



Completely Solving the Quintic by Iteration

Scott Crass¹ 

Received: 27 May 2021 / Revised: 2 March 2022 / Accepted: 12 April 2022 /
Published online: 27 May 2022
© The Author(s) 2022

Abstract

In the late nineteenth century, Felix Klein revived the problem of solving the quintic equation from the moribund state into which Galois had placed it. Klein's approach was a mix of algebra and geometry built on the structure of the regular icosahedron. His method's key feature is the connection between the quintic's Galois group and the rotational symmetries of the icosahedron. Roughly a century after Klein's work, Doyle and McMullen developed an algorithm for solving the quintic that also exploited icosahedral symmetry. Their innovation was to employ a symmetrical dynamical system in one complex variable. In effect, the dynamical behavior provides for a partial breaking of the polynomial's symmetry and the extraction of two roots following one iterative run of the map. The recent discovery of a map whose dynamics breaks *all* of the quintic's symmetry allows for five roots to emerge from a single pass. After sketching some algebraic and geometric background, the discussion works out an explicit procedure that deploys the special map in order to solve the quintic in a complete sense.

Keywords Icosahedral symmetry · Quintic equation · Equivariant map · Critically-finite dynamics

Mathematics Subject Classification 37F10

1 Overview

Solving a polynomial equation calls for a means to overcome the polynomial's symmetry. In the case of the fifth-degree equation, the general symmetry group is the symmetric group \mathcal{S}_5 . In terms of Galois theory, we can reduce the symmetry to that of the alternating group \mathcal{A}_5 by adjoining the square root of the polynomial's discriminant to the coefficient field. Our reward for this reduction is that we can realize \mathcal{A}_5

✉ Scott Crass
scrass@csulb.edu

¹ Department of Mathematics and Statistics, California State University, Long Beach, CA 90840, USA

as the rotational symmetries of the regular icosahedral configuration on the complex projective line \mathbf{CP}^1 —that is, the Riemann sphere.

Exploiting icosahedral structure, Doyle and McMullen constructed a quintic algorithm at the core of which is a map ϕ on \mathbf{CP}^1 that respects the \mathcal{A}_5 symmetry [1]. The map is strongly critically finite, meaning that its critical set \mathcal{C}_ϕ —the twenty face-centers of the icosahedron—is ϕ -invariant; that is, $\phi(\mathcal{C}_\phi) = \mathcal{C}_\phi$. In particular, each superattracting critical point has period two. It follows that ϕ 's global dynamics is *reliable*—meaning that a full measure's worth of points in \mathbf{CP}^1 belongs to the basins of attraction associated with the two-cycles in \mathcal{C}_ϕ . Their procedure employs ϕ 's dynamics in a way that partially breaks the \mathcal{A}_5 symmetry and, with one iterative run, computes two roots.

Recent computational work [2, 3] determined all maps with icosahedral symmetry whose critical sets have size 60 and are *internally periodic*—meaning that the map acts on its critical set as a permutation. The article [2] gave a cursory discussion of how an iterative procedure can rely on such a map in order to approximate all of a quintic's roots. Here, we provide a detailed account of how to design a quintic-solving device around the dynamics of one such map g whose critical points have period five. Since the superattracting set has generic size, the dynamics of g effectively breaks *all* of an equation's \mathcal{A}_5 symmetry. Accordingly, the algorithm produces all five roots with a single iterative run. Unlike the method developed in [4], the current approach explicitly draws upon icosahedral symmetry in algebraic, geometric, and dynamical settings.

The discussion first lays out the necessary ingredients derived from the icosahedral action on \mathbf{CP}^1 . Then from an understanding of the \mathcal{A}_5 -invariant forms and \mathcal{A}_5 -equivariant maps, the special map g emerges. In conclusion, the study assembles the moving parts from icosahedral algebra and the reliable dynamics of g into a procedure that solves equations in a specific family of quintics.

Computational results, whether exact or approximate, are products of *Mathematica*, as was some graphical content. The exposition makes clear which computations are exact or approximate. The program *Dynamics 2* generated basins-of-attraction plots [5].

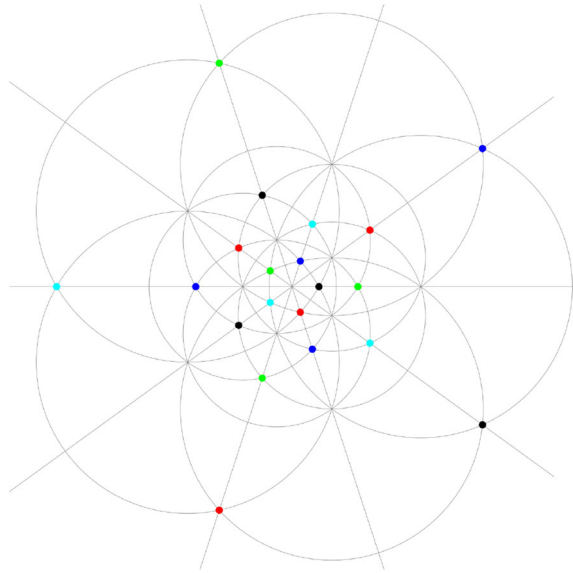
2 Icosahedral Algebra: Invariants and Equivariants

An account of the algebraic objects that arise from the icosahedral action on \mathbf{CP}^1 appears in other places. [1, 2, 6] Here, results relevant to the task at hand appear without discussion.

Denote by \mathcal{I} the \mathcal{A}_5 -isomorphic group of 60 rotational symmetries of the regular icosahedron as a graph structure on the sphere. We can express such a rotation as either a Möbius transformation or a member of $\mathbf{PSL}_2(\mathbf{C})$, that is, a 2×2 matrix whose determinant equals 1 [6] (Fig. 1 renders this graph on \mathbf{C}). Three polynomials generate the ring of \mathcal{I} -invariants:

$$\mathbf{C}[x, y]^{\mathcal{I}} = \langle F(x, y), H(x, y), T(x, y) \rangle$$

Fig. 1 Configuration of five sets of tetrahedral vertices: each color corresponds to four vertices of a tetrahedron when projected onto the sphere. Taking the tetrahedrons' face-centers instead would produce the same set of points, but a different color scheme



where (x, y) are homogeneous coordinates on \mathbf{CP}^1 . The forms $F, H,$ and T vanish at the special \mathcal{I} -orbits: 12 vertices $v_k,$ 20 face-centers $f_k,$ and 30 edge-midpoints e_k respectively. For ease of reference, call the members of these sets “12-points, 20-points, and 30-points.” In coordinates where a pair of antipodal vertices reside at the north and south poles $(0$ and ∞ in $\mathbf{C} \cup \{\infty\})$, we can express the generating invariants as products over special orbits:

$$\begin{aligned}
 F &= \prod_{k=1}^{12} (x - v_k y) = xy (x^{10} - 11x^5 y^5 - y^{10}) \\
 H &= \prod_{k=1}^{20} (x - f_k y) = x^{20} + 228x^{15}y^5 + 494x^{10}y^{10} - 228x^5y^{15} + y^{20} \\
 T &= \prod_{k=1}^{30} (x - e_k y) = x^{30} - 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} \\
 &\quad + 522x^5y^{25} + y^{30}.
 \end{aligned}$$

Accordingly, F and H are algebraically independent, while an algebraic combination of the two generators in degree 60 vanishes (with multiplicity two) at the 30-points:

$$T^2 = H^3 - 1728 F^5.$$

We also need the system of invariants for each of five tetrahedral subgroups denoted $\mathcal{T}_1, \dots, \mathcal{T}_5$ in \mathcal{I} . Each \mathcal{T}_k is a rotational symmetry group of a regular tetrahedron and acts as an alternating group \mathcal{A}_4 on two sets of four 20-points whose union forms eight

vertices of a \mathcal{T}_k -invariant cube. Overall, these disjoint sets of four points occupy the vertices of five regular tetrahedra that the icosahedron circumscribes in two chirally distinct ways. Figure 1 shows an icosahedral net in \mathbf{C} and one way to decompose the vertices into tetrahedral sets distinguished by color. Indices for the tetrahedral groups correspond to the colors. Under \mathcal{I} the five tetrahedra (and cubes) realize \mathcal{A}_5 behavior.

Definition 1 A relative invariant is a form Φ for which a non-trivial multiplicative character λ_T appears under a group action \mathcal{G} . That is, for all $T \in \mathcal{G}$,

$$\Phi \circ T = \lambda_T \Phi \quad \text{such that } \lambda_T \neq 1 \text{ for some } T \in \mathcal{G}.$$

An invariant is absolute when $\lambda_T = 1$ for all $T \in \mathcal{G}$.

Using \mathcal{T}_5 as a reference group, there are two degree-four relative \mathcal{T}_5 invariants: one, q_5 , given by the product that involves the tetrahedral vertices and the other, \hat{q}_5 , given by the product that uses the tetrahedral face-centers (antipodes of the vertices). The results are complex conjugate forms

$$q_5 = \frac{1}{2}(2x^4 + (1 + i\sqrt{15})x^3y + (3 - i\sqrt{15})x^2y^2 - (1 + i\sqrt{15})xy^3 + 2y^4)$$

$$\hat{q}_5 = \frac{1}{2}(2x^4 + (1 - i\sqrt{15})x^3y + (3 + i\sqrt{15})x^2y^2 - (1 - i\sqrt{15})xy^3 + 2y^4).$$

The rationale for using the index 5 will be evident presently. To illustrate relative invariance, recall that $\mathcal{T}_5 \simeq \mathcal{A}_4$ and that \mathcal{A}_4 contains a Klein-4 subgroup \mathcal{V} . As for the action of \mathcal{T}_5 on q_5 :

$$q_5(A(x, y)) = \begin{cases} q_5(x, y) & A \in \mathcal{V} \subset \mathcal{T}_5 \\ -q_5(x, y) & A \in \mathcal{T}_5 - \mathcal{V} \end{cases}$$

and similarly for \hat{q}_5 .

An absolute \mathcal{T}_5 invariant results when we take the form that vanishes at the six-point tetrahedral orbit associated with edges:

$$t_5 = x^6 - 2x^5y - 5x^4y^2 - 5x^2y^4 + 2xy^5 + y^6.$$

In this case, $t_5(A(x, y)) = t_5(x, y)$ for all $A \in \mathcal{T}_5$. Furthermore, the product of the degree-four forms yields a degree-eight \mathcal{T}_5 invariant:

$$u_5 = q_5\hat{q}_5 = x^8 + x^7y + 7x^6y^2 - 7x^5y^3 + 7x^3y^5 + 7x^2y^6 - xy^7 + y^8.$$

Since the eight zeroes of u_5 have order-three symmetry, they are vertices of an inscribed cube as well as eight of the icosahedron’s face-centers. Note that the vertices of such a cube coincide with the vertices and face-centers of a tetrahedron. Alternatively, these

points are vertices of two tetrahedra in opposite chiral systems. Hence, the icosahedral invariant H is divisible by u_5 and the quotient is a \mathcal{T}_5 invariant of degree $12 = 20 - 8$:

$$m_5 = \frac{H}{u_5} = x^{12} - x^{11}y - 6x^{10}y^2 + 20x^9y^3 + 15x^8y^4 + 24x^7y^5 + 11x^6y^6 - 24x^5y^7 + 15x^4y^8 - 20x^3y^9 - 6x^2y^{10} + xy^{11} + y^{12}.$$

The tetrahedral invariants satisfy relations in degrees 12 and 24:

$$q_5^3 = 80t_5^2 - (95 - 9\sqrt{15}i)m_5$$

$$64m_5^2 = 95t_5^2m_5 - 40t_5^4 + 9u_5^3.$$

Applying powers of an order-five element $P \in \mathcal{I}$ manufactures the remaining \mathcal{T}_k invariants:

$$t_k = t_5 \circ P^k \quad u_k = u_5 \circ P^k \quad m_k = m_5 \circ P^k \quad k = 1, \dots, 4.$$

In the chosen coordinates, we can take $P(x, y) = (\epsilon^3x, \epsilon^2y)$ where $\epsilon = e^{2\pi i/5}$. Significantly, the action of \mathcal{I} permutes each of these three sets of five tetrahedral invariants as \mathcal{A}_5 objects.

Given a group action \mathcal{G} on a space X , a natural symmetry for a map $f : X \rightarrow X$ is known as \mathcal{G} -equivariance and satisfies

$$f \circ \gamma = \gamma \circ f \quad \text{for all } \gamma \in \mathcal{G}.$$

Using standard invariant-theoretic techniques, a generating \mathcal{G} -invariant gives rise to a \mathcal{G} -equivariant (or \mathcal{G} -map) of one less degree.

For present purposes, set

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A bit later, we call on these coordinates again. There is a simple commutation rule for J and a linear change of coordinate $x = Aw$ as follows.

Lemma 2.1 *Let A be the matrix of a linear transformation on \mathbf{C}^2 with A^T its transpose and $\det A = 1$. Then*

$$J(A^T)^{-1} = AJ.$$

Now, for the equivariance claim.

Theorem 2.2 *Let $P(x)$ be an invariant of degree r under the action on \mathbf{CP}^1 of a group $\mathcal{G} \subset \mathbf{PSL}_2(\mathbf{C})$. The “cross” operator*

$$\times P(x) = (-\partial_{x_2}P, \partial_{x_1}P)$$

yields a \mathcal{G} -equivariant whose degree is $r - 1$.

Proof Express the cross operator as

$$f(x) = \times_x P(x) = J \nabla_x P(x) \quad \text{where } \nabla_x = (\partial_{x_1}, \partial_{x_2}).$$

Take $\gamma \in \mathcal{G}$ with $x = \gamma w$. A calculation remains.

$$\begin{aligned} P(\gamma w) &= P(w) \quad [\gamma \in \mathcal{G} \text{ and } P \text{ is } \mathcal{G} - \text{invariant}] \\ r P(x) &= r P(w) \\ \nabla_x P(x)^T x &= \nabla_w P(w)^T w \quad [\text{homogeneity of } P(w)] \\ \nabla_x P(x)^T x &= \nabla_w P(w)^T \gamma^{-1} \gamma w \\ \nabla_x P(x)^T &= \nabla_w P(w)^T \gamma^{-1} \quad [x = \gamma w] \\ \nabla_x P(x) &= (\gamma^{-1})^T \nabla_w P(w) \\ J \nabla_x P(x) &= J (\gamma^T)^{-1} \nabla_w P(w) \\ J \nabla_x P(x) &= \gamma J \nabla_w P(w) \quad [\text{Lemma 2.1}] \\ \times_x P(x) &= \gamma (\times_w P(w)) \\ f(x) &= \gamma f(w) \\ f(\gamma w) &= \gamma f(w). \end{aligned}$$

□

Applying the operator to the generating invariants of \mathcal{I} produces maps that generate the module of \mathcal{I} -equivariants over \mathcal{I} -invariants:

$$\begin{aligned} \phi &= \times F = (-x^{11} + 66x^6y^5 + 11xy^{10}, 11x^{10}y - 66x^5y^6 - y^{11}) \\ \eta &= \times H = 20(-57x^{15}y^4 - 247x^{10}y^9 + 171x^5y^{14} - y^{19}, x^{19} + 171x^{14}y^5 \\ &\quad + 247x^9y^{10} - 57x^4y^{15}). \end{aligned}$$

These exceptional maps exhibit elegant behavior: ϕ twists and wraps a dodecahedral face \mathcal{F} onto the 11 faces in the complement of the face antipodal to \mathcal{F} while η does the analogous twisting and wrapping for a face of the icosahedron. For edges of the respective polyhedra we can take great circle arcs between vertices to obtain sets that are forward invariant under the respective map. Call this structure a *dynamical polyhedron*. Moreover, each map expands the internal angle of a face in its dynamical polyhedron onto an external angle of the antipodal face. The vertices are thereby periodic critical points and their superattracting basins are full-measure subsets of \mathbf{CP}^1 . The Doyle-McMullen iteration uses ϕ whose attracting set is a special orbit—namely, the face-centers. Hence, \mathcal{A}_5 symmetry is partially broken allowing for the extraction of two roots.

To break \mathcal{A}_5 symmetry fully, we look for a map g whose critical set \mathcal{C}_g is a generic 60-point \mathcal{I} -orbit that is permuted under the action of g . All maps of this sort have degree 31 and are classified in [3]. Excepting two cases, the dynamical polyhedra associated with these special “31-maps” derive from the icosahedral structure in the

following way: they consist of regions that correspond to twelve pentagons, twenty triangles, and thirty quadrilaterals. The resulting configuration is called a B_{62} . (It also goes by the awkward name rhombicosidodecahedron.)

3 A Special Map

The paper [3] discusses the discovery and geometric behavior of 24 \mathcal{I} -maps with period-five critical points as well as of other critically-finite maps in degree 31. Now, take for g an \mathcal{I} -map whose critical points belong to twelve five-cycles each of which resides at consecutive pentagonal vertices on a B_{62} . Figure 2 depicts a B_{62} configuration realized by the critical set.

Its analytic form appears once a root

$$(\alpha, \beta) \approx (19, -10.82535 - 1.09144i)$$

of a homogeneous degree-24 equation has been approximated:

$$\begin{aligned}
 g &= \alpha H \cdot \phi + \beta F \cdot \eta \\
 &\approx \left(-19x(x^{30} - (487.52150 + 65.48659i)x^{25}y^5 - (10234.85663 - 436.57731i)x^{20}y^{10} \right. \\
 &\quad - (1781.38888 - 3383.47417i)x^{15}y^{15} - (9016.01160 + 1878.43133i)x^{10}y^{20} \\
 &\quad + (618.781738 - 183.822026i)x^5y^{25} + (0.39511 + 1.14888i)y^{30}), \\
 &\quad - 19y((0.39511 + 1.14888i)x^{30} - (618.78173 - 183.82202i)x^{25}y^5 \\
 &\quad - (9016.01160 + 1878.43133i)x^{20}y^{10} + (1781.38888 - 3383.47417i)x^{15}y^{15} \\
 &\quad \left. - (10234.85662 - 436.57731i)x^{10}y^{20} + (487.52150 + 65.48659i)x^5y^{25} + y^{30}) \right).
 \end{aligned}$$

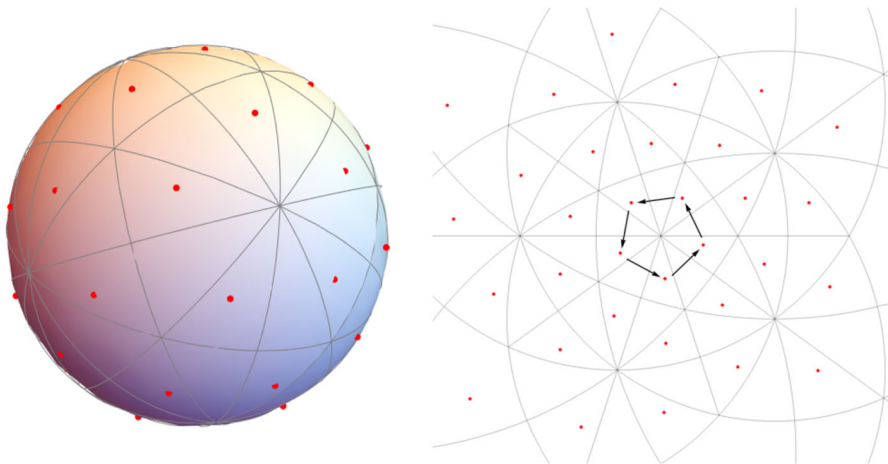


Fig. 2 A B_{62} arrangement of g 's critical set on both \mathbb{CP}^1 and \mathbb{C} is chiral, meaning the cluster of points has only rotational symmetry. Under iteration, a critical point superattracts as it cycles through consecutive vertices of the pentagon on which it resides

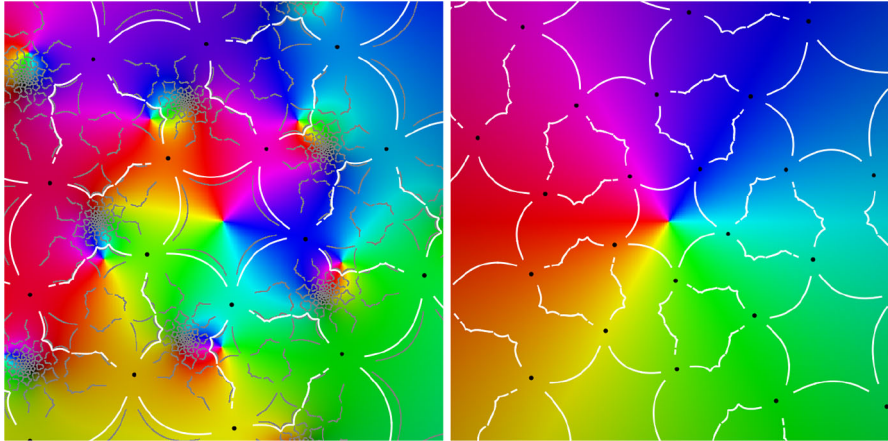


Fig. 3 On the right we see an approximation of a B_{62} in \mathbf{C} . The plot exhibits a color-luminosity (C, L) field in which each point z receives a $(C(z), L(z))$ coordinate determined by $(\arg(z), |z|)$. The plot on the left reveals the combinatorial behavior of g in which a point z is colored as $(C(g(z)), L(g(z)))$. So, the map's behavior is evident by matching color-luminosity values between left and right fields. The overlaid curves (gray) in the left plot outline the regions—call them “pre-faces”—that map to the respective types of face (outlined in white) on the right. The algorithm that generates the edges relies on backward iteration; hence, the appearance of gaps around the critical points where the pullback is highly expansive. Specifically, by noting which pre-faces reside in a face, we can discern that the image of a pentagon covers a pentagon, a triangle covers three pentagons, one triangle, and three quadrilaterals, while a quadrilateral covers ten pentagons, 20 triangles, and 29 quadrilaterals

Between the map's two components, a kind of duality appears in the coefficients as a manifestation of the \times operation when deriving the generating maps ϕ and η .

As discussed in [2] and [3], the combinatorial geometry of g 's behavior gives rise to a polyhedral system of forward invariant “edges” \mathcal{E}_g with vertices at the critical points. This collection of edges fills in the B_{62} structure whose faces consist of twelve pentagons, twenty triangles, and thirty quadrilaterals that realize five-fold, three-fold, and two-fold rotational symmetry respectively. Figure 3 shows the output of an algorithm worked out in [3] that constructs an approximation to the edge-system overlaid on a coloring scheme determined by the map's topological behavior.

Appearing in Fig. 4 are basin-of-attraction plots that reveal g 's symmetry and global dynamics.

By critical-finiteness, g is expanding relative to the hyperbolic metric on $\mathbf{CP}^1 - \mathcal{C}_g$. Hence, the orbit of almost every $p \in \mathbf{CP}^1$ tends to a critical five-cycle:

$$g^k(p) \xrightarrow{k \rightarrow \infty} (r_1, r_2, r_3, r_4, r_5) \subset \mathcal{C}_g.$$

Because g cycles the adjacent pentagonal vertices (r_1, \dots, r_5) , we can take each r_k to be a vertex of the tetrahedron invariant under \mathcal{T}_k —evident in Fig. 1. This dynamical outcome lies at the core of a quintic-solving procedure. The presence of a period-five attracting set makes for an elegant algorithm.

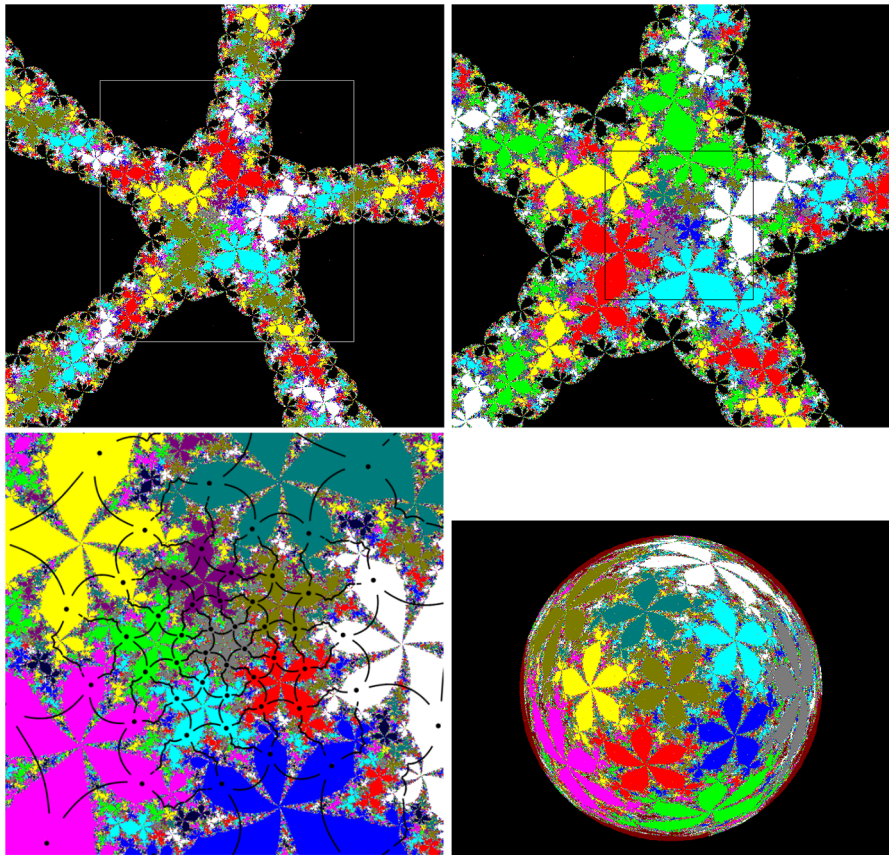


Fig. 4 Basins of attraction for twelve critical five-cycles. A point is colored (or shaded) according to which of the critical 5-cycles its orbit approaches asymptotically. The top right plot is a magnified view of the region in the square box shown in the top left plot while the bottom left image displays the region bounded by the square in the plot at top right and also displays the edge-system \mathcal{E}_g . Associated with a critical cycle are five large basin components in black that accumulate at infinity—a fixed 12-point. For a global view, at bottom right we see the basins under a projection of the plane onto a disk. Note that the coloring is inconsistent between the plots. The map’s chirality is evident

4 Solving the Quintic

Say that you want to solve the quintic

$$z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0.$$

L. Dickson reduced the general five-parameter equation to a resolvent using one parameter in such a way that, from a solution to the resolvent, you can recover a solution to the original equation. [7, Ch. XIII] The procedure about to be developed solves equations in Dickson’s special one-parameter family. Accordingly, it performs an essential role in the general solution. We now work out the steps that culminate in a quintic algorithm.

4.1 Resolvent

First, we create a parametrized family of quintic equations that our dynamical algorithm will solve. The parametrization makes use of the invariant-theoretic structure due to the icosahedral action \mathcal{I} on \mathbf{CP}^1 as described in Sect. 2. Let

$$\rho_k(x) = \frac{F(x)u_k(x)}{H(x)} \quad k = 1, \dots, 5$$

where, for clarity's sake, $x = (x_1, x_2)$ are homogeneous coordinates replacing the former (x, y) . By construction, the degree-eight tetrahedral forms u_k , hence, ρ_k , experience \mathcal{A}_5 permutation under the action of \mathcal{I} .

Next, take the degree-zero rational functions ρ_k as five roots of a polynomial:

$$R_x(v) = \prod_{k=1}^5 (v - \rho_k(x)) = \sum_{j=0}^5 b_j(x)v^j.$$

Being symmetric functions in the ρ_k , the coefficients b_j are \mathcal{I} -invariant and thereby expressible in terms of F and H .

Note that some of the coefficients vanish due to their degree. For instance, the coefficient of v^4 is

$$b_4 = \frac{F}{H} \sum_{k=1}^5 u_k.$$

Since $\sum_{k=1}^5 u_k$ is degree-eight and there are no such \mathcal{I} -invariants, it turns out that $b_4 = 0$. Similarly,

$$b_3 = \frac{F^2}{H^2} \sum_{1 \leq k < \ell \leq 5} u_k u_\ell,$$

but the degree of $u_k u_\ell$ is 16 for which no \mathcal{I} -invariant exists so that $b_3 = 0$. Continuing in this fashion,

$$b_2 = \frac{F^3}{H^3} \sum_{1 \leq k < \ell < m \leq 5} u_k u_\ell u_m.$$

In this case, the forms $u_k u_\ell u_m$ have degree 24 and so, the sum gives an \mathcal{I} -invariant $a F^2$. Accordingly, the coefficient admits expression in the icosahedral parameter $Z = \frac{F^5}{H^3}$:

$$b_2 = a \frac{F^5}{H^3} = a Z.$$

Next, we have

$$b_1 = \frac{F^4}{H^4} \sum_{1 \leq k < \ell < m < n \leq 5} u_k u_\ell u_m u_n$$

where the sum produces an \mathcal{I} -invariant of degree 32, a term expressible as $b F H$. Hence,

$$b_1 = \frac{F^4}{H^4} b F H = b \frac{F^5}{H^3} = b Z.$$

Finally, the zeroth-order term is

$$b_0 = \frac{F^5}{H^5} u_1 u_2 u_3 u_4 u_5.$$

The product of the u_i has degree 40, giving an \mathcal{I} -invariant $c H^2$ so that

$$b_0 = \frac{F^5}{H^5} c H^2 = c \frac{F^5}{H^3} = c Z.$$

Evaluating the factors a , b , and c yields a one-parameter family of quintic resolvents

$$R_Z(v) = v^5 - 40Zv^2 - 5Zv - Z$$

equivalent to Dickson’s fifth-degree resolvent.

Turning to the construction of a quintic-solving algorithm, the key step occurs when we connect a specific resolvent R_Z with a map g_Z that’s conjugate to the special \mathcal{I} -map g . That step is taken by self-parametrizing the icosahedral group. Finally, we build a function—also parametrized by Z —that converts the outcome of g_Z ’s superattracting behavior into the roots of a chosen R_Z .

4.2 Parametrization

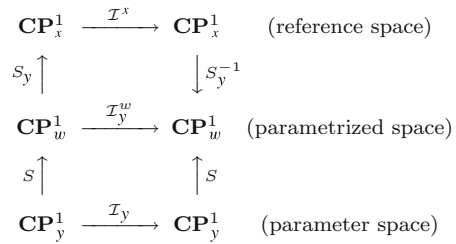
To begin the parametrization process, consider the family of transformations

$$x = S_y w = H(y)\phi(y)w_1 + F(y)\eta(y)w_2$$

that’s linear in $w = (w_1, w_2)$ and degree-31 in $y = (y_1, y_2)$. Using the substitution

$$(x_1, x_2) \rightarrow (y_1, y_2),$$

Fig. 5 Parametrizing the icosahedral action. From \mathbf{CP}_y^1 , the map S picks up a variable y as a parameter. That is, each y determines a change of variable transformation S_y from w -space to x -space. For a given y , the group \mathcal{I}_y^w is the icosahedral action on w -space



the coordinate y replaces x verbatim, giving an associated icosahedral group \mathcal{I}_y with y serving as a parameter. Accordingly, the transformation enjoys an equivariance property:

$$S_{Ay} = AS_y \quad \text{for all } A \in \mathcal{I}_y.$$

Figure 5 shows each S_y as a coordinate change from the y -parametrized w -space and icosahedral action \mathcal{I}_y^w to the fixed x -space with action \mathcal{I}^x .

With coordinate transformation S_y in hand, we can construct the generating invariants and equivariants under \mathcal{I}_y^w . Taking the degree-12 invariant

$$F(x) = F(S_y w) = \sum_{k=0}^{12} a_k(y) w_1^{12-k} w_2^k,$$

the result is a polynomial whose w -degree is 12 while each $a_k(y)$ has a y -degree of $12 \cdot 31$. Moreover, each $a_k(y)$ is invariant under \mathcal{I}_y and thereby expressible as a polynomial $\hat{a}_k(F(y), H(y))$. Hence, we get

$$F(S_y w) = \sum_{k=0}^{12} \hat{a}_k(F(y), H(y)) w_1^{12-k} w_2^k.$$

Note that, by degree considerations, the form $T(y)$, having degree 30, cannot appear to an odd power in the invariant expression for $a_k(y)$ whereas $T(y)$ raised to an even power converts to a polynomial in $F(y)$ and $H(y)$, courtesy of the relation mentioned in Sect. 2. Dividing by $F(y)^{31}$ “normalizes” $F(x)$ to a degree-zero rational function in y . Recalling that the coordinates x and y behave identically, we can take $Z = \frac{H(y)^3}{F(y)^5}$ to obtain a Z -parametrized function:

$$\begin{aligned}
 F_Z(w) &= \frac{F(x)}{F(y)^{31}} = \frac{F(S_y w)}{F(y)^{31}} \Big|_{H(y)^3 \rightarrow F(y)^5 Z} \\
 &= Z^{-6} (4096000000000000 w_1^{12} Z^3 (16Z(432Z(432Z - 95) - 437) + 57) \\
 &\quad - 2048000000000000 w_1^{11} w_2 Z^2 (132Z(864Z(216Z + 5) - 47) - 1) \\
 &\quad - 11264000000000000 w_1^{10} w_2^2 Z^2 (8Z(864Z(4104Z + 245) - 3443) - 11)
 \end{aligned}$$

$$\begin{aligned}
 & - 2816000000000w_1^9w_2^3Z^2(32Z(216Z(3456Z + 833) - 4961) - 121) \\
 & - 4224000000000w_1^8w_2^4Z^2(864Z(20952Z - 1147) - 1331) \\
 & - 337920000000w_1^7w_2^5Z^2(432Z(131328Z - 18053) - 18287) \\
 & - 704000000w_1^6w_2^6Z(48Z(432Z(138240Z - 76183) - 140479) + 1) \\
 & + 211200000w_1^5w_2^7Z(432Z(3314304Z + 28501) - 11) \\
 & + 26400000w_1^4w_2^8Z(432Z(4202496Z + 89177) - 121) \\
 & + 1760000w_1^3w_2^9Z(13824Z(138240Z + 11477) - 1331) \\
 & + 8553600w_1^2w_2^{10}Z(6027264Z - 113) \\
 & + w_1w_2^{11}(69120Z(84049920Z - 3077) - 20) \\
 & + w_2^{12}(1769472Z(172800Z - 11) - 11).
 \end{aligned}$$

To convey a sense of the result, the exact expression is quoted here. The lengthy formulas for subsequent computations are suppressed and can be found at [8]. Applying the same technique generates a function

$$H_Z(w) = \frac{H(x)}{H(y)^{31}} = \frac{H(S_y w)}{H(y)^{31}} \Big|_{H(y)^3 \rightarrow F(y)^5 Z}$$

whose w -degree is 20. Taking the cross of the parametrized forms produces the generating maps on \mathbf{CP}_w^1 :

$$\phi_Z(w) = \times_w F_Z(w) \quad \eta_Z(w) = \times_w H_Z(w).$$

Next, we develop a Z -parametrized version of g defined on \mathbf{CP}_w^1 the first step of which is to capture how the cross operator transforms under a linear change of coordinates on \mathbf{C}^2 . Let $|A|$ denote the determinant and operator subscripts specify differentiation variables.

Theorem 4.1 *The derivative operator \times satisfies a transformation rule under the coordinate change $x = Aw$:*

$$\times_x P(x) = |A|^{-1} A(\times_w P(Aw)).$$

Proof

$$\begin{aligned}
 \times_x P(x) &= J \nabla_x P(Aw) \\
 &= J |A|^{-1} (A^T)^{-1} \nabla_w P(Aw) \quad [\text{transformation rule for } \nabla] \\
 &= |A|^{-1} J (A^T)^{-1} \nabla_w P(Aw) \\
 &= |A|^{-1} A J \nabla_w P(Aw) \quad [\text{Lemma 2.1}] \\
 &= |A|^{-1} A (\times_w P(Aw)).
 \end{aligned}$$

□

We can regard the constant $|A|^{-1}$ as projectively meaningless so that the transformation rule establishes a semi-conjugacy:

$$f(Aw) = |A|^{-1}A\hat{f}(w) \quad \text{with } f(x) = \times_x P(x) \text{ and } \hat{f}(w) = \times_w P(Aw).$$

Applying the formula derived in Theorem 4.1 to the basic icosahedral maps yields

$$\begin{aligned} \phi(x) &= \times_x F(x) \\ &= \times_x F(S_y w) \\ &= |S_y|^{-1} S_y (\times_w (F(y)^{31} F_Z(w))) \\ &= F(y)^{31} |S_y|^{-1} S_y (\times_w F_Z(w)) \\ &= F(y)^{31} |S_y|^{-1} S_y (\phi_Z(w)) \end{aligned}$$

and

$$\begin{aligned} \eta(x) &= \times_x H(x) \\ &= \times_x H(S_y w) \\ &= |S_y|^{-1} S_y (\times_w (H(y)^{31} H_Z(w))) \\ &= H(y)^{31} |S_y|^{-1} S_y (\times_w H_Z(w)) \\ &= H(y)^{31} |S_y|^{-1} S_y (\eta_Z(w)). \end{aligned}$$

These transformation properties uncover a map on the w -space that is dynamically equivalent to $g(x)$.

Theorem 4.2 *With α and β as determined in Sect. 3, the map*

$$g_Z(w) := \alpha F_Z(w)\eta_Z(w) + \beta H_Z(w)\phi_Z(w)$$

satisfies a projective conjugacy with $g(x)$:

$$g(S_y w) = \lambda(y) S_y g_Z(w) \quad \lambda(y) \in \mathbf{C}.$$

Proof Using identities previously derived for the parametrization of basic invariants and equivariants,

$$\begin{aligned} g(x) &= \alpha F(x)\eta(x) + \beta H(x)\phi(x) \\ &= \alpha F(y)^{31} F_Z(w)H(y)^{31} |S_y|^{-1} S_y (\eta_Z(w)) \\ &\quad + \beta H(y)^{31} H_Z(w)F(y)^{31} |S_y|^{-1} S_y (\phi_Z(w)) \\ g(S_y w) &= F(y)^{31} H(y)^{31} |S_y|^{-1} S_y [\alpha F_Z(w)\eta_Z(w) + \beta H_Z(w)\phi_Z(w)] \\ g(S_y w) &= \lambda(y) S_y g_Z(w). \end{aligned}$$

□

In this formula, y is effectively fixed by the selection of Z . The conjugacy implies that the g_Z -orbit of a random initial condition w_0 in \mathbf{CP}_w^1 is asymptotic to a superattracting five-cycle

$$(\omega_1, \dots, \omega_5) = (S^{-1}r_1, \dots, S^{-1}r_5)$$

determined by \mathcal{T}_y^w . Naturally, (r_1, \dots, r_5) is a five-cycle of adjacent pentagonal vertices in \mathcal{C}_g under both $g(x)$ and the action of \mathcal{I}^x on \mathbf{CP}_x^1 .

4.3 Root-Selection

The final step is the assembly of an algorithm that uses the global attracting dynamics of g_Z and the random nature of the initial condition w_0 to effectively break \mathcal{A}_5 symmetry entirely—precisely the state required in order to obtain all of R_Z 's roots. To that end, we now fabricate a tool that's configured to output the roots of a chosen resolvent following a single iterative run of g_Z .

For each tetrahedral subgroup \mathcal{T}_k , consider the degree-12 family of \mathcal{T}_k invariants

$$h_k(x) = \gamma_k t_k(x)^2 + \theta_k m_k(x).$$

While $q_k(x)^3$ is also \mathcal{T}_k -invariant, we need not include it here in light of the relation that exists between the three tetrahedral forms in degree twelve. Let \mathcal{C}_k be the twelve-element subset of \mathcal{C}_g that \mathcal{T}_k preserves and tune one of the parameters γ_k or θ_k so that

$$h_k(p) = \begin{cases} 0 & p \in \mathcal{C}_k \\ c(p) \neq 0 & p \in \mathcal{C}_g - \mathcal{C}_k \end{cases}.$$

Note that $c(p)$ depends on the remaining parameter as well as which of the 48 points in $\mathcal{C}_g - \mathcal{C}_k$ is selected.

Next, define the “complementary” degree-48 form

$$\tilde{h}_k(x) = \frac{\prod_{\ell=1}^5 h_\ell(x)}{h_k(x)} = h_1(x) \dots h_{k-1}(x) h_{k+1}(x) \dots h_5(x).$$

Spending the parameters that remain, we obtain a normalized degree-zero function

$$B_k(x) = \frac{\tilde{h}_k(x)}{F(x)^4}$$

with specific behavior on the critical set of $g(x)$:

$$B_k(p) = \begin{cases} 1 & p \in \mathcal{C}_k \\ 0 & p \in \mathcal{C}_g - \mathcal{C}_k \end{cases}.$$

In practice, forcing \tilde{h}_k to have the desired properties is more convenient when working with undetermined coefficients in the degree-48 family of tetrahedral invariants in five homogeneous parameters:

$$\alpha_1 t_k(x)^8 + \alpha_2 t_k(x)^4 u_k(x)^3 + \alpha_3 u_k(x)^6 + \alpha_4 t_k(x)^6 m_k(x) + \alpha_5 t_k(x)^2 u_k(x) m_k(x).$$

Pairing the y -parametrized $B_k(S_y w)$ with the roots ρ_k of the resolvent R_Z leads to a *root-extraction* function in the Z parameter:

$$\begin{aligned} \Gamma_y(w) &= \sum B_k(x) \rho_k(y) = \sum B_k(S_y w) \frac{u_k(y) F(y)}{H(y)} \\ &= \sum \frac{\tilde{h}_k(S_y w) u_k(y) F(y)}{F(S_y w)^4 H(y)} \\ &= \sum \frac{\tilde{h}_k(S_y w) u_k(y) F(y)}{(F(y)^{31} F_Z(w))^4 H(y)} \quad [F(S_y w) = F(y)^{31} F_Z(w)] \\ &= \frac{F(y)}{F(y)^{124} F_Z(w)^4 H(y)} \sum \tilde{h}_k(S_y w) u_k(y) \\ &= \frac{\sum \tilde{h}_k(S_y w) u_k(y)}{F(y)^{123} H(y)} \frac{1}{F_Z(w)^4}. \end{aligned}$$

Here, we take a bare summation to run from 1 to 5.

Proposition 4.3 *The factor*

$$\Lambda(y, w) = \sum \tilde{h}_k(S_y w) u_k(y)$$

is \mathcal{T}_y -invariant.

Proof Let $A \in \mathcal{T}_y$. By the \mathcal{T}_y equivariance of $S_y w$ as well as the congruent permutation action of \mathcal{T}_y on $\tilde{h}_k(S_y w)$ and $u_k(y)$,

$$\begin{aligned} \Lambda(Ay, w) &= \sum \tilde{h}_k(S_{Ay} w) u_k(Ay) \\ &= \sum \tilde{h}_k(AS_y w) u_{\sigma(k)}(y) \\ &= \sum \tilde{h}_{\sigma(k)}(S_y w) u_{\sigma(k)}(y) \\ &= \Lambda(y, w) \end{aligned}$$

where σ is a permutation on $\{1, \dots, 5\}$. □

By \mathcal{T}_y -invariance, we obtain a Z -parametrized function

$$L_Z(w) = \frac{\Lambda(y, w)}{F(y)^{123} H(y)}$$

that’s degree-zero in y . Finally, we take

$$\Gamma_Z(w) = \frac{L_Z(w)}{F_Z(w)^4}$$

as a root-extractor associated with the resolvent R_Z . Note that this rational function is degree-zero in w .

To see how the extraction process works, fix a value for y and let $q \in \mathcal{C}_{gz} \subset \mathbf{CP}_w^1$ so that

$$q = S_y^{-1}p \text{ for some } p \in \mathcal{C}_j \subset \mathbf{CP}_x^1.$$

Evaluating the extraction function at q gives

$$\begin{aligned} \Gamma_Z(q) &= \frac{L_Z(q)}{F_Z(q)^4} = \frac{\sum \tilde{h}_k(S_y S_y^{-1}p)u_k(y)}{F(y)^{123}H(y)F_Z(q)^4} \\ &= \sum \frac{\tilde{h}_k(p)}{F(y)^{124}F_Z(q)^4} \frac{u_k(y)F(y)}{H(y)} \\ &= \sum \frac{\tilde{h}_k(p)}{F(y)^{31 \cdot 4}F_Z(q)^4} \rho_k(y) \\ &= \sum \frac{\tilde{h}_k(p)}{(F(y)^{31}F_Z(q))^4} \rho_k(y) \\ &= \sum \frac{\tilde{h}_k(p)}{F(p)^4} \rho_k(y) \quad [F(x) = F^{31}(y)F_Z(w)] \\ &= \sum B_k(p)\rho_k(y) \\ &= \rho_j \quad [p \in \mathcal{C}_j]. \end{aligned}$$

4.4 Algorithm

Machinery for the production of roots to the quintic resolvent R_Z is now in place. All that remains is to turn the crank.

1. Every mechanism relies on the icosahedral parameter $Z = \frac{F^5}{H^3}$. First, select a random or arbitrary value Z_0 for this parameter and so, from the family R_Z obtain a specific quintic

$$R_{Z_0} = v^5 - 40Z_0v^2 - 5Z_0v - Z_0$$

whose roots the procedure estimates.

Remark. Solving $R_{Z_0} = 0$ amounts to inverting the quotient map given by $Z(x) = Z_0$ in as much as the solutions form a single icosahedral orbit

$$Z^{-1}(Z_0) = \{x \in \mathbf{CP}^1 \mid F(x)^5 - Z_0H(x)^3 = 0\}.$$

That is, with the elements $x \in Z^{-1}(Z_0)$, we can produce the roots of R_{Z_0} by evaluating $\rho_k(x)$ for $k = 1, \dots, 5$. Such an inversion requires that 60-fold \mathcal{A}_5 symmetry fully breaks—an outcome that’s achieved dynamically. Ultimately, this result amounts to the inversion of the elementary symmetric functions that make up the coefficients of the general quintic. [9]

2. Compute the invariants $F_{Z_0}(w)$ and $H_{Z_0}(w)$ from which the equivariants $\phi_{Z_0}(w)$ and $\eta_{Z_0}(w)$ follow. Then determine the critically-finite map

$$g_{Z_0}(w) = \alpha F_{Z_0}(w)\eta_{Z_0}(w) + \beta H_{Z_0}(w)\phi_{Z_0}(w)$$

on \mathbf{CP}_w^1 .

Remark. The parameter Z is the harness that attaches R_Z to g_Z .

3. Randomly select an initial condition $w_0 \in \mathbf{CP}_w^1$ and compute the orbit $(g_{Z_0}^k(w_0))$ until it produces values $(\tilde{\omega}_1, \dots, \tilde{\omega}_5)$ that approximate to high precision a super-attracting five-cycle $(\omega_1, \dots, \omega_5)$.

Remark. Random selection of w_0 is the device that breaks \mathcal{A}_5 symmetry in a way that’s analogous to coloring g ’s basins. Since the critical points are simple, local convergence to a superattracting 5-cycle is on the order of squaring from one member of a cycle to the next.

4. The final mechanism is the root-extractor $\Gamma_{Z_0}(w)$ which returns an estimate to arbitrary precision for the roots r_k of R_{Z_0} :

$$r_k = \Gamma_{Z_0}(\tilde{\omega}_k), \quad k = 1, \dots, 5.$$

To implement the quintic-solving procedure, the interested reader can obtain a *Mathematica* notebook with supporting data at [8].

Acknowledgements The work described in this paper benefited significantly from discussions with Peter Doyle.

Declarations

Funding Not applicable.

Conflict of interest The author has no conflict of interest to declare that is relevant to the content of this article.

Data Availability Not applicable.

Code Availability *Mathematica* data and code for download: www.home.csulb.edu/scrass/math/quintic_comp_solve.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Doyle, P., McMullen, C.: Solving the quintic by iteration. *Acta Math.* **163**, 151–180 (1989)
2. Crass, S.: Dynamics of a soccer ball. *Exp. Math.* **23**(3), 261–270 (2014)
3. Crass, S.: Critically-finite dynamics on the icosahedron. *Symmetry* **12**(1), 177 (2020)
4. Hubbard, J., Schleicher, D., Sutherland, S.: How to find all roots of complex polynomials by Newton's method. *Invent. Math.* **146**(1), 1–33 (2001)
5. Nusse, H., Yorke, J.: *Dynamics: Numerical Explorations*, 2nd edn. Springer, Berlin (1998). (**Dynamics 2 (computer program) by B. Hunt and E. Kostelich**)
6. Klein, F.: *Lectures on the Icosahedron*. Dover, New York (1956)
7. Dickson, L.: *Modern Algebraic Theories*. Sanborn, Chicago (1926)
8. Crass, S.: home.csulb.edu/~scrass/math/quintic_comp/solve/ (2021)
9. Klein, F.: Ueber eine geometrische representation der resolventen algebraischer gleichungen. *Math. Ann.* **4**, 346–358 (1871)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.