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ORIGINAL RESEARCH ARTICLE

Three Conics Determine a Cubic

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Abstract
Given a cubic K in the real projective plane. Then for each point P there is a conic C_P associated to P. The conic C_P is called the *polar conic* of K with respect to the *pole* P. We investigate the situation when three conics C_1, C_2, and C_3 are polar conics of K with respect to the poles P_1, P_2, and P_3, respectively. In particular, we give an elementary proof—without using any results from algebraic geometry—that any three conics C_1, C_2, C_3 in general position, satisfying only a non-degeneracy condition, determine a unique cubic K and three points P_1, P_2, P_3, such that C_1, C_2, C_3 are polar conics of K with respect to the three poles P_1, P_2, P_3. This can be seen as a higher degree variant of von Staudt's Theorem.

Keywords Pencils · Conics · Polars · Polar conics of cubics

Mathematics Subject Classification 51A05 · 51A20

1 Introduction

This work proceeds the article [3], in which it is shown that two given conics C_0 and C_1 can always be considered as polar conics of a cubic K curve with respect to corresponding poles P_0 and P_1. However, even though P_1 is determined by P_0, neither the cubic nor the point P_0 is determined by the two conics C_0 and C_1. This changes if we start with three conics C_1, C_2, C_3 in general position. In this situation, there is a

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unique cubic K and uniquely determined points P_1, P_2, P_3 such that C_1, C_2, C_3 are the polar conics of K with respect to the three poles P_1, P_2, P_3. Instead of formulating the result in the abstract language of algebraic geometry, we propose an elementary and explicit approach that shows a concrete method to calculate the resulting cubic curve K and the poles P_1, P_2, P_3, starting from the three given conic sections C_1, C_2, C_3. In particular, the condition for uniqueness and existence becomes visible in this way.

Our result can be seen as a higher degree variant of von Staudt's Theorem which says that given three lines ℓ_1, ℓ_2, ℓ_3 and three points P_1, P_2, P_3 in perspective position determine a unique conic C such that the points P_i are the poles of the lines ℓ_i with respect to C (see [7, p. 135, Sect. 241]).

The setting in which we work is the same as in [3], but for the sake of completeness we recall the notation and terminology. We will work in the real projective plane $\mathbb{RP}^2 = \mathbb{R}^3\backslash\{0\}/\sim$, where $X \sim Y \in \mathbb{R}^3\backslash\{0\}$ are equivalent, if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by capital letters, the components with the corresponding small letter, and the equivalence class by $[X]$. However, since we mostly work with representatives, we often omit the square brackets in the notation. A non-degenerate conic in this setting is then given by an equation of the form $\langle X, AX\rangle = 0$, where A is a regular, real, symmetric 3×3-matrix with mixed signature, i.e., A has eigenvalues of both signs, and $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^3$.

Let f be a non-constant homogeneous polynomial in the variables x_1, x_2, x_3 of degree n. Then f defines a projective algebraic curve

$$C_f := \{[X] \in \mathbb{RP}^2 \colon f(X) = 0\},$$

of degree n. For a point $P \in \mathbb{RP}^2$,

$$Pf(X) := \langle P, \nabla f(X)\rangle,$$

is also a homogeneous polynomial in the variables x_1, x_2, x_3. If the homogeneous polynomial f is of degree n, then C_{Pf} is an algebraic curve of degree $n-1$. The curve C_{Pf} is called the *polar curve* of C_f with respect to the *pole* P; sometimes we call it the *polar curve* of P with respect to C_f. In particular, when C_f is a cubic curve (i.e., f is a homogeneous polynomial of degree 3), then C_{Pf} is a conic, which we call the *polar conic* of C_f with respect to the *pole* P, and when C_f is a conic, then C_{Pf} is a line, which we call the *polar line* of C_f with respect to the *pole* P (see, for example, the classical book of Wieleitner [8] or Dolgachev [2, Chap. 3] for a modern view). Note that C_{Pf} is defined and can be a regular curve even if C_f is singular or reducible. For some historical background, for the geometric interpretation of poles and polar lines, for the iterated construction of polar curves, as well as for the analytical method used today, see Monge [5, Sect. 3], Bobillier [1], and Joachimsthal [4, p. 373], or [3].

2 Algebraic Curves and Multilinear Forms

Let C_f be a conic given by the non-constant homogeneous polynomial

$$f(x_1, x_2, x_3) := \sum_{1 \le i \le j \le 3} c_{ij}\, x_i\, x_j.$$

Then, the symmetric matrix

$$T := \begin{pmatrix} c_{11} & c_{12}/2 & c_{13}/2 \\ c_{12}/2 & c_{22} & c_{23}/2 \\ c_{13}/2 & c_{23}/2 & c_{33} \end{pmatrix},$$

has the property that a point X belongs to C_f (i.e., $f(X) = 0$), if and only if $\langle X, T(X)\rangle = 0$. Thus, the conic C_f is represented by the matrix T. Since the expression $\langle X, T(Y)\rangle$ defines a bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $(X, Y) \mapsto \langle X, T(Y)\rangle$, we can consider the matrix T also as a purely covariant tensor of rank 2 (i.e., a tensor whose rank of covariance is 2 and whose rank of contravariance is 0). More precisely, if we consider the matrix T as a $(0, 2)$-tensor, where for $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ we define

$$T(X, Y) := \sum_{1 \le i, j \le 3} a_{ij}\, x_i\, y_j,$$

then the expression $\langle X, T(X)\rangle = 0$ is equivalent to $T(X, X) = 0$. In order to obtain the coefficients of the $(0, 2)$-tensor $T = (a_{ij})_{1 \le i, j \le 3}$ from a conic C_f defined by a non-constant homogeneous polynomial f, we just set

$$a_{ij} := \frac{1}{2!} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{for all } 1 \le i, j \le 3.$$

The next result shows that this relation between a conic C_f and the corresponding $(0, 2)$-tensor $T_f = (a_{ij})_{1 \le i, j \le 3}$ can be generalised to algebraic curves of arbitrary degree.

Lemma 2.1 *Let Γ_f be an algebraic curve of degree d given by the non-constant homogeneous polynomial*

$$f(x_1, x_2, x_3) := \sum_{1 \le i_1 \le \cdots \le i_d \le 3} c_{i_1 \ldots i_d} \cdot x_{i_1} \cdot \ldots \cdot x_{i_d},$$

and let $T_f = (a_{i_1 \ldots i_d})_{1 \le i_1, \ldots, i_d \le 3}$, where

$$a_{i_1 \ldots i_d} := \frac{1}{d!} \cdot \frac{\partial^d f}{\partial x_{i_1} \ldots \partial x_{i_d}} \quad \textit{for all } 1 \le i_1, \ldots, i_d \le 3.$$

Then T_f is a symmetric $(0, d)$-tensor and a point X is on the curve Γ_f if and only if

$$T_f(\underbrace{X, \ldots, X}_{d-\text{times}}) = 0.$$

Proof Since for every rearrangement π of the sequence $\langle i_1, \ldots, i_d \rangle$ we have

$$\frac{\partial^d f}{\partial x_{i_1} \ldots \partial x_{i_d}} = \frac{\partial^d f}{\partial x_{\pi(i_1)} \ldots \partial x_{\pi(i_d)}} \quad \text{and therefore} \quad a_{i_1 \ldots i_d} = a_{\pi(i_1) \ldots \pi(i_d)},$$

we get that the tensor T_f is symmetric. Furthermore, assume that the monomial $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$ appears in f. Then $n_1 + n_2 + n_3 = d$ and

$$\frac{1}{d!} \cdot \frac{\partial^d (c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} = \frac{n_1! \cdot n_2! \cdot n_3!}{d!} \cdot c_{n_1 n_2 n_3}.$$

Now, it is easy to see that the number of coefficients $a_{i_1 \ldots i_d}$ such that for $1 \leq i \leq 3$ the number i appears n_i-times in the sequence $\langle i_1, \ldots, i_d \rangle$ is given by the trinomial coefficient

$$\binom{d}{n_1, n_2, n_3} = \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any point X we have $T_f(X, \ldots, X) = 0$ if and only if $f(X) = 0$, or in other words, X is on the curve Γ. □

Let us turn our attention now to polar curves. For this, we consider first polar curves of conics C_f with corresponding $(0, 2)$-tensor $T_f = (a_{ij})_{1 \leq i, j \leq 3}$. Above we have seen that for a given point $P \in \mathbb{RP}^2$, a point X is on the polar curve $C_{Pf(X)}$ of C_f with respect to the pole P if and only if

$$Pf(X) := \langle P, \nabla f(X) \rangle = 0.$$

Now, for $P, X \in \mathbb{RP}^2$, a short calculation shows that $Pf(X) = 2 \cdot T_f(P, X)$, and hence, we get

$$Pf(X) = 0 \iff T_f(P, X) = 0.$$

Since T_f is symmetric, we have $T_f(P, X) = T_f(X, P)$, which shows that if X is a point on the polar curve of C_f with respect to the pole P, then P is a point on the polar curve of C_f with respect to the pole X. The next result shows that also this result can be generalised to algebraic curves of arbitrary degree.

Lemma 2.2 *Let Γ_f be an algebraic curve of degree d given by the non-constant homogeneous polynomial f, let T_f be the corresponding symmetric $(0, d)$-tensor, and let $P \in \mathbb{RP}^2$ be a point. Then*

$$Pf(X) = 0 \iff T_f(P, \underbrace{X, \ldots, X}_{(d-1)-\text{times}}) = 0.$$

In particular, a point $X \in \mathbb{RP}^2$ *is on the polar curve of* Γ_f *with respect to the pole* P *if and only if* $T_f(P, X, \ldots, X) = 0$.

Proof Notice first that for $P = (p_1, p_2, p_3)$ and $X = (x_1, x_2, x_3)$ we have:

$$\begin{aligned} T_f(P, X, \ldots, X) &= \sum_{j=1}^{3} p_j \cdot \left(\sum_{1 \le i_2, \ldots, i_d \le 3} a_{j\, i_2 \ldots i_d} \cdot x_{i_2} \cdot \ldots \cdot x_{i_d} \right) \\ &= \sum_{j=1}^{3} \sum_{1 \le i_2, \ldots, i_d \le 3} a_{j\, i_2 \ldots i_d} \cdot p_j \cdot x_{i_2} \cdot \ldots \cdot x_{i_d}. \end{aligned}$$

Now, assume again that the monomial $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$ appears in f. Then, for each $1 \le j \le 3$ we have

$$\frac{\partial (c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_j} = n_j \cdot c_{n_1 n_2 n_3} \cdot x_1^{n'_1} \cdot x_2^{n'_2} \cdot x_3^{n'_3},$$

where $n'_j = n_j - 1$ and $n'_i = n_i$ for $i \ne j$. Without loss of generality we assume that $j = 1$ and $n_1 \ge 1$. Now, it is easy to see that the number of coefficients $a_{1\, i_2 \ldots i_d}$ such that for $1 \le i \le 3$, the number i appears n_i-times in the sequence $\langle 1, \ldots, i_d \rangle$ is given by the trinomial coefficient

$$\binom{d-1}{n_1 - 1, n_2, n_3} = \frac{(d-1)!}{(n_1 - 1)! \cdot n_2! \cdot n_3!} = \frac{n_1}{d} \cdot \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any points $P, X \in \mathbb{RP}^2$ we have

$$d \cdot T_f(P, X, \ldots, X) = \big\langle P, \nabla f(X) \big\rangle,$$

in particular, we get

$$Pf(X) = 0 \iff T_f(P, X, \ldots, X) = 0.$$

□

It is obvious how the iterated construction of polar curves is carried out: If, for example, $P, Q, R \in \mathbb{RP}^2$ are given and Γ_f is an algebraic curve of degree $d \ge 3$, then the polar curve of the polar curve of the polar curve of Γ_f with respect to the points P, Q, R, respectively, is given by the zeros of the $(0, d-3)$-tensor $T_f(P, Q, R, X, \ldots, X)$. Notice that since T_f is symmetric, the order of P, Q, R is irrelevant. As a consequence, we obtain the following

Fact 2.3 *Let K be a cubic curve, let $P_1, P_2, P_3 \in \mathbb{RP}^2$, and for $1 \leq j \leq 3$ let T_j be the $(0,2)$-tensor of the polar conic of K with respect to the point P_j. Then for $1 \leq j_1, j_2 \leq 3$ we have*

$$T_{j_1}(P_{j_2}, X) = 0 \iff T_{j_2}(P_{j_1}, X) = 0,$$

in particular, if we consider the tensors T_j as 3×3-matrices, we obtain that

$$[P_{j_1}] = [(T_{j_2}^{-1} \cdot T_{j_1})\, P_{j_2}].$$

The question that we want to treat below, is embedded in a more general problem, namely the study of the relation of a hypersurface and its Hessian variety. In a recent work Sendra-Arranz [6] investigated the Hessian correspondence for the cases of hypersurfaces of degree 3 and 4 in an n-dimensional projective space. In particular, he showed that for degree 3 and dimension $n = 1$, the Hessian correspondence is two to one, and that for degree 3 and $n \geq 2$, and for degree 4, it is birational (see [6, Sects. 2.3, 2.4]). In particular, by introducing the variety of k-gradients as the variety of k-planes containing all the first order derivatives of a polynomial, he obtains algorithms which allow to reconstruct a hypersurface of degree 3 from its Hessian variety in the cases $n \geq 1$, and for degree 4 if n is even. More specifically, Sendra-Arranz proves in his Proposition 2.18 that for $n \geq 2$ a cubic can be recovered by the pencil spanned by its polars. Our main result in Theorem 2.4 is less general, but provides more specific information about the special case of degree 3 in 2 dimensions. Namely, what we show is that three conics in general position (i.e., three points of the Hessian variety) determine a unique cubic. More precisely, given three different conics C_1, C_2, C_3 which satisfy a non-degeneracy condition, we show how to construct the unique cubic K such that for three points $P_1, P_2, P_3 \in \mathbb{RP}^2$ determined by the three conics, the conic C_j (for $1 \leq j \leq 3$) is the polar conic of K with respect to the pole P_j. The construction we provide in the next section proves our main result, Theorem 2.4.

Theorem 2.4 *Let C_1, C_2, C_3 be three non-degenerate conics and let T_1, T_2, T_3 be the corresponding $(0,2)$-tensors given by 3×3-matrices. Assume that the matrices T_1, T_2, T_3 satisfy the following condition:*

(C) *For all $P \in \ker\big(T_3\, T_1^{-1}\, T_2 - T_2\, T_1^{-1}\, T_3\big)$, we have* $\det\big(T_1 P,\ T_2 P,\ T_3 P\big) \neq 0$.

Then there are exactly three points P_1, P_2, P_3, determined by the conics C_1, C_2, C_3, and a unique cubic curve K, such that for $1 \leq j \leq 3$, C_j is the polar conic of K with respect to the pole P_j. The cubic K only depends on the two-dimensional pencil

$$\mathcal{P} = \{\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 :\ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus (0,0,0)\},$$

generated by C_1, C_2, C_3: If C_1, C_2, C_3 are replaced by any other conics $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ in $\mathcal{P}$ satisfying condition (C)*, then the same cubic K results.*

Remark 1 With respect to condition (C), we would like to mention a few facts.

(a) First, condition (C) is symmetric in the three indices: To see this, notice that $P \in \ker\big(T_3\,T_1^{-1}\,T_2 - T_2\,T_1^{-1}\,T_3\big)$ is equivalent to

$$Q = T_1^{-1}\,T_2 P \in \ker\big(T_1\,T_2^{-1}\,T_3 - T_3\,T_2^{-1}\,T_1\big).$$

Replacing P in the determinant by the expression $T_2^{-1}\,T_1 Q$ yields

$$\begin{aligned} 0 \neq \det(T_1 P,\ T_2 P,\ T_3 P) &= \det(T_1\,T_2^{-1} T_1 Q,\ T_1 Q,\ T_3\,T_2^{-1}\,T_1 Q) \\ &= \det(T_1\,T_2^{-1} T_1 Q,\ T_1 Q,\ T_1\,T_2^{-1}\,T_3 Q) \\ &= \det(T_1\,T_2^{-1}) \det(T_1 Q,\ T_2 Q,\ T_3 Q). \end{aligned}$$

(b) Observe also that (C) implies that $T_3 T_1^{-1} T_2 \neq T_2 T_1^{-1} T_3$: Indeed, assume that $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3 = 0$. Then the kernel of $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3$ is $\mathbb{R}^3$. However, for $P = (x_1, x_2, x_3)$, $\det(T_1 P,\ T_2 P,\ T_3 P) = 0$ is a homogeneous cubic polynomial in the three variables x_1, x_2, x_3, which always has non-trivial solutions.

(c) Consider the following example:

$$T_1 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Notice that T_1 does not belong to the pencil spanned by T_2 and T_3. Here, we have that $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3 = 0$ and hence the kernel of $T_3 T_1^{-1} T_2 - T_2 T_1^{-1} T_3$ is $\mathbb{R}^3$. But $\det(T_1 P,\ T_2 P,\ T_3 P) = 0$ whenever the second coordinate of P is 0. So, the example shows that condition (C) can be violated even in the case when the pencil of T_1, T_2, T_3 is two-dimensional. On the other hand, it is easy to see that condition (C) implies that the pencil of T_1, T_2, T_3 is two-dimensional.

3 Constructing a Cubic from Three Conics

Let C_1, C_2, C_3 be three non-degenerate conics and let T_1, T_2, T_3 be the corresponding $(0, 2)$-tensors given by 3×3-matrices matrices T_1, T_2, T_3 which satisfy condition (C) of Theorem 2.4.

Example Let C_1, C_2, C_3 be given by the following three non-constant homogeneous polynomials f_1, f_2, f_3, respectively:

$$\begin{aligned} f_1(X) &= x_1^2 + x_2^2 + 4x_1x_3, \\ f_2(X) &= 2x_1^2 + 2x_1x_2 + 2x_2^2 + 6x_1x_3 + 6x_2x_3, \\ f_3(X) &= x_1^2 + 6x_1x_2 + x_2^2 + 2x_1x_3 - 6x_2x_3. \end{aligned}$$

Figure 1 shows these three conics. Notice that all three conics meet in the origin, which is not excluded by the condition (C), as we will see below. Notice also that one

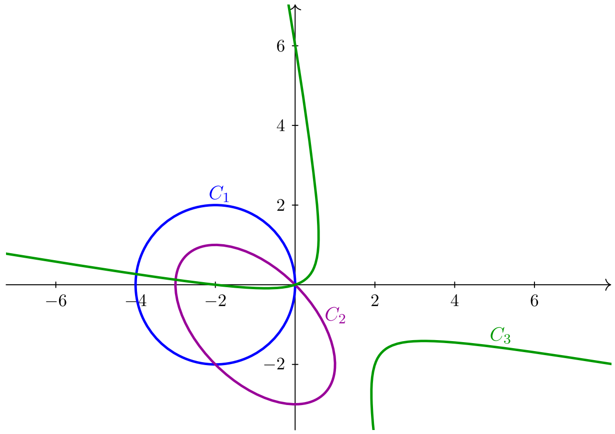

Fig. 1 The three conics C_1, C_2, and C_3 of the example

of the conics is a circle, which is not a restriction since we can transform any conic by a projective transformation into a circle.

Then the corresponding matrices are:

$$T_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & -3 \\ 1 & -3 & 0 \end{pmatrix}.$$

It is easy to verify that the matrices T_1, T_2, T_3 satisfy condition (C): Observe that $\ker\bigl(T_3\,T_1^{-1}\,T_2 - T_2\,T_1^{-1}\,T_3\bigr) = [P]$ for $P = \bigl(\frac{6}{5}, -\frac{24}{5}, 1\bigr)$.

Let us turn back to our general construction and construct the three points P_1, P_2, P_3: By Fact 2.3, the points P_1, P_2, P_3 satisfy the following three necessary conditions

$$T_2\,P_1 = T_1\,P_2, \qquad T_3\,P_2 = T_2\,P_3, \qquad T_1\,P_3 = T_3\,P_1,$$

which is equivalent to

$$(T_1^{-1}\,T_2)P_1 = P_2, \qquad (T_2^{-1}\,T_3)P_2 = P_3, \qquad (T_3^{-1}\,T_1)P_3 = P_1,$$

and implies that P_1 satisfies

$$(T_3^{-1}\,T_1)(T_2^{-1}\,T_3)(T_1^{-1}\,T_2)P_1 = P_1. \tag{1}$$

Since the matrices T_j are symmetric, for $M := T_3\,T_1^{-1}\,T_2$ we have $M^T = T_2\,T_1^{-1}\,T_3$. Therefore, Eq. (1) is equivalent to $M P_1 = M^T P_1$, which is equivalent to $(M - M^T)P_1 = 0$. Now, condition (C) ensures that $M \neq M^T$ (see Remark 1(b)).

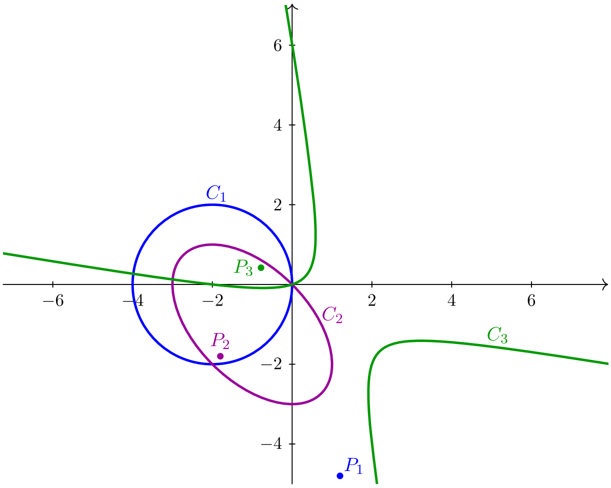

Fig. 2 The three conics C_1, C_2, C_3 of the example with the three poles P_1, P_2, P_3

Since $(M - M^T)$ is a non-zero, real, anti-symmetric 3×3-matrix, it has exactly one eigenvalue equal to zero. In fact, if

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

is an anti-symmetric matrix, then the eigenvalues of A are 0 and $\pm i\sqrt{a^2 + b^2 + c^2}$ and an eigenvector to the eigenvalue 0 is $(c, -b, a)^T$.

Hence, the pole P_1 is uniquely determined by Eq. (1), and we obtain $P_2 = (T_1^{-1} T_2) P_1$ and $P_3 = (T_1^{-1} T_3) P_1$. Before we proceed, let us compute the points P_1, P_2, P_3 in our example.

Example With respect to T_1, T_2, T_3 we get $P_1 = \left(\frac{6}{5}, -\frac{24}{5}, 1\right)$, $P_2 = \left(-\frac{27}{5}, -\frac{27}{5}, 3\right)$, and $P_3 = (\frac{39}{5}, -\frac{21}{5}, -10)$, which correspond to the affine points $\bar{P}_1 = \left(\frac{6}{5}, -\frac{24}{5}\right)$, $\bar{P}_2 = \left(-\frac{27}{15}, -\frac{27}{15}\right)$, and $\bar{P}_3 = \left(-\frac{39}{50}, \frac{21}{50}\right)$, respectively. Figure 2 shows the conics with their poles.

The goal of our construction is to find a $(0, 3)$-tensor T_K of a cubic K, such that we have

$$T_K(P_j, X, X) = T_j(X, X) \quad \text{for } 1 \leq j \leq 3.$$

Since by condition (C), the points P_1, P_2, P_3 are not incident with a projective line, we may choose $\{P_1, P_2, P_3\}$ as a new basis. In other words, for $\tilde{P}_1 = (1, 0, 0)$,

$\tilde{P}_2 = (0, 1, 0)$, and $\tilde{P}_3 = (0, 0, 1)$, we map $P_i \mapsto \tilde{P}_i$ (for $1 \le i \le 3$). For $1 \le i \le 3$, let $T_i = (a^i_{jk})_{1 \le j,k \le 3}$ and let $\tilde{T}_i$ be the (0, 2)-tensors (i.e., the conics $\tilde{C}_i$) in this new basis. Since for any $1 \le i, j, k \le 3$ we have $T_i(P_j, P_k) = T_i(P_k, P_j) = T_j(P_k, P_i)$, we also have

$$\tilde{T}_i(\tilde{P}_j, \tilde{P}_k) = \tilde{T}_i(\tilde{P}_k, \tilde{P}_j) = \tilde{T}_j(\tilde{P}_k, \tilde{P}_i). \tag{2}$$

Now, let $T_{\tilde{K}} = (\tilde{a}_{ijk})_{1 \le i,j,k \le 3}$ be a (0, 3)-tensor defined by stipulating

$$\tilde{a}_{ijk} := \tilde{T}_i(\tilde{P}_j, \tilde{P}_k) \quad \text{for } 1 \le i, j, k \le 3.$$

Then, by Eq. (2), the tensor $T_{\tilde{K}}$ is symmetric and has the property that for $1 \le i \le 3$,

$$T_{\tilde{K}}(\tilde{P}_i, X, X) = \tilde{T}_i(X, X).$$

For the corresponding cubic $\tilde{K}$ we therefore have that $\tilde{C}_i$ is the polar conic of $\tilde{K}$ with respect to the pole $\tilde{P}_i$.

Since every point $\tilde{Q} = (q_1, q_2, q_3) \in \mathbb{RP}^2$ can be written as $\tilde{Q} = q_1 P_1 + q_2 P_2 + q_3 P_3$, we have

$$\begin{aligned} T_{\tilde{K}}(\tilde{Q}, X, X) &= q_1\, T_{\tilde{K}}(\tilde{P}_1, X, X) + q_2\, T_{\tilde{K}}(\tilde{P}_2, X, X) + q_3\, T_{\tilde{K}}(\tilde{P}_3, X, X) \\ &= q_1\, \tilde{T}_1(X, X) + q_2\, \tilde{T}_2(X, X) + q_3\, \tilde{T}_3(X, X), \end{aligned}$$

which shows that the polar conic of $\tilde{K}$ with respect to the point $\tilde{Q}$ belongs to the pencil spanned by the conics $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}_3$.

Now, the re-transformed cubic K has the property that the conics C_1, C_2, C_3 are the polar conics of K with respect to the poles P_1, P_2, P_3, respectively. Furthermore, by the observation above, if, for example, the conic C_3 is replaced by a conic $\tilde{C}_3$ in the two-dimensional pencil of C_1, C_2, C_3 such that $C_1, C_2, \tilde{C}_3$ satisfy condition (C), then the conics C_1, C_2 and $\tilde{C}_3$ are the polar conics of K with respect to the poles P_1, P_2 and some point Q, where the three points P_1, P_2, Q are not collinear.

Example In our example, $\tilde{K}$ in the affine plane is given by

$$\begin{aligned} &-2192 - 2919x + 264x^2 + 122x^3 - 1557y + 3384xy + 198x^2y + 3726y^2 \\ &\quad -81xy^2 - 81y^3 = 0, \end{aligned}$$

and finally, the sought cubic K is

$$-13x^3 - 66x^2y - 27x^2 - 216xy - 39xy^2 - 27y^2 - 22y^3 = 0.$$

Figure 3 shows the cubic K together with the three polar conics C_i with respect to their three poles P_i. Recall that the lines connecting P_i and the points of intersection of K with the polar curve C_i are tangent to K.

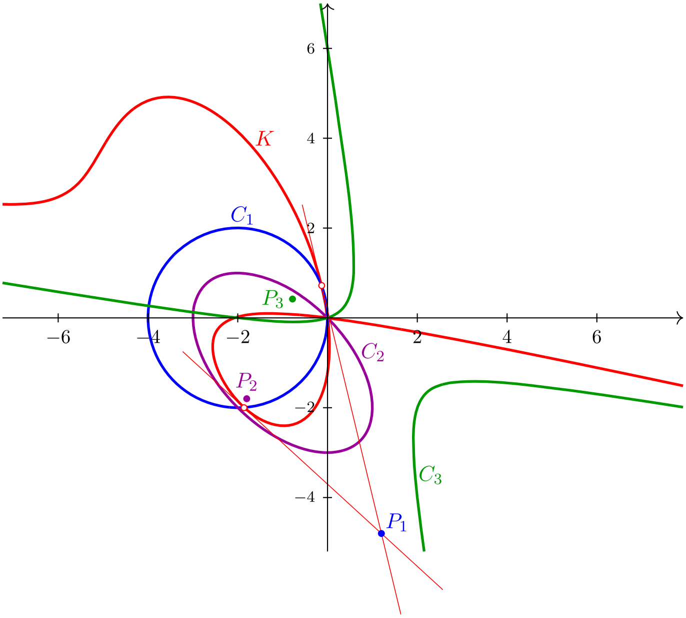

Fig. 3 The cubic K together with the three poles P_1, P_2, P_3 and the three polar conics C_1, C_2, C_3 of the example. The tangents from P_1 to K are also displayed

Remark 2 We close this discussion by considering the situation when condition (C) is violated for three given conics C_1, C_2, C_3. Suppose that K is a cubic such that C_j is the polar conic with respect to some pole P_j for $j = 1, 2, 3$. Then, $\det(T_1 P_1,\ T_2 P_1,\ T_3 P_1) = 0$ in condition (C) for $P_1 \in \ker(T_3\, T_1^{-1}\, T_2 - T_2\, T_1^{-1}\, T_3)$ means that the polar lines $g_1 = T_1 P_1$, $g_2 = T_2 P_1 = T_1 P_2$, $g_3 = T_3 P_1 = T_1 P_3$ of the conics C_1, C_2, C_3 with respect to the poles P_1, P_2, P_3 are concurrent, which in turn means that P_1, P_2, P_3 are collinear. Hence, C_1, C_2, C_3 are identical or span only a one-dimensional pencil. This shows that for the three conics in Remark 1(c), there is no cubic K with the property that C_1, C_2, C_3 are conic sections with respect to three poles. This means that (C) is a necessary condition in Theorem 2.4. On the other hand, if condition (C) is violated *and* C_1, C_2, C_3 span only a one-dimensional pencil, then a cubic K with the required properties exists, but this cubic is no longer unique: Just take an arbitrary conic $\tilde{C}_3$ such that C_1, C_2, $\tilde{C}_3$ satisfy condition (C) and apply Theorem 2.4 in order to obtain a cubic $\tilde{K}$ with respect to C_1, C_2 and $\tilde{C}_3$. Then there is a point P_3 on the line through P_1, P_2 and such that the polar conic of $\tilde{K}$ with respect to P_3 is C_3.

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