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On the spectrum of tridiagonal operators in the context of orthogonal polynomials

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Abstract. The basis for our studies is a large class of orthogonal polynomial sequences $(P_n)_{n\in\mathbb{N}_0}$, which is normalized by $P_n(x_0)=1$ for all $n\in\mathbb{N}_0$ where the coefficients in the three-term recurrence relation are bounded. The goal is to check if $x_0\in\mathbb{R}$ is in the support of the orthogonalization measure μ. For this purpose, we use, among other things, a result of G. H. Hardy concerning Cesàro operators on weighted l^2-spaces. These investigations generalize ideas from Lasser et al. (Arch Math 100:289–299, 2013).

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1. Orthogonal polynomials on the real line and tridiagonal operators

Let μ be a probability measure on the real line. We denote the support of μ by $\mathcal{S}$ and assume its cardinality $\#\mathcal{S}=\infty$. Let $(p_n)_{n\in\mathbb{N}_0}$ denote the unique orthonormal polynomial sequence with respect to μ, that is $\deg p_n = n$, $\int p_n p_m d\mu = \delta_{n,m}$, and p_n has a positive leading coefficient for all $n,m\in\mathbb{N}_0$. The orthonormal polynomial sequence $(p_n)_{n\in\mathbb{N}_0}$ satisfies a recurrence relation

$$xp_n(x)=\lambda_n p_{n+1}(x)+\beta_n p_n(x)+\lambda_{n-1}p_{n-1}(x) \tag{1}$$

with $p_{-1}(x)=0$, $p_0(x)=1$, $\lambda_{-1}=0$, $\lambda_n>0$ and $\beta_n\in\mathbb{R}$ for all $n\in\mathbb{N}_0$.

Conversely, if $(p_n)_{n\in\mathbb{N}_0}$ is defined by (1), there is a probability measure μ such that $(p_n)_{n\in\mathbb{N}_0}$ is the orthonormal polynomial sequence with respect to μ, see e.g. [2].

In the case $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are bounded $\mathcal{S}$ is compact and vice versa. The boundedness implies also that the orthogonalization measure μ is uniquely determined. The smallest interval containing $\mathcal{S}$ is called the true interval of orthogonality, see e.g. [2].

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Now, let $x_0 \in \mathbb{R}\backslash\mathcal{N}$, where $\mathcal{N} = \{x \in \mathbb{C} : \exists n \in \mathbb{N} \text{ with } p_n(x) = 0\}$ is the set of zeros of all orthonormal polynomials. It is well known that $\mathcal{N} \subset \mathbb{R}$, see e.g. [2]. The normalized polynomials

$$P_n(x) = \frac{p_n(x)}{p_n(x_0)} \tag{2}$$

form an orthogonal polynomial sequence $(P_n)_{n\in\mathbb{N}_0}$ with respect to μ, that is

$$\int P_n P_m d\mu = \frac{\delta_{n,m}}{h_n} \tag{3}$$

with $h_n > 0$. We call x_0 a normalizing point. The corresponding three-term recurrence relation is

$$xP_n(x) = \gamma_n P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) \tag{4}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$,

$$\gamma_n = \frac{p_{n+1}(x_0)}{p_n(x_0)}\lambda_n, \tag{5}$$

$$\alpha_n = \frac{p_{n-1}(x_0)}{p_n(x_0)}\lambda_{n-1}, \quad \text{and} \tag{6}$$

$$\alpha_n + \beta_n + \gamma_n = x_0 \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{7}$$

Note that (6) implies $\alpha_0 = 0$. It is also important to emphasize that (7) applies if and only if x_0 is a normalization point and that our investigations heavily depend on Eq. (7).

Moreover, $\gamma_n\alpha_{n+1} = \lambda_n^2 > 0$. One easily shows

$$h_{n+1}\alpha_{n+1} = h_n\gamma_n \tag{8}$$

which implies

$$h_n = \frac{\gamma_0 \cdots \gamma_{n-1}}{\alpha_1 \ldots \alpha_n} = p_n^2(x_0) \quad \text{for all } n \in \mathbb{N}_0. \tag{9}$$

Note that (9) also applies in the case $n = 0$, where the nominator and denominator are empty products that means they are set equal 1 by default. Therefore, (3) as well as (9) yields $h_0 = 1$.

The so called Christoffel–Darboux formula is given by

$$\sum_{k=0}^{n} P_k(x)P_k(y)h_k = \gamma_n h_n \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}, \tag{10}$$

see [2]. Hence,

$$\sum_{k=0}^{n} P_k(x)^2 h_k = \gamma_n h_n (P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x)), \tag{11}$$

and in particular setting $x = x_0$ we get

$$P'_{n+1}(x_0) - P'_n(x_0) = \frac{H_n}{\gamma_n h_n} \tag{12}$$

with

$$H_n = \sum_{k=0}^{n} h_k \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{13}$$

Definition 1.1. If $\{P'_{n+1}(x_0) - P'_n(x_0) : n \in \mathbb{N}_0\}$ is bounded, then we call x_0 a normalizing point with bounded growth of derivatives.

Note that further on speaking about x_0 as a normalizing point of bounded growth of derivatives is the same as to speak about the boundedness of $\{\frac{H_n}{\gamma_n h_n} : n \in \mathbb{N}_0\}$.

Subsequently we deal with the case $\mathcal{S} = \operatorname{supp}\mu$ is compact which is equivalent with $(\gamma_n \alpha_{n+1})_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ are bounded sequences. Then the true interval of orthogonality is $[\min \mathcal{S}, \max \mathcal{S}]$.

Lemma 1.1. *In the case $x_0 \geq \max \mathcal{S}$ we have $\alpha_{n+1}, \gamma_n > 0$ for all $n \in \mathbb{N}_0$ and in the case $x_0 \leq \min \mathcal{S}$ we have $\alpha_{n+1}, \gamma_n < 0$ for all $n \in \mathbb{N}_0$.*

Proof. Since $\mathcal{N} \subset (\min \mathcal{S}, \max \mathcal{S})$ and the leading coefficient of all orthonormal polynomials is positive we have in the case $x_0 \geq \max \mathcal{S}$ that $p_n(x_0) > 0$ for all $n \in \mathbb{N}_0$. Whereas in the case $x_0 \leq \min \mathcal{S}$ the sign of $p_n(x_0)$ is alternating. □

On the set of complex-valued sequences there acts a linear operator $T : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ determined by the recurrence relation (4). More precisely, for $\xi \in \mathbb{C}^{\mathbb{N}_0}$ put

$$(T\xi)_n = T\xi_n = \gamma_n \xi_{n+1} + \beta_n \xi_n + \alpha_n \xi_{n-1} \quad \text{for all} \quad n \in \mathbb{N}_0, \tag{14}$$

where $\xi_{-1} = 0$. Written as tridiagonal matrix the operator T has the form

$$T = \begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots \\ 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{15}$$

Note that in our investigations T acts on the different spaces $\mathbb{C}^{\mathbb{N}_0}$, $l^1(h)$ and $l^2(h)$ which is clear from the respective context. First let us study T as an operator on

$$l^1(h) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} \mid \xi_n \mid h_n < \infty \right\} \tag{16}$$

with norm $\|\xi\|_1 = \sum_{n=0}^{\infty} \mid \xi_n \mid h_n$ for all $\xi \in l^1(h)$.

Proposition 1.1. *In the case $\mid \alpha_n \mid, \mid \beta_n \mid$ and $\mid \gamma_n \mid \leq B$ for all $n \in \mathbb{N}_0$ the operator T: $l^1(h) \to l^1(h)$ is well defined and continuous. Especially we have*

$$\sum_{n=0}^{\infty} T\xi_n h_n = x_0 \sum_{n=0}^{\infty} \xi_n h_n \quad \textit{and} \quad \|T\xi\|_1 \leq C\|\xi\|_1 \tag{17}$$

for all $\xi \in l^1(h)$, where $C = \min(3B, \mid x_0 \mid + 2B)$.

Proof. Set $\gamma_{-1} = \xi_{-1} = h_{-1} = 0$.

Applying (8) and the assumed absolute convergence of the series we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} T\xi_n h_n &= \sum_{n=0}^{\infty}(\gamma_n \xi_{n+1} + \beta_n \xi_n + \alpha_n \xi_{n-1}) h_n \\ &= \sum_{n=0}^{\infty} \alpha_{n+1}\xi_{n+1}h_{n+1} + \beta_n \xi_n h_n + \gamma_{n-1}\xi_{n-1}h_{n-1} \\ &= \sum_{n=0}^{\infty}(\alpha_n + \beta_n + \gamma_n)\xi_n h_n = x_0 \sum_{n=0}^{\infty} \xi_n h_n.\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \mid T\xi_n \mid h_n &\leq \sum_{n=0}^{\infty}(\mid \gamma_n \mid\mid \xi_{n+1} \mid + \mid \beta_n \mid\mid \xi_n \mid + \mid \alpha_n \mid\mid \xi_{n-1} \mid) h_n \\ &= \sum_{n=0}^{\infty} \mid \alpha_{n+1} \mid\mid \xi_{n+1} \mid h_{n+1} + \mid \beta_n \mid\mid \xi_n \mid h_n \\ &\quad + \mid \gamma_{n-1} \mid\mid \xi_{n-1} \mid h_{n-1} \\ &= \sum_{n=0}^{\infty}(\mid \alpha_n \mid + \mid \beta_n \mid + \mid \gamma_n \mid) \mid \xi_n \mid h_n \leq 3B \sum_{n=0}^{\infty} \mid \xi_n \mid h_n.\end{aligned}$$

At least two of the coefficients in $\mid \alpha_n \mid + \mid \beta_n \mid + \mid \gamma_n \mid$ do have the same sign. For instance, if $\mathrm{sign}\alpha_n = \mathrm{sign}\beta_n$, then $\mid \alpha_n \mid + \mid \beta_n \mid + \mid \gamma_n \mid = \mid \alpha_n + \beta_n + \gamma_n - \gamma_n \mid + \mid \gamma_n \mid \leq \mid x_0 \mid +2 \mid \gamma_n \mid \leq \mid x_0 \mid +2B$. Proceeding the same way with all the other possibilities one gets alternatively $\sum_{n=0}^{\infty} \mid T\xi_n \mid h_n \leq (\mid x_0 \mid +2B) \sum_{n=0}^{\infty} \mid \xi_n \mid h_n$, which completes the proof. □

We focus on the weighted Hilbert space

$$l^2(h) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} \mid \xi_n \mid^2 h_n < \infty \right\} \tag{18}$$

with scalar product $\langle \xi, \upsilon \rangle = \sum_{n=0}^{\infty} \xi_n \overline{\upsilon_n} h_n$ and norm $\|\xi\|_2 = \sqrt{\langle \xi, \xi \rangle}$ for all $\xi, \upsilon \in l^2(h)$.

Proposition 1.2. *In the case* $\mid \alpha_n \mid, \mid \beta_n \mid$ *and* $\mid \gamma_n \mid \leq B$ *for all* $n \in \mathbb{N}_0$ *the operator* T*:* $l^2(h) \to l^2(h)$ *is a well defined, self-adjoint and continuous operator with*

$$\|T\xi\|_2 \leq C\|\xi\|_2, \tag{19}$$

where $C = \min(3B, \mid x_0 \mid +2B)$.

Proof. Set $\gamma_{-1} = \xi_{-1} = h_{-1} = \upsilon_{-1} = 0$.

Now let $\xi \in l^2(h)$. Since

$$\begin{aligned}\mid T\xi_n \mid \leq& \sqrt{\mid \gamma_n \mid}\sqrt{\mid \gamma_n \mid} \mid \xi_{n+1} \mid + \sqrt{\mid \beta_n \mid}\sqrt{\mid \beta_n \mid} \mid \xi_n \mid \\ &+ \sqrt{\mid \alpha_n \mid}\sqrt{\mid \alpha_n \mid} \mid \xi_{n-1} \mid\end{aligned}$$

the Cauchy–Schwarz inequality implies

$$\mid T\xi_n \mid^2 \leq (\mid \gamma_n \mid + \mid \beta_n \mid + \mid \alpha_n \mid)(\mid \gamma_n \mid\mid \xi_{n+1} \mid^2 + \mid \beta_n \mid\mid \xi_n \mid^2 + \mid \alpha_n \mid\mid \xi_{n-1} \mid^2).$$

Therefore, proceeding like in the proof of Proposition 1.1

$$\begin{aligned}\sum_{n=0}^{\infty} \mid T\xi_n \mid^2 h_n &\leq C\sum_{n=0}^{\infty}(\mid \gamma_n \mid\mid \xi_{n+1} \mid^2 + \mid \beta_n \mid\mid \xi_n \mid^2 + \mid \alpha_n \mid\mid \xi_{n-1} \mid^2)h_n \\ &= C\sum_{n=0}^{\infty} \mid \alpha_{n+1} \mid\mid \xi_{n+1} \mid^2 h_{n+1} + \mid \beta_n \mid\mid \xi_n \mid^2 h_n \\ &\quad + \mid \gamma_{n-1} \mid\mid \xi_{n-1} \mid^2 h_{n-1} \\ &= C\sum_{n=0}^{\infty} (\mid \alpha_n \mid + \mid \beta_n \mid + \mid \gamma_n \mid) \mid \xi_n \mid^2 h_n \leq C^2 \sum_{n=0}^{\infty} \mid \xi_n \mid^2 h_n,\end{aligned}$$

which implies $\|T\xi\|_2 \leq C\|\xi\|_2$, where $C = \min(3B, \mid x_0 \mid + 2B)$.

For arbitrary $\xi, \upsilon \in l^2(h)$ one gets due to the absolute convergence

$$\begin{aligned}\langle T\xi, \upsilon\rangle &= \sum_{n=0}^{\infty}(\gamma_n\xi_{n+1} + \beta_n\xi_n + \alpha_n\xi_{n-1})\overline{\upsilon_n}h_n \\ &= \sum_{n=0}^{\infty} \xi_n(\gamma_{n-1}\overline{\upsilon_{n-1}}h_{n-1} + \beta_n\overline{\upsilon_n}h_n + \alpha_{n+1}\overline{\upsilon_{n+1}}h_{n+1}) \\ &= \sum_{n=0}^{\infty} \xi_n(\alpha_n\overline{\upsilon_{n-1}}h_n + \beta_n\overline{\upsilon_n}h_n + \gamma_n\overline{\upsilon_{n+1}}h_n) = \langle \xi, T\upsilon\rangle.\end{aligned}$$

□

Corollary 1.1. *In the case* $\mid \alpha_n \mid, \mid \beta_n \mid$ *and* $\mid \gamma_n \mid \leq B$ *for all* $n \in \mathbb{N}_0$ *the spectrum* $\sigma(T)$ *is a subset of* $[-C, C]$, *where* $C = \min(3B, \mid x_0 \mid + 2B)$.

The numerical range of T is the set

$$W(T) = \left\{\langle T\xi, \xi\rangle : \xi \in l^2(h), \|\xi\|_2 = 1\right\}. \tag{20}$$

Since T is self-adjoint we have

$$\{m(T), M(T)\} \subseteq \sigma(T) \subseteq \mathrm{co}(\sigma(T)) \subseteq \overline{W(T)} = [m(T), M(T)], \tag{21}$$

where $\mathrm{co}(\sigma(T))$ is the convex hull of $\sigma(T)$, $m(T) = \inf W(T)$ and $M(T) = \sup W(T)$, see [5, Intro]. Moreover, $\|T\| = \max(\mid m(T) \mid, \mid M(T) \mid)$.

Proposition 1.3. *In the case* $\mid \alpha_n \mid, \mid \beta_n \mid$ *and* $\mid \gamma_n \mid \leq B$ *for all* $n \in \mathbb{N}_0$ *one gets*

$$\langle (x_0 id - T)\xi, \xi\rangle = \sum_{n=0}^{\infty} \gamma_n \mid \xi_n - \xi_{n+1} \mid^2 h_n \quad \textit{for all} \quad \xi \in l^2(h). \tag{22}$$

Proof. Set $\gamma_{-1} = \xi_{-1} = h_{-1} = 0$. Using (8) and the absolute convergence of the series one gets for an arbitrary $\xi \in l^2(h)$ that

$$\begin{aligned}\sum_{n=0}^{\infty}(x_0\xi_n - T\xi_n)\overline{\xi_n}h_n &= \sum_{n=0}^{\infty}((\alpha_n + \beta_n + \gamma_n)\xi_n - \alpha_n\xi_{n-1} \\ &\quad - \beta_n\xi_n - \gamma_n\xi_{n+1})\overline{\xi_n}h_n \\ &= \sum_{n=0}^{\infty}(\gamma_n\xi_n\overline{\xi_n} - \gamma_n\xi_{n+1}\overline{\xi_n})h_n \\ &\quad + (\gamma_{n-1}\xi_n\overline{\xi_n} - \gamma_{n-1}\xi_{n-1}\overline{\xi_n})h_{n-1} \\ &= \sum_{n=0}^{\infty}\gamma_n(\xi_n\overline{\xi_n} - \xi_{n+1}\overline{\xi_n} + \xi_{n+1}\overline{\xi_{n+1}} - \xi_n\overline{\xi_{n+1}})h_n \\ &= \sum_{n=0}^{\infty}\gamma_n \mid \xi_n - \xi_{n+1} \mid^2 h_n.\end{aligned}$$

□

Lemma 1.2. *The following statements apply.*

(i) $\sum_{n=0}^{\infty}\gamma_n \mid \xi_n - \xi_{n+1} \mid^2 h_n \geq 0$ *for all* $\xi \in l^2(h)$ *with* $\|\xi\|_2 = 1$ *if and only if* $\gamma_n > 0$ *for all* $n \in \mathbb{N}_0$.

(ii) $\sum_{n=0}^{\infty}\gamma_n \mid \xi_n - \xi_{n+1} \mid^2 h_n \leq 0$ *for all* $\xi \in l^2(h)$ *with* $\|\xi\|_2 = 1$ *if and only if* $\gamma_n < 0$ *for all* $n \in \mathbb{N}_0$.

Proof. If $\gamma_n > 0$ for all $n \in \mathbb{N}_0$ then $\sum_{n=0}^{\infty}\gamma_n \mid \xi_n - \xi_{n+1} \mid^2 h_n \geq 0$ for all $\xi \in l^2(h)$. In the case we have not $\gamma_n > 0$ for all $n \in \mathbb{N}_0$ there is an index $m \in \mathbb{N}_0$ such that $\gamma_m < 0$ and $\gamma_n > 0$ for all $n \in \{0, \ldots, m-1\}$. Define $\zeta \in l^2(h)$ by $\zeta_n = (\sum_{k=0}^{m} h_k)^{-1/2}$ for all $n \in \{0, \ldots, m\}$ and $\zeta_n = 0$ for all $n \in \{m+1, m+2 \ldots\}$. Then $\|\zeta\|_2 = 1$ and $\sum_{n=0}^{\infty}\gamma_n \mid \zeta_n - \zeta_{n+1} \mid^2 h_n = \gamma_m \mid \zeta_m \mid^2 h_m < 0$.

The second statement is shown quite analogue. □

Corollary 1.2. *If* $\mid \alpha_n \mid, \mid \beta_n \mid, \mid \gamma_n \mid \leq B$ *for all* $n \in \mathbb{N}_0$ *and* $C = \min(3B, \mid x_0 \mid + 2B)$, *then the following statements apply.*

(i) *If* $\gamma_n > 0$ *for all* $n \in \mathbb{N}_0$, *then* $\overline{W(T)} \subseteq [-C, x_0]$. *In particular,* $\sigma(T) \subseteq [-C, x_0]$.

(ii) *If* $\gamma_n < 0$ *for all* $n \in \mathbb{N}_0$, *then* $\overline{W(T)} \subseteq [x_0, C]$. *In particular,* $\sigma(T) \subseteq [x_0, C]$.

(iii) *If there exist* $k, l \in \mathbb{N}_0$ *with* $\gamma_k\gamma_l < 0$, *then* $x_0 \in (\min \mathcal{S}, \max \mathcal{S})$.

Note that in the following $L^2(\mathbb{R}, \mu)$ is as usual a set of equivalence classes and a function used in this context represents an equivalence class. This is also expressed by using the formulation 'for μ-almost all $x \in \mathbb{R}$'.

Define $\epsilon^{(k)} \in l^2(h)$ by

$$\epsilon_n^{(k)} = \frac{\delta_{k,n}}{h_k} \quad \text{for all} \quad n, k \in \mathbb{N}_0. \tag{23}$$

Then obviously

$$\|\epsilon^{(k)}\|_2^2 = \frac{1}{h_k} = \int P_k^2 d\mu \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{24}$$

Extending the map $\epsilon^{(k)} \mapsto P_k$ linearly to the linear span of $\{\epsilon^{(k)} : k \in \mathbb{N}_0\}$ and finally to the closure of the linear span we get the so-called Plancherel isomorphism

$$\mathcal{P} : l^2(h) \to L^2(\mathbb{R}, \mu),$$

which is an isometric isomorphism from $l^2(h)$ onto $L^2(\mathbb{R}, \mu)$. It is completely determined by

$$\mathcal{P}(\epsilon^{(k)}) = P_k \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Note that

$$T\epsilon^{(k)} = \alpha_k \epsilon^{(k-1)} + \beta_k \epsilon^{(k)} + \gamma_k \epsilon^{(k+1)} \quad \text{for all} \quad k \in \mathbb{N}_0, \tag{25}$$

where $\epsilon_n^{(-1)} = 0$ for all $n \in \mathbb{N}_0$. Now we define an operator M on $L^2(\mathbb{R}, \mu)$ by

$$M(f) = \mathcal{P} \circ T \circ \mathcal{P}^{-1}(f) \quad \text{for all} \quad f \in L^2(\mathbb{R}, \mu), \tag{26}$$

where $\mathcal{P}^{-1}$ denotes the inverse operator of $\mathcal{P}$. Then $M \in B(L^2(\mathbb{R}, \mu))$ with $\|M\| \leq \min(3B, \mid x_0 \mid + 2B)$. Taking into account the three-term recurrence relation (4) we deduce that

$$M(P_k)(x) = \mathcal{P}(T\epsilon^{(k)})(x) = \mathcal{P}(\alpha_k \epsilon^{(k-1)} + \beta_k \epsilon^{(k)} + \gamma_k \epsilon^{(k+1)})(x) = xP_k(x) \tag{27}$$

for μ-almost all $x \in \mathbb{R}$ and for all $k \in \mathbb{N}_0$. If g is a function in the linear span of $\{P_k : k \in \mathbb{N}_0\}$, then the linearity of M yields

$$M(g)(x) = xg(x) \quad \text{for} \quad \mu\text{-almost all } x \in \mathbb{R}. \tag{28}$$

Since M is bounded and the closure of the linear span of $\{P_k : k \in \mathbb{N}_0\}$ is $L^2(\mathbb{R}, \mu)$ we get by standard arguments that

$$M(f)(x) = xf(x) \text{ for } \mu\text{-almost all } x \in \mathbb{R} \text{ and for all } f \in L^2(\mathbb{R}, \mu). \tag{29}$$

By [4, Definition 2.61 and Corollary 4.24] the spectrum $\sigma(M)$ is exactly the essential range

$$\mathcal{R} = \{\lambda \in \mathbb{R} : \mu(\{x \in \mathbb{R} : \mid x - \lambda \mid < \epsilon\}) > 0 \text{ for all } \epsilon > 0\}, \tag{30}$$

Obviously $\mathcal{R} = \text{supp}\mu$ and $\sigma(M) = \sigma(T)$. Hence, we can add to Corollary 1.2 the following result.

Corollary 1.3. *For orthogonal polynomials $(P_n)_{n \in \mathbb{N}_0}$ which are defined by (4) with $\mid \alpha_n \mid$, $\mid \beta_n \mid$ and $\mid \gamma_n \mid \leq B$ for all $n \in \mathbb{N}_0$ we have*

$$\mathcal{S} = supp\mu = \sigma(T). \tag{31}$$

2. A characterization of $x_0 \notin \mathcal{S}$

In the whole section we assume that $|\alpha_n|$, $|\beta_n|$ and $|\gamma_n| \leq B$ for all $n \in \mathbb{N}_0$.

The main result of this paper will be a necessary and sufficient condition for $x_0 \in \mathcal{S}$. Moreover, in the case of $x_0 \notin \mathcal{S}$ we will present an explicit form of the inverse $(x_0\mathrm{id} - T)^{-1}$, which is based on a weighted Cesàro operator $C \in B(l^2(h))$.

Define $C\eta = ((C\eta)_n)_{n\in\mathbb{N}_0} = (C\eta_n)_{n\in\mathbb{N}_0}$ by

$$C\eta_n = \frac{1}{H_n}\sum_{k=0}^{n} \eta_k h_k \quad \text{for all} \quad \eta \in l^2(h). \tag{32}$$

Then C is a bounded linear operator on $l^2(h)$ with $\|C\| \leq 2$, see [3, Theorem A]. It is straightforward to show that the adjoint operator $C^* \in B(l^2(h))$ is defined by

$$C^*\eta_n = \sum_{k=n}^{\infty} \eta_k \frac{h_k}{H_k} \quad \text{for all} \quad \eta \in l^2(h). \tag{33}$$

Theorem 2.1. *If $x_0 \notin \mathcal{S} = \sigma(T)$, then x_0 is a normalizing point with bounded growth of derivatives.*

Proof. Given $n \in \mathbb{N}_0$ denote by $\chi^{(n)}$ the sequence with $\chi_k^{(n)} = 1$ for $k \in \{0, \ldots, n\}$ and $\chi_k^{(n)} = 0$ for $k \in \{n+1, n+2, \ldots\}$. An easy computation shows that

$$\begin{aligned}(x_0\mathrm{id} - T)\chi_k^{(n)} &= 0 \quad \text{for all} \quad k \in \mathbb{N}_0 \backslash \{n, n+1\},\\ (x_0\mathrm{id} - T)\chi_n^{(n)} &= \gamma_n, \quad \text{and}\\ (x_0\mathrm{id} - T)\chi_{n+1}^{(n)} &= -\alpha_{n+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\|(x_0\mathrm{id} - T)\chi^{(n)}\|_2^2 &= \gamma_n^2 h_n + \alpha_{n+1}^2 h_{n+1} = \gamma_n(\gamma_n + \alpha_{n+1})h_n\\ &= |\gamma_n||\gamma_n + \alpha_{n+1}|h_n.\end{aligned}$$

Since $x_0 \notin \sigma(T)$, there exists $A = (x_0\mathrm{id} - T)^{-1} \in B(l^2(h))$. Then

$$\begin{aligned}\|A \circ (x_0\mathrm{id} - T)\chi^{(n)}\|_2^2 &= \|\chi^{(n)}\|_2^2 = \sum_{k=0}^{n} h_k = H_n, \quad \text{and}\\ \|A \circ (x_0\mathrm{id} - T)\chi^{(n)}\|_2^2 &\leq \|A\|^2\|(x_0\mathrm{id} - T)\chi^{(n)})\|_2^2\\ &= \|A\|^2 \, |\gamma_n| \, h_n \, |\gamma_n + \alpha_{n+1}|\\ &\leq 2B\|A\|^2 \, |\gamma_n| \, h_n,\end{aligned}$$

which implies

$$H_n \leq 2B\|A\|^2 \, |\gamma_n| \, h_n \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Therefore, $\left\{\frac{H_n}{\gamma_n h_n} : n \in \mathbb{N}_0\right\}$ is bounded. □

In order to prove the converse implication we start with determining a sequence $\omega = (\omega_n)_{n\in\mathbb{N}_0}$ such that $(x_0\mathrm{id} - T)(\omega) = \epsilon^{(0)}$. Note that in the following lemma the operator T acts on $\mathbb{C}^{\mathbb{N}_0}$.

Lemma 2.1. *A sequence* $\omega = (\omega_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ *satisfies* $(x_0 id - T)(\omega) = \epsilon^{(0)}$ *if and only if*

$$\omega_{n+1} = \omega_0 - \sum_{k=0}^{n} \frac{1}{\gamma_k h_k} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{34}$$

Proof. We have $((x_0\mathrm{id} - T)\omega)_0 = 1/h_0 = 1$ if and only if $\omega_0 - \omega_1 = \frac{1}{\gamma_0}$. For $n \geq 1$ we see that $((x_0\mathrm{id} - T)\omega)_n = \omega_n - (\gamma_n\omega_{n+1} + \beta_n\omega_n + \alpha_n\omega_{n-1}) = 0$ if and only if $\gamma_n(\omega_{n+1} - \omega_n) = \alpha_n(\omega_n - \omega_{n-1})$. Now, by iteration we get

$$\omega_{n+1} - \omega_n = \frac{\alpha_n}{\gamma_n}(\omega_n - \omega_{n-1}) = \frac{\alpha_n\alpha_{n-1}\cdots\alpha_1}{\gamma_n\gamma_{n-1}\cdots\gamma_1}\frac{-1}{\gamma_0} = \frac{-1}{\gamma_n h_n}.$$

□

Next we investigate under which assumptions a sequence $\omega = (\omega_n)_{n\in\mathbb{N}_0}$ of Lemma 2.1 is a member of $l^2(h)$.

Lemma 2.2. *If* x_0 *is a normalizing point with bounded growth of derivatives, then*

$$\sum_{k=0}^{\infty} \frac{1}{\mid \gamma_k \mid h_k} < \infty, \tag{35}$$

and consequently the series $\sum_{k=0}^{\infty} \frac{1}{\gamma_k h_k}$ *is convergent.*

Proof. Due to the assumption there exists a $D > 0$ with

$$\sum_{k=0}^{n} \left(\frac{1}{\gamma_k h_k}\right)^2 h_k \leq D \sum_{k=0}^{n} \left(\frac{1}{H_k}\right)^2 h_k \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Since $C\epsilon^{(0)} = \left(\frac{1}{H_n}\right)_{n\in\mathbb{N}_0} \in l^2(h)$, we have $\left(\frac{1}{\gamma_n h_n}\right)_{n\in\mathbb{N}_0} \in l^2(h)$, that is $\sum_{k=0}^{\infty} \frac{1}{\gamma_k^2 h_k} < \infty$. Finally, $\gamma_k^2 \leq B \mid \gamma_k \mid$ yields $\sum_{k=0}^{\infty} \frac{1}{|\gamma_k| h_k} < \infty$, which implies the series $\sum_{k=0}^{\infty} \frac{1}{\gamma_k h_k}$ is convergent. □

Now with respect to Lemma 2.1, if the series $\sum_{k=0}^{\infty} \frac{1}{\gamma_k h_k}$ is convergent, then the sequence $\omega = (\omega_n)_{n\in\mathbb{N}_0}$ is defined by

$$\omega_n = \sum_{k=n}^{\infty} \frac{1}{\gamma_k h_k} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{36}$$

In order to prove that $\omega \in l^2(h)$ whenever $\left\{\frac{H_n}{\gamma_n h_n} : n \in \mathbb{N}_0\right\}$ is bounded, we use the adjoint weighted Cesàro operator $C^* \in B(l^2(h))$. Define a sequence $\eta = (\eta_n)_{n\in\mathbb{N}_0}$ by

$$\eta_n = \frac{H_n}{\gamma_n h_n^2} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{37}$$

Lemma 2.3. *If* x_0 *is a normalizing point with bounded growth of derivatives, then* $\eta \in l^2(h)$.

Proof. We have $\mid \eta_n \mid \leq D\frac{1}{h_n}$ for all $n \in \mathbb{N}_0$. Hence, we have to show that $\left(\frac{1}{h_n}\right)_{n\in\mathbb{N}_0} \in l^2(h)$. According to Lemma 2.2 it follows

$$\sum_{k=0}^{\infty} \frac{1}{h_k^2} h_k \leq B \sum_{k=0}^{\infty} \frac{1}{\mid \gamma_k \mid h_k} < \infty.$$

□

Since $C^*\eta = \omega$, Lemmas 2.3 and 2.1 yield the following proposition.

Proposition 2.1. *If x_0 is a normalizing point with bounded growth of derivatives, then $\omega \in l^2(h)$ (defined by (36)) satisfies $(x_0 id - T)\omega = \epsilon^{(0)}$.*

Assuming that x_0 is a normalizing point with bounded growth of derivatives our next goal is to find sequences $\omega^{(m)} \in l^2(h)$ with $(x_0\mathrm{id} - T)\omega^{(m)} = \epsilon^{(m)}$ for all $m \in \mathbb{N}$.

To that end, we introduce a sequence of operators $S_m \in B(l^2(h))$ by setting

$$S_{m+1} = \frac{1}{\gamma_m}\left(T \circ S_m - \beta_m S_m - \alpha_m S_{m-1}\right) \quad \text{for all} \quad m \in \mathbb{N}_0, \tag{38}$$

where $S_{-1} = 0$ and $S_0 = \mathrm{id}$.

Proposition 2.2. *The following two statements apply.*

(i)

$$S_m \epsilon^{(0)} = \epsilon^{(m)} \quad \textit{for all} \quad m \in \mathbb{N}_0. \tag{39}$$

(ii)

$$(S_m\omega)_k = \begin{cases} \omega_m & \text{if} \quad k = 0, \ldots, m, \\ \omega_k & \text{if} \quad k = m+1, m+2, \ldots \end{cases} \quad \textit{for all}\, m \in \mathbb{N}_0. \tag{40}$$

Proof. In any case the proof is done by induction.

(i): By trivial means we have $S_0\epsilon^{(0)} = \epsilon^{(0)}$. Since $S_1 = \frac{1}{\gamma_0}(T - \beta_0\mathrm{id})$ we have $(S_1\epsilon^{(0)})_0 = \frac{1}{\gamma_0}(\beta_0 - \beta_0) = 0$, $(S_1\epsilon^{(0)})_1 = \frac{\alpha_1}{\gamma_0}\frac{1}{h_0} = \frac{1}{h_1}$ and $(S_1\epsilon^{(0)})_k = 0$ for all $k \geq 2$. Therefore, $S_1\epsilon^{(0)} = \epsilon^{(1)}$.

Assume that $S_m\epsilon^{(0)} = \epsilon^{(m)}$ and $S_{m-1}\epsilon^{(0)} = \epsilon^{(m-1)}$ for $m \in \mathbb{N}_0$ is already shown. Then

$$\begin{aligned} S_{m+1}\epsilon^{(0)} &= \frac{1}{\gamma_m}\left(T \circ S_m - \beta_m S_m - \alpha_m S_{m-1}\right)\epsilon^{(0)} \\ &= \frac{1}{\gamma_m}\left(T\epsilon^{(m)} - \beta_m\epsilon^{(m)} - \alpha_m\epsilon^{(m-1)}\right) \\ &= \frac{1}{\gamma_m}\left(\alpha_m\epsilon^{(m-1)} + \beta_m\epsilon^{(m)} + \gamma_m\epsilon^{(m+1)} - \beta_m\epsilon^{(m)} - \alpha_m\epsilon^{(m-1)}\right) \\ &= \frac{1}{\gamma_m}\left(\gamma_m\epsilon^{(m+1)}\right) = \epsilon^{(m+1)}. \end{aligned}$$

(ii): By trivial means we have $(S_0\omega)_k = \omega_k$ for all $k \in \mathbb{N}_0$. Moreover, since $S_1 = \frac{1}{\gamma_0}(T - \beta_0 id)$, $T\omega = x_0 id - \epsilon^{(0)}$ and $x_0 = \beta_0 + \gamma_0$ we get

$$S_1\omega = \frac{1}{\gamma_0}T\omega - \frac{\beta_0}{\gamma_0}\omega = \frac{1}{\gamma_0}((\beta_0 + \gamma_0)\omega - \epsilon^{(0)}) - \frac{\beta_0}{\gamma_0}\omega = \omega - \frac{\epsilon^{(0)}}{\gamma_0}.$$

Hence, $(S_1\omega)_0 = \omega_1$, $(S_1\omega)_1 = \omega_1$, and $(S_1\omega)_k = \omega_k$ for all $k \geq 2$.

Assume again that the statement is already shown for $m \in \mathbb{N}$ and $m-1$. Then for $k = 0, \ldots, m-1$ we have

$$\begin{aligned}(S_{m+1}\omega)_k &= \frac{1}{\gamma_m}((T \circ S_m\omega)_k - \beta_m\omega_m - \alpha_m\omega_{m-1})\\ &= \frac{1}{\gamma_m}(x_0\omega_m - \beta_m\omega_m - \alpha_m\omega_{m-1}) = \omega_m + \frac{\alpha_m}{\gamma_m}(\omega_m - \omega_{m-1})\\ &= \omega_m - \frac{\alpha_m}{\gamma_m}\frac{1}{\gamma_{m-1}h_{m-1}} = \omega_m - \frac{1}{\gamma_m h_m} = \omega_{m+1}.\end{aligned}$$

For $k = m$ it follows

$$(S_{m+1}\omega)_k = \frac{1}{\gamma_m}(\alpha_m\omega_m + \beta_m\omega_m + \gamma_m\omega_{m+1} - \beta_m\omega_m - \alpha_m\omega_m) = \omega_{m+1}.$$

Finally, for $k = m+1, m+2, \ldots$ we have

$$\begin{aligned}(S_{m+1}\omega)_k &= \frac{1}{\gamma_m}\left(\alpha_k\omega_{k-1} + \beta_k\omega_k + \gamma_k\omega_{k+1} - \beta_m\omega_k - \alpha_m\omega_k\right)\\ &= \frac{1}{\gamma_m}\left(\gamma_k\left(\omega_k - \frac{1}{\gamma_k h_k}\right) + \beta_k\omega_k\right.\\ &\quad \left. + \alpha_k\left(\omega_k + \frac{1}{\gamma_{k-1}h_{k-1}}\right) - \beta_m\omega_k - \alpha_m\omega_k\right)\\ &= \frac{1}{\gamma_m}\left(x_0\omega_k - \frac{1}{h_k} + \frac{1}{h_k} - \beta_m\omega_k - \alpha_m\omega_k\right) = \omega_k.\end{aligned}$$

□

Now our goal is met by setting $\omega^{(m)} = S_m\omega$ for all $m \in \mathbb{N}_0$.

Proposition 2.3. *If x_0 is a normalizing point with bounded growth of derivatives, then $S_m\omega \in l^2(h)$ satisfies $(x_0 id - T)S_m\omega = \epsilon^{(m)}$ for all $m \in \mathbb{N}_0$.*

Proof. Obviously S_m commutes with $x_0\mathrm{id} - T$. Hence,

$$(x_0\mathrm{id} - T)S_m\omega = S_m(x_0\mathrm{id} - T)\omega = S_m\epsilon^{(0)} = \epsilon^{(m)} \quad \text{for all} \quad m \in \mathbb{N}_0.$$

□

For $m \in \mathbb{N}_0$ define the sequence $\eta^{(m)}$ by

$$\eta_k^{(m)} = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, m-1, \\ \frac{H_k}{\gamma_k h_k^2} & \text{if} \quad k = m, m+1, \ldots \end{cases}. \tag{41}$$

Note that $\eta^{(0)} = \eta$. If $\left\{\frac{H_n}{\gamma_n h_n} : n \in \mathbb{N}_0\right\}$ is bounded, then according to Lemma 2.3 we know that $\eta^{(m)} \in l^2(h)$ for all $m \in \mathbb{N}_0$. Moreover,

$$C^*\eta_n^{(m)} = \begin{cases} \sum_{k=m}^{\infty} \frac{\eta_k^{(m)} h_k}{H_k} = \sum_{k=m}^{\infty} \frac{1}{\gamma_k h_k} = \omega_m & \text{if} \quad n \leq m, \\ \sum_{k=n}^{\infty} \frac{1}{\gamma_k h_k} = \omega_n & \text{if} \quad n > m \end{cases}. \tag{42}$$

By Proposition 2.2(ii) we have $C^*\eta^{(m)} = S_m\omega$ for all $m \in \mathbb{N}_0$. Now we can combine the results above to determine the inverse operator of $x_0\mathrm{id} - T$. Define a sequence $\varphi = (\varphi_n)_{n\in\mathbb{N}_0}$ by

$$\varphi_n = \frac{H_n^2}{\gamma_n h_n^2} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{43}$$

Note that $\frac{H_n}{|\gamma_n| h_n} \leq D$ for all $n \in \mathbb{N}_0$ implies $| \varphi_n | \leq D^2 B$ for all $n \in \mathbb{N}_0$.

The multiplication with $\varphi \in l^\infty$ defines a bounded operator M_φ on $l^2(h)$, where $M_\varphi(\xi)_n = \varphi_n \xi_n$ for all $\xi \in l^2(h)$, $n \in \mathbb{N}_0$.

Theorem 2.2. *If x_0 is a normalizing point with bounded growth of derivatives, then $C^* \circ M_\varphi \circ C$ is the inverse of the operator $x_0\mathrm{id} - T$, where φ is the sequence in (43).*

Proof. Let $m \in \mathbb{N}_0$. We know that $C\epsilon_k^{(m)} = 0$ for all $k = 0, \ldots, m-1$ and $C\epsilon_k^{(m)} = \frac{1}{H_k}$ for all $k = m, m+1, \ldots$. Hence, $M_\varphi \circ C\epsilon^{(m)} = \eta^{(m)}$ and $C^* \circ M_\varphi \circ C\epsilon^{(m)} = S_m\omega$. In particular

$$(x_0\mathrm{id} - T) \circ (C^* \circ M_\varphi \circ C)\epsilon^{(m)} = \epsilon^{(m)} \quad \text{for all} \quad m \in \mathbb{N}_0.$$

Furthermore, we obtain

$$\begin{aligned} &C^* \circ M_\varphi \circ C \circ (x_0\mathrm{id} - T)\epsilon^{(m)} \\ &\quad = (C^* \circ M_\varphi \circ C)(x_0\epsilon^{(m)} - (\gamma_m \epsilon^{(m+1)} + \beta_m \epsilon^{(m)} + \alpha_m \epsilon^{(m-1)})) \\ &\quad = x_0 S_m\omega - (\gamma_m S_{m+1}\omega + \beta_m S_m\omega + \alpha_m S_{m-1}\omega) \\ &\quad = x_0 S_m\omega - T \circ S_m\omega = (x_0\mathrm{id} - T)S_m\omega = \epsilon^{(m)} \quad \text{for all} \quad m \in \mathbb{N}_0. \end{aligned}$$

Therefore,

$$C^* \circ M_\varphi \circ C \circ (x_0\mathrm{id} - T) = \mathrm{id} \ = (x_0\mathrm{id} - T) \circ C^* \circ M_\varphi \circ C,$$

i.e. $(x_0\mathrm{id} - T)^{-1} = C^* \circ M_\varphi \circ C$. □

Summing up the results we gain the following theorem.

Theorem 2.3. *$x_0 \notin \mathrm{supp}\mu = \sigma(T)$ if and only if x_0 is a normalizing point with bounded growth of derivatives.*

Finally, we want to show the relationship of the results here with [1, Theorem 2.3]. For this we use the terms $\mathcal{A}$, A and $\mathcal{D}(A)$ with the same meaning as in [1]. Let

$$\mathcal{A} = \begin{pmatrix} \beta_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \lambda_0 & \beta_1 & \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_1 & \beta_2 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_2 & \beta_3 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{44}$$

where $(\beta_n)_{n\in\mathbb{N}_0}$ and $(\lambda_n)_{n\in\mathbb{N}_0}$ are the coefficients of (1). Then $\mathcal{A}$ can be regarded as a linear operator $\mathcal{A} : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$, $\xi \mapsto \mathcal{A}\xi = (\mathcal{A}\xi_n)_{n\in\mathbb{N}_0}$, where

$$\mathcal{A}\xi_n = \lambda_{n-1}\xi_{n-1} + \beta_n\xi_n + \lambda_n\xi_{n+1} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{45}$$

Note that $\lambda_{-1} = 0$ and ξ_{-1} can be chosen arbitrary.

Moreover, let $l^2 = l^2(h)$ with $h_n = 1$ for all $n \in \mathbb{N}_0$, and

$$c_{00} = \{\xi \in \mathbb{C}^{\mathbb{N}_0} : \#\{n \in \mathbb{N}_0 : \xi_n \neq 0\} < \infty\}. \tag{46}$$

Of course, $(c_{00}, \|\|_2)$ is a subspace of the Hilbert space $(l^2, \|\|_2)$. As mentioned in [1] the linear operator

$$\mathcal{A} : c_{00} \to l^2, \quad \xi \mapsto \mathcal{A}\xi \tag{47}$$

is closable and its closure is given by

$$A : \mathcal{D}(A) \to l^2, \quad \xi \mapsto A\xi, \quad \text{where} \tag{48}$$

$$\mathcal{D}(A) = \left\{\xi \in l^2 : \exists (\xi^{(k)})_{k\in\mathbb{N}_0} \subset c_{00} : \lim_{k\to\infty} \xi^{(k)} = \xi \wedge \lim_{k\to\infty} \mathcal{A}\xi^{(k)} \text{ exists}\right\} \tag{49}$$

$$\text{and} \quad A\xi = \lim_{k\to\infty} \mathcal{A}\xi^{(k)}. \tag{50}$$

According to [1, Theorem 2.3.], the following statements hold true.
If $x_0 \in \Omega(A) = \mathbb{R}\backslash\sigma(A)$ *then*

$$\sup_{n\geq 0} \frac{\sum_{k=0}^{n} p_k^2(x_0)}{\lambda_n^2(p_n^2(x_0) + p_{n+1}^2(x_0))} < \infty. \tag{51}$$

Provided A *is bounded, also the converse is true.*

Note that the assumptions made at the beginning of Sect. 2 imply the boundedness of $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$. One can show that the boundedness of $(\lambda_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ imply that A is a bounded operator.

The relationship with our result can be derived from

$$\begin{aligned}
&(\alpha_{n+1} + \gamma_n) \frac{\sum_{k=0}^{n} p_k^2(x_0)}{\lambda_n^2(p_n^2(x_0) + p_{n+1}^2(x_0))} \\
&\quad = \alpha_{n+1} \frac{p_n^2(x_0) + p_{n+1}^2(x_0)}{p_n^2(x_0)} \frac{\sum_{k=0}^{n} p_k^2(x_0)}{\lambda_n^2(p_n^2(x_0) + p_{n+1}^2(x_0))} \\
&\quad = \frac{\sum_{k=0}^{n} p_k^2(x_0)}{\frac{\lambda_n^2}{\alpha_{n+1}} p_n^2(x_0)} = \frac{\sum_{k=0}^{n} p_k^2(x_0)}{\gamma_n p_n^2(x_0)} = \frac{H_n}{\gamma_n h_n}
\end{aligned} \tag{52}$$

and $\operatorname{sign} \alpha_{n+1} = \operatorname{sign} \gamma_n$.

In [1] there is no formula of the inverse as in Theorem 2.2 but there is no restriction $x_0 \in \mathbb{R}\backslash\mathcal{N}$.

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