Connectivity, Traceability and Hamiltonicity

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Abstract

Let G be a simple, connected, triangle-free graph with minimum degree δ , order n and leaf number $L(G)$. If G has a cut-vertex, we prove that $L(G) \geq 4\delta - 4$ and $n \geq 4\delta - 1$. Both lower bounds are sharp. The lower bound on the leaf number strengthens a result by Mukwembi for triangle-free graphs. As corollaries, we deduce sufficient conditions for connectivity, traceability and Hamiltonicity in triangle-free graphs. As an easy extension of a result by Goodman and Hedetniemi, we show that a simple, connected, claw-free, paw-free graph G is Hamiltonian if and only if G is not a path. We consider only simple graphs, that is, graphs with neither loops nor multiple edges.

Keywords Leaf number · Minimum degree · Order · Connectivity · Cycles · Paths

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1 Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *connectivity* $\kappa(G)$ of G refers to the minimum number of vertices whose deletion

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disconnects G . G is said to be k -connected if $\kappa(G) \geq k$. A vertex $u \in V(G)$ is called a *cut-vertex* or *articulation vertex* if the deletion of u disconnects G . It follows that every 2-connected graph has no cut-vertex. The degree of a vertex $y \in V(G)$, denoted $\deg_G(y)$, refers to the number of edges in G incident with y . The smallest of the degrees of vertices in G is called the *minimum degree*, denoted $\delta = \delta(G)$. A leaf is a vertex of degree one. A tree refers to a connected graph without cycles. The *leaf number* $L(G)$, whose applications in network designs are pertinent, is the maximum number of leaf vertices contained in a spanning tree of G . We denote by C_n , a cycle of length n . A triangle is a C_3 graph. G is called a C_n -free if it has no cycle of length n . Thus, G is triangle-free if it is a C_3 -free graph. A cycle C is said to be *dominating* if no pair of vertices outside C are adjacent. A *claw* is a $K_{1,3}$ bipartite graph. Let $K_{1,3} = \{xx_1, xx_2, xx_3\}$. Then the addition of just one of the edges x_1x_2, x_1x_3 and x_2x_3 yields a graph called a *paw*, see also [6,9,10] for more details on claws and paws.

Let H be a graph. Then, H is a *subgraph* of G if $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$. The graph H is a *spanning subgraph* of G , if $V(H) = V(G)$. A subgraph, $G[S]$ of G , whose vertex set is S is *induced* if its edge set consists of all edges in $E(G)$ which have all their endpoints in S . If $H \neq C_n$, then G is H -free if it has no induced subgraph isomorphic to H . Equivalently, H is called a *forbidden subgraph* of G . The graph G is *Hamiltonian* if it contains a spanning cycle and *traceable* if it has a spanning path. Thus, every Hamiltonian graph is traceable. For vertex disjoint graphs $G_1, G_2, \dots, G_k$, the *sequential join* $G_1 + G_2 + \dots + G_k$ is the graph obtained from the union of $G_1, G_2, \dots, G_k$ by joining every vertex of G_i to every vertex of G_{i+1} for $i = 1, 2, \dots, k - 1$.

Existence of spanning paths and cycles in graphs helps to solve data communication problems among other applications (see for instance [7,20]). The problem of determining the leaf number and whether spanning paths or cycles exists in a graph is NP -complete (see for instance [8]). Sufficient conditions for a graph to be Hamiltonian has a long history [4,5,18,19]. The search for sufficient conditions for a graph to be Hamiltonian remains an on going research area (see for instance [12,16,17]). It is known that for a graph to be Hamiltonian necessarily, it must be 2-connected. Hence, it becomes natural to ask for conditions that guarantee a graph to be 2-connected. For graphs in general, Mukwambi [16] established a sufficient condition for a graph to be 2-connected based on leaf number and minimum degree. Precisely, he proved the following result:

Theorem 1 [16] *Let G be a connected graph with minimum degree δ and leaf number $L(G)$. If G has a cut-vertex, then $L(G) \geq 2\delta$. Moreover, the bound is tight.*

Theorem 1 was shown to be best in the sense that, for each δ , the graph $G_{2\delta+1} = K_\delta + K_1 + K_\delta$ contains a cut-vertex and has leaf number $L(G_{2\delta+1}) = 2\delta$. For graphs in general, Theorem 1 has been used as a powerful tool in the establishment of sufficient conditions for traceability or Hamiltonicity based on leaf number and minimum degree (see for instance [11–14,16]). This motivated us to establish a similar bound for triangle-free graphs with the hope that the results will in the near future play an important role in providing sufficient conditions for triangle-free graphs to be either Hamiltonian or traceable. In fact, some of the corollaries and theorems in this paper demonstrate how useful the main results are in providing sufficient conditions for

Hamiltonicity or traceability in triangle-free graphs. Thus, in this paper, we strengthen Theorem 1 for triangle-free graphs.

Apart from this, we establish a similar lower bound on the order in terms of minimum degree. In addition, we show that the results are sharp. Further, we deduce sufficient conditions for a triangle-free graph to be 2-connected, traceable or Hamiltonian. We mention here that similar ideas used to prove results in this paper can be used to provide similar results for graphs of given girth.

The study of path- and cycle-related properties in triangle-free graphs has been dealt with and is still an ongoing research area (see for instance [1–3, 15]). The next theorem is crucial in settling some of the corollaries in this paper.

Theorem 2 [2] *Let G be a 2-connected, triangle-free graph of order n and minimum degree δ . If C is a longest cycle in G , then $|V(C)| \geq \min\{4\delta - 4, n\}$ or C is a dominating cycle.*

Goodman and Hedetneimi [9] proved the following result:

Theorem 3 *Let G be a 2-connected, claw-free, paw-free graph. Then G is Hamiltonian.*

Following Theorem 3, we note that it has gone unnoticed that the only non-Hamiltonian claw-free, paw-free graph is a path P_n . One of the results in this paper proves that every connected claw-free, paw-free graph G is Hamiltonian unless $G = P_n$. This improves Theorem 3 for claw-free, paw-free graphs.

Apart from notation already defined, we use the following: For vertices $u, v \in V(G)$, the *distance*, $d_G(u, v)$ is the length of a shortest path between u and v in G . The *neighbourhood* or *closed-neighbourhood*, $N_G[v]$, of a vertex v in G is given by

$$N_G[v] = \{u \in V(G) : d_G(u, v) \leq 1\}.$$

The *open-neighbourhood*, $N_G(v)$, of a vertex v in G is given by

$$N_G(v) = \{u \in V(G) : d_G(u, v) = 1\}.$$

Thus, $N_G[v] = \{v\} \cup N_G(v)$. The k th *neighbourhood* of v , $N_G^k[v]$, is defined as

$$N_G^k[v] = \{u \in V(G) : d_G(u, v) \leq k\}.$$

Here, the k th *open-neighbourhood* of v , $N_G^k(v)$, is defined as

$$N_G^k(v) = \{u \in V(G) : d_G(u, v) = k\}.$$

Further, $G \setminus \{v\}$ denotes the graph obtained from G by deleting a vertex v . If H is a subgraph of G , we denote by $N_H^k(v)$ the k th open-neighbourhood of v in H . Where there is no ambiguity, we drop the argument G .

2 Main Results

We now prove the main result of this paper and we deduce from it sufficient conditions for a triangle-free graph to be 2-connected. The following result strengthens Theorem 1 for triangle-free graphs.

Theorem 4 *Let G be a connected, triangle-free graph with minimum degree δ , order n and leaf number $L(G)$. If G has a cut-vertex, then $L(G) \geq 4\delta - 4$ and $n \geq 4\delta - 1$. Moreover, the bounds are sharp.*

Proof The proof is trivial for $\delta = 1$. For $\delta = 2$, the bound on the leaf number follows by an application of Theorem 1. To establish the bound on the order, assume to the contrary that $n \leq 6$. Since $\delta = 2$, G has a cycle. Let C_k be a longest cycle in G . Then $4 \leq k \leq 6$, since G is a triangle-free graph and $n \leq 6$. If $k = 5$ or 6 , then G is 2-connected, since there is at most one vertex outside C_k and $\delta = 2$, a contradiction. Similar arguments holds if $k = 4$ and $n \leq 5$. So, it is adequate to consider $k = 4$ and $n = 6$.

In this case, let $V(C_4) = \{v_1, v_2, v_3, v_4\}$, $E(C_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ and, $u_1, u_2 \notin V(C_4)$. Assume first that $u_1u_2 \in E(G)$. Since $\delta = 2$, apart from u_2 , we may assume that u_1 is adjacent v_1 . Similarly, u_2 has a neighbour on C_4 which is not v_1 , since G is a triangle-free graph. Thus, the second neighbour of u_2 is in the set $\{v_2, v_3, v_4\}$. But this yields a cycle of length at least 5, a contradiction. So this case fails.

Assume that $u_1u_2 \notin E(G)$. Then each u_i has at least two neighbours on C_4 which are not consecutive. We may assume that $u_1v_1, u_1v_3 \in E(G)$. Then, if we join u_2 to at least two of its neighbours in $V(C_4)$ we obtain a 2-connected graph, which is not allowed. In either case the supposition that $n \leq 6$ yields a contradiction. Thus, we conclude that $n \geq 7$ as desired.

Consider $\delta \geq 3$. Let u be a cut-vertex of G . Further, let x and y be neighbours of u that are in different components of $G \setminus \{u\}$. In addition, let G_1 and G_2 be the components of $G \setminus \{u\}$ containing x and y , respectively. By our choice of u, x and y , there following property holds:

Property 1 *Let x and y be as defined above. Then $N_G(x) \cap N_G(y) = \{u\}$ and for all $s, k; s \geq 1, k \geq 2$, we have $N_{G \setminus \{u\}}^s(x) \cap N_{G \setminus \{u\}}^k(y) = N_{G \setminus \{u\}}^k(x) \cap N_{G \setminus \{u\}}^s(y) = \emptyset$.*

Let $x_1, x_2, \dots, x_{\delta-1} \in N(x) \setminus \{u\}$ and $y_1, y_2, \dots, y_{\delta-1} \in N(y) \setminus \{u\}$. By Property 1, elements of $N(x) \setminus \{u\}$ are different from those in $N(y) \setminus \{u\}$. By the same argument for all i, j , x_i cannot be adjacent to y_j . Also, since G is triangle-free, for $i \neq j$, x_i cannot be adjacent to x_j and y_i cannot be adjacent to y_j . Following these arguments, let $w_1, w_2, \dots, w_{\delta-1}$ and $z_1, z_2, \dots, z_{\delta-1}$ be neighbours of x_1 and y_1 , respectively, apart from x and y . Then G has a tree T shown in Fig. 1, with $4\delta - 6$ leaves and order $4\delta - 1$, where

Fig. 1 The tree T

$$V(T) = \{u, x, y, w_1, w_2, \dots, w_{\delta-1}, x_1, x_2, \dots, x_{\delta-1}, y_1, y_2, \dots, y_{\delta-1}, z_1, z_2, \dots, z_{\delta-1}\}$$

and

$$E(T) = \{ux, uy, xx_i, yy_i, x_1w_i, y_1z_i \mid 1 \leq i \leq \delta - 1\}.$$

Thus,

$$n \geq |V(T)| = 4\delta - 1.$$

To settle the bound on the leaf number, we prove the following lemmas using the tree T defined above.

Lemma 1 *Assuming the above notation, if u has at least one neighbour outside T , then G has a tree with at least $4\delta - 4$ leaves.*

Proof of Lemma 1 If u has at least two neighbours outside T then we are done. So, we can assume that u has only one neighbour, say, z outside T . Let $T_1 = T \cup \{uz\}$. Then T_1 has $4\delta - 5$ leaves. We can also assume that no vertex in G has at least two neighbours outside T_1 , otherwise, we get the required tree and we are done. Hence, z has at least two neighbours in T , which are neither x nor y , otherwise, we obtain a triangle, which is not allowed.

By Property 1, z cannot have neighbours in both components G_1 and G_2 . We can assume that z has a neighbour in G_1 , the other case is treated similarly. Now, we look at possible neighbours of z'_i s, apart from, y_1 . Note that by the above arguments, the only possible neighbours of z'_i in T_1 are $y_1, y_2, \dots, y_{\delta-1}$ and u . If each z_i is adjacent to u , then the set $\{w_1, w_2, \dots, w_{\delta-1}, x_2, \dots, x_{\delta-1}, y_1, y_2, \dots, y_{\delta-1}, z, z_1, z_2, \dots, z_{\delta-1}\}$ forms $4\delta - 4$ leaves of a tree in G , whose interior vertices are u, x, y and x_1 .

So, we can assume that at least one z_i , say z_1 , is not adjacent to u . Since z_1 is adjacent to at most one vertex outside T_1 and G is triangle-free, it follows by our choice of u, x and y that z_1 must be adjacent to $y_1, y_2, \dots, y_{\delta-1}$ in T_1 . Since $\deg(z_1) \geq \delta$, let $t_1 \notin V(T_1)$ be a neighbour of z_1 . Then t_1 has at least $\delta - 1$ neighbours in T_1 which can only be $z_1, z_2, \dots, z_{\delta-1}$, otherwise, we are done or we contradict our choice of G as a triangle-free graph. Therefore, t_1 has a neighbour, say, t_2 not in T_1 which again must have at least $\delta - 1$ neighbours in T_1 , which must be $y_1, y_2, \dots, y_{\delta-1}$. In particular, t_2

must be adjacent to y_1 . Hence, the tree $T_1 \cup \{y_1 t_2\}$ has $4\delta - 4$ leaves and Lemma 1 holds. Therefore, in what follows, we assume that u has no neighbour outside T . $\square$

Lemma 2 *Let $A = \{w_1, w_2, \dots, w_{\delta-1}\}$ and $B = \{z_1, z_2, \dots, z_{\delta-1}\}$. If each element of A or each element of B is adjacent to u , then G has a tree with at least $4\delta - 4$ leaves.*

Proof of Lemma 2 Assume first that each element in $A \cup B$ is adjacent to u . Then the set $A \cup B \cup \{x_1, x_2, \dots, x_{\delta-1}, y_1, y_2, \dots, y_{\delta-1}\}$ forms $4\delta - 4$ leaves of a tree in G .

Consider the case where each element of A is adjacent to u , the case for elements of B can be treated similarly. Clearly, with all elements of A attached to u we have a tree with $4\delta - 5$ leaves and we are short of one leaf. We create that extra leaf as we did in the last part of the proof of Lemma 1. That is, we consider possible neighbours of z'_i s, apart from y_1 . Note that the only possible neighbours of z_i in T are $y_1, y_2, \dots, y_{\delta-1}$ and u . Further, at least one z_i , say z_1 , is not adjacent to u , otherwise, we are done, see previous paragraph. Now z_1 must be adjacent to $y_1, y_2, \dots, y_{\delta-1}$, since z_1 has at most one neighbour outside T and the aforementioned vertices are the only remaining possible neighbours of z_1 in T . Since $\deg(z_1) \geq \delta$, let $t_1 \notin V(T)$ be a neighbour of z_1 . Then t_1 has at least $\delta - 1$ neighbours in T which can only be $z_1, z_2, \dots, z_{\delta-1}$, otherwise, we are done or we contradict our choice of G as a triangle-free graph. Therefore, t_1 has a neighbour, say, t_2 not in T which again must have at least $\delta - 1$ neighbours in T , possibly, $y_1, y_2, \dots, y_{\delta-1}$. In particular, t_2 must be adjacent to y_1 . Hence, t_2 is an extra leaf gained and we are done. Therefore, Lemma 2 holds.

To complete the proof, it is adequate to consider the case where at least one element of A is not adjacent to u and at least one element of B is not adjacent to u . $\square$

Lemma 3 *Let A and B be as defined in Lemma 2. If at least one element of A is not adjacent to u and at least one element of B is not adjacent to u , then G has a tree with at least $4\delta - 4$ leaves.*

Proof of Lemma 3 Assume without loss of generality that w_1 and z_1 are not adjacent to u . Consider first the case where there is a leaf of T that is adjacent to at least two neighbours outside T . We may assume that w_1 has two neighbours outside T , say r_1 and r_2 , other cases are treated likewise. Note that it cannot have at least three neighbours out, otherwise, we are done. Since z_1 is not adjacent to u and cannot have two neighbours outside T , z_1 must be adjacent to $y_1, y_2, \dots, y_{\delta-1}$. Since $\deg(z_1) \geq \delta$, let $t_1 \notin V(T)$ be a neighbour of z_1 . Then t_1 has at least $\delta - 1$ neighbours in T which can only be $z_1, z_2, \dots, z_{\delta-1}$, otherwise, we are done or we contradict our choice of G as a triangle-free graph. Therefore, t_1 has a neighbour, say, t_2 not in T which again must have at least $\delta - 1$ neighbours in T , namely, $y_1, y_2, \dots, y_{\delta-1}$. In particular, t_2 must be adjacent to y_1 . So, the tree $T \cup \{w_1 r_1, w_1 r_2, y_1 t_2\}$ has $4\delta - 4$ leaves as desired. The same argument applies if there is a vertex of G that has two neighbours outside T . So, we assume no vertex of G has two neighbours outside T .

Consider the case where no vertex of G has two neighbours outside T . We proceed as in the previous lemmas to show that there are vertices, s_2 and t_2 not in T which are adjacent to x_1 and y_1 , respectively. Then the tree $T \cup \{x_1 s_2, y_1 t_2\}$ has $4\delta - 4$ leaves and the result follows.

Fig. 2 The graph $G_{4\delta}$

Consider possible neighbours of z'_i s, apart from, y_1 . As before, the only possible neighbours of z_i in T are $y_1, y_2, \dots, y_{\delta-1}$ and u . Then z_1 cannot have two neighbours out, otherwise, we are done. Since z_1 is not adjacent to u and cannot have two neighbours outside T , z_1 must be adjacent to $y_1, y_2, \dots, y_{\delta-1}$. Since $\deg(z_1) \geq \delta$, let $t_1 \notin V(T)$ be a neighbour of z_1 . Then t_1 has at least $\delta - 1$ neighbours in T which can only be $z_1, z_2, \dots, z_{\delta-1}$, otherwise, we are done or we contradict our choice of G as a triangle-free graph. Therefore, t_1 has a neighbour, say, t_2 not in T which again must have at least $\delta - 1$ neighbour in T , namely, $y_1, y_2, \dots, y_{\delta-1}$. In particular, t_2 must be adjacent to y_1 . By similar arguments, if we look at possible neighbours of w'_i s, apart from x_1 , we end up having a vertex $s_2 \notin V(T)$ which is adjacent to x_1 . Hence, Lemma 3 is proven.

Thus, by following lemmas 1–3, all cases have been exhausted. Further, in each case whenever u is a cut-vertex we obtain a tree with at least $4\delta - 4$ leaves as required. Therefore, the theorem holds. $\square$

To show that the bound on the leaf number in Theorem 4 is best in a sense, for each δ , let $G_{4\delta}$ be a graph isomorphic to a graph formed by taking two complete $K_{\delta, \delta}$ bipartite graphs and join them by a bridge, say $e = uz$, see Fig. 2. Further, to see that Theorem 4 is sharp for both the bound on the order and the bound on the leaf number, let $G_{4\delta-1}$ be a graph isomorphic to a graph formed from $G_{4\delta}$ by collapsing the bridge $e = uz$ such that vertex u becomes the same as vertex z , see Fig. 3.

The following corollaries are consequences of Theorem 4.

Corollary 1 *Let G be a connected, triangle-free graph with minimum degree δ and leaf number $L(G)$ such that $L(G) \leq 4\delta - 5$. Then G is 2-connected.*

Corollary 2 *Let G be a connected, triangle-free graph with minimum degree δ and order n such that $n \leq 4\delta - 2$. Then G is 2-connected.*

For graphs in general, the algorithm used to prove Theorem 4 can easily be employed to show that every simple, connected graph G that has a cut-vertex has order $n \geq 2\delta + 1$ and the bound is tight as revealed by the family $G_{2\delta+1} = K_\delta + K_1 + K_\delta$.

Theorem 5 *Let G be a connected, triangle-free graph with minimum degree δ and order n such that $n \leq 4\delta - 2$. Then G is Hamiltonian or every longest cycle is a dominating cycle.*

Fig. 3 The graph $G_{4\delta-1}$

Proof By Corollary 2, G is 2-connected. So, if $n \leq 4\delta - 3$, then the result follows by employing Theorem 2. Assume that $n = 4\delta - 2$. Following Theorem 2, it is enough to consider the case where every longest cycle has length $4\delta - 4$ and where there are exactly two vertices outside the longest cycle. Let $C_{4\delta-4}$ be a longest cycle in G and $u_1, u_2 \notin V(C_{4\delta-4})$ be distinct vertices not on $C_{4\delta-4}$. To establish the result, assume to the contrary that $C_{4\delta-4}$ is neither a spanning cycle of G nor a dominating cycle. Then $u_1 u_2 \in E(G)$. The following arguments yield a contradiction showing that $u_1 u_2 \notin E(G)$ and the result is established.

We note first that for $i = 1, 2$ each u_i has at least $\delta - 1$ neighbours on $C_{4\delta-4}$. Let $v_{t_1}, v_{t_2}, v_{t_3}, \dots, v_{t_{\delta-1}} \in N(u_1)$ and $v_{s_1}, v_{s_2}, v_{s_3}, \dots, v_{s_{\delta-1}} \in N(u_2)$ be neighbours of u_1 and u_2 on $C_{4\delta-4}$, respectively. Since G is a triangle-free graph and $u_1 u_2 \in E(G)$, u_1 cannot share a neighbour with u_2 . Thus, the set

$$A = \{v_{t_1}, v_{t_2}, v_{t_3}, \dots, v_{t_{\delta-1}}, v_{s_1}, v_{s_2}, v_{s_3}, \dots, v_{s_{\delta-1}}\}$$

has $2\delta - 2$ distinct vertices of $C_{4\delta-4}$. This implies that the set,

$$B = \{v_{t_1+1}, v_{t_2+1}, v_{t_3+1}, \dots, v_{t_{\delta-1}+1}, v_{s_1+1}, v_{s_2+1}, v_{s_3+1}, \dots, v_{s_{\delta-1}+1}\}$$

has $2\delta - 2$ distinct vertices of $C_{4\delta-4}$. Further, both u_1 and u_2 cannot have neighbours in B , otherwise, we obtain a cycle longer than $C_{4\delta-4}$, which is not allowed. Thus, A and B are disjoint sets. Hence $V(C_{4\delta-4}) = A \cup B$. The aforementioned arguments imply that $\deg(u_1) = \deg(u_2) = \delta$.

We claim first that no two consecutive vertices on $C_{4\delta-4}$ are such that both of them are not in A . To prove this, assume contrary to the claim that there exist two consecutive vertices, say, v_i and v_{i+1} on $C_{4\delta-4}$ such that $v_i, v_{i+1} \notin A$. Then $v_{i+1} \notin B$, since $v_i \notin A$. This means there is a vertex on $C_{4\delta-4}$ which is neither in A nor in B , a contradiction to $V(C_{4\delta-4}) = A \cup B$. So, the first claim holds.

Secondly, we claim that whenever, v_{t_i} is a neighbour of u_1 on $C_{4\delta-4}$ then $v_{t_i+2} \notin N(u_2)$. To prove this, suppose on the contrary that $v_{t_i+2} \in N(u_2)$. Then the cycle obtain from $C_{4\delta-4}$ by deleting edges $v_{t_i} v_{t_i+1}, v_{t_i+1} v_{t_i+2}$ and replace them by edges $u_1 v_{t_i}, u_1 u_2, u_2 v_{t_i+2}$ is longer than $C_{4\delta-4}$, a contradiction. Therefore, the second claim is established.

Lastly, we claim that for all i , $v_{i+2} \in N(u_1)$ whenever $v_{i_1} \in N(u_1)$. To settle this, assume to the contrary that $v_{i+2} \notin N(u_1)$ for some i . Then by previous arguments, v_{i+1} and v_{i+2} are to consecutive vertices not in A . This contradicts the first claim. Hence this claim holds.

The last claim implies that $\deg(u_1) \geq \delta + 1$, a contradiction to the fact that $\deg(u_1) = \delta$. Thus the assumption that $u_1 u_2 \in E(G)$ is false.

Therefore, G must be Hamiltonian or every longest cycle is a dominating cycle in all cases. $\square$

Theorem 5 is best in the sense that each graph $G_{4\delta-1}$ described before is a simple, connected, non-Hamiltonian, triangle-free graph in which every longest cycle is not dominating.

Corollary 3 *Let G be a connected, triangle-free graph with minimum degree δ and order n such that $n \leq 4\delta - 1$. Then G is traceable or every longest cycle is a dominating cycle.*

Proof If $\delta = 1$ then $n \leq 3$. Hence, G is traceable for $\delta = 1$.

Consider $\delta \geq 2$. Assume first that G has a cut-vertex, say u . Let u , x and y be as defined in the proof of Theorem 4. By our choice of u , x and y together with the fact that δ is the minimum degree and G is triangle-free, we obtain a tree isomorphic to T , using exactly the arguments we used to construct T in the proof of Theorem 4. So, it suffices to consider the tree to be T as given in the proof of Theorem 4, see Fig. 1. Since $|V(T)| = 4\delta - 1$ and $n \leq 4\delta - 1$, T spans G . That is, $|V(G)| = |V(T)|$. Note that for all $i, j : 1 \leq j \leq \delta - 1$, p_i cannot be adjacent to p_j for $p \in \{w, x, y, z\}$, since G is triangle-free. This together with our choice of u , x , y and the fact that G is triangle-free imply that for each i , the δ neighbours of z_i must be $u, y_1, y_2, \dots, y_{\delta-1}$. Similarly, for all i , the δ neighbours of w_i must be $u, x_1, x_2, \dots, x_{\delta-1}$. Therefore, $G = G_{4\delta-1}$ and is traceable.

Consider the case where G is 2-connected. If $n \leq 4\delta - 2$ then the result is an immediate consequence of Theorem 5. Assume that $n = 4\delta - 1$. Following Theorem 2, it is adequate to consider the case where every longest cycle C is of length $4\delta - 4$ and $G - C = K_2 \cup K_1$, otherwise, we are done easily. Let $u_1, u_2, u_3 \notin V(C)$. We may assume that $u_1 u_2 \in E(G)$ and $u_1 u_3, u_2 u_3 \notin E(G)$, other cases are treated likewise. Since $u_1 u_2 \in E(G)$ and $|V(C)| = 4\delta - 4$, one can easily use similar arguments as the proof of Theorem 5 for $n = 4\delta - 2$ to get a contradiction. Hence the result follows. $\square$

We now improve Theorem 3 by proving the following result:

Theorem 6 *Let G be a connected, claw-free, paw-free graph. Then G is Hamiltonian if and only if G is not a path.*

Proof Assume first that G is a tree. Then G is a path, otherwise, G contains a claw, which is not allowed. So, assume that G is not a tree. Then G contains at least one cycle. Let $C_k = v_1, v_2, v_3, \dots, v_k, v_1$ be a longest cycle in G . It suffices to show that C_k spans G . Assume contrary to the claim that C_k does not span G . Then, since G is connected there is a vertex, say, v which is not on C_k and v is adjacent to a vertex,

say, $v_i \in V(C_k)$. Now, v cannot be adjacent to v_{i-1} or v_{i+1} , otherwise, we contradict our choice of C_k as a longest cycle in G . Thus, if v_{i-1} is not adjacent to v_{i+1} , then the set $\{v, v_{i-1}, v_i, v_{i+1}\}$ forms a claw, which is prohibited. If on the other hand, v_{i-1} is adjacent to v_{i+1} , then the set $\{v, v_{i-1}, v_i, v_{i+1}\}$ yields a paw in G , which is not permitted. Hence, in either case we get a contradiction. So the supposition that C_k does not span G is false. Consequently, G is Hamiltonian. $\square$

From Theorem 6, one can easily deduce that every simple, connected, claw-free, paw-free graph is traceable.

3 Conclusion

Sufficient conditions for a connected, triangle-free graph to be 2-connected based on minimum degree and leaf number or minimum degree and order have been presented. The results are crucial in the establishment of sufficient conditions for a triangle-free graph to be Hamiltonian. In fact, the results have been used to deduce sufficient conditions for the existence of spanning paths or cycles in triangle-free graphs. The algorithm used to establish sufficient conditions for a triangle-free graph to be 2-connected can also be used for C_4 -free graphs and graphs of given girth. It would be interesting if sharp or asymptotically tight conditions on 2-connectedness can be obtained for C_4 -free graphs or graphs of given girth based on the aforementioned parameters.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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