A Whipple ${}_7F_6$ Formula Revisited

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Abstract

A well-known formula of Whipple relates certain hypergeometric values ${}_7F_6(1)$ and ${}_4F_3(1)$. In this paper, we revisit this relation from the viewpoint of the underlying hypergeometric data HD , to which there are also associated hypergeometric character sums and Galois representations. We explain a special structure behind Whipple's formula when the hypergeometric data HD are primitive and self-dual. If the data are also defined over $\mathbb{Q}$, by the work of Katz, Beukers, Cohen, and Mellit, there are compatible families of ℓ -adic representations of the absolute Galois group of $\mathbb{Q}$ attached to HD . For specialized choices of HD , these Galois representations are shown to be decomposable and automorphic. As a consequence, the values of the corresponding hypergeometric character sums can be explicitly expressed in terms of Fourier coefficients of certain modular forms. We further relate the hypergeometric values ${}_7F_6(1)$ in Whipple's formula to the periods of these modular forms.

Keywords Hypergeometric functions · Whipple's ${}_7F_6$ formula · Hypergeometric character sums · Galois representations and modular forms

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1 Introduction

1.1 Hypergeometric Data

Hypergeometric functions have a long history in Mathematics, and they occur in many different research branches. In recent years, rapid progress has been made in hypergeometric character sums under the framework of hypergeometric motives. It gives a unifying perspective to many hypergeometric structures and identities, see [8,12,18,19,27–29,32,38,42,49]. Classical hypergeometric formulas such as Clausen’s formula have a finite field analogue by Evans and Greene [22]. McCarthy [42] obtained several finite field analogues of well-posed hypergeometric values with argument being ± 1 . Based on this work, McCarthy and Papanikolas relate hypergeometric sums with Siegel eigenforms in [43]. Dembélé, Panchishkin, Voight, and Zudilin in [19] provided numerical evidence relating special values of certain hypergeometric L-functions and Asai L-functions of Hilbert modular forms. In [27], the authors interpreted hypergeometric character sums in a manner parallel to the development of classical hypergeometric functions.

Given a hypergeometric datum $HD = \{\alpha = \{a_1, a_2, \dots, a_n\}, \beta = \{1, b_2, \dots, b_n\}; \lambda\}$ with $a_i, b_j \in \mathbb{Q}$, satisfying $a_i - b_j \notin \mathbb{Z}$ for all i, j (in which case the pair α, β is called *primitive*), and nonzero λ in a cyclotomic field,¹ one can

- (1) Define a classical (generalized) hypergeometric series (HGS) $F(\alpha, \beta; \lambda)$ by

$$F(\alpha, \beta; \lambda) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_n)_k}{(1)_k (b_2)_k \dots (b_n)_k} \lambda^k$$

if it converges, where $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\dots(a+k-1)$ is the Pochhammer symbol. It satisfies a Fuchsian differential equation of order n with singularities at 0, 1, and ∞ . Solutions at nonsingular points form a local system and give rise to a monodromy representation of the fundamental group of $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ with a base point, see [9,32,61] for example. We use $F(\alpha, \beta; \lambda)_n := \sum_{k=0}^n \frac{(a_1)_k (a_2)_k \dots (a_n)_k}{(1)_k (b_2)_k \dots (b_n)_k} \lambda^k$ to denote its truncation at n , which plays a vital role in Dwork’s unit root theory for hypergeometric functions [21]; see [10,11] by Beukers and Vlasenko for recent developments.

- (2) Compute hypergeometric character sums as hypergeometric functions over finite fields $\mathbb{F}_q$. In this paper, we focus on the finite period function $\mathbb{P}(HD; q)$ in [27] and the finite hypergeometric function $H_q(HD)$ in [12].
- (3) Associate ℓ -adic Galois representations, such as $\rho_{HD,\ell}$ in [32,33] by Katz for the Galois group of a cyclotomic field, and its extension $\rho_{HD,\ell}^{BCM}$ in [12] by Beukers, Cohen, and Mellit to a representation of the absolute Galois group of $\mathbb{Q}$, when possible. The traces of these representations are essentially given by $\mathbb{P}(HD; q)$ (see (11) below) or $H_p(HD)$ (see (21) below), respectively. These are the ℓ -adic counterparts of (1).

¹ As an element in a global field, λ in different settings below is suitably interpreted: in (1), λ is viewed as a complex number via an embedding of the cyclotomic field in $\mathbb{C}$, while in (2), λ is viewed as an element in the residue field of a nonarchimedean place of the cyclotomic field, when appropriate.

These perspectives are interconnected and are compatible with each other. For an overview, see [29] by Hoffman and Tu. As we will see from the elaboration below, (2) and (3) in the enumeration above only depend on the multi-sets α and $\beta \bmod \mathbb{Z}$, while $\mathbb{P}(HD; q)$ and $\rho_{HD, \ell}$ also depend on the order of the elements in the multi-sets.

There are different versions of hypergeometric character sums. One of the most convenient choices for our purpose is the period function ${}_n\mathbb{P}_{n-1}$ (see (11) below) given in [27, Sect. 4]. For any finite field $\mathbb{F}_q$ of characteristic $p > 2$, let ω be a generator of the group $\widehat{\mathbb{F}_q^\times} := \text{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$ of multiplicative characters of $\mathbb{F}_q$. Let $\alpha = \{a_1, \dots, a_n\}$, $\beta = \{1, b_2, \dots, b_n\}$ be two multi-sets with entries in $\mathbb{Q}$. Denote by $M := \text{lcd}(\alpha \cup \beta)$ the positive least common denominator of all a_i, b_j . For a finite field $\mathbb{F}_q$ containing a primitive M th root of 1 and any $\lambda \in \mathbb{F}_q$, we write

$$\mathbb{P}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) := {}_n\mathbb{P}_{n-1} \left[\begin{matrix} \omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n} \\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n} \end{matrix} ; \lambda; q \right]. \quad (1)$$

The value depends on a_i and b_j modulo $\mathbb{Z}$, as well as their order. All characters $\omega^{(q-1)a_i}$ and $\omega^{(q-1)b_j}$ have orders dividing M , so the above $\mathbb{P}$ function depends only on $\omega^{(q-1)/M}$, which has order M . When the pair $\{\alpha, \beta\}$ is primitive, that is, $a_i - b_j \notin \mathbb{Z}$ for all i, j , as shown in [27], the above $\mathbb{P}$ function can be normalized to rid of the dependence on the order of the a_i and b_j . For any number field K , denote by $G_K = \text{Gal}(\mathbb{Q}/K)$ the absolute Galois group of K . When $K = \mathbb{Q}(\zeta_M)$ is the cyclotomic field obtained by adjoining a primitive M th root of unity ζ_M , we further abbreviate $G_{\mathbb{Q}(\zeta_M)}$ by $G(M)$. Throughout this paper, a prime ideal will always be nonzero. For a prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/M]$, $\zeta_M \bmod \wp$ in the residue field $\kappa_\wp$ of $\wp$ has order M , and it generates the cyclic group $(\kappa_\wp^\times)^{(N(\wp)-1)/M}$. Put

$$\mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp) = \mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp; \omega_\wp)$$

where $\omega_\wp$ is a generator of $\kappa_\wp^\times$ so that

$$\omega_\wp(\zeta_M \bmod \wp) = \zeta_M^{-1}.$$

Note that $\mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp)$ is independent of the choice of $\omega_\wp$.

The connection between $\mathbb{P}(HD, \cdot)$ and Galois representation $\rho_{HD, \ell}$ is the following result of Katz.

Theorem 1 (Katz [32,33]) *Let ℓ be a prime. Given a primitive pair of multi-sets $\alpha = \{a_1, \dots, a_n\}$, $\beta = \{1, b_2, \dots, b_n\}$ with $M = \text{lcd}(\alpha \cup \beta)$, for any datum $HD = \{\alpha, \beta; \lambda\}$ with $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \setminus \{0\}$ the following hold.*

- (i) *There exists an ℓ -adic Galois representation $\rho_{HD, \ell} : G(M) \rightarrow GL(W_\lambda)$ unramified almost everywhere such that at each prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/(M\ell\lambda)]$ with norm $N(\wp) = |\kappa_\wp|$,*

$$\text{Tr} \rho_{HD, \ell}(\text{Frob}_\wp) = (-1)^{n-1} \omega_\wp^{(N(\wp)-1)a_1} (-1) \mathbb{P}(\alpha, \beta; 1/\lambda; \kappa_\wp), \quad (2)$$

- where $\text{Frob}_{\wp}$ stands for the geometric Frobenius conjugacy class of $G(M)$ at $\wp$.
- (iia) When $\lambda \neq 1$, the dimension $d := \dim_{\overline{\mathbb{Q}}_{\ell}} W_{\lambda}$ equals n and all roots of the characteristic polynomial of $\rho_{HD,\ell}(\text{Frob}_{\wp})$ are algebraic numbers and have the same absolute value $N(\wp)^{(n-1)/2}$ under all archimedean embeddings.
 - (iib) When $\lambda \neq 1$ and HD is self-dual (see Definition 1 below), then W_{λ} admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even).
 - (iii) When $\lambda = 1$, dimension of W_{λ} equals $n - 1$. In this case if HD is self-dual, then $\rho_{HD,\ell}$ has a subrepresentation $\rho_{HD,\ell}^{\text{prim}}$ of dimension $2 \lfloor \frac{n-1}{2} \rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even). All roots of the characteristic polynomial of $\rho_{HD,\ell}^{\text{prim}}(\text{Frob}_{\wp})$ have absolute value $N(\wp)^{(n-1)/2}$, the same as (iia).

Here and in what follows, when $\text{ord}_{\wp} \lambda \geq 0$, the λ in $\mathbb{P}(\alpha, \beta; \lambda; \kappa_{\wp})$ is viewed as an element in the residue field $\kappa_{\wp}$. The reader is referred to [32,33,40] for a discussion related to the formulation of Theorem 1. The relation (2) sometimes appears as equating certain hypergeometric character sums with either coefficients of modular forms or Jacobi sums, see [18,23–26,40,45,50].

Remark 1 The sign $\omega_{\wp}^{(N(\wp)-1)a_1}(-1)$, as $\wp$ varies, defines a character $\phi(M, a_1)$ of $G(M)$, which is trivial unless $\text{ord}_2 M = -\text{ord}_2 a_1 = r \geq 1$. In the latter case, it is the quadratic character corresponding to the Hilbert's quadratic norm residue symbol $\left(\frac{\zeta_M}{\cdot}\right)_2$ on the field $\mathbb{Q}(\zeta_M)$.

1.2 The Purpose of This Paper

A well-known formula of Whipple relates ${}_7F_6(1)$ to ${}_4F_3(1)$. When specialized to the self-dual form with a fixed prime p , it reads as follows:

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(\frac{p}{2})\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)\Gamma(\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)} \\ &\quad \times \left(p \cdot {}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c \end{matrix} ; 1 \right] \right). \end{aligned} \quad (3)$$

The reader is referred to Sect. 3.1 for the derivation. The hypergeometric datum corresponds to the left-hand side (after deleting the pair $\frac{5}{4}$ and $\frac{1}{4}$ that correspond to the same character) is

$$\begin{aligned} HD_1(c, f) &:= \left\{ \alpha_6(c, f) := \left\{ \frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f \right\}, \right. \\ &\quad \left. \beta_6(c, f) := \left\{ 1, \frac{3}{2}-c, \frac{1}{2}+c, 1, \frac{3}{2}-f, \frac{1}{2}+f \right\}; 1 \right\}; \end{aligned}$$

while that corresponds to the right-hand side is

$$HD_2(c, f) := \left\{ \alpha_4(f) := \left\{ \frac{1}{2}, \frac{1}{2}, f, 1-f \right\}, \beta_4(c) := \left\{ 1, 1, \frac{3}{2} - c, \frac{1}{2} + c \right\}; 1 \right\}.$$

The formula (3) says a period of the hypergeometric datum $HD_1(c, f)$ is “degenerate” in the sense that it can be represented by a period of the hypergeometric datum $HD_2(c, f)$. The first main result of this paper is to give a Galois representation-theoretic interpretation of the above Whipple formula for special pairs of (c, f) .

Theorem 2 *For (c, f) such that both $HD_1(c, f)$ and $HD_2(c, f)$ are primitive, let $M(c, f) := \text{lcd}(HD_2(c, f))$, and $N(c, f) := \text{lcd}(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4})$. Then either $N(c, f) = 2M(c, f)$ or $N(c, f) = M(c, f)$. Given any prime ℓ , the semisimplification $\rho_{HD_1(c, f), \ell}^{ss}$ of $\rho_{HD_1(c, f), \ell}$ decomposes as*

$$\rho_{HD_1(c, f), \ell}^{ss}|_{G(N(c, f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c, f), \ell})|_{G(N(c, f))} \oplus \sigma_{\text{sym}(c, f), \ell}$$

where ϵ_ℓ is the ℓ -adic cyclotomic character, and $\sigma_{\text{sym}(c, f), \ell}$ is a 2-dimensional representation of $G(N(c, f))$ that can be computed explicitly.

There are only seven un-ordered pairs of $(c, f) \in \mathbb{Q}^2$ for which $HD_1(c, f)$ is defined over $\mathbb{Q}$ (see next section for definition) and primitive. They are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{1}{10}, \frac{3}{10}\right). \quad (4)$$

Among them, $HD_2(c, f)$ is primitive for all pairs, but is defined over $\mathbb{Q}$ only for the first five pairs. Direct computations show that $\rho_{HD_1(c, f), \ell}$ and $\rho_{HD_2(c, f), \ell}$ are semisimple (cf. Theorems 5 and 4 in Sect. 1.4). Thus we obtain the following specialization of Theorem 2.

Theorem 3 *For the seven pairs of (c, f) listed in (4), $N(c, f) = 2M(c, f)$ except for $(c, f) = (\frac{1}{2}, \frac{1}{3})$, in which case $N(c, f) = M(c, f)$. Further*

$$\rho_{HD_1(c, f), \ell}|_{G(N(c, f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c, f), \ell})|_{G(N(c, f))} \oplus \sigma_{\text{sym}(c, f), \ell},$$

where $\sigma_{\text{sym}(c, f), \ell}$ is a 2-dimensional representation of $G(N(c, f))$.

For the pairs in (4), the representations $\rho_{HD_i(c, f), \ell}$, $i = 1, 2$, can be extended to $G_{\mathbb{Q}}$. One of the extensions, $\rho_{HD_i(c, f), \ell}^{BCM}$, was studied by Beukers, Cohen, and Mellit [12], as explained in Sect. 2.3.

Our second purpose is to address, for the seven cases, three natural questions, explained below, connecting different aspects of the hypergeometric data, extending the result in [38] by Long.

- (a) **Modularity** A Galois representation is called *modular* or *automorphic* if it corresponds to an automorphic representation of a general linear group as described in

the Langlands program. As both $\rho_{HD,\ell}$ and $\rho_{HD,\ell}^{BCM}$ are nice Galois representations arising from geometry and unramified almost everywhere, they are expected to be automorphic. Can this be established when they are representations of $G_{\mathbb{Q}}$, such as $\rho_{HD,\ell}^{BCM}$?

- (b) **Truncated HGS and p -adic periods** Assume that at p , the characteristic polynomial of $\rho_{HD,\ell}^{BCM}(\text{Frob}_p)$ has a **unique** unit root γ_p . When $\beta = \{1, 1, \dots, 1\}$, Dwork showed that, when $F(\alpha, \beta; \lambda)_{p-1} \not\equiv 0 \pmod{p}$, such a γ_p exists and can be computed from truncated HGS

$$\gamma_p := \gamma_p(HD) \equiv F(\alpha, \beta; \lambda)_{p^s-1} / F(\alpha, \beta; \lambda^p)_{p^s-1} \pmod{p^s}.$$

If $\rho_{HD,\ell}^{BCM}$ is modular, is there a 2-dimensional component corresponding to a Hecke eigenform g such that at such p , the Hecke eigenvalue $a_p(g)$ of g satisfies the congruence relation $a_p(g) \equiv \gamma_p \pmod{p}$?

- (c) **HGS and Complex periods** If the answers to both questions (a) and (b) are positive, is the untruncated hypergeometric value $F(\alpha, \beta; \lambda)$ related to a period of the normalized Hecke eigenform g which affords the unit roots in (b)?

1.3 An Example

We illustrate an example of Katz representation $\rho_{HD,\ell}$ of $G_{\mathbb{Q}}$, and discuss the three questions (a), (b), (c) raised above.

Let $\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, $\beta = \{1, 1, 1\}$, which form a primitive pair with $M = \text{lcd}(\alpha, \beta) = 2$. For any $\lambda \in \mathbb{Q}^\times$, $p \nmid \text{lcd}(\{\frac{1}{2}, \lambda\})$, by definition,

$$\begin{aligned} \mathbb{P}(HD; p) &= \mathbb{P}(\alpha, \beta; \lambda; \mathbb{F}_p) = {}_3\mathbb{P}_2 \left[\begin{matrix} \omega_p^{(p-1)/2} & \omega_p^{(p-1)/2} & \omega_p^{(p-1)/2} \\ & \varepsilon & \varepsilon \end{matrix} ; \lambda; p \right] \\ &= \sum_{x, y \in \mathbb{F}_p} \phi_p(xy(y-1)(1-x)(x-\lambda y)), \end{aligned}$$

where ε is the trivial character and $\phi_p = \omega_p^{(p-1)/2}$ is the quadratic character of $\mathbb{F}_p^\times$ extended to $\mathbb{F}_p$ by $\phi_p(0) = 0$.

Now consider the following surface defined by the affine equation

$$\mathcal{A}_\lambda : s^2 = xy(x - \lambda y)(y - 1)(1 - x). \quad (5)$$

It is a double cover of $\mathbb{C}P^2$ ramified at 6 lines. Its smooth model is a K3 surface obtained by blowing up the intersection points as described in [54, Sect. 8] by Stienstra and Beukers. The smooth model has good reduction at p . The number of affine $\mathbb{F}_p$ -rational points of $\mathcal{A}_\lambda$ is equal to

$$\sum_{x, y \in \mathbb{F}_p} (1 + \phi_p(xy(x - \lambda y)(y - 1)(1 - x))) = p^2 + \mathbb{P}(HD; p).$$

For a generic λ , the Picard number of the K3 surface $\mathcal{A}_\lambda$ is 19, as shown in [2] by Ahlgren, Ono, and Penniston or [37] by Long. The associated representation $\rho_{HD,\ell}$ of $G_{\mathbb{Q}}$ is 3-dimensional which is the symmetric square of a 2-dimensional Galois representation arising from an explicit elliptic curve E_λ , see [2, (2)]. This essentially follows from the Clausen formula, see (39) and Theorem 7 for the finite field version by Evans and Greene. When $\lambda = 1$, $\mathcal{A}_1$ has Picard number 20 and $\rho_{HD,\ell}$ is 2-dimensional. As shown in [59, Sect. 5] by the first two authors and Liu, the action of the Galois group $G_{\mathbb{Q}}$ on the 2-dimensional subspace of the cohomological space $H_{\text{ét}}^2(\mathcal{A}_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ spanned by the transcendental cycles on $\mathcal{A}_1$ has the same trace as $\rho_{HD,\ell}$ twisted by $\phi(2, 1/2) = \left(\frac{-1}{p}\right) = \chi_{-1}$. In this paper, for a square-free integer d , χ_d denotes the quadratic character of $G_{\mathbb{Q}}$ with kernel $G_{\mathbb{Q}(\sqrt{d})}$; it is also identified with the quadratic character of $\mathbb{Q}$ associated with the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. On the other hand, since $\rho_{HD,\ell}(\text{Frob}_p)$ has trace zero at $p \equiv 3 \pmod{4}$ where $\left(\frac{-1}{p}\right) = -1$, $\rho_{HD,\ell}$ is invariant under twist by $\phi(2, 1/2)$ introduced in Remark 1 and we get the representation $\rho_{HD,\ell}$.

It is explained in [54] by Stienstra and Beukers (see also [4, Sect. 3] and [59, Sect. 5]) that $\mathcal{A}_1$ is birationally isomorphic to an elliptic surface fibered over a genus 0 curve X over $\mathbb{Q}$. The above $G_{\mathbb{Q}}$ action on the 2-dimensional subspace of $H_{\text{ét}}^2(\mathcal{A}_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ can also be realized on H^1 of $X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ with coefficients in an ℓ -adic sheaf on X . This is the prototype of the Katz representation $\rho_{HD,\ell}$ in Theorem 1 for a general hypergeometric datum HD.

Ahlgren in [1] showed that for any prime $p > 2$,

$$\mathbb{P}(HD; p) = \mathbb{P}\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \{1, 1, 1\}; 1; \mathbb{F}_p\right) = a_p(\eta(4\tau)^6),$$

the p th coefficient of $\eta(4\tau)^6$, which admits complex multiplication (CM) by χ_{-1} . Here $\eta(\tau)$ is the Dedekind eta function. In other words, $\rho_{HD,\ell}$ is isomorphic to the Galois representation attached to $\eta(4\tau)^6$, hence it is modular. This is a). For b), Dwork's theorem holds for the ordinary prime $p \equiv 1 \pmod{4}$. In particular, the unit root can be written as $\gamma_p = -\Gamma_p(\frac{1}{4})^4$ where $\Gamma_p(\cdot)$ denotes the p -adic Gamma function. In fact, Ahlgren showed that a stronger congruence, called supercongruence, holds. Namely for each prime $p > 2$,

$$F\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \{1, 1, 1\}; 1\right)_{p-1} \equiv a_p(\eta(4\tau)^6) \pmod{p^2}. \quad (6)$$

In [39], Long and Ramakrishna further showed that when $p > 2$,

$$F\left(\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \{1, 1, 1\}; 1\right)_{p-1} \equiv \begin{cases} -\Gamma_p(\frac{1}{4})^4 & \text{if } p \equiv 1 \pmod{4} \\ -\frac{p^2}{16}\Gamma_p(\frac{1}{4})^4 & \text{if } p \equiv 3 \pmod{4} \end{cases} \pmod{p^3}. \quad (7)$$

An archimedean version for the untruncated hypergeometric value is proved in our paper [58]:

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; 1 \right] = \frac{16}{\pi^2} L(\eta(4\tau)^6, 2) = \frac{\Gamma(\frac{1}{4})^4}{4\pi^3}, \quad (8)$$

which is (c). Similar recent discussions relating L -values of hypergeometric motives to L -functions of modular forms or Hilbert modular forms include Osbrun and Straub [46], and Dembélé et al. [19].

1.4 Main Results on Modularity, Congruences, and Periods

To each of the seven pairs (c, f) in (4), by Theorem 1 there are associated representations $\rho_{HD_1(c,f),\ell}$ and $\rho_{HD_2(c,f),\ell}$ of the Galois group $G(M(c, f))$, where $M(c, f) = \text{lcd}(\alpha_6(c, f) \cup \beta_6(c, f)) = \text{lcd}(\alpha_4(f), \beta_4(c))$, of degrees 5 and 3, respectively. Moreover, $\rho_{HD_1(c,f),\ell}$ can be extended to $\rho_{HD_1(c,f),\ell}^{BCM}$ of $G_{\mathbb{Q}}$ by Theorem 8. The same holds for $\rho_{HD_2(c,f),\ell}$ for the first five pairs; and as shown in Sect. 3.3, this is also true for the remaining two pairs.

It follows from Theorem 8 that, for the seven pairs (resp. the first five pairs) of (c, f) in (4), $\rho_{HD_1(c,f),\ell}^{BCM}$ (resp. $\rho_{HD_2(c,f),\ell}^{BCM}$) has a 4-dimensional (resp. 2-dimensional) subrepresentation $\rho_{HD_1(c,f),\ell}^{BCM, \text{prim}}$ (resp. $\rho_{HD_2(c,f),\ell}^{BCM, \text{prim}}$) and a 1-dimensional complement $\rho_{HD_1(c,f),\ell}^{BCM, 1}$ (resp. $\rho_{HD_2(c,f),\ell}^{BCM, 1}$).

We begin with the modularity question (a). First consider $\rho_{HD_2(c,f),\ell}^{BCM}$.

Theorem 4 *For each of the first five pairs of (c, f) such that $HD_2(c, f)$ is defined over $\mathbb{Q}$, both components of $\rho_{HD_2(c,f),\ell}^{BCM}$ are modular, which are explicitly identified by the information at primes $p > 7$ in the table below. For each of the remaining two pairs, $M(c, f) = 10$ and the representation $\rho_{HD_2(c,f),\ell}$ of $G(10)$ can be extended to $G_{\mathbb{Q}}$, with both components modular. For each pair, we list extensions with integral traces under “ $\rho_{HD_2(c,f),\ell}^{BCM, \text{prim}}$ ” and “ $\rho_{HD_2(c,f),\ell}^{BCM, 1}$ ”, resp.*

(c, f)	α	β	$w(HD_2)$	$\text{Tr } \rho_{HD_2(c,f),\ell}^{BCM, \text{prim}}(\text{Frob}_p)$	$\rho_{HD_2(c,f),\ell}^{BCM, 1}$
$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(1, 1, 1, 1)$	4	$a_p(f_{8.4.a.a})$	ϵ_{ℓ}
$(\frac{1}{2}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3})$	$(1, 1, 1, 1)$	4	$a_p(f_{36.4.a.a}) = \left(\frac{-3}{p}\right) a_p(f_{12.4.a.a})$	$\chi_3 \cdot \epsilon_{\ell}$
$(\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3})$	$(1, 1, \frac{7}{6}, \frac{5}{6})$	4	$a_p(f_{6.4.a.a}) = \left(\frac{-3}{p}\right) a_p(f_{18.4.a.a})$	$\chi_{-3} \cdot \epsilon_{\ell}$
$(\frac{1}{2}, \frac{1}{6})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6})$	$(1, 1, 1, 1)$	4	$a_p(f_{72.4.a.b}) = \left(\frac{-3}{p}\right) a_p(f_{24.4.a.a})$	ϵ_{ℓ}
$(\frac{1}{6}, \frac{1}{6})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6})$	$(1, 1, \frac{4}{3}, \frac{2}{3})$	2	$a_p(f_{24.2.a.a}) \cdot p = \left(\frac{-3}{p}\right) a_p(f_{72.2.a.a}) \cdot p$	$\chi_{-3} \cdot \epsilon_{\ell}$
$(\frac{1}{5}, \frac{2}{5})$	$(\frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{3}{5})$	$(1, 1, \frac{13}{10}, \frac{7}{10})$	4	$a_p(f_{50.4.a.b}) = \left(\frac{5}{p}\right) a_p(f_{50.4.a.a})$	$\epsilon_{\ell} \text{ or } \chi_5 \cdot \epsilon_{\ell}$
$(\frac{1}{10}, \frac{3}{10})$	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{10}, \frac{7}{10})$	$(1, 1, \frac{7}{5}, \frac{3}{5})$	2	$a_p(f_{200.2.a.b}) \cdot p = \left(\frac{5}{p}\right) a_p(f_{200.2.a.d}) \cdot p$	$\epsilon_{\ell} \text{ or } \chi_5 \cdot \epsilon_{\ell}$

The modular forms above are denoted by their LMFDB labels; for instance, $f_{8.4.a.a}$ means a normalized cuspidal newform of level 8 and weight 4. See (24) for the definition of $w(HD_2)$.

Our next theorem records the modularity of $\rho_{HD_1(c,f),\ell}^{BCM}$.

Theorem 5 For each pair (c, f) in (4), $\rho_{HD_1(c,f),\ell}^{BCM,prim}$ decomposes as a sum of two 2-dimensional $G_{\mathbb{Q}}$ -modules, both of these two subrepresentations are modular. There are two normalized cuspidal newforms $\mathfrak{f}_{c,f}$ and $\mathfrak{g}_{c,f}$ (expressed in their LMFDB labels) such that for each prime $p \geq 7$, $\text{Tr } \rho_{HD_1(c,f),\ell}^{BCM}(\text{Frob}_p)$ is explicitly described by the Fourier coefficients $a_p(\mathfrak{f}_{c,f})$ and $a_p(\mathfrak{g}_{c,f})$ in the following table, where $\mathfrak{f}_{c,f}$ (resp. $\mathfrak{g}_{c,f}$) is the first (resp. second) modular form listed in the last column.

(c, f)	$\mathcal{J}(HD_1(c, f); \mathbb{F}_p)$	$w(HD_1)$	$w(HD_2)$	$\text{Tr } \rho_{HD_1(c,f),\ell}^{BCM}(\text{Frob}_p)$
$(\frac{1}{2}, \frac{1}{2})$	$-\left(\frac{-1}{p}\right)$	6	4	$a_p(f_{8.6.a.a}) + p \cdot a_p(f_{8.4.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{2}, \frac{1}{3})$	$-\left(\frac{-1}{p}\right) p$	6	4	$a_p(f_{4.6.a.a}) + p \cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right) p^2$
$(\frac{1}{3}, \frac{1}{3})$	$-\left(\frac{-1}{p}\right) p^2$	6	4	$a_p(f_{6.6.a.a}) + p \cdot a_p(f_{18.4.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{2}, \frac{1}{6})$	$-\left(\frac{-1}{p}\right) p$	4	4	$p \cdot a_p(f_{8.4.a.a}) + p \cdot a_p(f_{24.4.a.a}) + \left(\frac{3}{p}\right) p^2$
$(\frac{1}{6}, \frac{1}{6})$	$-\left(\frac{-1}{p}\right) p^2$	2	2	$p^2 \cdot a_p(f_{24.2.a.a}) + p^2 \cdot a_p(f_{72.2.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{5}, \frac{2}{5})$	$-\left(\frac{-1}{p}\right) p^2$	4	4	$p \cdot a_p(f_{10.4.a.a}) + p \cdot a_p(f_{50.4.a.a}) + \left(\frac{-5}{p}\right) p^2$
$(\frac{1}{10}, \frac{3}{10})$	$-\left(\frac{-1}{p}\right) p^2$	2	2	$p^2 \cdot a_p(f_{40.2.a.a}) + p^2 \cdot a_p(f_{200.2.a.a}) + \left(\frac{-5}{p}\right) p^2$

In the above Table, the term $\mathcal{J}(HD_1(c, f); \mathbb{F}_p)$ is defined by (20), and the weights $w(HD_1)$ and $w(HD_2)$ are defined in (24).

As shown in (23), $\text{Tr } \rho_{HD_1(c,f),\ell}^{BCM}(\text{Frob}_p)$ is expressed in terms of the finite field hypergeometric function H_p introduced in [12], see (21) for its definition by Gauss sums. Its relationship to the $\mathbb{P}$ -function is given in (18). Hence the above two theorems can be reformulated in terms of character sums. For example, the case $(\frac{1}{2}, \frac{1}{3})$ is equivalent to, for $p > 5$,

$$H_p\left(HD_2\left(\frac{1}{2}, \frac{1}{3}\right)\right) = a_p(f_{36.4.a.a}) + \left(\frac{3}{p}\right) p$$

and

$$pH_p\left(HD_1\left(\frac{1}{2}, \frac{1}{3}\right)\right) = a_p(f_{4.6.a.a}) + pa_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right) p^2.$$

When $(c, f) = (\frac{1}{2}, \frac{1}{2})$, the conclusion above is equivalent to a conjecture of Koike [35], proved in [24] by Frechette, Ono and Papanikolas. Comparing Theorems 4 and 5, we see that the extension to $G_{\mathbb{Q}}$ of $\epsilon_{\ell}|_{G(M(c,f))} \otimes \rho_{HD_2(c,f),\ell}^{prim}$ in $\rho_{HD_1(c,f),\ell}^{BCM}$ differs from $\epsilon_{\ell} \otimes \rho_{HD_2(c,f),\ell}^{BCM,prim}$ by a character of $G_{\mathbb{Q}}/G(N(c,f))$. The same phenomenon occurs for the 1-dimensional component.

For the question (b), we see from the above table that only when $(c, f) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}), \rho_{HD_1(c,f),\ell}^{BCM}$ could possibly have a unique unit root γ_p at an unramified prime p . In each of these cases, there is a congruence relation modulo p between a truncated hypergeometric series and the coefficients of $f_{c,f}$, see Theorem 4 of [38]. Numerical computations suggest the following supercongruences for primes $p > 5$:

$$\begin{aligned} (c, f) = \left(\frac{1}{2}, \frac{1}{2}\right) : \quad {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} &\stackrel{?}{\equiv} a_p(f_{8.6.a.a}) \pmod{p^5}. \\ (c, f) = \left(\frac{1}{2}, \frac{1}{3}\right) : \quad p \cdot {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} &\stackrel{?}{\equiv} a_p(f_{4.6.a.a}) \pmod{p^5}. \\ (c, f) = \left(\frac{1}{3}, \frac{1}{3}\right) : \quad p^2 \cdot {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & \frac{7}{6} & 1 & \frac{5}{6} & \frac{7}{6} & \frac{5}{6} \end{matrix} ; 1 \right]_{p-1} &\stackrel{?}{\equiv} a_p(f_{6.6.a.a}) \pmod{p^4}. \end{aligned}$$

Note that these supercongruences hold for both ordinary and supersingular primes.

In the above formulas, each ${}_6F_5$ is multiplied by a suitable p power to ensure the resulting product is p -adically integral. Of these, the first congruence is shown to hold mod p^3 by Osburn, Straub, and Zudilin in [47]. It would be interesting to know the truth. The reader is referred to [34,38,40,41,48] for related discussions on hypergeometric supercongruences.

Finally, we provide an archimedean version of these congruences for $(c, f) = (\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{3})$. Namely, we give precise descriptions of the untruncated hypergeometric values as periods of $f_{c,f}$, which addresses the question (c). Interestingly, for $(c, f) = (\frac{1}{2}, \frac{1}{6})$, we find an expression of $F(HD_1(c, f))$ in terms of a period of $f_{8.4.a.a}$, see (10).

Similar to the identification of $f_{c,f}$ at the archimedean place, we will also explain in the $(\frac{1}{2}, \frac{1}{2})$ case the natural appearance of $g_{c,f}$ from the perspective of modular forms. Meanwhile, numerical data suggest for any prime $p > 3$

$${}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} p \cdot a_p(f_{8.4.a.a}) \pmod{p^4}. \quad (9)$$

The next result expresses the untruncated hypergeometric ${}_6F_5(1)$ and ${}_7F_6(1)$ values in terms of periods of modular forms $f_{\frac{1}{2}, \frac{1}{2}}$ and $g_{\frac{1}{2}, \frac{1}{2}}$, respectively. For a general frame work for periods in terms of integrals, see [36] by Kontsevich and Zagier, in which the authors pointed out that “Periods appear surprisingly often in various formulas and conjectures in mathematics, and often provide a bridge between problems coming from different disciplines.”

Theorem 6 *The following two equalities hold:*

$$\begin{aligned} {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right] &= 16 \oint_{|t_2(\tau)|=1} \tau^2 f_{8.6.a.a} \left(\frac{\tau}{2} \right) d\tau \\ &= 16 \int_{1/2+i/2}^{-1/2+i/2} \tau^2 f_{8.6.a.a} \left(\frac{\tau}{2} \right) d\tau \end{aligned}$$

and

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right] &= \frac{32i}{\pi} \oint_{|t_2(\tau)|=1} \tau f_{8.4.a.a} \left(\frac{\tau}{2} \right) d\tau \\ &= \frac{32i}{\pi} \int_{1/2+i/2}^{-1/2+i/2} \tau f_{8.4.a.a} \left(\frac{\tau}{2} \right) d\tau, \end{aligned} \quad (10)$$

where $t_2(\tau) = -64 \frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}}$ is a Hauptmodul for $\Gamma_0(2)$, and the path in τ such that $|t_2(\tau)| = 1$ is the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$ going counter clockwise.

We believe the appearance of $\mathfrak{g}_{c,f}$ in the $(\frac{1}{2}, \frac{1}{3})$ case can be explained in a similar way. This is left to any interested reader.

1.5 Geometric Perspective

Given $k \in \mathbb{C} \setminus \{0, \pm 1\}$, the 1st de Rham cohomology $H_{DR}^1(L(k)/\mathbb{C})$ of the elliptic curve $L(k) : y^2 = (1-x^2)(1-k^2x^2)$ is isomorphic to the differentials on $L(k)$ with at most a double pole at infinity (see [31, Appendix 1] by Katz). It is a 2-dimensional vector space over $\mathbb{C}$ generated by $\mathfrak{w}_1(k) = \frac{dx}{y}$ and $\mathfrak{w}_2(k) = (1-k^2x^2)\frac{dx}{y}$. Here $\mathfrak{w}_1(k)$ is holomorphic, which is unique up to scalar. Typically, there is no a priori ‘functorial’ choice for the second generator of $H_{DR}^1(L(k)/\mathbb{C})$ unless $L(k)$ admits complex multiplication (CM), see [31]. The two periods

$$K(k) = \int_0^1 \mathfrak{w}_1(k) = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; k^2 \right]$$

and

$$E(k) = \int_0^1 \mathfrak{w}_2(k) = \frac{\pi}{2} {}_2F_1 \left[\begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; k^2 \right]$$

are complete elliptic integrals of first and second kind, respectively, see [13, Chapter 1] by Borwein–Borwein. As k varies, these two period functions are related by

$$E(k) = k(1-k^2) \frac{d}{dk} K(k) + (1-k^2) K(k).$$

In our situation, the hypergeometric functions

$$F(\lambda) := F(\alpha_6(c, f), \beta_6(c, f); \lambda)$$

and

$${}_7F_6(\lambda) := {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1}{2} & f & 1-f \\ \frac{1}{4} & \frac{3}{2} & -c & \frac{1}{2} + c & 1 & \frac{3}{2} - f & \frac{1}{2} + f \end{matrix} ; \lambda \right] = 4 \frac{d}{d\lambda} F(\lambda) + F(\lambda)$$

play roles parallel to $K(k)$ and $E(k)$, respectively. Geometrically, for any given primitive pair α, β of length n and $\lambda \in \overline{\mathbb{Q}}^\times$, one can write down (using the inductive formula via Euler integral (see [53, Sect. 2.1]) for ${}_nF_{n-1}$) an affine equation, which defines a variety $V(\lambda) := V_{\alpha, \beta}(\lambda)$. The surface $\mathcal{A}_\lambda$ in Sect. 1.3 is such an example. The variation of the de Rham cohomology $H_{DR}^{n-1}(V(\lambda)/\mathbb{C})$ can be described by the hypergeometric differential equation satisfied by $F(\alpha, \beta; \lambda)$. By the Hodge filtration, we have $H_{DR}^{n-1}(V(\lambda)) \cong \bigoplus_{i=0}^{n-1} H^i(V(\lambda), \Omega_{V(\lambda)}^{n-1-i})$. For example, $H^0(V(\lambda), \Omega_{V(\lambda)}^{n-1})$ consists of holomorphic differential $n - 1$ st forms on $V(\lambda)$. For $HD_1(c, f) = \{\alpha_6(c, f), \beta_6(c, f); 1\}$ with (c, f) in (4), $\dim H^0(V_{\alpha_6(c, f), \beta_6(c, f)}(1), \Omega^5) = 1$ only occurs for the first three cases of Theorem 5.

From this perspective, Whipple's formula (3) suggests that when $\lambda = 1$, up to a Tate twist, there is a correspondence between $H_{DR}^5(V_{\alpha_6(c, f), \beta_6(c, f)}(1))$ and $H_{DR}^3(V_{\alpha_4(c, f), \beta_4(c, f)}(1))$. Firstly, such a correspondence only exists for special fibers like $\lambda = 1$, similar to $H_{DR}^1(L(k))$ admitting a canonical splitting only when $L(k)$ has CM. Secondly, such a correspondence, when it exists, can be recognized simultaneously from different aspects in compatible ways, including the statement of Whipple's formula (3), the statements of Theorems 3 and 5, the conjectural supercongruence (9), as well as (10). In particular, the factor $\frac{1}{\pi}$ on the right-hand side of (10) and the p on the right-hand side of (9) both reflect a Tate twist by 1.²

Finally, we comment on the role of Theorem 6. From transcendence theory, given a normalized weight- k cusp form $f(\tau)$, the values $\oint_\gamma f(\tau) p(\tau) d\tau$ for all polynomials $p(\tau) \in \overline{\mathbb{Q}}[\tau]$ of degree at most $k - 1$ and all closed paths γ on the underlying modular curve generate a finite dimensional vector space over $\overline{\mathbb{Q}}(\pi)$. Thus the equalities in Theorem 6 are not coincidental by nature. On the other hand, from the perspective of de Rham cohomology, one expects that untruncated hypergeometric values arise from line integrals of some differentials in $H_{DR}^{n-1}(V(\lambda))$. The challenge becomes how to realize hypergeometric values as periods.

This paper is organized in the following way. In Sect. 2, we recall basic notation, relevant results for hypergeometric data, character sums, liftings of Katz representations, and modular forms. Section 3 is devoted to the proofs of Theorems 2, 3, 4 and 5. In Sect. 4, Theorem 6 and other computations for untruncated hypergeometric functions are obtained using properties of modular forms and hypergeometric functions. It is worth pointing out that in the proofs of Theorems 2 and 6, the classical Clausen

² The Legendre relation for $L(k)$ is $E(k)K(\sqrt{1-k^2}) + K(k)E(\sqrt{1-k^2}) - K(k)K(\sqrt{1-k^2}) = \frac{\pi}{2}$, which is a pairing between $H^0(L(k), \Omega_{V(1)})$ and $H^1(L(k), \mathcal{O}_{V(1)})$.

formula (13) and its finite field analogue (Theorem 7) play a prominent role. The finite field version of the well posedness of the ${}_6F_5(HD_1)$ under consideration gives rise to Lemma 2, using which we decompose the representation $\rho_{HD_1, \ell}$.

2 Preliminaries

2.1 Hypergeometric Functions Over Finite Fields

We recall the period functions for finite fields with odd characteristic. All multiplicative characters A are extended to 0 by letting $A(0) = 0$. In what follows, denote by ε the trivial character, by ϕ_p or simply ϕ the quadratic character, and by $\bar{A}$ the inverse of the character A . For any characters $A_i, B_i, i = 1, \dots, n+1$ with $B_1 = \varepsilon$ in $\widehat{\mathbb{F}_q^\times}$, the ${}_{n+1}\mathbb{P}_n$ -function is defined as follows: set

$${}_1\mathbb{P}_0 \left[\begin{matrix} A_1 \\ ; \lambda; q \end{matrix} \right] := \bar{A}_1(1 - \lambda)$$

and construct ${}_{n+1}\mathbb{P}_n$ inductively by

$$\begin{aligned} & {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \cdots & A_{n+1} \\ B_2 & \cdots & B_{n+1} \end{matrix} ; \lambda; q \right] \\ & := \sum_{x \in \mathbb{F}_q} A_{n+1}(x) \bar{A}_{n+1} B_{n+1}(1-x) \cdot {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_1 & A_2 & \cdots & A_n \\ B_2 & \cdots & B_n \end{matrix} ; \lambda x; q \right]. \quad (11) \end{aligned}$$

For simplicity, we may leave out the q in the notation when there is no danger of ambiguity in the underlying finite field $\mathbb{F}_q$.

This $\mathbb{P}$ -function is defined as an analog of the integral expression of the classical hypergeometric series. See [27, 28]. Note that for any character A in $\widehat{\mathbb{F}_q^\times}$, we have

$$\bar{A}(1 - \lambda) = \delta(\lambda) + \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} J(A\chi, \bar{\chi}) \chi(-\lambda) = \delta(\lambda) + \frac{-1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \chi(\lambda),$$

where

$$\begin{aligned} \delta(t) &:= \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise,} \end{cases} \\ J(A, B) &:= \sum_{t \in \mathbb{F}_q} A(t) B(1-t), \quad \text{and} \quad \binom{A}{B} := -B(-1) J(A, \bar{B}). \end{aligned}$$

Thus, we can rewrite the $\mathbb{P}$ -function in terms of the finite field version of binomial coefficients $\binom{A}{B}$ defined above (see [28, Theorem 3.6] by Greene for example):

$$\begin{aligned}
 & {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \cdots & A_{n+1} \\ B_2 & \cdots & B_{n+1} \end{matrix} ; \lambda; q \right] \\
 &= \frac{(-1)^{n+1}}{q-1} \cdot \left(\prod_{i=2}^{n+1} A_i B_i (-1) \right) \cdot \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_1 \chi}{\chi} \binom{A_2 \chi}{B_2 \chi} \\
 &\quad \cdots \binom{A_{n+1} \chi}{B_{n+1} \chi} \chi(\lambda) + \delta(\lambda) \prod_{i=2}^{n+1} J(A_i, \overline{A_i B_i}). \tag{12}
 \end{aligned}$$

Here are some properties of these character sums that are useful for our later discussion.

Proposition 1 (Theorem 4.2, [28]) *For any fixed $\mathbb{F}_q$ of characteristic $p > 2$, $A_i, B_j \in \widehat{\mathbb{F}_q^\times}$, and $t \neq 0$,*

$$\begin{aligned}
 & {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_1 & A_2 & \cdots & A_n \\ B_2 & \cdots & B_n \end{matrix} ; \frac{1}{t} \right] \\
 &= A_1(-t) \left(\prod_{i=2}^n A_i B_i (-1) \right) \cdot {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_1 & A_1 \overline{B_2} & \cdots & A_1 \overline{B_n} \\ A_1 \overline{A_2} & \cdots & A_1 \overline{A_n} \end{matrix} ; t \right] \\
 &= A_2(t) \left(\prod_{i=3}^n A_i B_i (-1) \right) \cdot {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_2 \overline{B_2} & A_2 & A_2 \overline{B_3} & \cdots & A_2 \overline{B_n} \\ A_2 \overline{A_1} & A_2 \overline{A_3} & \cdots & A_2 \overline{A_n} \end{matrix} ; t \right].
 \end{aligned}$$

This proposition is comparable to the following classical result.

Proposition 2 *Given $\alpha = \{a_1, a_2, \dots, a_n\}$, $\beta = \{1, b_2, b_3, \dots, b_n\}$, if we denote $1+a_j-\beta := \{a_j, 1+a_j-b_2, \dots, 1+a_j-b_n\}$ and $1+a_j-\alpha := \{1+a_j-a_1, \dots, 1+a_j-a_n\}$ then the functions $F(\alpha, \beta; z)$ and $(-z)^{-a_j} F(1+a_j-\beta, 1+a_j-\alpha; 1/z)$ for any $j = 1, \dots, n$, satisfy the same differential equation*

$$[\theta(\theta + b_2 - 1) \cdots (\theta + b_n - 1) - z(\theta + a_1) \cdots (\theta + a_n)] F = 0, \quad \text{where } \theta = z \frac{d}{dz}.$$

Next we recall a version of the classical Clausen formula ([3, p. 184]):

$$\begin{aligned}
 & {}_2F_1 \left[\begin{matrix} a & b \\ a+b+\frac{1}{2} \end{matrix} ; x \right] {}_2F_1 \left[\begin{matrix} \frac{1}{2}-a & \frac{1}{2}-b \\ \frac{3}{2}-a-b \end{matrix} ; x \right] \\
 &= {}_3F_2 \left[\begin{matrix} \frac{1}{2} & a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; x \right]. \tag{13}
 \end{aligned}$$

Below is the Clausen formula over finite fields obtained by Evans and Greene.

Theorem 7 (Evans–Greene [22]) Assume $\eta, K \in \widehat{\mathbb{F}_q^\times}$ such that none of $\eta, K\phi, \eta K, \eta\bar{K}$ is trivial. Suppose $\eta K = S^2$ is a square. When $t \neq 0, 1$, we have

$$\begin{aligned} {}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; t \right] &= \phi(1-t) \left(\phi(t-1)K(t) \frac{J(\phi S, \phi K \bar{S})}{J(S, K \bar{S})} {}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right]^2 - q \right) \\ &= \phi(1-t) \left({}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right] {}_2\mathbb{P}_1 \left[\begin{matrix} \phi \bar{K} S & \bar{S} \\ & \bar{K} \end{matrix} ; t \right] - q \right), \end{aligned}$$

where

$${}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right] = \bar{K}(t)\phi(t-1) \frac{J(S, K \bar{S})}{J(\phi K \bar{S}, \phi S)} {}_2\mathbb{P}_1 \left[\begin{matrix} \phi \bar{K} S & \bar{S} \\ & \bar{K} \end{matrix} ; t \right]. \quad (14)$$

When $t = 1$, we have

$$\begin{aligned} {}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; 1 \right] &= \phi\eta(-1)q \frac{J(\eta K, \bar{\eta} K)J(\phi, \phi K)}{J(\phi S, \bar{K})J(S, \bar{K})} \left(\frac{J(S, \phi \bar{S})}{J(S\bar{\eta}, \phi \bar{S}\eta)} + \frac{J(\phi S, \bar{S})}{J(\eta \bar{S}, \phi S\bar{\eta})} \right) \\ &= \frac{J(\eta K, \bar{\eta} K)}{J(\phi, \bar{K})} (J(S\bar{K}, \phi \bar{S})^2 + J(\phi S\bar{K}, \bar{S})^2). \end{aligned} \quad (15)$$

In particular, when $K = \varepsilon$, this gives

$${}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & \varepsilon & \varepsilon \end{matrix} ; 1 \right] = J(S, \phi \bar{S})^2 + J(\bar{S}, \phi S)^2. \quad (16)$$

When ηK is not a square, we have

$${}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; 1 \right] = 0. \quad (17)$$

The formulation given in [22] is different from the above self-dual form. We will explain in the proof below how to bridge both versions.

Proof When $t \neq 0$,

$$\begin{aligned} {}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; t \right] &= \frac{-1}{q-1} \sum_{\chi} \binom{\phi\chi}{\chi} \binom{\eta\chi}{K\chi} \binom{\bar{\eta}\chi}{\bar{K}\chi} \chi(t) \\ &\stackrel{\chi \mapsto K\chi}{=} K(t) \frac{-1}{q-1} \sum_{\chi} \binom{\phi K\chi}{K\chi} \binom{\eta K\chi}{K^2\chi} \binom{\bar{\eta} K\chi}{\chi} \chi(t) \\ &= \phi(-1)K(t) {}_3\mathbb{P}_2 \left[\begin{matrix} \bar{\eta} K & \eta K & \phi K \\ & K^2 & K \end{matrix} ; t \right]. \end{aligned}$$

Apply the Clausen's formula over $\mathbb{F}_q$ in [22, Theorem 1.5] with $C = K$ and $S^2 = \eta K$, when $t \neq 1$, one has

$$\begin{aligned} & {}_3\mathbb{P}_2 \left[\begin{matrix} \bar{\eta}K & \eta K & \phi K \\ & K^2 & K \end{matrix} ; t \right] \\ &= \frac{J(\phi S, \phi K \bar{S})}{J(S, K \bar{S})} \left({}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right]^2 - \phi(1-t) \bar{K}(t) \cdot \phi(-1)q \cdot \frac{J(S, K \bar{S})}{J(\phi S, \phi K \bar{S})} \right). \end{aligned}$$

On the other hand, the claim (14) follows from [27, Proposition 8.7 (2)]. Therefore, when $t \neq 0, 1$,

$$\begin{aligned} & {}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; t \right] = \phi(-1)K(t) \frac{J(\phi S, \phi K \bar{S})}{J(S, K \bar{S})} {}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right]^2 - \phi(1-t) \cdot q \\ &= \phi(1-t) {}_2\mathbb{P}_1 \left[\begin{matrix} \phi K \bar{S} & S \\ & K \end{matrix} ; t \right] {}_2\mathbb{P}_1 \left[\begin{matrix} \phi \bar{K} S & \bar{S} \\ & \bar{K} \end{matrix} ; t \right] - \phi(1-t) \cdot q \end{aligned}$$

When $t = 1$, according to [28, Theorem 4.38] (or [27, Eq. (7.2)]), the evaluation is

$$\begin{aligned} & {}_3\mathbb{P}_2 \left[\begin{matrix} \phi & \eta & \bar{\eta} \\ & K & \bar{K} \end{matrix} ; 1 \right] = \phi(-1) {}_3\mathbb{P}_2 \left[\begin{matrix} \bar{\eta}K & \eta K & \phi K \\ & K^2 & K \end{matrix} ; 1 \right] \\ &= \phi(-1)J(\eta K, \bar{\eta}K)J(\phi, \phi K) \left(\sum_{D=S, \phi S} \frac{J(D, K \bar{S}^2)J(\phi \bar{D}, \bar{K} S^2)}{J(\phi S, \bar{K})J(S, \bar{K})} \right) \end{aligned}$$

which can be converted to the right-hand side of (15) using properties of Jacobi sums. The identity (16) can be verified directly, and (17) follows from Theorem 4.38(ii) of [28] by Greene. $\square$

Lemma 1 For q odd, the product of two summands of (15) equals q^2 , namely

$$\left(\phi \eta (-1)q \frac{J(\eta K, \bar{\eta}K)J(\phi, \phi K)}{J(\phi S, \bar{K})J(S, \bar{K})} \right)^2 \frac{J(S, \phi \bar{S})}{J(S \bar{\eta}, \phi \bar{S} \bar{\eta})} \frac{J(\phi S, \bar{S})}{J(\eta \bar{S}, \phi S \bar{\eta})} = q^2.$$

Proof It is a straightforward verification. Using the double angle formula

$$\mathfrak{g}(A)\mathfrak{g}(\phi A) = \mathfrak{g}(A^2)\mathfrak{g}(\phi)\bar{A}(4),$$

and the reflection formula

$$\mathfrak{g}(A)\mathfrak{g}(\bar{A}) = A(-1)q, \quad A \neq \varepsilon$$

on Gauss sums $g(A)$, we have

$$\begin{aligned} \left(\frac{J(\eta K, \bar{\eta} K) J(\phi, \phi K)}{J(\phi S, \bar{K}) J(S, \bar{K})} \right)^2 &= \left(\frac{g(\eta K) g(\bar{\eta} K) g(\phi S \bar{K}) g(\phi) g(\phi K) g(S \bar{K})}{g(K^2) g(\phi S) g(\bar{K}) g(K) g(S) g(\bar{K})} \right)^2 \\ &= \left(\frac{g(\eta K) g(\bar{\eta} K) g(\phi)^2 g(\phi K) g(S^2 \bar{K}^2)}{g(\phi K) g(\bar{K})^2 g(K)^2 g(S^2)} \right)^2 \\ &= \left(\frac{g(\eta K) g(\bar{\eta} K) g(\phi)^2 g(\phi K) g(\eta \bar{K})}{g(\phi K) g(\bar{K})^2 g(K)^2 g(\eta K)} \right)^2 = 1 \end{aligned}$$

and

$$\frac{J(S, \phi \bar{S})}{J(S \bar{\eta}, \phi \bar{S} \eta)} \frac{J(\phi S, \bar{S})}{J(\eta \bar{S}, \phi S \bar{\eta})} = \frac{q^2}{q^2} = 1.$$

The claim hence follows. $\square$

Lemma 2 For a finite field $\mathbb{F}_q$ of odd characteristic and $A, B, C, D, E \in \widehat{\mathbb{F}_q^\times}$, we have

$$\begin{aligned} {}_6\mathbb{P}_5 \left[\begin{matrix} A & B & C & A & D & E \\ A\bar{D} & A\bar{E} & \varepsilon & A\bar{B} & A\bar{C} \end{matrix}; 1 \right] &= BCDE(-1) \sum_{t \in \mathbb{F}_q} A(t) {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ A\bar{D} & A\bar{E} \end{matrix}; t \right]^2 \\ &= BCDE(-1) \sum_{t \in \mathbb{F}_q} A(t) {}_3\mathbb{P}_2 \left[\begin{matrix} A & D & E \\ A\bar{B} & A\bar{C} \end{matrix}; t \right]^2. \end{aligned}$$

Proof

$$\begin{aligned} &BCDE(-1) \sum_{t \in \mathbb{F}_q} A(t) {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ A\bar{D} & A\bar{E} \end{matrix}; t \right]^2 \\ &\stackrel{\text{Prop. 1}}{=} A(-1) \sum_{t \in \mathbb{F}_q^\times} {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ A\bar{D} & A\bar{E} \end{matrix}; t \right] {}_3\mathbb{P}_2 \left[\begin{matrix} A & D & E \\ A\bar{B} & A\bar{C} \end{matrix}; 1/t \right] \\ &= A(-1) \sum_{t \in \mathbb{F}_q^\times} \frac{-1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{B\chi}{A\bar{D}\chi} \binom{C\chi}{A\bar{E}\chi} \\ &\quad \chi(t) \frac{-1}{q-1} \sum_{\psi \in \widehat{\mathbb{F}_q^\times}} \binom{A\psi}{\psi} \binom{D\psi}{A\bar{B}\psi} \binom{E\psi}{A\bar{C}\psi} \psi(1/t) \\ &= A(-1) \frac{1}{(q-1)^2} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi} \binom{B\chi}{A\bar{D}\chi} \binom{C\chi}{A\bar{E}\chi} \end{aligned}$$

$$\sum_{\psi \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A\psi \\ \psi \end{pmatrix} \begin{pmatrix} D\psi \\ A\overline{B}\psi \end{pmatrix} \begin{pmatrix} E\psi \\ A\overline{C}\psi \end{pmatrix} \sum_{t \in \mathbb{F}_q^\times} \chi \psi^{-1}(t).$$

By the following orthogonality relation

$$\sum_{t \in \mathbb{F}_q^\times} \chi \psi^{-1}(t) = \begin{cases} 0, & \text{if } \chi \neq \psi, \\ q-1, & \text{if } \chi = \psi, \end{cases}$$

we obtain that

$$\begin{aligned} & BCDE(-1) \sum_{t \in \mathbb{F}_q} A(t) {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ A\overline{D} & A\overline{E} & t \end{matrix}; t \right]^2 \\ &= A(-1) \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \begin{pmatrix} A\chi \\ \chi \end{pmatrix} \begin{pmatrix} B\chi \\ A\overline{D}\chi \end{pmatrix} \begin{pmatrix} C\chi \\ A\overline{E}\chi \end{pmatrix} \begin{pmatrix} A\chi \\ \chi \end{pmatrix} \begin{pmatrix} D\chi \\ A\overline{B}\chi \end{pmatrix} \begin{pmatrix} E\chi \\ A\overline{C}\chi \end{pmatrix} \\ &= {}_6\mathbb{P}_5 \left[\begin{matrix} A & B & C & A & D & E \\ A\overline{D} & A\overline{E} & \varepsilon & A\overline{B} & A\overline{C} & 1 \end{matrix}; 1 \right]. \end{aligned}$$

□

2.2 Hypergeometric Character Sums and Galois Representations

To a hypergeometric datum $HD = \{\alpha, \beta; \lambda\}$ consisting of a primitive pair of multi-sets $\alpha = \{a_1, \dots, a_n\}$, $\beta = \{b_1, b_2, \dots, b_n\}$ with $M = \text{lcd}(\alpha \cup \beta)$ and $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \setminus \{0\}$, and a prime ℓ , we associate the ℓ -adic representation $\rho_{HD,\ell}$ of $G(M) = G_{\mathbb{Q}(\zeta_M)}$ as in Theorem 1. We are interested in when it can be extended to a representation of G_K for a subfield K of $\mathbb{Q}(\zeta_M)$, in particular $K = \mathbb{Q}$. This happens if and only if $\rho_{HD,\ell}$ is isomorphic to its conjugation by each element in $\text{Gal}(\mathbb{Q}(\zeta_M)/K)$. This is a standard result by the Clifford theory [15] and Frobenius reciprocity. For related discussions, see [59, Sect. 2.6] by Li, Liu, and Long. If this happens, we say $\{\alpha, \beta; \lambda\}$ can be *extended* to K .

We now explain the Galois action. An element $\tau \in \text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ sends ζ_M to ζ_M^c for some $c = c(\tau) \in (\mathbb{Z}/M\mathbb{Z})^\times$. The conjugation of $\rho_{HD,\ell}$ by τ , denoted by $\rho_{HD,\ell}^\tau$, satisfies

$$\text{Tr} \rho_{HD,\ell}^\tau(\text{Frob}_{\wp}) = \text{Tr} \rho_{HD,\ell}(\text{Frob}_{\tau(\wp)}) = \text{Tr} \rho_{c \cdot HD,\ell}(\text{Frob}_{\wp})$$

for almost all prime ideals $\wp$ of $\mathbb{Z}[\zeta_M]$. Regard the pair α, β in HD as a multi-set of column vectors $\left\{ \begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}$. Here $c \cdot HD$ denotes the datum

$$\{c \cdot \alpha, c \cdot \beta; \lambda\} = \left\{ \begin{pmatrix} ca_1 \\ c \end{pmatrix}, \dots, \begin{pmatrix} ca_n \\ cb_n \end{pmatrix}; \lambda \right\}.$$

This motivates the following definitions.

Definition 1 For a primitive pair α, β with $M = lcd(\alpha \cup \beta)$ as above and $c \in (\mathbb{Z}/M\mathbb{Z})^\times$, the pairs α, β and $c \cdot \alpha, c \cdot \beta$ are *congruent mod $\mathbb{Z}$* if the two multi-sets $\left\{ \binom{ca_1}{c}, \binom{ca_2}{cb_2}, \dots, \binom{ca_n}{cb_n} \right\}$ and $\left\{ \binom{a_1}{1}, \binom{a_2}{b_2}, \dots, \binom{a_n}{b_n} \right\}$ are congruent mod $\mathbb{Z}$. The pair α, β is said to be *self-dual* if it is congruent to the pair $-\alpha, -\beta \bmod \mathbb{Z}$, and it is *defined over a subfield K of $\mathbb{Q}(\zeta_M)$* if it is congruent to $c \cdot \alpha, c \cdot \beta \bmod \mathbb{Z}$ for all $c = c(\tau)$, where $\tau \in \text{Gal}(\mathbb{Q}(\zeta_M)/K)$. A hypergeometric datum $HD = \{\alpha, \beta; \lambda\}$ is said to be *self-dual* if the pair α, β is self-dual and λ is totally real; it is *defined over $\mathbb{Q}$* if the pair α, β is defined over $\mathbb{Q}$ and $\lambda \in \mathbb{Q}^\times$.

Clearly, if the pair α, β in HD above is defined over K and $\lambda \in K^\times$, then the representation $\rho_{HD, \ell}$ can be extended to G_K .

Remark 2 Let K be a number field with the ring of integers $\mathbb{Z}_K$. Since the prime ideals of $\mathbb{Z}_K$ of degree 1 have density one among all prime ideals of $\mathbb{Z}_K$, by the Chebotarev density theorem, if two ℓ -adic representations of G_K ramified only at finitely many prime ideals have the same traces at $\text{Frob}_\wp$ for almost all prime ideals $\wp$ of $\mathbb{Z}_K$ of degree 1, then they are isomorphic as G_K representations provided that they are semisimple. In particular, if $K = \mathbb{Q}(\zeta_M)$ is a cyclotomic field, the semisimplification of an ℓ -adic representation of G_K ramified only at finitely many prime ideals, such as those in Theorems 1 and 8, is determined by its traces at $\text{Frob}_\wp$ for almost all prime ideals $\wp$ of $\mathbb{Z}_K$ above primes $p \equiv 1 \pmod{M}$.

2.3 BCM Representations

When $HD = \{\alpha, \beta; \lambda\}$ with $M = lcd(\alpha \cup \beta)$ is defined over $\mathbb{Q}$, the representation $\rho_{HD, \ell}$ of $G(M)$ can be extended to $G_{\mathbb{Q}}$. The extensions differ by twists by characters of $G_{\mathbb{Q}}$ trivial on $G(M)$. The next result provides such an extension expressed in terms of the hypergeometric sums $H_q(HD)$ defined in [12, Definition 1.1] by Beukers, Cohen and Mellit using Gauss sums (see (21) for the case when HD is defined over $\mathbb{Q}$). They depend only on the elements in α and $\beta \bmod \mathbb{Z}$ and are independent of their order. For $\alpha = \{a_1, \dots, a_n\}$, $\beta = \{1, b_2, \dots, b_n\}$ forming a primitive pair and λ a nonzero element in a finite field $\mathbb{F}_q$ containing M th roots of 1, H_q and $\mathbb{P}$ are related by

$$\mathbb{P}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) = H_q(\alpha, \beta; \lambda; \omega) \cdot \prod_{i=2}^n \omega^{(q-1)a_i} (-1) J(\omega^{(q-1)a_i}, \omega^{(q-1)(-b_i)}), \quad (18)$$

where $J(A, B) = \sum_{x \in \mathbb{F}_q} A(x)B(1-x)$ is the Jacobi sum of characters A and B .

Given a hypergeometric datum $HD' = \{\alpha', \beta'; 1\}$, where $\alpha' = \{c, a\}$, $\beta' = \{1, b\}$ is a primitive pair with $lcd(\alpha' \cup \beta') = M'$, for a finite field $\mathbb{F}_q$ containing M' th roots of 1, we have

$$\mathbb{P}(HD'; \mathbb{F}_q; \omega) = J(\omega^{(q-1)a}, \omega^{(q-1)(b-c-a)}) \quad (19)$$

(from either the definition of the $\mathbb{P}$ -function or (6.11) of [27]). By Theorem 1, the representation $\rho_{HD',\ell}$ is a character of $G(M')$. The relation (18) shows that when the elements in the multi-sets α and β are permuted, the associated representations in Theorem 1 differ by a character of $G(M)$. On the other hand, the representation remains the same if the elements in α and β are shuffled by the same permutation on $\{2, \dots, n\}$. For this reason, we shall regard the pair α, β in HD as a multi-set of column vectors $\left\{ \begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}$ from now on.

Since the Jacobi sum $J(A, B)$ for any $A, B \in \widehat{\mathbb{F}_q^\times}$ has absolute value $q^{1/2}$ as long as A, B and AB are nontrivial, and is equal to -1 if one of A, B is trivial, for the primitive pair α, β above, we have

$$\begin{aligned} \mathcal{J}(\alpha, \beta; \mathbb{F}_q; \omega) &:= \prod_{i=2}^n \omega^{(q-1)a_i} (-1) J(\omega^{(q-1)a_i}, \omega^{(q-1)(-b_i)}) \\ &= (-1)^{m-1} \chi(\alpha, \beta; \mathbb{F}_q) q^{(n-m)/2}, \end{aligned} \quad (20)$$

where m is the number of elements in β which are in $\mathbb{Z}$ and $\chi(\alpha, \beta; \mathbb{F}_q)$ is a root of unity. If, moreover, the pair α, β is self-dual (see Sect. 2.2 for definition), then $\chi(\alpha, \beta; \mathbb{F}_q) = \prod_{i=2}^n \omega^{(q-1)a_i} (-1) = \pm 1$ and $n - m$ is even.

Moreover, if $HD = \{\alpha, \beta; \lambda\}$ is defined over $\mathbb{Q}$, writing $\prod_{j=1}^n \frac{X - e^{2\pi i a_j}}{X - e^{2\pi i b_j}} = \frac{\prod_{j=1}^r (X^{p_j} - 1)}{\prod_{k=1}^s (X^{q_k} - 1)}$ where $p_j, q_k \in \mathbb{Z}_{>0}$ and $p_j \neq q_k$ for all j, k , by Theorem 1.3 of [12], the character sum $H_q(\alpha, \beta; \lambda)$ can be defined in the following way for any finite field $\mathbb{F}_q$ as long as q is coprime to $\text{lcd}(\{1/M, \lambda\})$

$$H_q(\alpha, \beta; \lambda) := \frac{(-1)^{r+s}}{1-q} \sum_{m=0}^{q-2} q^{-s(0)+s(m)} \prod_{j=1}^r \mathbf{g}(\omega^{mp_j}) \prod_{k=1}^s \mathbf{g}(\omega^{-mq_k}) \omega(N^{-1}\lambda), \quad (21)$$

where $\mathbf{g}(\chi)$ denotes the Gauss sum of the character χ , $N = (-1)^{q_1+\dots+q_s} \frac{p_1^{p_1} \dots p_r^{p_r}}{q_1^{q_1} \dots q_s^{q_s}}$ and $s(m)$ is the multiplicity of $X - e^{2\pi i m/(q-1)}$ in the factorization of $\gcd(\prod_{j=1}^r (X^{p_j} - 1), \prod_{k=1}^s (X^{q_k} - 1))$. Further, since H_q is $\mathbb{Q}$ -valued, it is independent of the choice of ω .

Theorem 8 (Katz, Beukers–Cohen–Mellit) *Let $\alpha = \{a_1, \dots, a_n\}$, $\beta = \{1, b_2, \dots, b_n\}$ be a primitive pair with $M = \text{lcd}(\alpha \cup \beta)$ and let $\lambda \in \mathbb{Z}[1/M] \setminus \{0\}$. Suppose the hypergeometric datum $HD = \{\alpha, \beta; \lambda\}$ is defined over $\mathbb{Q}$ (see Definition 1). Assume that exactly m elements in β are in $\mathbb{Z}$. Then, for each prime ℓ , there exists an ℓ -adic representation $\rho_{HD,\ell}^{BCM}$ of $G_{\mathbb{Q}}$ with the following properties:*

- (i) $\rho_{HD,\ell}^{BCM}|_{G(M)} \cong \rho_{HD,\ell}$.

(ii) For any prime $p \nmid \ell \cdot M$ such that $\text{ord}_p \lambda = 0$,

$$\text{Tr } \rho_{HD,\ell}^{BCM}(\text{Frob}_p) = \phi(M, a_1)(\text{Frob}_p) \chi(\alpha, \beta; \mathbb{F}_p) H_p(\alpha, \beta; 1/\lambda) \cdot p^{(n-m)/2} \in \mathbb{Z}. \quad (22)$$

(iii) When $\lambda = 1$, $\rho_{HD,\ell}^{BCM}$ is $(n-1)$ -dimensional and it has a subrepresentation, denoted by $\rho_{HD,\ell}^{BCM, \text{prim}}$, of dimension $2 \lfloor \frac{n-1}{2} \rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even). All roots of the characteristic polynomial of $\rho_{HD,\ell}^{BCM, \text{prim}}(\text{Frob}_p)$ have absolute value $p^{(n-1)/2}$.

Note that $n-m$ is even, resulting from the primitive and self-dual assumptions on the pair α, β .

The character $\phi(M, a_1)$ in Remark 1 extends to a character of $G_{\mathbb{Q}}$. The character $\phi(M, a_1)$ in Theorem 8 refers to the extension with minimal conductor. When it is nontrivial, that is, when $\text{ord}_2 M = -\text{ord}_2 a_1 = r \geq 1$, it has conductor 2^{r+1} . In particular, when $\text{ord}_2 M = -\text{ord}_2 a_1 = 1$, $\phi(M, a_1)(\text{Frob}_p) = \left(\frac{-1}{p}\right)$ is given by the Legendre symbol at odd primes p .

For the pairs (c, f) in (4), the characters χ and ϕ in Theorem 8 at unramified odd primes p are given by $\phi(M(c, f), a_1)(\text{Frob}_p) = \left(\frac{-1}{p}\right) = \chi(\alpha_6(c, f), \beta_6(c, f); \mathbb{F}_p) = \chi(\alpha_4(f), \beta_4(c); \mathbb{F}_p)$ so that

$$\text{Tr } \rho_{HD_i(c,f),\ell}^{BCM}(\text{Frob}_p) = H_p(HD_i(c, f)) \cdot p^{(n-m)/2} \quad \text{for } i = 1, 2. \quad (23)$$

2.4 Modularity

To obtain the modularity result for degree-2 Galois representations of $G_{\mathbb{Q}}$, we shall use the following version of the modularity theorem; see Theorem 2.1.4 of [5].

Theorem 9 *Given a prime ℓ and a 2-dimensional absolutely irreducible representation ρ of $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_{\ell}$ that is odd, unramified at almost all primes, and its restriction to a decomposition subgroup D_{ℓ} at ℓ is crystalline with Hodge–Tate weight $\{0, r\}$ where $1 \leq r \leq \ell-2$ and $\ell+1 \nmid 2r$, then ρ is modular and corresponds to a weight $r+1$ holomorphic Hecke eigenform.*

The actual identification of the target modular form is carried out by computing the trace of the representation at various Frobenius conjugacy classes Frob_p . In the next subsection, we discuss key issues involved in computation. Since representations from Theorem 8 have integral traces, we use the following theorem of Serre (cf. [51, Sect. 4.8] by Serre or [20, Theorem 2.2] by Dieulefait) to narrow the search for the corresponding modular forms.

Theorem 10 (Serre) *Suppose the trace of the representation ρ of $G_{\mathbb{Q}}$ in Theorem 9 is $\mathbb{Z}$ -valued. Then the p -exponents of the conductor of ρ are bounded by 8 for $p = 2$, by 5 for $p = 3$, and by 2 for all other bad primes.*

2.5 Some Computational Issues

Given $\alpha = \{a_1, \dots, a_n\}$ and $\beta = \{b_1, \dots, b_n\}$ with $a_i, b_j \in \mathbb{Q} \cap [0, 1)$ and $M = \text{lcd}(\alpha \cup \beta)$, the following step function on the interval $[0, 1)$ is introduced in [38] by the second author:

$$e_{\alpha, \beta}(x) := \sum_{i=1}^n -\lfloor a_i - x \rfloor - \lfloor x + b_i \rfloor.$$

The value of $e_{\alpha, \beta}(x)$ jumps up (resp. down) only at a_i (resp. $1 - b_j$). When p is an odd prime not dividing M , $0 \leq k < p - 1$ is an integer and the pair α, β is defined over $\mathbb{Q}$, $e_{\alpha, \beta}(\frac{k}{p-1})$ gives the collective exponent of p in the k th summand of $H_p(\alpha, \beta; \lambda)$ recalled in (21). Moreover when the pair α, β is defined over $\mathbb{Q}$, the graph of $e_{\alpha, \beta}(x)$ is independent of any integer c prime to M .

The weight $w(HD)$ of a datum $HD = \{\alpha, \beta; \lambda\}$ is defined as

$$w(HD) = w(\alpha, \beta) := \max e_{\alpha, \beta}(x) - \min e_{\alpha, \beta}(x). \quad (24)$$

When $\rho_{HD, \ell}$ can be extended to $G_{\mathbb{Q}}$, $w(c\alpha, c\beta)$ is independent of c , although the graph of $e_{c\alpha, c\beta}(x)$ may depend on c .

From the definition, one can implement $H_p(\alpha, \beta; \lambda)$ in any computational package of choice. In some of our computation and posted data, we use Sagemath. For HD defined over $\mathbb{Q}$, there is an efficient Magma program called “Hypergeometric Motives over $\mathbb{Q}$ ” implemented by Watkins which computes the characteristic polynomial of $\rho_{\{\alpha, \beta; \lambda\}, \ell}^{BCM}[t]$ (resp. $\rho_{\{\alpha, \beta; \lambda\}, \ell}^{BCM, \text{prim}}[t]$) at Fr_p , the inverse of Frob_p , for $p \nmid M\ell$ efficiently when $\lambda \neq 0, 1$ (resp. $\lambda = 1$), where

$$t = -\min\{e_{\alpha, \beta}(x) \mid 0 \leq x < 1\} - \frac{n-m}{2}, \quad (25)$$

which is an integer. We let $\rho[t]$ denote the weight- t Tate twist of a representation ρ of $G_{\mathbb{Q}}$.

Example 1 For $\alpha = \{\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$, $\beta = \{1, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\}$, the graph of the function is as follows.

In this case, $\min e_{\alpha, \beta} = -1$, $\max e_{\alpha, \beta} = 2$, $w(HD) = 2 - (-1) = 3$, $n = 5$, $m = 1$, so $t = -\min e_{\alpha, \beta} - \frac{n-m}{2} = 1 - \frac{5-1}{2} = -1$. When $\lambda = 1$, $\dim \rho_{HD, \ell}^{BCM, \text{prim}} =$

$2\lfloor \frac{5-1}{2} \rfloor = 4$. We exhibit below the commands used in the Magma program to compute the characteristic polynomial of $\rho_{HD,\ell}^{BCM,prim}[-1]$ at Fr_7 :

```
H:=HypergeometricData([1/2,1/5,2/5,3/5,4/5],[1,1/10,3/10,7/10,9/10]);
```

```
Factorization(EulerFactor(H,1,7));
```

The output is

```
< 2401*$.1^4 - 88*$.1^2 + 1, 1 >
```

It confirms that $\dim \rho_{HD,\ell}^{BCM,prim} = 4$ and the roots of the characteristic polynomial of $\rho_{HD,\ell}^{BCM,prim}[-1]$ at Fr_7 all have absolute value $1/7$ under any embedding, which is $1/p^{(n-1)/2+t}$ at $p = 7$.

Example 2 We now explain how to identify the weight-4 modular form corresponding to $HD_2(\frac{1}{2}, \frac{1}{2})$. Using Magma, we obtain the following output of the first few characteristic polynomials of Fr_p .

```
[3, 27*$.1^2 + 4*$.1 + 1],
[5, 125*$.1^2 + 2*$.1 + 1],
[7, 343*$.1^2 - 24*$.1 + 1],
[11, 1331*$.1^2 + 44*$.1 + 1],
```

From the list, we know the corresponding modular form has weight 4. As $\text{lcd}(HD_2(\frac{1}{2}, \frac{1}{2})) = 2$, the level of the modular form is a power of 2 from which we identify the target modular form

$$f_{8.4.a.a}(\tau) = q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - 44q^{11} + \dots$$

Example 3 Here we consider another example $HD = \{\{\frac{1}{3}, \frac{2}{3}\}, \{1, \frac{1}{2}\}; -1\}$, in which each of $\{\frac{1}{3}, \frac{2}{3}\}$ and $\{1, \frac{1}{2}\}$ is defined over $\mathbb{Q}$ as a multi-set, but it is not so as a pair (see Definition 1).

A partial output is

```
[7, $.1^2 + $.1 + 1],
[11, $.1^2 + $.1 + 1],
[13, -$.1^2 + 1],
[17, -$.1^2 + 1],
[19, -$.1^2 + 1],
[23, -$.1^2 + 1],
[29, $.1^2 - 2*$.1 + 1]
```

From the above data, we first read off the determinant being χ_{-6} . Looking more closely we see that it corresponds to the weight-1 Hecke eigenform

$$f_{216.1.h.b}(\tau) = q + q^2 + q^4 - q^5 - q^7 + q^8 - q^{10} - q^{11}$$

$$-q^{14} + q^{16} - q^{20} - q^{22} - q^{28} + 2q^{29} + \dots$$

More explicitly, by Theorems 8.4 and 8.11 of [27] when $p \equiv 1 \pmod{3}$

$$\begin{aligned} H_p \left(\left\{ \frac{1}{3}, \frac{2}{3} \right\}, \left\{ 1, \frac{1}{2} \right\}; -1; \omega \right) \\ = \overline{\omega}^{\frac{p-1}{3}} (2) H_p \left(\left\{ \frac{1}{3}, \frac{5}{6} \right\}, \left\{ 1, \frac{1}{2} \right\}; 1/2; \omega \right) \\ = \begin{cases} 0, & \text{if } p \equiv 13, 19 \pmod{24}, \\ \overline{\omega}^{\frac{p-1}{3}} (2+a) + \overline{\omega}^{\frac{p-1}{3}} (2-a), & \text{if } p \equiv 1, 7 \pmod{24}, \end{cases} \end{aligned}$$

where a is a fixed square root of 2 in $\mathbb{F}_p$; and from the conversion between $\mathbb{P}$ and H_p we get, for $p \equiv 1 \pmod{3}$

$$\mathbb{P} \left(\left\{ \frac{1}{3}, \frac{2}{3} \right\}, \left\{ 1, \frac{1}{2} \right\}; -1; p; \omega \right) = J(\overline{\omega}^{\frac{p-1}{3}}, \omega^{\frac{p-1}{2}}) H_p \left(\left\{ \frac{1}{3}, \frac{2}{3} \right\}, \left\{ 1, \frac{1}{2} \right\}; -1; \omega \right).$$

It indicates that $\rho_{HD,\ell}$, which is a representation of $G(3)$, can not be directly extended to $G_{\mathbb{Q}}$; however, after twisting by a suitable character of $G(3)$, an extension exists.

3 Galois Representations Associated with Hypergeometric Data

3.1 The Whipple ${}_7F_6(1)$ Formula

We explain how the special form of the Whipple formula in Sect. 1.2 is derived. The well-known Whipple ${}_7F_6(1)$ formula asserts that

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} a & 1 + \frac{a}{2} & c & d & e & f & g \\ \frac{a}{2} & 1 + a - c & 1 + a - d & 1 + a - e & 1 + a - f & 1 + a - g \end{matrix}; 1 \right] \\ = \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)} \\ \cdot {}_4F_3 \left[\begin{matrix} a & e & f & g \\ e+f+g-a & 1+a-c & 1+a-d \end{matrix}; 1 \right], \end{aligned} \quad (26)$$

when both sides terminate; see Theorem 3.4.4 of the book [3] by Andrews, Askey, and Roy for backgrounds and details. The ${}_7F_6(1)$ on the left-hand side is well posed, meaning the upper and lower parameters sum to the same value (equal to $1+a$ in our case) in each column. By Bailey [6], the left-hand side can be also computed by a contour integral of Barnes type, which is related to the irrationality of $\zeta(3)$, see [64] by Zudilin for the related discussions. To the corresponding hypergeometric systems, Katz [32] attached ℓ -adic sheaves on the projective line with three points $0, 1, \infty$ removed.

For the hypergeometric sheaves arising from the data in the Whipple formula to be self-dual, we impose additional assumptions:

$$a = \frac{1}{2}, \quad c + d = 1, \quad f + g = 1, \quad \text{and } e = \frac{1-p}{2}$$

for an odd natural number p so that both sides converge.³ Whipple's formula now becomes (3) in the Introduction.

Recall that the hypergeometric data HD_1 and HD_2 are given by

$$\begin{aligned} HD_1(c, f) &= \left\{ \alpha_6(c, f) := \left\{ \frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f \right\}, \right. \\ &\quad \left. \beta_6(c, f) := \left\{ 1, \frac{3}{2} - c, \frac{1}{2} + c, 1, \frac{3}{2} - f, \frac{1}{2} + f \right\}; 1 \right\} \\ HD_2(c, f) &= \left\{ \alpha_4(f) := \left\{ \frac{1}{2}, \frac{1}{2}, f, 1-f \right\}, \beta_4(c) := \left\{ 1, 1, \frac{3}{2} - c, \frac{1}{2} + c \right\}; 1 \right\} \end{aligned}$$

with $c, f \in \mathbb{Q} \cap (0, 1)$ as in §1.2. They are both self-dual from the construction. Let

$$M = M(c, f) = lcd(\alpha_6(c, f) \cup \beta_6(c, f)) = lcd(\alpha_4(c, f) \cup \beta_4(c, f)).$$

Then both $\alpha_6(c, f)$ and $\beta_6(c, f) = 1 + \frac{1}{2} - \alpha_6(c, f)$ in HD_1 are defined over $\mathbb{Q}(\zeta_M + \zeta_M^{-1})$. In particular, only 14 un-ordered couples (c, f) give rise to HD_1 defined over $\mathbb{Q}$:

$$\text{either } c, f \in \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \right\}, \quad \text{or } (c, f) = \left(\frac{1}{5}, \frac{2}{5} \right), \left(\frac{1}{8}, \frac{3}{8} \right), \left(\frac{1}{10}, \frac{3}{10} \right)$$

as enumerated in [40]. Among them, only seven pairs of (c, f) give rise to primitive HD_1 , as listed in (4).

3.2 Proof of Theorems 2 and 3

Let

$$\alpha_3(f) = \left\{ \frac{1}{2}, f, 1-f \right\}, \quad \beta_3(c) = \left\{ 1, \frac{3}{2} - c, \frac{1}{2} + c \right\} \quad (27)$$

which form a primitive pair with $M(c, f) = lcd(\alpha_3(f) \cup \beta_3(c))$. Motivated by the finite field Clausen formula (Theorem 7), we define the multi-sets

$$\alpha_2(c, f) := \left\{ \frac{1+2f-2c}{4}, \frac{3-2f-2c}{4} \right\}, \quad \beta_2(c) := \left\{ 1, \frac{3}{2} - c \right\}$$

³ Here, e is set to be $\frac{1-p}{2}$ instead of the target $\frac{1}{2}$ so that the Gamma quotient on the right-hand side of (26) is a finite value.

with $N(c, f) = lcd(\alpha_2(c, f) \cup \beta_2(c))$. Then either $N(c, f) = 2M(c, f)$ or $N(c, f) = M(c, f)$.

Fix a prime ℓ . By Theorem 1, for any nonzero $\lambda \in \mathbb{Z}[\zeta_{M(c, f)}, 1/M(c, f)]$, there exists an ℓ -adic representation

$$\rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell} : G(M(c, f)) \rightarrow GL_{\overline{\mathbb{Q}}_\ell}(W_\lambda(c, f)) \quad (28)$$

such that at any prime ideal $\wp$ of $\mathbb{Z}[\zeta_{M(c, f)}, 1/(N(c, f)\ell)]$ with $\text{ord}_\wp \lambda = 0$

$$\text{Tr} \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell}(\text{Frob}_\wp) = \phi_\wp(-1) \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_\wp). \quad (29)$$

Moreover, $\dim W_1(c, f) = 2$ and $\dim W_\lambda(c, f) = 3$ when $\lambda \neq 0, 1$.

We first deal with the $\lambda = 1$ case. Let $HD_3(c, f) = \{\alpha_3(f), \beta_3(c); 1\}$, and write $\rho_{HD_3(c, f), \ell}$ for the 2-dimensional representation $\rho_{\{\alpha_3(f), \beta_3(c); 1\}, \ell}$.

Let $\wp$ be a prime ideal of $\mathbb{Z}[\zeta_{N(c, f)}, 1/(N(c, f)\ell)]$. A similar formula for $\text{Tr} \rho_{HD_3(c, f), \ell}(\text{Frob}_\wp)$ holds. The character $\omega_\wp^{(N(\wp)-1)(f+\frac{3}{2}-c)}$ is a square in $\widehat{\kappa_\wp^\times}$ by the definition of $N(c, f)$. Hence, we can apply Theorem 7 to get more information on $\mathbb{P}(\alpha_3(f), \beta_3(c); 1; \kappa_\wp)$. It follows from (15) and (19) that $\rho_{HD_3(c, f), \ell}|_{G(N(c, f))}$ is a direct sum of two explicit characters ξ_1 and ξ_2 of $G(N(c, f))$ associated with Jacobi sums, see [57]. Further, Lemma 1 implies $\det \rho_{HD_3(c, f), \ell}|_{G(N(c, f))} = \epsilon_\ell^2$. Since $G(N(c, f))$ has index at most 2 in $G(M(c, f))$, there is a character $\psi_{c, f}$ of $G(M(c, f))$ of order dividing 2 such that $\det \rho_{HD_3(c, f), \ell} = \psi_{c, f} \epsilon_\ell^2$. We record this in

Proposition 3 *There is a character $\psi_{c, f}$ of $G(M(c, f))$ of order dividing 2 such that*

$$\det \rho_{HD_3(c, f), \ell} = \psi_{c, f} \epsilon_\ell^2.$$

Moreover, $\det \rho_{HD_3(c, f), \ell}|_{G(N(c, f))} = \epsilon_\ell^2$.

For the seven pairs (c, f) listed in (4), we have more detailed information about the representation $\rho_{HD_3(c, f), \ell}$, as shown below. Among them, $N(c, f) = M(c, f)$ only happens when $(c, f) = (\frac{1}{2}, \frac{1}{3})$.

Proposition 4 *Let ℓ be a prime. For each pair (c, f) in (4), $\rho_{HD_3(c, f), \ell}$ is invariant under the twist by the quadratic character $\chi_{-1}|_{G(M(c, f))}$ when $(c, f) \neq (\frac{1}{2}, \frac{1}{3})$. Further, each $\rho_{HD_3(c, f), \ell}$ can be extended to a representation of $G_{\mathbb{Q}}$, which is modular and invariant under a quadratic twist. Each extension has determinant $\psi_{c, f} \epsilon_\ell^2$, where $\psi_{c, f}$ is a quadratic character of $G_{\mathbb{Q}}$ given in the table below, extending the character $\psi_{c, f}$ of $G(M(c, f))$ in Prop. 3. Listed below are the modular forms corresponding to $\rho_{HD_3(c, f), \ell}^{BCM}$ when $HD_3(c, f)$ is defined over $\mathbb{Q}$ and one of the extensions for the last two cases.*

(c, f)	$w(HD_3)$	$Tr\rho_{\{\alpha_3(f), \beta_3(c); 1\}, \ell}(Frob_p)$	$\det \rho_{\{\alpha_3(f), \beta_3(c); 1\}, \ell} = \psi_{c, f} \epsilon_\ell^2$	CM
$(\frac{1}{2}, \frac{1}{2})$	3	$a_p(\eta(4\tau)^6)$	$\chi_{-1}\epsilon_\ell^2$	χ_{-1}
$(\frac{1}{2}, \frac{1}{3})$	3	$a_p(\eta(2\tau)^3\eta(6\tau)^3)$	$\chi_{-3}\epsilon_\ell^2$	χ_{-3}
$(\frac{1}{3}, \frac{1}{3})$	3	$a_p(f_{36,3,d,b})$	$\chi_{-1}\epsilon_\ell^2$	χ_{-1}
$(\frac{1}{2}, \frac{1}{6})$	3	$\left(\frac{-3}{p}\right)a_p(\eta(4\tau)^6)$	$\chi_{-1}\epsilon_\ell^2$	χ_{-1}
$(\frac{1}{6}, \frac{1}{6})$	1	$p \cdot a_p(\eta(12\tau)^2)$	$\chi_{-1}\epsilon_\ell^2$	χ_{-1}, χ_{-3}
$(\frac{1}{5}, \frac{2}{5})$	3	$a_p(f_{20,3,d,a})$	$\chi_{-5}\epsilon_\ell^2$	χ_{-5}
$(\frac{1}{10}, \frac{3}{10})$	1	$p \cdot a_p(\eta(4\tau)\eta(20\tau))$	$\chi_{-5}\epsilon_\ell^2$	χ_{-1}, χ_{-5}

Proof Write N for $N(c, f)$ and M for $M(c, f)$ when there is no ambiguity. As noted above, $\rho_{HD_3, \ell}|_{G(N)}$ is a direct sum of two characters ξ_1 and ξ_2 of $G(N)$. When $(c, f) \neq (\frac{1}{6}, \frac{1}{6})$ or $(\frac{1}{10}, \frac{3}{10})$, ξ_1 is not equal to ξ_2 . Furthermore, by (17), when $N = 2M$, that is, $(c, f) \neq (\frac{1}{2}, \frac{1}{3})$, $\mathbb{P}(HD_3, q) = 0$ when $q \equiv 3 \pmod{4}$. This implies that $\rho_{HD_3, \ell}$ is invariant under twist by $\chi_{-1}|_{G(M)}$.

For the first five pairs, HD_3 is defined over $\mathbb{Q}$. The determinant of $\rho_{HD_3, \ell}^{BCM}$, by Theorem 8, can be determined explicitly as it is ϵ_ℓ^2 times a character of order at most 2 and conductor dividing 24. They turn out to be either $\chi_{-1}\epsilon_\ell^2$ or $\chi_{-3}\epsilon_\ell^2$, as listed in the table. For example, when $(c, f) = (\frac{1}{2}, \frac{1}{3})$, the determinant at $Frob_p$ equals $-25, 49, -121, 169$ for $p = 5, 7, 11, 13$, respectively. One sees that the determinant is $\chi_{-3}\epsilon_\ell^2$. This shows that $\rho_{HD_3, \ell}^{BCM}$ is odd. When $(c, f) \neq (\frac{1}{6}, \frac{1}{6})$, from $\xi_1 \neq \xi_2$, we know $\rho_{HD_3, \ell}^{BCM}$ is irreducible. When $(c, f) = (\frac{1}{6}, \frac{1}{6})$, by (21) and the reflection formula for Gauss sum, we have for any prime p not dividing 6

$$H_p(HD_3) = \frac{1}{p-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \frac{g(\chi^6)g(\overline{\chi}^3)}{g(\chi^3)} \chi(-2^{-6}).$$

When $p \equiv 2 \pmod{3}$, the map $\chi \mapsto \chi^3$ gives an automorphism of $\widehat{\mathbb{F}_p^\times}$. It follows that

$$\begin{aligned} H_p(HD_3) &= \frac{1}{p-1} \sum_{\chi} \frac{g(\chi^2)g(\overline{\chi})}{g(\chi)} \chi(-2^{-2}) = \frac{1}{p-1} \sum_{\chi} \frac{g(\chi)g(\phi_p \chi)g(\overline{\chi})}{g(\phi_p \chi)g(\chi)} \chi(-1) \\ &= \frac{1}{p-1} \sum_{\chi} J(\phi_p \chi, \overline{\chi}) \chi(-1) = \frac{1}{p-1} \sum_x \phi_p(x) \sum_{\chi} \chi(-x) \overline{\chi}(1-x) \\ &= 0. \end{aligned}$$

To obtain the second equality above, we use the duplication formula for Gauss sums. If $\rho_{HD_3, \ell}^{BCM}$ is reducible, then it is isomorphic to $\xi \oplus \xi\chi_{-3}$ for some 1-dimensional representation ξ of $G_{\mathbb{Q}}$. Meanwhile, as $\rho_{HD_3, \ell}$ is invariant when twisted by χ_{-1} , we have $(\xi \oplus \xi\chi_{-3})|_{G(6)} \cong (\xi \oplus \xi)|_{G(6)} \cong (\xi\chi_{-1} \oplus \xi\chi_{-1})|_{G(6)}$, which is impossible.

For the remaining two pairs, we have $\rho_{HD_3, \ell} \cong (\chi_{-1})|_{G(M)} \otimes \rho_{HD_3, \ell}$ as noted before. In these two cases $M = 10$ and the field $\mathbb{Q}(\zeta_M)$ has degree 4 over $\mathbb{Q}$. We show that $\rho_{HD_3, \ell}$ can be extended to $G_{\mathbb{Q}}$ for the case $(c, f) = (\frac{1}{5}, \frac{2}{5})$; the same conclusion holds for $(c, f) = (1/10, 3/10)$ by a similar argument. The group $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is

generated by the automorphism $\tau : \zeta_5 \mapsto \zeta_5^3$. It maps $\alpha_3(\frac{2}{5}) = \{\frac{1}{2}, \frac{2}{5}, \frac{3}{5}\}$ to $\{\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\}$ and $\beta_3(\frac{1}{5}) = \{1, \frac{13}{10}, \frac{7}{10}\}$ to $\{1, \frac{9}{10}, \frac{11}{10}\}$. By Proposition 1, for any prime power q coprime to M ,

$$\mathbb{P}(\alpha_3(f), \beta_3(c); 1; \mathbb{F}_q) = \phi_q(-1) \mathbb{P}\left(\left\{\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right\}, \left\{1, \frac{9}{10}, \frac{1}{10}\right\}; 1; \mathbb{F}_q\right).$$

In terms of representations, this means that $\rho_{HD_3, \ell} \cong \chi_{-1}|_{G(M)} \otimes \rho_{HD_3, \ell}^\tau$, where $\rho_{HD_3, \ell}^\tau$ is the conjugate of $\rho_{HD_3, \ell}$ by τ . As shown above, $\rho_{HD_3, \ell} \otimes \chi_{-1}|_{G(M)} \cong \rho_{HD_3, \ell}$, therefore $\rho_{HD_3, \ell} \cong \rho_{HD_3, \ell}^\tau$. This proves that $\rho_{HD_3, \ell}$ can be extended to $G_{\mathbb{Q}}$, denoted by $\rho_{HD_3, \ell}^{BCM}$ by abuse of notation.

For $(\frac{1}{5}, \frac{2}{5})$, $\xi_1 \neq \xi_2$, which implies that $\rho_{HD_3, \ell}^{BCM}$ is irreducible. By Clifford theory, ξ_1 extends to an index-2 subgroup G_K of $G_{\mathbb{Q}}$ so that $\rho_{HD_3, \ell}^{BCM} = \text{Ind}_{G_K}^{G_{\mathbb{Q}}} \xi_1$. Since the map $\iota : \zeta_4 \mapsto -\zeta_4$ in $\text{Gal}(\mathbb{Q}(\zeta_{20})/\mathbb{Q}(\zeta_{10}))$ swaps ξ_1 and ξ_2 , K is not contained in $\mathbb{Q}(\zeta_{10})$. On the other hand, $\mathbb{Q}(\zeta_{20})$ contains three quadratic subfields, so K is one of the two imaginary extensions outside $\mathbb{Q}(\zeta_{10})$. This shows that $\rho_{HD_3, \ell}^{BCM}$ is odd.

For $(\frac{1}{10}, \frac{3}{10})$, by formula (15) and simplification using the reflection and multiplication formulas of Gauss sums, we get $\xi_1 = \xi_2 = \left(\frac{\sqrt{5}}{\cdot}\right)_2 \epsilon_{\ell}|_{G(20)}$, where $\left(\frac{\sqrt{5}}{\cdot}\right)_2$ is a Hilbert quadratic norm residue symbol on $\mathbb{Q}(\zeta_{20})$. Clearly, ξ_1 extends to $G_{\mathbb{Q}(\sqrt{5})}$ and not $G_{\mathbb{Q}}$. Hence, $\rho_{HD_3, \ell}^{BCM}$ is irreducible and it equals $\text{Ind}_{G_{\mathbb{Q}(\sqrt{5})}}^{G_{\mathbb{Q}}} \xi_1$. Moreover, viewed as an idele class character of $\mathbb{Q}(\sqrt{5})$ by class field theory, $\left(\frac{\sqrt{5}}{\cdot}\right)_2$ is trivial at one real place of $\mathbb{Q}(\sqrt{5})$ and is the sign character at the other real place, resulting from the two real embeddings of $\sqrt{5}$. This shows that $\rho_{HD_3, \ell}^{BCM}$ is odd.

In conclusion, for each of the seven cases, $\rho_{HD_3, \ell}^{BCM}$ is odd, irreducible, and, by Clifford theory, induced from a character of an index-2 subgroup of $G_{\mathbb{Q}}$. Hence, it is modular by the work of Hecke. When HD_3 is defined over $\mathbb{Q}$, we compute the characteristic polynomials of each representation at suitable Frobenius conjugacy classes and identify the corresponding modular forms using the method described in Sect. 2.5. For the other two cases, from (15), we identify the corresponding modular forms. For example, for primes $p < 200$ and $p \equiv 1 \pmod{10}$, $\mathbb{P}(HD_3(\frac{1}{5}, \frac{2}{5}); 1; \mathbb{F}_p) = a_p(f_{20.3.d.a}) = a_p(f_{100.3.b.c})$. Here, $f_{20.3.d.a}$ is a cusp form of character χ_{-5} with coefficients in $\mathbb{Z}$ while the coefficients of $f_{100.3.b.c}$ are no longer in $\mathbb{Z}$. $\square$

As explained above Proposition 3, $\rho_{HD_3, \ell}|_{G(N(c, f))}$ is a direct sum of two 1-dimensional representations and $\det \rho_{HD_3, \ell}(\text{Frob}_{\wp}) = \psi_{c, f}(\text{Frob}_{\wp}) N(\wp)^2$ at prime ideals $\wp$ of $\mathbb{Z}[\zeta_{M(c, f)}, 1/M(c, f)\ell]$. Consequently,

$$\text{Tr Alt}^2 \rho_{HD_3, \ell}(\text{Frob}_{\wp}) = \det \rho_{HD_3, \ell}(\text{Frob}_{\wp}) = \psi_{c, f}(\text{Frob}_{\wp}) N(\wp)^2; \quad (30)$$

$$\text{Tr Sym}^2 \rho_{HD_3, \ell}(\text{Frob}_{\wp}) + \det \rho_{HD_3, \ell}(\text{Frob}_{\wp}) = \mathbb{P}(HD_3; \kappa_{\wp})^2, \quad (31)$$

where $\text{Alt}^2 \rho$ and $\text{Sym}^2 \rho$ stand for the alternating square and symmetric square of a representation ρ , respectively.

Next, we consider the case when $\lambda \neq 1$.

Proposition 5 *Let ℓ be a prime and $c, f \in \mathbb{Q}^\times$ be such that $\alpha_3(f) = \{\frac{1}{2}, f, 1 - f\}$, $\beta_3(c) = \{1, \frac{3}{2} - c, \frac{1}{2} + c\}$ form a primitive pair. Let $N(c, f) = \text{lcd}(\alpha_2(c, f) \cup \beta_2(c))$ as before. Then for any $\lambda \in \mathbb{Z}[\zeta_{M(c, f)}] \setminus \{0, 1\}$ the 3-dimensional ℓ -adic Galois representation $\rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}$ as in (28) possesses the following properties at a prime ideal $\wp$ of $\mathbb{Z}[\zeta_{N(c, f)}, 1/N(c, f)\ell]$:*

$$\begin{aligned} & \text{Tr Alt}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}(\text{Frob}_\wp) \\ &= \phi_\wp(1 - 1/\lambda)N(\wp) \cdot \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_\wp) \end{aligned} \quad (32)$$

and

$$\begin{aligned} & \text{Tr Sym}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell} \\ &= \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_\wp)^2 - \phi_\wp(1 - 1/\lambda)N(\wp) \cdot \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_\wp), \end{aligned} \quad (33)$$

where $\phi_\wp(\cdot)$ is the quadratic character of $\kappa_\wp$. Further, $\text{Sym}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}$ contains a 1-dimensional subrepresentation isomorphic to $\epsilon_\ell^2 | G(N(c, f))$.

Proof Write ρ for the 2-dimensional ℓ -adic representation $\rho_{\{\alpha_2(c, f), \beta_2(c); \lambda\}, \ell}$ of $G(N(c, f))$ and by $\bar{\rho}$ the representation $\rho_{\{\bar{\alpha}_2(c, f), \bar{\beta}_2(c); \lambda\}, \ell}$ of the same group, where $\bar{\alpha}_2(c, f) = 1 - \alpha_2(c, f)$ and $\bar{\beta}_2(c) = \{1, \frac{1}{2} + c\}$. Then at a prime ideal $\wp$ of $\mathbb{Z}[\zeta_{N(c, f)}, 1/N(c, f)\ell]$ such that $\text{ord}_\wp \lambda = 0$, we have

$$\text{Tr} \rho(\text{Frob}_\wp) = -\omega^{(q-1)\frac{1+2f-2c}{4}}(-1)\mathbb{P}(\alpha_2(c, f), \beta_2(c); 1/\lambda; \kappa_\wp)$$

and a similar conclusion for $\bar{\rho}$. The eigenvalues $u_\wp, v_\wp$ of $\rho(\text{Frob}_\wp)$ have absolute value $N(\wp)^{1/2}$, and those of $\bar{\rho}(\text{Frob}_\wp)$ are $N(\wp)/u_\wp$ and $N(\wp)/v_\wp$. Thus the 4 eigenvalues of $\rho \otimes \bar{\rho}(\text{Frob}_\wp)$ are $N(\wp)u_\wp/v_\wp, N(\wp)v_\wp/u_\wp, N(\wp), N(\wp)$. Write $\det \rho = \chi_\rho \epsilon_\ell$ for a finite-order character χ_ρ of $G(N(c, f))$. Then $u_\wp v_\wp = \chi_\rho(\text{Frob}_\wp)N(\wp)$ implies $N(\wp)/u_\wp = \chi_\rho(\text{Frob}_\wp)^{-1}v_\wp$ and $N(\wp)/v_\wp = \chi_\rho(\text{Frob}_\wp)^{-1}u_\wp$. Since the eigenvalues of $\text{Sym}^2 \rho(\text{Frob}_\wp)$ are $u_\wp^2, v_\wp^2, u_\wp v_\wp$, we get

$$\chi_\rho^{-1}(\text{Frob}_\wp) \text{Tr Sym}^2 \rho(\text{Frob}_\wp) = \text{Tr} \rho \otimes \bar{\rho}(\text{Frob}_\wp) - N(\wp).$$

Theorem 7 then implies that

$$\text{Tr} \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}(\text{Frob}_\wp) = \phi_\wp(1 - 1/\lambda) \chi_\rho^{-1}(\text{Frob}_\wp) \text{Tr Sym}^2 \rho(\text{Frob}_\wp).$$

In other words, as representations of $G(N(c, f))$, $\rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}$ is isomorphic to $\chi_\rho^{-1} \otimes \text{Sym}^2 \rho$ twisted by the Hilbert quadratic norm residue symbol $\left(\frac{1-1/\lambda}{\cdot}\right)_2$ on $\mathbb{Q}(\zeta_{N(c, f)})$. Hence the eigenvalues of $\rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}(\text{Frob}_\wp)$ are

$$c \cdot N(\wp)u_\wp/v_\wp, \quad c \cdot N(\wp)v_\wp/u_\wp, \quad c \cdot N(\wp), \quad \text{where } c = \phi_\wp(1 - 1/\lambda).$$

Therefore,

$$\mathrm{Sym}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))} \simeq \mathrm{Sym}^2(\chi_\rho^{-1} \otimes \mathrm{Sym}^2 \rho) \simeq \chi_\rho^{-2} \otimes \mathrm{Sym}^2(\mathrm{Sym}^2 \rho)$$

and

$$\mathrm{Alt}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))} \simeq \mathrm{Alt}^2(\chi_\rho^{-1} \otimes \mathrm{Sym}^2 \rho).$$

The last relation implies that the eigenvalues of $\mathrm{Alt}^2 \rho_{\{\alpha_3(c), \beta_3(f); \lambda\}, \ell | G(N(c, f))}(\mathrm{Frob}_\wp)$ are

$$N(\wp)^2, \quad N(\wp)^2 u_\wp / v_\wp, \quad \text{and} \quad N(\wp)^2 v_\wp / u_\wp.$$

This proves (32). The equality (33) follows from the general identity for a representation ρ' (see [17, (67.8)]):

$$\mathrm{Tr} \mathrm{Alt}^2 \rho' + \mathrm{Tr} \mathrm{Sym}^2 \rho' = \mathrm{Tr} \rho' \otimes \rho' = (\mathrm{Tr} \rho')^2.$$

For the last assertion, note the following relation for 2-dimensional semisimple representations ρ which can be proved by comparing the characters of both sides :

$$\mathrm{Sym}^2(\mathrm{Sym}^2 \rho) = \mathrm{Sym}^4 \rho \oplus (\det \rho)^2.$$

It follows that the degree-6 representation $\mathrm{Sym}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))}$ decomposes into the sum of a degree-5 and a degree-1 subrepresentations:

$$\mathrm{Sym}^2 \rho_{\{\alpha_3(f), \beta_3(c); \lambda\}, \ell | G(N(c, f))} \simeq (\chi_\rho^{-2} \otimes \mathrm{Sym}^4 \rho) \oplus \chi_\rho^{-2} (\det \rho)^2. \quad (34)$$

Since $\det \rho = \chi_{\rho \in \ell | G(N(c, f))}$, the degree-1 subrepresentation of $\mathrm{Sym}^2 \rho_{\{\alpha_3(c), \beta_3(f); \lambda\}, \ell | G(N(c, f))}$ is $\epsilon_\ell^2 | G(N(c, f))$. $\square$

Proof of Theorems 2 and 3 Below we fix a pair (c, f) such that $HD_1(c, f)$ and $HD_2(c, f)$ are primitive and drop the reference to c, f when there is no danger of confusion. Fix a prime ℓ . Using Lemma 2, we have, at a prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/M\ell]$,

$$\mathbb{P}(HD_1; \kappa_\wp) = \sum_{t \in \kappa_\wp} \phi_\wp(t) \cdot \mathbb{P}(\alpha_3(f), \beta_3(c); t; \kappa_\wp)^2. \quad (35)$$

Recall that finite dimensional semisimple representations over fields of characteristic zero of a group of any size are determined by their traces (see [17, (27.13)] by Curtis and Reiner). We will use (35) to obtain an isomorphism between two Galois representations of $G(N)$ with different origins. For the left-hand side, recall from Theorem 1 that $\mathrm{Tr} \rho_{HD_1, \ell}(\mathrm{Frob}_\wp) = -\phi_\wp(-1) \mathbb{P}(HD_1; \kappa_\wp)$.

For the right-hand side, by Katz [32, 33], there exists an ℓ -adic sheaf $\mathcal{H}$ of the affine line $\mathbb{A}$ over $R = \mathbb{Z}[\zeta_N, 1/\ell N]$ such that the action of $G(N)$ on the stalk $\mathcal{H}_\lambda$ at each nonzero $\lambda \in R$ is a generically 3-dimensional representation, characterized by the following trace formula: At any prime ideal $\wp$ of R , and $\lambda \in R$ with $\mathrm{ord}_\wp \lambda = 0$

$$\mathrm{Tr} \mathrm{Frob}_{\wp}(\mathcal{H}_{\lambda}) = \phi_{\wp}(-1) \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_{\wp}).$$

Here and thereafter, to ease our notation, we sometimes identify a representation ρ' with the underlying space V on which it acts, and use $\mathrm{Frob}_{\wp}(V)$ to denote $\rho'(\mathrm{Frob}_{\wp})$. By [17, (30.9)], $\mathrm{Tr} \mathrm{Frob}_{\wp}(\mathcal{H}_{\lambda} \otimes \mathcal{H}_{\lambda}) = \mathbb{P}(\alpha_3(f), \beta_3(c); 1/\lambda; \kappa_{\wp})^2$. Now consider the representation $\tilde{\rho}$ obtained from $G(N)$ acting on $V := H_{\acute{e}t}^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathcal{H}^{\otimes 2})$ where $\mathcal{L}_{\phi}$ is a rank-1 sheaf on $\mathbb{A}$ corresponding to the quadratic character ϕ . At a prime ideal $\wp$ where the representation is unramified, the trace of $\tilde{\rho}(\mathrm{Frob}_{\wp})$ on V can be computed from the action of the Frobenius element in $\mathrm{Gal}(\overline{\kappa_{\wp}}/\kappa_{\wp})$ on $H_{\acute{e}t}^1(\mathbb{A} \otimes_{\kappa_{\wp}} \overline{\kappa_{\wp}}, \mathcal{L}_{\phi} \otimes \mathcal{H}^{\otimes 2})$ since $H_{\acute{e}t}^i(\mathbb{A} \otimes_{\kappa_{\wp}} \overline{\kappa_{\wp}}, \mathcal{L}_{\phi} \otimes \mathcal{H}^{\otimes 2}) = 0$ for $i \neq 1$. Expressed in terms of the action of the Frobenius F_x at each point $x \in \kappa_{\wp}$, the Lefschetz formula gives $\mathrm{Tr} \tilde{\rho}(\mathrm{Frob}_{\wp}) = -\sum_{x \in \kappa_{\wp}^{\times}} \mathrm{Tr}(x)$ where

$$\mathrm{Tr}(x) = \mathrm{Tr} F_x((\mathcal{L}_{\phi} \otimes \mathcal{H}^{\otimes 2})_x) = \phi_{\wp}(x) \mathbb{P}(\alpha_3(f), \beta_3(c); 1/x; \kappa_{\wp})^2.$$

Putting all these together, we get from (35) that, up to semisimplification, $(\chi_{-1} \otimes \rho_{HD_1, \ell})|_{G(N)} \cong \tilde{\rho}$.

Next, we will use this isomorphism to obtain a decomposition of $\rho_{HD_1, \ell}^{ss}|_{G(N)}$. Note that $\mathcal{H}^{\otimes 2} = \mathcal{H} \otimes \mathcal{H}$ decomposes naturally into $\mathrm{Sym}^2 \mathcal{H} \oplus \mathrm{Alt}^2 \mathcal{H}$ as $G(N)$ -invariant sheaves. From (34) and the proof of Proposition 5, we further know that $\mathrm{Sym}^2 \mathcal{H} = (\mathcal{L}_{\chi_{\rho}^{-2}} \otimes \mathrm{Sym}^4 \rho) \oplus (\mathcal{L}_{\chi_{\rho}^{-2}} \otimes (\det \rho)^2)$, where ρ stands for the underlying ℓ -adic sheaf on $\mathbb{A} \setminus \{0\}$ for the representation in Theorem 1 associated to the pair $\alpha_2(c, f)$, $\beta_2(c)$; and $(\det \rho)^2$ is the rank-1 sheaf associated to the square of the determinant character $\chi_{\rho \in \ell}|_{G(N)}$. Accordingly $H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathcal{H}^{\otimes 2})$ decomposes into $H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathrm{Alt}^2 \mathcal{H}) \oplus H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathcal{L}_{\epsilon_{\ell}^2|_{G(N)}}) \oplus H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathcal{L}_{\chi_{\rho}^{-2}} \otimes \mathrm{Sym}^4 \rho)$ as $G(N)$ -modules, and we write the corresponding decomposition of representations of $G(N)$ on these cohomology spaces in order as

$$\tilde{\rho} = \tilde{\rho}_{AS} \oplus \tilde{\rho}_{ES} \oplus \tilde{\rho}_{SS}.$$

This naturally yields the decomposition of the trace at each prime ideal $\wp$ of R :

$$\mathrm{Tr} \tilde{\rho}(\mathrm{Frob}_{\wp}) = -\mathbb{P}(HD_1, \wp) = AS_{c, f}(\wp) + ES_{c, f}(\wp) + SS_{c, f}(\wp)$$

where

$$\begin{aligned} AS_{c, f}(\wp) &:= \mathrm{Tr} \tilde{\rho}_{AS}(\mathrm{Frob}_{\wp}) \\ &= -\mathrm{Tr} F_1((\mathrm{Alt}^2 \mathcal{H})_1) - \sum_{t \in \kappa_{\wp}^{\times}, t \neq 1} \phi_{\wp}(t) \mathrm{Tr} F_t((\mathrm{Alt}^2 \mathcal{H})_t) \\ &\stackrel{(30), (32)}{=} -\psi_{c, f}(\mathrm{Frob}_{\wp}) N(\wp)^2 - \sum_t \phi_{\wp}(t) \phi_{\wp}(1 - 1/t) N(\wp) \\ &\quad \cdot \mathbb{P}(\alpha_3(f), \beta_3(c); 1/t; \kappa_{\wp}) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(11)}{=} -\psi_{c,f}(\text{Frob}_{\wp})N(\wp)^2 - N(\wp) \cdot \mathbb{P}\left(\left\{\frac{1}{2}, \frac{1}{2}, f, 1-f\right\}, \right. \\
&\quad \left. \left\{1, 1, \frac{3}{2}-c, \frac{1}{2}+c\right\}; 1; \kappa_{\wp}\right) \\
&\stackrel{\text{Prop. 3}}{=} -N(\wp)^2 - N(\wp) \cdot \mathbb{P}(HD_2; \kappa_{\wp}) \\
&= -N(\wp)^2 + \phi_{\wp}(-1)N(\wp)\text{Tr } \rho_{HD_2, \ell}(\text{Frob}_{\wp}); \\
&ES_{c,f}(\wp) := \text{Tr } \tilde{\rho}_{ES}(\text{Frob}_{\wp}) = - \sum_{t \neq 0,1} \phi_{\wp}(t)N(\wp)^2 = N(\wp)^2; \\
&SS_{c,f}(\wp) := \text{Tr } \tilde{\rho}_{SS}(\text{Frob}_{\wp}).
\end{aligned}$$

From the above expressions, we see that the representation of $G(N)$ on $H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \text{Alt}^2 \mathcal{H}) \oplus H^1(\mathbb{A} \otimes_R \overline{\mathbb{Q}}, \mathcal{L}_{\phi} \otimes \mathcal{L}_{\epsilon_{\ell}^2|G(N)})$ is isomorphic to $\epsilon_{\ell} \otimes \chi_{-1} \otimes \rho_{HD_2, \ell}|G(N)$. Consequently, we obtain the following isomorphism:

$$\rho_{HD_1, \ell}^{ss}|G(N) \simeq (\epsilon_{\ell} \otimes \rho_{HD_2, \ell})|G(N) \oplus (\tilde{\rho}_{SS} \otimes \chi_{-1})|G(N), \quad (36)$$

showing the decomposition of $\rho_{HD_1, \ell}^{ss}|G(N)$ into a sum of a degree-3 and a degree-2 representations. This proves Theorem 2.

When the pair (c, f) is among those listed in (4), explicit computations as explained in Sect. 2.5 show that the representation $\rho_{HD_1, \ell}|G(N)$ contains no irreducible subrepresentations of multiplicity larger than 1. Hence $\rho_{HD_1, \ell}$ and $\rho_{HD_2, \ell}$ are semisimple and (36) holds without semisimplification. This is Theorem 3. $\square$

By Theorem 1, (iii), we can decompose the degree-3 representation $\rho_{HD_2, \ell}$ as $\rho_{HD_2, \ell}^{prim} \oplus \rho_{HD_2, \ell}^1$, and similarly, $\rho_{HD_1, \ell} = \rho_{HD_1, \ell}^{prim} \oplus \rho_{HD_1, \ell}^1$. Contained in the proof above is the information on $\rho_{HD_2, \ell}^1$ and $\rho_{HD_1, \ell}^1$.

Corollary 1 For any pair (c, f) in (4) and any prime ℓ ,

$$\begin{aligned}
\rho_{HD_2(c,f), \ell}^1|G(N(c,f)) &= \chi_{-1}\epsilon_{\ell}|G(N(c,f)) \quad \text{and} \\
\rho_{HD_1(c,f), \ell}^1|G(N(c,f)) &= \chi_{-1}\epsilon_{\ell}^2|G(N(c,f)).
\end{aligned}$$

As a corollary, we can write down the following finite field analogue of (3).

Corollary 2 For any pair (c, f) in (4) and any prime ideal $\wp$ of $\mathbb{Z}[\zeta_{N(c,f)}, 1/N(c, f)]$ we have

$$-\mathbb{P}(HD_1; \kappa_{\wp}) = -N(\wp) \cdot \mathbb{P}(HD_2; \kappa_{\wp}) + SS_{c,f}(\wp). \quad (37)$$

3.3 A Proof of Theorem 4

For any (c, f) as in (4), $HD_2(c, f)$ and $HD_2(f, c)$ may not agree by definition. Nonetheless, the representations $\rho_{HD_2(c,f), \ell}$ and $\rho_{HD_2(f,c), \ell}$ are isomorphic because of the following Lemma.

Lemma 3 Given any (c, f) in (4), for any prime power $q \equiv 1 \pmod{M(c, f)}$,

$$\mathbb{P}(\alpha_4(f), \beta_4(c); 1; \mathbb{F}_q) = \mathbb{P}(\alpha_4(c), \beta_4(f); 1; \mathbb{F}_q).$$

Proof

$$\mathbb{P}(\alpha_4(f), \beta_4(c); 1; \mathbb{F}_q) = {}_4\mathbb{P}_3 \left[\begin{matrix} \phi & \phi & \omega^{(q-1)f} & \omega^{(q-1)(1-f)} \\ \varepsilon & \omega^{(q-1)(\frac{1}{2}-c)} & \omega^{(q-1)(\frac{1}{2}+c)} & \end{matrix} ; 1; q \right]$$

By Proposition 1, the right-hand side coincides with

$${}_4\mathbb{P}_3 \left[\begin{matrix} \phi & \phi & \omega^{(q-1)c} & \omega^{(q-1)(1-c)} \\ \varepsilon & \omega^{(q-1)(\frac{1}{2}-f)} & \omega^{(q-1)(\frac{1}{2}+f)} & \end{matrix} ; 1; q \right] \text{ which is } \mathbb{P}(\alpha_4(c), \beta_4(f); 1; \mathbb{F}_q). \quad \square$$

It is clear that when (c, f) is one of the first five pairs, HD_2 is also defined over $\mathbb{Q}$. When (c, f) is either $(\frac{1}{5}, \frac{2}{5})$ or $(\frac{1}{10}, \frac{3}{10})$, one has

Lemma 4 Let (c, f) be either $(\frac{1}{5}, \frac{2}{5})$ or $(\frac{1}{10}, \frac{3}{10})$. Then the representation $\rho_{HD_2, \ell}$ attached to the hypergeometric datum $HD_2 = \{\alpha_4(f), \beta_4(c); 1\}$ can be extended to $G_{\mathbb{Q}}$.

Proof When $(c, f) = (\frac{1}{5}, \frac{2}{5})$, $M = 10$ and $\rho_{HD_2, \ell}$ is a representation of $G(10) = G(5)$. Since HD_2 is self-dual and defined over $K = \mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$, $\rho_{HD_2, \ell}$ is invariant under the Galois conjugation sending ζ_5 to ζ_5^{-1} so that it can be extended to G_K , the absolute Galois group of K . The Galois group $\text{Gal}(K/\mathbb{Q})$ is generated by $\tau : \zeta_5 \mapsto \zeta_5^3$, which maps $\alpha_4(f) = \{\frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{3}{5}\}$ to $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{4}{5}\}$ and $\beta_4(c) = \{1, 1, \frac{3}{10}, \frac{7}{10}\}$ to $\{1, 1, \frac{9}{10}, \frac{1}{10}\}$. For all primes $p \equiv 1 \pmod{5}$, we have $p \equiv 1 \pmod{10}$ and by Lemma 3,

$$\begin{aligned} \mathbb{P}(\alpha_4(f), \beta_4(c); 1; \mathbb{F}_p) &= \mathbb{P}(\alpha_4(c), \beta_4(f); 1; \mathbb{F}_p) \\ &= \mathbb{P} \left(\left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{4}{5} \right\}, \left\{ 1, 1, \frac{9}{10}, \frac{1}{10} \right\}; 1; \mathbb{F}_p \right). \end{aligned}$$

This shows that $\rho_{HD_2, \ell}$ of G_K is invariant under conjugation by τ , and hence, it can be extended to $G_{\mathbb{Q}}$. The case of $(c, f) = (\frac{1}{10}, \frac{3}{10})$ can be handled in a parallel manner. $\square$

Now that we know each $\rho_{HD_2, \ell}$ has an extension to $G_{\mathbb{Q}}$, we proceed to prove the assertions about $\rho_{HD_2, \ell}^{BCM}$ in Theorem 4. For $(c, f) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{6})$, this has been established in [40]. We prove the remaining cases. If HD_2 is defined over $\mathbb{Q}$, from Theorems 1 and 8, we know each $\rho_{HD_2, \ell}^{BCM} \cong \rho_{HD_2, \ell}^{BCM, \text{prim}} \oplus \rho_{HD_2, \ell}^{BCM, 1}$ where $\rho_{HD_2, \ell}^{BCM, 1}$ is 1-dimensional. When HD_2 is not defined over $\mathbb{Q}$, we continue to use $\rho_{HD_2, \ell}^{BCM, \text{prim}}$ and $\rho_{HD_2, \ell}^{BCM, 1}$ to denote an extension of the degree-2 and degree-1 pieces of $\rho_{HD_2, \ell}$ to $G_{\mathbb{Q}}$ with integral traces.

We first deal with $\rho_{HD_2, \ell}^{BCM, prim}$ by using Theorem 9 to prove the modularity of $\rho_{HD_2, \ell}^{BCM, prim}[t]$ where t is computed by (25) and identify the corresponding cuspidal Hecke eigenform. Then, we determine $\rho_{HD_2, \ell}^{BCM, 1}$.

Using the Magma program called “Hypergeometric Motives over $\mathbb{Q}$ ” implemented by Watkins [56], the characteristic polynomial of $\rho_{HD_2, \ell}^{BCM, prim}[t](\text{Fr}_p)$ for $p \geq 5$ can be computed efficiently by choosing $\ell \neq p$ as mentioned in Sect. 2.5. Thus for each $p > 3$, the trace and determinant of $\rho_{HD_2, \ell}^{BCM, prim}[t](\text{Fr}_p)$ can be read off. When HD_2 is defined over $\mathbb{Q}$, the characteristic polynomials of $\rho_{HD_2, \ell}^{BCM, prim}[t](\text{Fr}_p)$ all have coefficients in $\mathbb{Z}$. The representation $\rho_{HD_2, \ell}^{BCM, prim}$ is unramified outside $\{2, 3, \ell\}$ for $(c, f) = (\frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{6}, \frac{1}{6})$, and these ℓ -adic representations are compatible as ℓ varies. Hence, $\rho_{HD_2, \ell}^{BCM, prim}$ and its determinant can ramify only at 2 and 3. To compute it, we may choose $\ell = 2$ or 3. As $n = 4$ and dimension of $\rho_{HD_2, \ell}^{BCM, prim}$ is 2, by (iii) of Theorem 8, and the absolute value of the roots of $\rho_{HD_2, \ell}^{BCM, prim}(\text{Frob}_p)$ are $p^{3/2}$ for $p > 3$, one sees that $\det \rho_{HD_2, \ell}^{BCM, prim} = \chi_{c, f} \epsilon_\ell^3$ where ϵ_ℓ denotes the unramified ℓ -adic cyclotomic character with $\epsilon_\ell(\text{Frob}_p) = p$ and $\chi_{c, f}$ has order at most 2 because of integer coefficients. Since $\chi_{c, f}$ is unramified outside 2 and 3, its conductor divides 24. From computing the first few characteristic polynomials of $\rho_{HD_2, \ell}^{BCM, prim}[t](\text{Fr}_p)$, we confirm that $\chi_{c, f}$ is the trivial character for each case. So $\det \rho_{HD_2, \ell}^{BCM, prim} = \epsilon_\ell^3$ which implies $\rho_{HD_2, \ell}^{BCM, prim}$ and hence each $\rho_{HD_2, \ell}^{BCM, prim}[t]$ (which is 2-dimensional) is odd for one and hence all ℓ . The irreducibility of $\rho_{HD_2, \ell}^{BCM, prim}$ follows from the lemma below since it has degree 2 and determinant ϵ_ℓ^3 .

Lemma 5 *Suppose the hypergeometric datum $HD = \{\alpha, \beta; 1\}$ in Theorem 1 is such that, for all primes ℓ , the representation $\rho_{HD, \ell}^{prim}$ of $G(M)$ has a degree-2 subrepresentation which can be extended to a representation ρ_ℓ of $G_{\mathbb{Q}}$ and the ρ_ℓ form a compatible family. If $\det \rho_\ell$, up to a finite twist, is ϵ_ℓ^m for an odd natural number m , then ρ_ℓ is irreducible.*

Proof Suppose otherwise. Then the semisimplification of ρ_ℓ is of the form $\chi_1 \epsilon_\ell^a \oplus \chi_2 \epsilon_\ell^b$, where a, b are non-negative integers with $a + b = m$ and χ_1, χ_2 are characters of $G_{\mathbb{Q}}$ of finite order. Thus the eigenvalues of $\rho_{HD, \ell}^{prim}(\text{Frob}_\wp)$ would not have the same absolute values at almost all prime ideals $\wp$, contradicting Theorem 1, iii). $\square$

To apply Theorem 9 to conclude the modularity of $\rho_{HD_2, \ell}^{BCM, prim}[t]$, choose $\ell > 5$ to be a prime which splits completely in $\mathbb{Q}(\zeta_M)$. Then the action of $\rho_{HD_2, \ell}^{BCM, prim}[t]$ restricted to a decomposition group D_ℓ at ℓ is isomorphic to the action of $\rho_{HD_2, \ell}^{prim}[t]$ restricted to a decomposition group $D_\wp$ at a prime $\wp$ of $\mathbb{Q}(\zeta_M)$ above ℓ , which is known to be crystalline by Katz [32].

For $(c, f) = (\frac{1}{3}, \frac{1}{3})$, the graph of $e_{\alpha, \beta}$ with $\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$, $\beta = \{1, 1, \frac{7}{6}, \frac{5}{6}\}$ is as follows.

In this case, $\min e_{\alpha, \beta} = -1$, $t = 1 - \frac{4-2}{2} = 0$, and $w(HD_2) = 3 - (-1) = 4$. This means $\rho_{HD_2, \ell}^{BCM, prim}$ has Hodge–Tate weights $\{0, 3\}$. By Theorem 9, $\rho_{HD_2, \ell}^{BCM, prim}$ is modular and the corresponding modular form is of weight-4 with integer coefficients and by Theorem 10, its level divides $2^8 \cdot 3^5$. From the list of cuspidal Hecke eigenforms of weight-4 and level dividing $2^8 \cdot 3^5$ listed in the database. We see that only $f_{6.4.a.a}$ has the matching p -coefficients for all primes within the range $5 \leq p < 100$.

When $(c, f) = (\frac{1}{6}, \frac{1}{6})$, $\min = 0$, $\max = 2$, $w(HD_2) = 2$ and $t = -1$. By a similar analysis, we know $\rho_{HD_2, \ell}^{BCM, prim}[-1]$ is modular and is isomorphic to the ℓ -adic Galois representation attached to $f_{24.2.a.a}$.

Next we determine $\rho_{HD_2, \ell}^{BCM, 1}$. We implemented a SageMath program to compute $H_p(\alpha, \beta; 1)$. From Theorem 8, we know $\rho_{HD_2, \ell}^{BCM, 1}$ is 1-dimensional and its value at Frob_p can be computed by

$$\text{Tr} \rho_{HD_2, \ell}^{BCM, 1}(\text{Frob}_p) = p^{(4-m)/2} H_p(\alpha, \beta; 1) - \text{Tr} \rho_{HD_2, \ell}^{BCM, prim}(\text{Frob}_p),$$

where m is the number of elements in $\{1, 1, 3/2 - c, 1/2 + c\}$ that are integers, in these cases $m = 2$ or 4 and $n - m$ is even. So $\rho_{HD_2, \ell}^{BCM, 1}(\text{Frob}_p) = \varphi_{c, f}(p)p$ for some character $\varphi_{c, f}$ of order at most 2 and conductor dividing 24. By computing the first few primes p , $\varphi_{c, f}$ is identified quickly. For example, when $\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$, $\beta = \{1, 1, \frac{7}{6}, \frac{5}{6}\}$, for $p \in \{7, 11, 13, 17, 19\}$, the values of $\rho_{HD_2, \ell}^{BCM, 1}(\text{Frob}_p)$ computed from $pH_p(\alpha, \beta; 1) - \text{Tr} \rho_{HD_2, \ell}^{BCM, prim}(\text{Frob}_p)$ are 7, -11, 13, -17, 19, respectively, which agree with $\left(\frac{-3}{p}\right)p$.

For the last two cases $(c, f) = (\frac{1}{5}, \frac{2}{5})$ and $(\frac{1}{10}, \frac{3}{10})$, each $\rho_{HD_2, \ell} = \rho_{HD_2, \ell}^{prim} + \rho_{HD_2, \ell}^1$ is a representation of $G(10)$. Since $\rho_{HD_2, \ell}$ and $\rho_{HD_2, \ell}^1$ can be extended to $G_{\mathbb{Q}}$, so can $\rho_{HD_2, \ell}^{prim}$, and each extension is unique up to twists by the 4 characters of $G_{\mathbb{Q}}$ trivial on $G(10)$. Unlike the previous 5 cases, there is no preferred choice; we shall choose extensions with integral traces. From (iii) of Theorem 1 and numeric computation, we know $\det \rho_{HD_2, \ell}^{prim} = (\epsilon_{\ell}^3)|_{G(10)}$. Thus any extension of $\rho_{HD_2, \ell}^{prim}$ from $G(10)$ to $G_{\mathbb{Q}}$ has determinant either ϵ_{ℓ}^3 or $\chi_5 \otimes \epsilon_{\ell}^3$, and hence is odd. It is also irreducible by Lemma 5. By Corollary 1, $\rho_{HD_2, \ell}^1|_{G(20)} = \epsilon_{\ell}|_{G(20)}$. Thus we can obtain the trace of $\rho_{HD_2, \ell}^{BCM, prim}$ at any Frob_p with $p \equiv 1 \pmod{20}$ from computing $\text{Tr} \rho_{HD_2, \ell}(\text{Frob}_p)$ at any prime ideal $\wp$ of $\mathbb{Z}[\zeta_{20}]$ above such p and coprime to ℓ which, by Remark 2, allows us to

identify the corresponding modular form up to quartic twists, parallel to the first 5 cases. The degree-1 piece can be determined accordingly.

3.4 A Proof of Theorem 5

For each pair (c, f) listed in (4), we will proceed in a way similar to the proof of Theorem 4 by dealing with $\rho_{HD_1, \ell}^{BCM, prim}$ and then its 1-dimensional complement $\rho_{HD_1, \ell}^{BCM, 1}$ which can be determined the same way as the end of the proof of Theorem 4.

Proposition 6 *For each pair (c, f) in (4), the 4-dimensional representation $\rho_{HD_1, \ell}^{BCM, prim}$ is the sum of two odd degree-2 irreducible automorphic representations of $G_{\mathbb{Q}}$.*

Proof Fix a pair (c, f) in (4). Since HD_1 is defined over $\mathbb{Q}$ and HD_2 can be extended to $\mathbb{Q}$, the representations $\rho_{HD_1, \ell}|_{G(N)}$ and $(\epsilon_{\ell} \otimes \rho_{HD_2, \ell})|_{G(N)}$ can be extended to $G_{\mathbb{Q}}$ as $\rho_{HD_1, \ell}^{BCM}$ and $\epsilon_{\ell} \otimes \rho_{HD_2, \ell}^{BCM}$, respectively. Hence by (36), $(\tilde{\rho}_{SS} \otimes \chi_{-1})|_{G(N)}$ can be extended to $\tilde{\rho}_{SS}^{BCM} \otimes \chi_{-1}$ of $G_{\mathbb{Q}}$. Therefore $\rho_{HD_1, \ell}^{BCM}$ decomposes as the sum of a finite twist of $\epsilon_{\ell} \otimes \rho_{HD_2, \ell}^{BCM}$ and a finite twist of $\tilde{\rho}_{SS}^{BCM} \otimes \chi_{-1}$, written as

$$\rho_{HD_1, \ell}^{BCM, prim} \cong \sigma_{HD_1, alt, \ell} \oplus \sigma_{HD_1, sym, \ell} \quad (38)$$

where $\sigma_{HD_1, sym, \ell}$ and $\sigma_{HD_1, alt, \ell}$ are both 2-dimensional. Here we use *sym* and *alt* in the subscripts in order to keep track of their origin. It is useful to keep in mind that they will play asymmetric roles in general. From Theorem 4, $\det \sigma_{HD_1, alt, \ell} = \det(\epsilon_{\ell} \otimes \rho_{HD_2, \ell}^{prim}) = \epsilon_{\ell}^5 \otimes \chi_5 \epsilon_{\ell}^5$. We also determine $\det \rho_{HD_1, \ell}^{BCM, prim} = \epsilon_{\ell}^{10}$ the same way as the proof of Theorem 4. Hence $\sigma_{HD_1, sym, \ell}$ and $\sigma_{HD_1, alt, \ell}$ have the same determinant and are both odd. They are irreducible by Lemma 5. Since $\rho_{HD_1, \ell}^{BCM, prim}$ and $\sigma_{HD_1, alt, \ell}$ are both crystalline for a large prime $\ell \equiv 1 \pmod{N}$ by Katz [32], so is $\sigma_{HD_1, sym, \ell}$. By Theorem 9, both $\sigma_{HD_1, alt, \ell}$ and $\sigma_{HD_1, sym, \ell}$ are modular. $\square$

Our next task is to determine the modular forms $f_{c, f}$ and $g_{c, f}$ corresponding to $\sigma_{HD_1(c, f), sym, \ell}$ and $\sigma_{HD_1(c, f), alt, \ell}$, respectively, in the decomposition (38) of $\rho_{HD_1, \ell}^{BCM, prim}$.

As each HD_1 is defined over $\mathbb{Q}$, using Magma, one can compute the factorization of the characteristic polynomial of $\rho_{HD_1, \ell}^{BCM, prim}[t](\text{Fr}_p)$ over $\mathbb{Z}$ for any unramified prime p , which is always a product of two degree-2 polynomials over $\mathbb{Z}$. For example,

```
H1:=HypergeometricData([1/2, 1/2, 1/2, 1/2, 1/3, 2/3],
[1, 1, 1, 1, 1/6, 5/6]);
Factorization(EulerFactor(H1, 1, 7));
The output is
```

$$\begin{aligned} &< 16807 * \$.1^2 - 56 * \$.1 + 1, 1 >, \\ &< 16807 * \$.1^2 + 88 * \$.1 + 1, 1 >. \end{aligned}$$

Our next task is to assign the degree-2 factors to $\sigma_{HD_1, sym, \ell}$ and $\sigma_{HD_1, alt, \ell}$, respectively. By Theorem 3, $\sigma_{HD_1, alt, \ell}|_{G(N)}$ and $(\epsilon_{\ell} \otimes \rho_{HD_2, \ell}^{BCM, prim})|_{G(N)}$ agree. This

implies that at each unramified prime p , $\text{Tr}\sigma_{HD_1,alt,\ell}(\text{Frob}_p)$ and $p \cdot \text{Tr}\rho_{HD_2,\ell}^{BCM,prim}(\text{Frob}_p)$ differ by a root of unity. Since they both have integer coefficients, they differ at most by a quadratic character. This allows us to recognize which quadratic factor is from $\sigma_{HD_1,alt,\ell}|_{G(N)}$. For instance,

```
H2:=HypergeometricData([1/2,1/2,1/3,2/3],[1,1,1,1]);
Factorization(EulerFactor(H2,1,7));
The output is
```

$$< 343 * \$.1^2 - 8 * \$.1 + 1, 1 >$$

from which we know the first quadratic factor computed from H1 is from $\sigma_{HD_1,alt,\ell}$. The separation of the quadratic factors allows us to compute the trace and determinant of $\sigma_{HD_1,sym,\ell}$ and $\sigma_{HD_1,alt,\ell}$, respectively. Similar to the proof of Theorem 4, the determination of $f_{c,f}$ and $g_{c,f}$ boils down to the application of Theorem 10 as outlined in Sect. 2.5.

4 Hypergeometric Values and Periods of Modular Forms

In this section, we study untruncated hypergeometric series. Our main purpose is to relate hypergeometric values and periods of modular forms.

4.1 Modular Forms and Differential Equations

In [63], Zagier used

$${}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{matrix} ; 1 \right] = \frac{1}{2\pi i} \oint_{|t|=1} {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; t \right] {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; 1/t \right] \frac{dt}{t}$$

with $t = \lambda(\tau)$, the modular lambda function, to obtain

$${}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{matrix} ; 1 \right] = \frac{16}{\pi^2} L(f_{8.4.a.a}, 2).$$

In [55], Stiller studied periods of modular forms and their relation to the Eichler–Shimura parabolic cohomology theory. It is known in the literature (see [55] by Stiller and [60, Theorem 1] by Yang) that

Theorem 11 *Let Γ be a Fuchsian group commensurable with $SL_2(\mathbb{Z})$ and let t be a non-constant modular function of Γ . Given a weight- k meromorphic modular form f on Γ , write $f(\tau) = F(t(\tau))$ locally. Then $F(t), \tau F(t), \dots, \tau^k F(t)$ satisfy an order $k+1$ differential equation in variable t which has only regular singularities.*

Because of this reason, the period formulas below often involve polynomials in τ .

To prove Theorem 6, we first recall some relevant properties of hypergeometric functions [44, Chapters 15, 16] (For more detail, see [3,53] for example.):

(P.1) [44, (16.8.7)] The functions

$$(-z)^{-a_1} {}_3F_2 \left[\begin{matrix} a_1 & 1+a_1-b_2 & 1+a_1-b_3 \\ & 1+a_1-a_2 & 1+a_1-a_3 \end{matrix} ; \frac{1}{z} \right] \text{ and } {}_3F_2 \left[\begin{matrix} a_1 & a_2 & a_3 \\ & b_2 & b_3 \end{matrix} ; z \right]$$

satisfy the same 3rd-order differential equation.

(P.2) [44, §15.10, §16.8] The following four functions

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z \right]^2, \quad z^{\frac{1}{2}-a-b} {}_2F_1 \left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z \right] \\ & {}_2F_1 \left[\begin{matrix} \frac{1}{2}-a & \frac{1}{2}-b \\ & \frac{3}{2}-a-b \end{matrix} ; z \right], \\ & {}_3F_2 \left[\begin{matrix} 2a & 2b & a+b \\ & 2a+2b & a+b+\frac{1}{2} \end{matrix} ; z \right], \\ & z^{\frac{1}{2}-a-b} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ & a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; z \right] \end{aligned}$$

satisfy the same 3rd-order differential equation (ODE) and hence are linearly dependent.

(P.3) [44, (16.12.2)] The Clausen's formulas give two identities related to the functions in (P.2):

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z \right]^2 = {}_3F_2 \left[\begin{matrix} 2a & 2b & a+b \\ & 2a+2b & a+b+\frac{1}{2} \end{matrix} ; z \right] \\ & {}_2F_1 \left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \frac{1}{2}-a & \frac{1}{2}-b \\ & \frac{3}{2}-a-b \end{matrix} ; z \right] \\ & = {}_3F_2 \left[\begin{matrix} \frac{1}{2}a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ & a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; z \right]. \end{aligned} \quad (39)$$

Moreover, ${}_2F_1 \left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z \right]$ and $z^{\frac{1}{2}-a-b} {}_2F_1 \left[\begin{matrix} \frac{1}{2}-a & \frac{1}{2}-b \\ & \frac{3}{2}-a-b \end{matrix} ; z \right]$ satisfy the same ordinary differential equation.

From (P.1) above, we denote by L the order-3 ordinary differential operator which annihilates

$$z^{-1} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ & a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; 1/z \right] \text{ and } {}_3F_2 \left[\begin{matrix} \frac{1}{2}a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ & a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; z \right],$$

then $L \circ L$ annihilates both

$$z^{-1} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a - b + \frac{1}{2}b - a + \frac{1}{2} \\ a + b + \frac{1}{2}\frac{3}{2} - a - b \end{matrix} ; 1/z \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2}a - b + \frac{1}{2}b - a + \frac{1}{2} \\ a + b + \frac{1}{2}\frac{3}{2} - a - b \end{matrix} ; z \right]$$

and

$$z^{-1/2} {}_3F_2 \left[\begin{matrix} \frac{1}{2}a - b + \frac{1}{2}b - a + \frac{1}{2} \\ a + b + \frac{1}{2}\frac{3}{2} - a - b \end{matrix} ; z \right]^2.$$

Letting $z = u^2$, we conclude that

$${}_3F_2 \left[\begin{matrix} \frac{1}{2}a - b + \frac{1}{2}b - a + \frac{1}{2} \\ a + b + \frac{1}{2}\frac{3}{2} - a - b \end{matrix} ; u^2 \right]^2 \quad \text{and} \quad {}_2F_1 \left[\begin{matrix} a & b \\ a + b + \frac{1}{2} \end{matrix} ; u^2 \right]^4$$

satisfy the same differential equation.

4.2 The Occurrence of $\{c, f\}$ in $w(HD_1) = 6$ Cases

We show how $\{c, f\}$ in Theorem 5 for the three cases where $w(HD_1) = 6$ arises naturally in connection with a ${}_2F_1$ function.

Theorem 12 When $(c, f) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})$, the monodromy group $\Gamma(c, f)$ of the order-2 ODE satisfied by ${}_2F_1 \left[\begin{matrix} \frac{1+2f-2c}{4} & \frac{3-2c-2f}{4} \\ \frac{3}{2} - c \end{matrix} ; z^2 \right]$ is a genus 0 arithmetic Fuchsian group. The space of weight-6 holomorphic modular forms for $\Gamma(c, f)$ is 1-dimensional, generated by a Hecke eigenform $h_{c,f}$ with eigenvalue $a_p(\{c, f\})$ for almost all primes $p > 7$.

Proof We first compute the monodromy group $G(c, f)$ of the 2nd-order ODE satisfied by

$${}_2F_1(x) := {}_2F_1 \left[\begin{matrix} \frac{1+2f-2c}{4} & \frac{3-2c-2f}{4} \\ \frac{3}{2} - c \end{matrix} ; x \right] = {}_2F_1 \left[\begin{matrix} a & b \\ a + b + \frac{1}{2} \end{matrix} ; x \right], \quad (40)$$

(namely letting $a = \frac{1+2f-2c}{4}$, $b = \frac{3-2c-2f}{4}$) which is an arithmetic triangle group with angles $\pi|a + b - \frac{1}{2}|, \frac{\pi}{2}, \pi|a - b|$. For related background, see [27, Sect. 3.2.4]. Then under the change of variable $x = z^2$, the singularities of the ODE in the variable z satisfied by ${}_2F_1(z^2)$ are $\{\infty, 0, 1, -1\}$ and the monodromy group is an index-2 subgroup $\Gamma(c, f)$ of $G(c, f)$. We will give a description for $\Gamma(c, f)$ as a matrix group for each case and then describe the space $S_6(\Gamma(c, f))$ of weight-6 holomorphic modular forms on $\Gamma(c, f)$ which vanish at each cusp if there is any.

In what follows, we use the signature $(e_1, e_2, \dots, e_r)$ to denote a genus-zero group Γ which has r in-equivalent elliptic points and cusps together of respective order e_1 ,

$e_2, \dots, e_r$, where $e_i \in \mathbb{Z}_{>1} \cup \{\infty\}$. According to [52, Theorem 2.23] by Shimura, when $k \geq 4$ is even, the dimension of the space of weight- k holomorphic cusp forms on Γ is

$$\dim S_k(\Gamma) = -k + 1 + \sum_{i=1}^r \lfloor \frac{k}{2} (1 - 1/e_i) \rfloor - \#\{\text{cusps}\}.$$

It follows that $\dim S_6(\Gamma(c, f)) = 1$ when $w(HD_1) = 6$.

For $(c, f) = (\frac{1}{2}, \frac{1}{2})$, by (40), ${}_2F_1(x) = {}_2F_1\left[\begin{smallmatrix} \frac{1}{4} & \frac{1}{4} \\ 1 \end{smallmatrix}; x\right]$. In this case, the signature of $G(\frac{1}{2}, \frac{1}{2})$ is $(\infty, \infty, 2)$, and $G(\frac{1}{2}, \frac{1}{2})$ is isomorphic to $\Gamma_0(2)$. The group $\Gamma(\frac{1}{2}, \frac{1}{2})$ has signature $(\infty, \infty, 2, 2)$ and is isomorphic to the level 4 index-6 subgroup of $SL_2(\mathbb{Z})$ labeled by $4C^0$ in [16]. The group $4C^0$ is a supergroup of $\Gamma_0(8)$. By the above dimension formula, $\dim S_6(\Gamma(\frac{1}{2}, \frac{1}{2})) = -5 + 2\lfloor \frac{6}{4} \rfloor + 2(3 - 1) = 1$. The space is generated by $f_{8.6.a.a}(\tau)$, which is $f_{\frac{1}{2}, \frac{1}{2}}(\tau)$ in Theorem 5.

For $(c, f) = (\frac{1}{2}, \frac{1}{3})$, ${}_2F_1(x) = {}_2F_1\left[\begin{smallmatrix} \frac{1}{6} & \frac{1}{3} \\ 1 \end{smallmatrix}; x\right]$. In this case, the signature of $G(\frac{1}{2}, \frac{1}{3})$ is $(\infty, 6, 2)$. The group $G(\frac{1}{2}, \frac{1}{3})$ can be realized as

$$\Gamma_0^+(3) = \left\langle w_3 := \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}, T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

and its index-2 subgroup $\Gamma(\frac{1}{2}, \frac{1}{3})$ has signature $(2, 2, 3, \infty)$. This group and $\Gamma_0(3)$ are generated by

$$\left\{ w_3, (w_3 T^{-1})^2 = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, T^2 \right\} \quad \text{and} \quad \left\{ T, \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \right\},$$

respectively. Their intersection has signature $(3, 3, \infty, \infty)$, generated by

$$\left\{ T^2, \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} = T^{-1} \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} T \right\}.$$

This intersection group has a supergroup, which is the index-2 subgroup of $SL_2(\mathbb{Z})$ with signature $(3, 3, \infty)$. It has a weight-6 cuspform $f_{4.6.a.a}(\tau) = \eta(2\tau)^{12}$, which is $f_{\frac{1}{2}, \frac{1}{3}}(\tau)$ in Theorem 5. So $S_6(\Gamma(\frac{1}{2}, \frac{1}{3})) = \langle f_{4.6.a.a}(\tau) \rangle$.

For $(c, f) = (\frac{1}{3}, \frac{1}{3})$, ${}_2F_1(x) = {}_2F_1\left[\begin{smallmatrix} \frac{1}{4} & \frac{5}{12} \\ \frac{7}{6} \end{smallmatrix}; x\right]$. Its monodromy group has signature $(6, 6, 2)$, and the index-2 subgroup $\Gamma(\frac{1}{3}, \frac{1}{3})$ has signature $(3, 3, 2, 2)$, which can be realized as the subgroup $\Gamma_0^6(1)$ of $PSL_2(\mathbb{R})$ arising from the norm 1 group of the maximal order of the quaternion algebra over $\mathbb{Q}$ with discriminant 6. By the Jacquet–Langlands correspondence [30], $\mathfrak{h}_{c,f}$ the generator of $S_6(\Gamma(\frac{1}{3}, \frac{1}{3}))$ and the

Hecke eigenform $f_{6,6,a,a}(\tau)$, which is $f_{\frac{1}{3},\frac{1}{3}}(\tau)$ in Theorem 5, are both Hecke eigenforms with the same eigenvalue at each prime $p > 3$. $\square$

4.3 The Case $(c, f) = (\frac{1}{2}, \frac{1}{2})$: A Proof of Theorem 6

The goal of this subsection is to prove Theorem 6. To continue, we first recall some known results below. The Jacobi theta functions

$$\begin{aligned}\theta_2(\tau) &= \sum_{k \in \mathbb{Z}} q^{(2k+1)^2/8} = 2 \frac{\eta(2\tau)^2}{\eta(\tau)}, \\ \theta_3(\tau) &= \sum_{k \in \mathbb{Z}} q^{k^2/2} = \frac{\eta(\tau)^5}{\eta^2(\tau/2)\eta^2(2\tau)}, \\ \theta_4(\tau) &= \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2/2} = \frac{\eta^2(\tau/2)}{\eta(\tau)}\end{aligned}$$

satisfy the identities $\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4$ and $\theta_2^4(\tau) + \theta_3^4(\tau) = 2E_2(\tau) - E_2(\tau/2)$, where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$ with $q = e^{2\pi i \tau}$ is the normalized weight-2 Eisenstein series. Let $\lambda(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4$ be the modular λ -function. The following three derivative formulas hold (see [62, Proposition 7], [58, Lemma 1]):

$$\begin{aligned}\frac{d\eta(\tau)}{d\tau} &= \eta(\tau) \cdot \frac{2\pi i}{24} E_2(\tau), \\ \frac{d\theta_4(\tau)}{d\tau} &= \theta_4(\tau) \cdot \frac{2\pi i}{24} \cdot (E_2(\tau/2) - E_2(\tau)), \\ \frac{d\lambda(\tau)}{d\tau} &= \lambda(\tau) \cdot \pi i \cdot \theta_4^4(\tau).\end{aligned}$$

We now return to hypergeometric functions.

Let $F(t) = {}_3F_2 \left[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{smallmatrix}; t \right]$. Then

$${}_6F_5 \left[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}; 1 \right] = \frac{1}{2\pi i} \oint_{|t|=1} F(t) F(1/t) \frac{dt}{t}. \quad (41)$$

Using the Clausen formula (39) with $a = b = \frac{1}{4}$ and the following Kummer quadratic formula

$${}_2F_1 \left[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{smallmatrix}; z \right] = (1-z)^{-1/2} {}_2F_1 \left[\begin{smallmatrix} \frac{1}{4} & \frac{1}{4} \\ 1 \end{smallmatrix}; \frac{-4z}{(1-z)^2} \right],$$

we have the following identity

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 \end{matrix} ; \frac{-4z}{(1-z)^2} \right] = (1-z) {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; z \right]^2.$$

We now relate the discussion to modular forms by letting $z = \lambda(2\tau)$ and

$$t_2(\tau) = \frac{-4\lambda(2\tau)}{(1-\lambda(2\tau))^2} = -64 \frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}}, \quad (42)$$

which is a Hauptmodul of $\Gamma_0(2)$. It satisfies

$$t_2 \left(-\frac{1}{2\tau} \right) = \frac{1}{t_2(\tau)},$$

$$\frac{dt_2(\tau)}{d\tau} = \frac{4(\lambda(2\tau) + 1)}{(\lambda(2\tau) - 1)^3} \frac{d\lambda(2\tau)}{d\tau} = \frac{8\pi i \lambda(2\tau)(\lambda(2\tau) + 1)}{(1 - \lambda(2\tau))^3} \theta_4^4(2\tau).$$

A classical result says

$$\theta_3(\tau)^4 = {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; \lambda(\tau) \right]^2, \quad \theta_4(\tau)^4 = (1 - \lambda(\tau)) {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} ; \lambda(\tau) \right]^2.$$

It follows that

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; t_2(\tau) \right] = \theta_4^4(2\tau) = \frac{\eta(\tau)^8}{\eta(2\tau)^4} \quad \text{and}$$

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; 1/t_2(\tau) \right] = 16\tau^2 \frac{\eta(2\tau)^8}{\eta(\tau)^4}.$$

Also, we have $\frac{1 + \lambda(2\tau)}{1 - \lambda(2\tau)} = 1 + 32 \left(\frac{\eta(4\tau)}{\eta(\tau)} \right)^8$, and thus

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; t_2(\tau) \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; 1/t_2(\tau) \right] \frac{dt_2}{t_2} = 16 \cdot 2\pi i \tau^2 \cdot f_{8.6.a.a}(\tau/2) d\tau,$$

where $f_{8.6.a.a}(\tau/2) = \eta^{12}(\tau) + 32\eta^4(\tau)\eta(4\tau)^8 = -\left(\eta^{12}(\tau) - 2\frac{\eta(2\tau)^{24}}{\eta(\tau)^4\eta(4\tau)^8}\right)$. Using (41), we get the first claim of Theorem 6.

Next we demonstrate the appearance of $\mathfrak{g}_{c,f} = f_{8.4.a.a}(\tau) = \eta(2\tau)^4\eta(4\tau)^4$ in the second claim of Theorem 6 by showing

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{matrix} ; t_2(\tau) \right] {}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 \end{matrix} ; 1/t_2(\tau) \right] \frac{dt_2}{t_2} \sim_e -64\tau f_{8.4.a.a}(\tau) d\tau,$$

where the notation $\sim_e$ means the two differentials differ by an exact differential form.

Observe that

$${}_4F_3 \left[\begin{matrix} \frac{5}{4} & a & b & c \\ & \frac{1}{4} & d & e \end{matrix} ; 1/z \right] = -4z \frac{d}{dz} {}_3F_2 \left[\begin{matrix} a & b & c \\ & d & e \end{matrix} ; 1/z \right] + {}_3F_2 \left[\begin{matrix} a & b & c \\ & d & e \end{matrix} ; 1/z \right]$$

provided that the series converge. Using

$${}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right] = \frac{1}{2\pi i} \oint_{|z|=1} {}_4F_3 \left[\begin{matrix} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{matrix} ; 1/z \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; z \right] \frac{dz}{z},$$

and taking $z = t_2$ as (42), we have

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right] &= \frac{1}{2\pi i} \oint_{|t_2|=1} {}_4F_3 \left[\begin{matrix} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{matrix} ; 1/t_2 \right] F(t_2) \frac{dt_2}{t_2} \\ &= \frac{1}{2\pi i} \oint_{|t_2|=1} \left(-4t_2 \frac{d}{dt_2} F(1/t_2) + F(1/t_2) \right) F(t_2) \frac{dt_2}{t_2}, \end{aligned}$$

where $F(t) = {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; t \right]$ as before.

In the computation below, we use the following notations: $\theta_i := \theta_i(2\tau)$, $k := \frac{\theta_2^2}{\theta_3^2}$, $\lambda = k^2$. We rewrite F -functions as

$$F(t_2(\tau)) = \theta_4^4, \quad F\left(\frac{1}{t_2(\tau)}\right) = 4\tau^2 \theta_3^4 k.$$

Firstly,

$$-4t_2 F(t_2) \frac{d}{dt_2} F(1/t_2) \frac{dt_2}{t_2} = -16\theta_4^4 \frac{d(\tau^2 \theta_3^4 k)}{d\tau} d\tau,$$

and

$$\begin{aligned} F(1/t_2) F(t_2) \frac{dt_2}{t_2} &= 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot \theta_4^4 \frac{\lambda + 1}{1 - \lambda} d\tau = 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot \left(\theta_4^4 + 32 \frac{\eta^8(4\tau)}{\eta^4(2\tau)} \right) \\ &= 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot (\theta_4^4 + 2\theta_2^4) d\tau = 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot (\theta_3^4 + \theta_2^4) d\tau. \end{aligned}$$

Noting

$$d(\tau^2 k \theta_3^4 \theta_4^4) = \theta_4^4 d(\tau^2 k \theta_3^4) + 4\tau^2 k \theta_3^4 \theta_4^4 \frac{d\theta_4}{\theta_4}$$

and

$$\frac{d\theta_4}{\theta_4} = \frac{2\pi i}{24} \cdot 2(E_2(\tau) - E_2(2\tau)) d\tau,$$

one has

$$\begin{aligned} -4t_2 F(t_2) \frac{d}{dt_2} F(1/t_2) \frac{dt_2}{t_2} &\sim_e -16 \left(-4\tau^2 k \theta_3^4 \theta_4^4 \frac{d\theta_4}{\theta_4} \right) = 64\tau^2 k \theta_3^4 \theta_4^4 \left(\frac{d\theta_4}{\theta_4} \right) \\ &= 64\tau^2 k \theta_3^4 \theta_4^4 \frac{2\pi i}{24} \cdot 2 (E_2(\tau) - E_2(2\tau)) d\tau \\ &= 8\pi i \tau^2 k \theta_3^4 \theta_4^4 \cdot \frac{4}{3} (E_2(\tau) - E_2(2\tau)) d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} F(1/t_2) F(t_2) \frac{dt_2}{t_2} - 4t_2 F(t_2) \frac{d}{dt_2} F(1/t_2) \frac{dt_2}{t_2} \\ \sim_e 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot (\theta_3^4 + \theta_2^4 + \frac{4}{3} (E_2(\tau) - E_2(2\tau))) d\tau \\ = 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot (2E_2(2\tau) - E_2(\tau) + \frac{4}{3} (E_2(\tau) - E_2(2\tau))) d\tau \\ = 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot \frac{1}{12} (8E_2(2\tau) + 4E_2(\tau)) d\tau \\ = 8\pi i \tau^2 \theta_3^4 \theta_4^4 k \cdot \frac{1}{12} \frac{24}{2\pi i} \frac{d(\eta(2\tau)^4 \eta(\tau)^4)}{\eta(2\tau)^4 \eta(\tau)^4} = 8\tau^2 \cdot 4d(\eta(2\tau)^4 \eta(\tau)^4). \end{aligned}$$

Thus,

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix} ; 1 \right] &= \frac{1}{2\pi i} \oint_{|t_2|=1} 32\tau^2 d\eta(2\tau)^4 \\ &= \frac{32}{2\pi i} \left(-2 \oint_{|t_2|=1} \tau \eta(2\tau)^4 \eta(\tau)^4 d\tau \right). \end{aligned}$$

Finally we consider the path $|t_2| = 1$. As a Hauptmodul of $\Gamma_0(2)$, $t_2(\tau)$ has a simple pole at 0 and satisfies $t_2\left(\frac{i}{\sqrt{2}}\right) = -1$ and $t_2\left(\frac{1+i}{2}\right) = 1$, where $i/\sqrt{2}$ is fixed by the Atkin–Lehner involution w_2 on $\Gamma_0(2)$. We interpret $|t_2| = 1$ as the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$. Wrapping up the above discussion, we conclude the two identities stated in Theorem 6.⁴

⁴ If we denote $g(\tau) = f_{8,6,a,a}(\tau/2)$, then

$$g(\tau \pm 1/2) = i^{\pm} \left(\left(\frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)} \right)^{12} - 32\eta(2\tau)^{12} \left(\frac{\eta(4\tau)}{\eta(\tau)} \right)^4 \right) = i^{\pm} f_{16,6,a,a} \left(\frac{\tau}{2} \right),$$

where $f_{16,6,a,a}$ is the quadratic twist of $f_{8,6,a,a}$ by χ_{-1} . It follows that

$${}_6F_5(HD_1\left(\frac{1}{2}, \frac{1}{2}\right)) = -4i \left(\int_{i/2}^{i\infty} f_{16,6,a,a} \left(\frac{\tau}{2} \right) (4\tau^2 + 1) d\tau \right). \quad (43)$$

4.4 The Case $(c, f) = (\frac{1}{2}, \frac{1}{3})$

In this subsection, we will prove the following identity

$$\begin{aligned} & \frac{1}{\pi} {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 \end{matrix} ; 1 \right] \\ &= 6i \oint_{|t_3|=1} \left(\frac{1}{3} + \tau + \tau^2 \right) \cdot (f_{4.6.a.a}(\tau/2) - 27f_{4.6.a.a}(3\tau/2)) d\tau, \end{aligned}$$

where

$$t_3(\tau) = 4 \left(\frac{1}{3\sqrt{3}} \frac{\eta^6(\tau)}{\eta^6(3\tau)} + 3\sqrt{3} \frac{\eta^6(3\tau)}{\eta^6(\tau)} \right)^{-2} \quad (44)$$

is a Hauptmodul of $\Gamma_0^+(3)$ taking values 0, 1, ∞ at the elliptic points of order 6, 2, and at the cusp (of order ∞), respectively. To proceed, we use

$${}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 \end{matrix} ; 1 \right] = \frac{1}{2\pi i} \oint_{|t|=1} {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{6} & \frac{7}{6} \end{matrix} ; 1/t \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; t \right] \frac{dt}{t}$$

The second ${}_3F_2$ term can be handled by the Clausen formula (39)

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; 4z(1-z) \right] = {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; z \right]^2.$$

To deal with the first one, we use Property (P.2). By a straightforward verification, one has

$$\begin{aligned} & \frac{\sqrt{3}}{\pi} (-4z(1-z))^{-1/2} {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{6} & \frac{5}{6} \end{matrix} ; \frac{1}{4z(1-z)} \right] \\ &= \frac{1}{3} {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; z \right]^2 + \frac{i}{\sqrt{3}} {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; 1-z \right] \\ & \quad - \frac{1}{3} {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; 1-z \right]^2. \end{aligned}$$

When $z = (1 + \eta(\tau)^{12}/27\eta(3\tau)^{12})^{-1}$, we can express a point on the upper-half plane by

$$\tau = \frac{i}{\sqrt{3}} {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; 1-z \right] / {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; z \right].$$

Such results are part of Ramanujan's alternative theory of elliptic functions. The details can be found in [7] by Berndt, Bhargava, and Garvan, for example. The previous equation becomes

$$\begin{aligned} & \frac{\sqrt{3}}{\pi} (-4z(1-z))^{-1/2} {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{6} & \frac{5}{6} \end{matrix} ; \frac{1}{4z(1-z)} \right] \\ &= \left(\frac{1}{3} + \tau + \tau^2 \right) {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; 4z(1-z) \right]. \end{aligned}$$

Note that $(1 + \eta(\tau)^{12}/27\eta(3\tau)^{12})^{-1}$ is the modular function s^3 in the work [14], where $s = c/a$ and a, c are two weight-1 modular forms recalled in [58, Sect. 2]. From the formula (44),

$$\frac{dt_3}{t_3} = \frac{3(2s^3 - 1)}{s(s^3 - 1)} ds = -2\pi i \cdot a^2(2s^3 - 1) d\tau$$

and

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; t_3 \right] = {}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 \end{matrix} ; s^3 \right]^2 = a^2.$$

We conclude that

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{6} & \frac{7}{6} \end{matrix} ; 1/t_3 \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; t_3 \right] \frac{dt_3}{t_3} \\ &= \left(\frac{1}{3} + \tau + \tau^2 \right) \frac{\pi}{\sqrt{3}} (-t_3)^{1/2} {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ 1 & 1 \end{matrix} ; t_3 \right]^2 \frac{dt_3}{t_3} \\ &= \left(\frac{1}{3} + \tau + \tau^2 \right) \frac{4\pi^2}{\sqrt{3}} (s^3(1-s^3))^{1/2} a^6(2s^3 - 1) d\tau \\ &= -12\pi^2 \cdot \left(\frac{1}{3} + \tau + \tau^2 \right) \cdot (f_{4.6.a.a}(\tau/2) - 27f_{4.6.a.a}(3\tau/2)) d\tau. \end{aligned}$$

4.5 The Case $(c, f) = (\frac{1}{2}, \frac{1}{6})$

In this case, the unit root of the local zeta-function for HD_1 may not be unique. Despite it, we still have the following conjectural congruence in [38] for any prime $p \geq 5$,

$${}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ 1 & 1 & 1 & 1 & \frac{4}{3} & \frac{2}{3} \end{matrix} ; 1 \right] \Big|_{p-1} \stackrel{?}{\equiv} a_p(f_{8.4.a.a}) \pmod{p^3},$$

which is verified for the primes between 4 and 500, showing the exponent 3 sharp.

The archimedean counterpart is

$${}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ 1 & 1 & 1 & 1 & \frac{4}{3} & \frac{2}{3} \end{matrix} ; 1 \right] = 16 \oint_{|t_2|=1} \left(f_{8.4.a.a} \left(\frac{\tau}{2} \right) + 27 f_{8.4.a.a} \left(\frac{3\tau}{2} \right) \right) d\tau, \quad (45)$$

where t_2 is given in (42). To prove (45), we start with the following Lemma.

Lemma 6 *The following identity holds near $x = 0$ when both sides converge:*

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ \frac{4}{3} & \frac{2}{3} \end{matrix} ; \frac{x(x+4)^3}{4(2x-1)^3} \right] = \frac{4(1-2x)^{1/2}}{4+x}.$$

Proof Letting $a = \frac{1}{4}$, $b = \frac{7}{12}$ in (13), we get

$${}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{7}{12} \\ \frac{4}{3} \end{matrix} ; x \right] {}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{-1}{12} \\ \frac{2}{3} \end{matrix} ; x \right] = {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ \frac{4}{3} & \frac{2}{3} \end{matrix} ; x \right].$$

By (9.25) of [27]

$${}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{-1}{12} \\ \frac{2}{3} \end{matrix} ; f(x) \right] = \frac{1}{(1-2x)^{1/4}},$$

where $f(x) = \frac{x(x+4)^3}{4(2x-1)^3}$ and $f(-\frac{2}{x}) = \frac{1}{f(x)}$. Using (3.10) of [27], we get

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{-1}{12} \\ \frac{2}{3} \end{matrix} ; f(x) \right] &= \frac{1}{\sqrt{24}} (-f(x))^{-1/4} {}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{7}{12} \\ \frac{4}{3} \end{matrix} ; \frac{1}{f(x)} \right] \\ &\quad + \sqrt{\frac{2}{3}} (-f(x))^{1/12} {}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{-1}{12} \\ \frac{2}{3} \end{matrix} ; \frac{1}{f(x)} \right] \end{aligned}$$

from which we deduce

$${}_2F_1 \left[\begin{matrix} \frac{1}{4} & \frac{7}{12} \\ \frac{4}{3} \end{matrix} ; f(x) \right] = \frac{4(1-2x)^{3/4}}{4+x}.$$

The claim of the Lemma follows. $\square$

Let x be a Hauptmodul $-\frac{1}{2} \frac{\eta(\tau)^5 \eta(3\tau)}{\eta(2\tau) \eta(6\tau)^5} - 4$ on $\Gamma_0(6)$ and $f(x) = \frac{x(x+4)^3}{4(2x-1)^3}$ as above. Then $f(x) = 1/t_2$ and hence $f(-\frac{2}{x}) = t_2$. By Lemma 6,

$$\begin{aligned}
& {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; 1/t_2 \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ & \frac{4}{3} & \frac{2}{3} \end{matrix} ; t_2 \right] \frac{dt_2}{t_2} \\
&= {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; 1/t_2 \right] {}_3F_2 \left[\begin{matrix} \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ & \frac{4}{3} & \frac{2}{3} \end{matrix} ; f \left(\frac{-2}{x} \right) \right] \frac{dt_2}{t_2} \\
&= -16\tau^2 \frac{\eta(2\tau)^8}{\eta(\tau)^4} \cdot \left(\frac{x+4}{x} \right)^{1/2} \cdot \frac{4(x^2 - 10x - 2)}{(1-2x)^2} \frac{d(-2(x+4))}{-2(x+4)} \\
&= 32\pi i \left(f_{8.4.a.a} \left(\frac{\tau}{2} \right) + 27 f_{8.4.a.a} \left(\frac{3\tau}{2} \right) \right) \tau^2 d\tau,
\end{aligned}$$

where

$$\frac{d(-2(x+4))}{-2(x+4)} = \frac{2\pi i}{24} (5E_2(\tau) + 3E_2(3\tau) - 30E_2(6\tau) - 2E_2(2\tau)) d\tau.$$

4.6 Other Numerical Conjectures

Our numerical data for primes up to 500 suggest the following supercongruences for primes $p > 11$, with the corresponding p powers sharp for generic primes p in each case:

$$\begin{aligned}
(c, f) = \left(\frac{1}{2}, \frac{1}{3} \right) : \quad & {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{1}{4} & \frac{7}{6} & \frac{1}{6} & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{12.4.a.a}) \pmod{p^3}. \\
(c, f) = \left(\frac{1}{2}, \frac{1}{6} \right) : \quad & {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{5}{6} \\ & \frac{1}{4} & \frac{4}{3} & \frac{2}{3} & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{24.4.a.a}) \pmod{p^3}. \\
(c, f) = \left(\frac{1}{3}, \frac{1}{3} \right) : \quad & p^2 \cdot {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & \frac{5}{6} & \frac{7}{6} \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{6.6.a.a}) \pmod{p^4}. \\
(c, f) = \left(\frac{1}{3}, \frac{1}{3} \right) : \quad & p \cdot {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{1}{4} & \frac{5}{6} & \frac{7}{6} & 1 & \frac{5}{6} & \frac{7}{6} \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{18.4.a.a}) \pmod{p^3}. \\
(c, f) = \left(\frac{1}{5}, \frac{2}{5} \right) : \quad & p \cdot {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{5} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{4}{5} \\ & \frac{11}{10} & \frac{9}{10} & 1 & \frac{13}{10} & \frac{7}{10} \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{10.4.a.a}) \pmod{p^3}. \\
(c, f) = \left(\frac{1}{5}, \frac{2}{5} \right) : \quad & p \cdot {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{5} & \frac{2}{5} & \frac{1}{2} & \frac{4}{5} & \frac{4}{5} \\ & \frac{1}{4} & \frac{11}{10} & \frac{9}{10} & 1 & \frac{13}{10} & \frac{7}{10} \end{matrix} ; 1 \right]_{p-1} \stackrel{?}{\equiv} a_p(f_{50.4.a.d}) \pmod{p^3}.
\end{aligned}$$

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