{G}}$ on Z as:

$$\hat{\mathcal{G}}(x) = \begin{cases} [\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m]) & \text{if } x \in \{0, \iota\}, \\ [\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m]) & \text{if } x \in \{j, \kappa, \ell\}. \end{cases}$$

It is straightforward to check that $\hat{\mathcal{G}}$ is an IVmPF p -ideal of Z but not an IVmPF a -ideal because $[\rho, \sigma] = \hat{\mathcal{G}}(\ell * j) \not\geq \hat{\mathcal{G}}(j * 0) * (0 * \ell) \wedge \hat{\mathcal{G}}(0) \wedge \hat{\mathcal{G}}(0) = \hat{\mathcal{G}}(0) = [\theta, \lambda]$. $\square$

Theorem 5.5 IVmPF a -ideals are IVmPF q -ideals in Z , conversely IVmPF q -ideals need not be IVmPF a -ideals.

Proof Let $\hat{\mathcal{G}}$ be an IVmPF a -ideal of Z . By Theorems 5.4 and 3.3, we have

$$\hat{\pi}_i \circ \hat{\mathcal{G}}(v * \kappa) \geq \hat{\pi}_i \circ \hat{\mathcal{G}}(0 * (0 * (v * \kappa))). \quad (9)$$

By properties of BCI-algebra Z , we have

$$\begin{aligned} &(0 * (0 * (v * \kappa))) * (v * (0 * \kappa)) \\ &= (0 * ((0 * v) * (0 * \kappa))) * (v * (0 * \kappa)) \\ &= ((0 * (0 * v)) * (0 * (0 * \kappa))) * (v * (0 * \kappa)) \\ &\leq (v * \kappa) * (v * (0 * \kappa)) \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
& \widehat{\pi}_i \circ \widehat{\mathcal{G}}(0 * (0 * (v * \kappa))) \\
& \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \kappa) * (v * (0 * \kappa))) \\
& \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa)) \\
& \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa)).
\end{aligned}$$

Thus, $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0 * (0 * (v * \kappa))) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa))$. So from (9), we get

$$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa)).$$

It follows that $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * \kappa) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v * (0 * \kappa))$. Hence by Theorem 4.3, $\widehat{\mathcal{G}}$ is an IVmPF q -ideal.

To show the converse, consider the BCI-algebra defined in Example 2 by Table 2. Let $[\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m])$, $[\psi, \phi] = ([\psi_1, \phi_1], [\psi_2, \phi_2], \dots, [\psi_m, \phi_m])$, $[\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m])$ and $[\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D[0, 1]^m$ such that $[\theta, \lambda] \geq [\alpha, \beta] \geq [\rho, \sigma] \geq [\psi, \phi]$. Define an IVmPF set $\widehat{\mathcal{G}}$ on Z as:

$$\widehat{\mathcal{G}}(x) = \begin{cases} [\theta, \lambda] = ([\theta_1, \lambda_1], [\theta_2, \lambda_2], \dots, [\theta_m, \lambda_m]) & \text{if } x = 0, \\ [\psi, \phi] = ([\psi_1, \phi_1], [\psi_2, \phi_2], \dots, [\psi_m, \phi_m]) & \text{if } x \in \{i, \ell\}, \\ [\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) & \text{if } x = j, \\ [\rho, \sigma] = ([\rho_1, \sigma_1], [\rho_2, \sigma_2], \dots, [\rho_m, \sigma_m]) & \text{if } x = \kappa. \end{cases}$$

It is straightforward to verify that $\widehat{\mathcal{G}}$ is an IVmPF q -ideal of Z but not an IVmPF a -ideal because $[\psi, \phi] = \widehat{\mathcal{G}}(\ell * j) \not\geq \widehat{\mathcal{G}}((j * 0) * (0 * \ell)) \wedge \widehat{\mathcal{G}}(0) = \widehat{\mathcal{G}}(\ell) = [\rho, \sigma]$. $\square$

6 Correspondence Between $p(q$ and a)-Ideals and IVmPF $p(q$ and a)-Ideals

In this section, we discuss correspondence between $p(q$ and a)-ideals and IVmPF $p(q$ and a)-ideals.

Definition 6.1 Let $\widehat{\mathcal{G}}$ be any IVmPF set on Z . For $[\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D[0, 1]^m$ define a level subset $U(\widehat{\mathcal{G}}; [\alpha, \beta])$ as follows:

$$U(\widehat{\mathcal{G}}; [\alpha, \beta]) = \{v \in Z \mid \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq [\alpha_i, \beta_i] \text{ for all } i \in \{1, 2, \dots, m\}\}.$$

Theorem 6.2 An IVmPF set $\widehat{\mathcal{G}}$ is an IVmPF p -ideal (resp. IVmPF q -ideal and IVmPF a -ideal) of $Z \Leftrightarrow$ each level subset $(\emptyset \neq) U(\widehat{\mathcal{G}}; [\alpha, \beta])$ is a p -ideal (resp. q -ideal and a -ideal) of $Z \forall [\alpha, \beta] = ([\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_m, \beta_m]) \in D[0, 1]^m$.

Proof ($\Rightarrow$) Let us suppose that $\widehat{\mathcal{G}}$ is an IVmPF p -ideal of Z and $v \in U(\widehat{\mathcal{G}}; [\alpha, \beta])$. Then $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq [\alpha_i, \beta_i]$. By hypothesis,

$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq [\alpha_i, \beta_i]$. Thus, $0 \in U(\widehat{\mathcal{G}}; [\alpha, \beta])$. Next, take any $v, \kappa, \tau \in Z$ such that $(v * \tau) * (\kappa * \tau) \in U(\widehat{\mathcal{G}}; [\alpha, \beta])$ and $\kappa \in U(\widehat{\mathcal{G}}; [\alpha, \beta])$. Then $\widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \geq [\alpha_i, \beta_i]$ and $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa) \geq [\alpha_i, \beta_i]$. As $\widehat{\mathcal{G}}$ is an IVmPF p -ideal, we have

$$\begin{aligned}
& \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa) \\
& \geq [\alpha_i, \beta_i] \wedge [\alpha_i, \beta_i] \\
& = [\alpha_i, \beta_i]
\end{aligned}$$

Therefore, $v \in U(\widehat{\mathcal{G}}; [\alpha, \beta])$. Hence, $U(\widehat{\mathcal{G}}; [\alpha, \beta])$ is a p -ideal of Z .

($\Leftarrow$) Assume that $U(\widehat{\mathcal{G}}; [\alpha, \beta])$ is a p -ideal of $Z \forall [\alpha, \beta] \in D[0, 1]^m$. If $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) < \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v)$ for some $v \in Z$. So $\exists [\gamma, \lambda] = ([\gamma_1, \lambda_1], [\gamma_2, \lambda_2], \dots, [\gamma_m, \lambda_m]) \in D[0, 1]^m$ such that $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) < [\gamma_i, \lambda_i] \leq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v)$ for each $i \in \{1, 2, \dots, m\}$ implies $v \in U(\widehat{\mathcal{G}}; [\gamma, \lambda])$ but $0 \notin U(\widehat{\mathcal{G}}; [\gamma, \lambda])$, which is a contradiction. Therefore, $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v)$ for all $v \in Z$ and $i \in \{1, 2, \dots, m\}$. Again, if $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) < \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$ for some $v, \kappa, \tau \in Z$. So $\exists [\gamma, \lambda] = ([\gamma_1, \lambda_1], [\gamma_2, \lambda_2], \dots, [\gamma_m, \lambda_m]) \in D[0, 1]^m$ such that $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) < [\gamma_i, \lambda_i] \leq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$ for each $i \in \{1, 2, \dots, m\}$ implies $(v * \tau) * (\kappa * \tau) \in U(\widehat{\mathcal{G}}; [\gamma, \lambda])$ and $\kappa \in U(\widehat{\mathcal{G}}; [\gamma, \lambda])$ but $v \notin U(\widehat{\mathcal{G}}; [\gamma, \lambda])$, gives a contradiction. Therefore, $\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$ for all $v, \kappa, \tau \in Z$ and $i \in \{1, 2, \dots, m\}$.

A similar argument is used with respect to IVmPF q -ideal and IVmPF a -ideal. $\square$

Theorem 6.3 An IVmPF set $\widehat{\mathcal{G}} = ([\mathcal{G}_1^-, \mathcal{G}_1^+], [\mathcal{G}_2^-, \mathcal{G}_2^+], \dots, [\mathcal{G}_m^-, \mathcal{G}_m^+])$ in Z is an IVmPF p -ideal (resp. IVmPF q -ideal and IVmPF a -ideal) of $Z \Leftrightarrow \mathcal{G}_i^-$ and $\mathcal{G}_i^+$ for all $i \in \{1, 2, \dots, m\}$ are fuzzy p -ideal (resp. fuzzy q -ideal and fuzzy a -ideal) of Z .

Proof ($\Rightarrow$) Suppose that $\widehat{\mathcal{G}}(v) = ([\mathcal{G}_1^-(v), \mathcal{G}_1^+(v)], [\mathcal{G}_2^-(v), \mathcal{G}_2^+(v)], \dots, [\mathcal{G}_m^-(v), \mathcal{G}_m^+(v)])$ in Z is an IVmPF p -ideal. For any $v \in Z$, we have

$$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(0) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \forall i \in \{1, 2, \dots, m\}$$

implies for each $i \in \{1, 2, \dots, m\}$,

$$[\mathcal{G}_i^-(0), \mathcal{G}_i^+(0)] \geq [\mathcal{G}_i^-(v), \mathcal{G}_i^+(v)].$$

It follows that $\mathcal{G}_i^-(0) \geq \mathcal{G}_i^-(v)$ and $\mathcal{G}_i^+(0) \geq \mathcal{G}_i^+(v)$. Take any $v, \kappa, \tau \in Z$. By hypothesis, we have for all $i \in \{1, 2, \dots, m\}$

$$\widehat{\pi}_i \circ \widehat{\mathcal{G}}(v) \geq \widehat{\pi}_i \circ \widehat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \widehat{\pi}_i \circ \widehat{\mathcal{G}}(\kappa)$$

implies

$$\begin{aligned}
& [\mathcal{G}_i^-(v), \mathcal{G}_i^+(v)] \\
& \geq [\mathcal{G}_i^-((v * \tau) * (\kappa * \tau)), \mathcal{G}_i^+((v * \tau) * (\kappa * \tau))] \wedge [\mathcal{G}_i^-(\kappa), \mathcal{G}_i^+(\kappa)] \\
& = [\mathcal{G}_i^-((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^-(\kappa), \mathcal{G}_i^+((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^+(\kappa)].
\end{aligned}$$

Therefore, $\mathcal{G}_i^-(v) \geq \mathcal{G}_i^-((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^-(\kappa)$ and $\mathcal{G}_i^+(v) \geq \mathcal{G}_i^+((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^+(\kappa)$. Hence, $\mathcal{G}_i^-$ and $\mathcal{G}_i^+$ are fuzzy p -ideals of Z .

($\Leftarrow$) Suppose that for all $i \in \{1, 2, \dots, m\}$, $\mathcal{G}_i^-$ and $\mathcal{G}_i^+$ are fuzzy p -ideals of Z .

$$\begin{aligned}
\hat{\pi}_i \circ \hat{\mathcal{G}}(0) &= [\mathcal{G}_i^-(0), \mathcal{G}_i^+(0)] \\
&\geq [\mathcal{G}_i^-(v), \mathcal{G}_i^+(v)] \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}(v)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\pi}_i \circ \hat{\mathcal{G}}(v) &= [\mathcal{G}_i^-(v), \mathcal{G}_i^+(v)] \\
&\geq [\mathcal{G}_i^-((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^-(\kappa), \mathcal{G}_i^+((v * \tau) * (\kappa * \tau)) \wedge \mathcal{G}_i^+(\kappa)] \\
&= [\mathcal{G}_i^-((v * \tau) * (\kappa * \tau)), \mathcal{G}_i^+((v * \tau) * (\kappa * \tau))] \wedge [\mathcal{G}_i^-(\kappa), \mathcal{G}_i^+(\kappa)] \\
&= \hat{\pi}_i \circ \hat{\mathcal{G}}((v * \tau) * (\kappa * \tau)) \wedge \hat{\pi}_i \circ \hat{\mathcal{G}}(\kappa).
\end{aligned}$$

Hence, $\hat{\mathcal{G}}$ is an IVmPF p -ideal.

A similar argument is used with respect to IVmPF q -ideal and IVmPF a -ideal. $\square$

7 Conclusion

Based on IVmPF structures, we developed the ideal theory of BCI-algebra. We introduced the concepts of IVmPF p -ideals, IVmPF q -ideals and IVmPF a -ideals in BCI-algebra. Then, we have shown that IVmPF $p(q$ and $a)$ ideals are IVmPF ideals but not the converse. We provided a condition under which IVmPF ideals become IVmPF $p(q$ and $a)$ ideals. Also, we proved that an IVmPF a -ideal is both an IVmPF p -ideal and an IVmPF q -ideal and that the converse implications are not true. Further, we characterize IVmPF $p(q$ and $a)$ ideals in terms of fuzzy $p(q$ and $a)$ ideals of BCI-algebra. Moreover, correspondence between IVmPF $p(q$ and $a)$ ideals of BCI-algebra and $p(q$ and $a)$ -ideals of BCI-algebra is investigated.

In our future work, we intend to apply the presented notions of this present paper to different algebras such as MTL-algebras, BL-algebras, $R0$ -algebras, EQ-algebras, MV-algebras and lattice implication algebras etc.

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