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# Integration questions related to fractional Brownian motion*

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**Abstract.** Let $\{B_H(u)\}_{u\in\mathbb{R}}$ be a fractional Brownian motion (fBm) with index $H \in (0, 1)$ and $\overline{Sp}(B_H)$ be the closure in $L^2(\Omega)$ of the span $Sp(B_H)$ of the increments of fBm $B_H$. It is well-known that, when $B_H = B_{1/2}$ is the usual Brownian motion (Bm), an element $X \in \overline{Sp}(B_{1/2})$ can be characterized by a unique function $f_X \in L^2(\mathbb{R})$, in which case one writes $X$ in an integral form as $X = \int_{\mathbb{R}} f_X(u)dB_{1/2}(u)$. From a different, though equivalent, perspective, the space $L^2(\mathbb{R})$ forms a class of integrands for the integral on the real line with respect to Bm $B_{1/2}$. In this work we explore whether a similar characterization of elements of $\overline{Sp}(B_H)$ can be obtained when $H \in (0, 1/2)$ or $H \in (1/2, 1)$. Since it is natural to define the integral of an elementary function $f = \sum_{k=1}^n f_k 1_{[u_k,u_{k+1})}$ by $\sum_{k=1}^n f_k(B_H(u_{k+1}) - B_H(u_k))$, we want the spaces of integrands to contain elementary functions. These classes of integrands are inner product spaces. If the space of integrands is not complete, then it characterizes only a strict subset of $\overline{Sp}(B_H)$. When $0 < H < 1/2$, by using the moving average representation of fBm $B_H$, we construct a complete space of integrands. When $1/2 < H < 1$, however, an analogous construction leads to a space of integrands which is not complete. When $0 < H < 1/2$ or $1/2 < H < 1$, we also consider a number of other spaces of integrands. While smaller and hence incomplete, they form a natural choice and are convenient to work with. We compare these spaces of integrands to the reproducing kernel Hilbert space of fBm.

## 1. Introduction

Fractional Brownian motion (fBm) $\{B_H(u)\}_{u\in\mathbb{R}}$ with index $0 < H < 1$ is a Gaussian, mean-zero and $H$-self-similar process with $B_H(0) = 0$ and stationary increments. $H$-self-similarity means that, for $a > 0$,

$$\{B_H(au), u \in \mathbb{R}\} \stackrel{d}{=} \{a^H B_H(u), u \in \mathbb{R}\},$$

where $\stackrel{d}{=}$ stands for the equality in the finite-dimensional distributions. The fBm $B_H$ with $H = 1/2$ is the usual Brownian motion (Bm). If $EB_H^2(1) = 1$, the fBm

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$B_H$ is called standard. Since we will deal with fractional integration, it will be more convenient in the sequel to use the parametrization

$$\kappa = H - \frac{1}{2}.$$

The range $0 < H < 1$ now corresponds to $-1/2 < \kappa < 1/2$. (The index $\kappa$ is usually denoted $d$ in the statistical literature, but in the context of fractional integration the letter $d$ may be confusing.)

We will denote the fBm $B_H = \{B_H(u)\}_{u\in\mathbb{R}}$ in terms of the parameter $\kappa$ as $B^\kappa = \{B^\kappa(u)\}_{u\in\mathbb{R}}$. By using stationarity of the increments and $\kappa + \frac{1}{2}$ – self-similarity of fBm $B^\kappa$, it is easy to see that the covariance of a standard fBm $B^\kappa$ is given by

$$\Gamma^\kappa(u, v) = EB^\kappa(u)B^\kappa(v) = \frac{1}{2}\left(|u|^{2\kappa+1} + |v|^{2\kappa+1} - |u-v|^{2\kappa+1}\right). \tag{1.1}$$

For more information on fBm see, for example, Chapter 7 and references to it in Samorodnitsky and Taqqu [15].

The fBm model is widely applied in telecommunications (see Leland et al. [13]) and is also of interest in finance, notwithstanding the fact that it is not a semimartingale. Integration with respect to fBm has thus many potential applications. The difficulty, however, lies in the following. On one hand, the paths of fBm are of unbounded variation and hence the usual Lebesgue-Stieltjes integration cannot be applied. One cannot use the usual Itô's stochastic calculus either because fBm $B^\kappa$ is not a semimartingale. Hence, some other approaches have to be taken. Recently, there were numerous attempts to develop some type of stochastic calculus for fBm. For example, Decreusefond and Üstünel [4] define a stochastic integral with respect to fBm by using the stochastic calculus of variations (also known as the Malliavin calculus) and the fact that fBm is a Gaussian process. The Malliavin calculus ideas along with approximation by stochastic integrals with respect to semimartingales are also employed in another two works by Carmona, Coutin and Montseny [3], and Alòs, Mazet and Nualart [1]. The paper by Duncan, Hu and Pasik-Duncan [6] defines a stochastic integral by using the Wick product but it is still in the spirit of Decreusefond and Üstünel [4]. A different approach, that of a pathwise integration, is taken by Dudley and Norvaiša [5], and Zähle [17]. They use specific path properties of fBm, namely, $p$-variation in Dudley and Norvaiša [5] and Hölder continuity in Zähle [17]. Eventually, we note the work by Kleptsyna, LeBreton and Roubaud [12], where an elementary approach to stochastic calculus for fBm is developed. The purpose of this paper is to explore some fundamental questions related to the definition of integrals with respect to fBm. As it will soon become apparent, these questions do not have trivial answers, even when the integrand $f$ is not random which is the case considered here.

In applications, one likes to view the integral $\int_{\mathbb{R}} f dB^\kappa$ as approximated by

$$\sum_{k=1}^{n} f_k \triangle B^\kappa(u_k) = \sum_{k=1}^{n} f_k \left(B^\kappa(u_{k+1}) - B^\kappa(u_k)\right), \tag{1.2}$$

where $f_k$ and $u_k < u_{k+1}$ are real numbers. This idea can be formalized mathematically as follows. Write the linear combination (1.2) in an integral form as

$$\mathscr{I}^\kappa(f) = \int_{\mathbb{R}} f(u)dB^\kappa(u), \tag{1.3}$$

where $f$ is the *elementary* (or *step*) function given by

$$f(u) = \sum_{k=1}^{n} f_k 1_{[u_k,u_{k+1})}(u), \ \ u \in \mathbb{R}. \tag{1.4}$$

$\mathscr{I}^\kappa(f)$, in (1.3), is a Gaussian random variable with zero mean and variance which can be easily evaluated by using (1.1). (We will also use in the sequel *simple* functions where the intervals in (1.4) are replaced by arbitrary bounded Borel sets.) If $\mathscr{E}$ denotes the set of elementary functions on the real line, then the linear space of Gaussian random variables $\{\mathscr{I}^\kappa(f),\ f \in \mathscr{E}\}$ is a subset of the larger linear space

$$\overline{Sp}(B^\kappa) = \{X : \mathscr{I}^\kappa(f_n) \overset{L^2}{\to} X,\ \text{for some } (f_n) \subset \mathscr{E}\}. \tag{1.5}$$

An element $X \in \overline{Sp}(B^\kappa)$ is also a Gaussian random variable with zero mean and variance

$$\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(\mathscr{I}^\kappa(f_n)).$$

We can associate with $X$ an equivalence class of *sequences* of elementary functions $(f_n)$ such that $\mathscr{I}^\kappa(f_n) \to X$ in the $L^2(\Omega)$-sense. If $f_X$ stands for this equivalence class, then $X$ is usually written in an integral form as

$$X = \int_{\mathbb{R}} f_X dB^\kappa \tag{1.6}$$

and the right-hand side of (1.6) is called the integral with respect to fBm on the real line (see, for example, Huang and Cambanis [11], p. 587). Thus the integral (1.6) is indeed a limit (in the $L^2(\Omega)$-sense) of the (1.2) type linear combinations.

It is well known, however, that when $\kappa = 0$ (in which case $B^\kappa$ corresponds to the usual Bm) there is a *simpler* characterization of elements of $\overline{Sp}(B^0)$. If we assume that Bm $B^0$ is standard, then independence of its increments implies that, for $f \in \mathscr{E}$,

$$\mathrm{Var}(\mathscr{I}^0(f)) = \int_{\mathbb{R}} f^2(u)du.$$

Hence, if $(f_n) \subset \mathscr{E}$ and if $\mathscr{I}^0(f_n)$ converges to $X \in \overline{Sp}(B^0)$ in the $L^2(\Omega)$-sense, there is a unique function $f_X \in L^2(\mathbb{R})$ such that

$$\mathrm{Var}(X) = \lim_{n\to\infty} \mathrm{Var}(\mathscr{I}^0(f_n)) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n^2(u)du = \int_{\mathbb{R}} f_X^2(u)du.$$

This is because the space $L^2(\mathbb{R})$ is complete. Thus, whereas $X \in \overline{Sp}(B^0)$ has been associated above with a *sequence of elementary functions* $(f_n)$, it can now be characterized by a *single function* $f_X \in L^2(\mathbb{R})$. One usually writes $X$ in an integral form as

$$X = \int_{\mathbb{R}} f_X(u)dB^0(u).$$

Conversely, since any function $f \in L^2(\mathbb{R})$ can be approximated in $L^2(\mathbb{R})$ by elementary functions, there is a random variable $X_f \in \overline{Sp}(B^0)$ such that

$$X_f = \int_{\mathbb{R}} f(u)dB^0(u).$$

This implies that the integral on the real line with respect to Bm can be defined for arbitrary functions $f \in L^2(\mathbb{R})$. Since, for $X, Y \in \overline{Sp}(B^0)$,

$$E(XY) = \int_{\mathbb{R}} f_X(u) f_Y(u)du,$$

we can also say that the Hilbert spaces $\overline{Sp}(B^0)$ and $L^2(\mathbb{R})$ are isometric, that is, there is a linear and onto map between these spaces which preserves inner products. Observe also that this map is an extension of the map $f \to \mathcal{I}^0(f)$, for $f \in \mathcal{E}$.

In this work we explore whether a similar characterization of the elements of the space $\overline{Sp}(B^\kappa)$ can be obtained when $-1/2 < \kappa < 0$ or $0 < \kappa < 1/2$. In particular, we are interested in the following question:

*Is there a Hilbert space $\mathcal{C}$ of functions on the real line which is isometric to $\overline{Sp}(B^\kappa)$?*

We impose throughout a natural restriction on this isometry, requiring it to be an extension of the map $f \to \mathcal{I}^\kappa(f)$, for $f \in \mathcal{E}$. We show in this paper that such a Hilbert space $\mathcal{C}$ exists when $-1/2 < \kappa < 0$. As in the case $\kappa = 0$, this means that:

(i) every element of the space $\overline{Sp}(B^\kappa)$ with $-1/2 < \kappa < 0$ can be expressed as an integral of a deterministic function with respect to fBm $B^\kappa$, and

(ii) the integral on the real line with respect to fBm $B^\kappa$ with $-1/2 < \kappa < 0$ is well-defined for functions from this Hilbert space $\mathcal{C}$.

When $0 < \kappa < 1/2$, however, we do not know the answer to the above question. This is quite surprising because typically the case $0 < \kappa < 1/2$ is (or, is thought to be) simpler than the case $-1/2 < \kappa < 0$. Instead, when $0 < \kappa < 1/2$, we pursue a simpler goal which is to characterize elements of some linear subspaces of $\overline{Sp}(B^\kappa)$ (the larger, the better). The relevant question in this context is:

*What are the inner product spaces of functions on the real line which are isometric to linear subspaces of $\overline{Sp}(B^\kappa)$?*

We think here again of those isometries that extend the map $f \to \mathcal{I}^\kappa(f)$, for $f \in \mathcal{E}$. The interest of this question lies in the following. If $\mathcal{C}$ is such an inner product space and $S_{\mathcal{C}}$ is the corresponding subspace of $\overline{Sp}(B^\kappa)$, then:

(i) every element of the linear subspace $S_{\mathcal{C}}$ can be expressed as an integral of a deterministic function with respect to fBm $B^\kappa$, and

(ii) the integral on the real line with respect to fBm $B^\kappa$ with $0 < \kappa < 1/2$ is well-defined for functions from $\mathcal{C}$.

Note that, by (ii), an inner product space $\mathcal{C}$ can then be viewed as a class of possible deterministic integrands. We will show that the spaces of deterministic integrands one usually considers, correspond to spaces $\mathscr{C}$ that are *not complete* and hence cannot be isometric to $\overline{Sp}(B^\kappa)$ when $0 < \kappa < 1/2$. If one completed them, one would fall back to the case discussed in (1.6), namely, the representative $f_X$ for a random variable $X \in \overline{Sp}(B^\kappa)$ would only be an (equivalent) class of *sequences* of functions.

A trivial example of an inner product space $\mathscr{C}$ is the set of elementary functions $\mathscr{E}$ itself, where the inner product is defined by $E(\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g))$, for $f, g \in \mathscr{E}$. Another example when $0 < \kappa < 1/2$ is the space of functions $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with the inner product defined by

$$\kappa(2\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)g(t)|s-t|^{2\kappa-1}dsdt \tag{1.7}$$

(see Gripenberg and Norros [9], p. 404. The kernel $|s-t|^{2\kappa-1}$ appears as $|s-t|^{2H-2}$ in the $H$ parametrization.) In this work we construct, analyze and compare a number of such inner product spaces (or classes of integrands). We also unite the various approaches for both $0 < \kappa < 1/2$ and $-1/2 < \kappa < 0$. For instance, depending on how fBm is represented, we can obtain a class of integrands in the "time domain" or the "spectral domain" (the "time domain" representation of fBm is given in (3.7) below and its "spectral domain" representation is given in (3.1) below).

The different spaces $\mathscr{C}$ we consider are denoted $\widetilde{\Lambda}^\kappa$, $\Lambda^\kappa$ and $|\Lambda|^\kappa$. The space

$$|\Lambda|^\kappa = \left\{ f : \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1}du\, dv < \infty \right\} \tag{1.8}$$

is used when $0 < \kappa < 1/2$. Observe that the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with the inner product (1.7) is a subspace of (1.8) because the functions are required to belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as well. We will also consider the larger inner product space $\Lambda^\kappa$, defined, using fractional integration, by (3.15) and (3.17) below when $0 < \kappa < 1/2$ and by (3.30) and (3.31) when $-1/2 < \kappa < 0$. We will also analyze a class of integrands in the "spectral domain", namely, the inner product space $\widetilde{\Lambda}^\kappa$ defined by (3.5) and (3.6). For the convenience of the reader, we gather at the end of the paper the definitions of the various spaces and their inner products.

We show that the "time domain" spaces $\Lambda^\kappa$ are complete only when $-1/2 < \kappa < 0$ and that the "spectral domain" spaces $\widetilde{\Lambda}^\kappa$ are strict subsets of $\Lambda^\kappa$ and not complete for all $-1/2 < \kappa < 1/2$, $\kappa \neq 0$. As to $|\Lambda|^\kappa$, $0 < \kappa < 1/2$, we show that it is also a strict subset of $\Lambda^\kappa$ and is not complete.

The paper is structured as follows. In Section 2 we recall how inner product spaces or classes of integrands $\mathscr{C}$ are constructed. Then, in Sections 3 and 4, we deal with specific classes $\mathscr{C}$. Most of the results are proved in Section 5. In Section 6 we compare the classes of integrands obtained in the previous sections with the so-called reproducing kernel Hilbert space of fBm. This provides a broader perspective on our results. The summary of the paper can be found in Section 7.

## 2. General construction of classes of integrands

For the sake of clarity and completeness, we recall, in the proposition below, how to construct classes of integrands $\mathscr{C}$. This scheme will be used many times in the sequel to obtain classes $\mathscr{C}$.

**Proposition 2.1.** *Let $\mathscr{E}$ be the set of elementary functions* (1.4), *$\mathscr{I}^\kappa(f)$ be an integral of $f \in \mathscr{E}$ with respect to fBm $B^\kappa$ and $-1/2 < \kappa < 1/2$. Suppose that $\mathscr{C}$ is a set of deterministic functions on the real line such that*

*(1) $\mathscr{C}$ is an inner product space with an inner product $(f, g)_{\mathscr{C}}$, for $f, g \in \mathscr{C}$,*

*(2) $\mathscr{E} \subset \mathscr{C}$ and $(f, g)_{\mathscr{C}} = E\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g)$, for $f, g \in \mathscr{E}$,*

*(3) the set $\mathscr{E}$ is dense in $\mathscr{C}$.*

*Then*

*(a) there is an isometry between the space $\mathscr{C}$ and a linear subspace of $\overline{Sp}(B^\kappa)$ which is an extension of the map $f \to \mathscr{I}^\kappa(f)$, for $f \in \mathscr{E}$,*

*(b) $\mathscr{C}$ is isometric to $\overline{Sp}(B^\kappa)$ itself if and only if $\mathscr{C}$ is complete.*

*Proof.* We first show part $(a)$. Let $f \in \mathscr{C}$. By (3), there is a sequence $(f_n) \subset \mathscr{E}$ such that $f_n \to f$ in $\mathscr{C}$. In particular, $(f_n)$ is Cauchy in $\mathscr{C}$ and hence, by (2), $(\mathscr{I}^\kappa(f_n))$ is a Cauchy sequence in $L^2(\Omega)$. Since the space $L^2(\Omega)$ is complete, there is $\mathscr{I}^\kappa(f) \in L^2(\Omega)$ such that

$$\mathscr{I}^\kappa(f) = \lim_n \mathscr{I}^\kappa(f_n),$$

in the $L^2(\Omega)$-sense. Moreover, since $(\mathscr{I}^\kappa(f_n)) \subset \overline{Sp}(B^\kappa)$ and $\overline{Sp}(B^\kappa)$ is a closed subset of $L^2(\Omega)$, we obtain that $\mathscr{I}^\kappa(f) \in \overline{Sp}(B^\kappa)$. We can thus define the map $\mathscr{I}^\kappa$ from the space $\mathscr{C}$ into the space $\overline{Sp}(B^\kappa)$. It is easy to verify that this definition does not depend on an approximating sequence $(f_n)$. This construction of $\mathscr{I}^\kappa$ and (2) imply that, for $f, g \in \mathscr{C}$,

$$(f, g)_{\mathscr{C}} = E\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g), \tag{2.1}$$

and, since the map $\mathscr{I}^\kappa$ is linear, we conclude that $\mathscr{I}^\kappa$ is, in fact, an isometry between the space $\mathscr{C}$ and a linear *subspace* of $\overline{Sp}(B^\kappa)$.

We now turn to part $(b)$. If $\mathscr{C}$ is isometric to $\overline{Sp}(B^\kappa)$ itself, then $\mathscr{C}$ is complete because the space $\overline{Sp}(B^\kappa)$ is complete (it is a closed subset of the complete space $L^2(\Omega)$). Conversely, if $\mathscr{C}$ is complete, then the map $\mathscr{I}^\kappa$ is onto because $\mathscr{E}$ is dense in $\mathscr{C}$ and hence $\mathscr{C}$ is isometric to $\overline{Sp}(B^\kappa)$ itself. □

The isometry map $\mathscr{I}^\kappa$ obtained in the proof above is also denoted

$$\mathscr{I}^\kappa(f) = \int_{\mathbb{R}} f(u) dB^\kappa(u), \tag{2.2}$$

for $f \in \mathscr{C}$, and the right-hand side of (2.2) is called the integral on the real line of $f$ with respect to fBm $B^\kappa$.

Observe that the isometry map $\mathscr{I}^\kappa$ might depend on the inner product space $\mathscr{C}$. In other words, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are two different classes of functions that satisfy the conditions of Proposition 2.1, then, a priori, it is not clear whether the corresponding isometry maps, say $\mathcal{I}_1^\kappa$ and $\mathcal{I}_2^\kappa$, are equal on $\mathcal{C}_1 \cap \mathcal{C}_2$. (A necessary and sufficient condition for this fact is $(f,g)_{\mathcal{C}_1} = (f,g)_{\mathcal{C}_2}$, $\forall f,g \in \mathcal{C}_1 \cap \mathcal{C}_2$.) For this reason, when we consider in the sequel specific classes of integrands, we will compare their corresponding isometry maps and use the same notation only if they are equal.

Finally, as an example in the simple case when $\kappa = 0$, we can take $\mathcal{C} = L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is complete, the map $\mathcal{I}^0$ is an isometry between the spaces $\mathcal{C}$ and $\overline{Sp}(B^0)$. The cases $-1/2 < \kappa < 0$ and $0 < \kappa < 1/2$, however, are neither immediately obvious nor simple as the following sections show.

## 3. Spectral and time domain integrands

We use different representations of fBm and the scheme described in Proposition 2.1 to construct different classes of integrands. The first class, denoted by $\widetilde{\Lambda}^\kappa$, is obtained in the context of the "spectral domain" and the second one, denoted by $\Lambda^\kappa$, arises in the context of the "time domain". We then show that the former class is a subset of the latter, that is $\widetilde{\Lambda}^\kappa \subset \Lambda^\kappa$, and also prove that their corresponding isometry maps $\mathcal{I}^\kappa$'s are equal almost surely on the smaller space $\widetilde{\Lambda}^\kappa$. The fBm $B^\kappa$, in the sequel, is assumed to be *standard*.

### *3.1. Class of integrands in the "spectral domain" for $-1/2 < \kappa < 1/2$*

Recall that a standard fBm $\{B^\kappa(t)\}_{t\in\mathbb{R}}$ with index $-1/2 < \kappa < 1/2$ has the spectral representation (see Samorodnitsky and Taqqu [15], p. 328)

$$\{B^\kappa(t)\}_{t\in\mathbb{R}} \overset{d}{=} \left\{ \frac{1}{c_2(\kappa)} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{-\kappa} d\widetilde{B}(x) \right\}_{t\in\mathbb{R}}, \tag{3.1}$$

where

$$c_2(\kappa) = \left( \frac{2\pi}{\Gamma(2\kappa+2)\sin(\pi(\kappa+1/2))} \right)^{1/2} \tag{3.2}$$

and $\widetilde{B} = B^1 + iB^2$ is a complex Gaussian measure such that $B^1(A) = B^1(-A)$, $B^2(A) = -B^2(-A)$ and $E(B^1(A))^2 = E(B^2(A))^2 = |A|/2$, for any Borel set $A$ of finite Lebesgue measure $|A|$ (see Section 7.2.2 in Samorodnitsky and Taqqu [15]). Observe now that $(e^{itx}-1)/ix = \widehat{1}_{[0,t)}(x)$, where $\widehat{f}$ denotes the Fourier transform of a function $f$, i.e. $\widehat{f}(x) = \int_{\mathbb{R}} e^{ixu} f(u)du$. Then, for $f \in \mathcal{E}$,

$$\mathcal{I}^\kappa(f) \overset{d}{=} \frac{1}{c_2(\kappa)} \int_{\mathbb{R}} \widehat{f}(x)|x|^{-\kappa} d\widetilde{B}(x) \tag{3.3}$$

and hence (see (7.2.9) in Samorodnitsky and Taqqu [15]) that, for $f, g \in \mathcal{E}$,

$$E(\mathcal{I}^\kappa(f)\mathcal{I}^\kappa(g)) = \frac{1}{c_2(\kappa)^2} \int_{\mathbb{R}} \widehat{f}(x)\overline{\widehat{g}(x)}|x|^{-2\kappa} dx. \tag{3.4}$$

Based on (3.4), we introduce

**Definition 3.1.**

$$\widetilde{\Lambda}^{\kappa} = \left\{ f : f \in L^2(\mathbb{R}), \ \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx < \infty \right\}, \tag{3.5}$$

for $-1/2 < \kappa < 1/2$.

The following theorem and (3.4) show that $\widetilde{\Lambda}^{\kappa}$ satisfies the conditions (1), (2) and (3) of Proposition 2.1. This theorem is proved in Section 5.1.

**Theorem 3.1.** *For* $-1/2 < \kappa < 1/2$, *the class of functions* $\widetilde{\Lambda}^{\kappa}$, *defined by* (3.5), *is a linear space with the inner product*

$$(f, g)_{\widetilde{\Lambda}^{\kappa}} = \frac{1}{c_2(\kappa)^2} \int_{\mathbb{R}} \widehat{f}(x) \overline{\widehat{g}(x)} |x|^{-2\kappa} dx. \tag{3.6}$$

*The set of elementary functions* $\mathscr{E}$ *is dense in* $\widetilde{\Lambda}^{\kappa}$. *Moreover, the space* $\widetilde{\Lambda}^{\kappa}$ *is not complete unless* $\kappa = 0$.

It follows from Proposition 2.1 that there is an isometry map between the space $\widetilde{\Lambda}^{\kappa}$ and a linear subspace of $\overline{Sp}(B^{\kappa})$ which is denoted by $\mathscr{I}^{\kappa}(f) = \int_{\mathbb{R}} f(u) dB^{\kappa}(u)$, for $f \in \widetilde{\Lambda}^{\kappa}$. This linear subspace is a strict subset of $\overline{Sp}(B^{\kappa})$ when $\kappa \neq 0$ because, by the above theorem, $\widetilde{\Lambda}^{\kappa}$ is not complete. The integral $\mathscr{I}^{\kappa}$ so defined satisfies the relations (3.3) and (3.4) for $f, g \in \widetilde{\Lambda}^{\kappa}$.

We conclude this section with a proposition which characterizes a large subset of $\widetilde{\Lambda}^{\kappa}$ in the case $0 < \kappa < 1/2$ (this result is mentioned in Barton and Poor [2], p. 945).

**Proposition 3.1.** *For* $0 < \kappa < 1/2$, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *implies* $f \in \widetilde{\Lambda}^{\kappa}$.

*Proof.* The proposition follows from the estimate

$$\begin{aligned}
\int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx &= \int_{|x| \le 1} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx + \int_{|x|>1} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx \\
&\le \|f\|^2_{L^1(\mathbb{R})} \int_{|x| \le 1} |x|^{-2\kappa} dx + \int_{\mathbb{R}} |\widehat{f}(x)|^2 dx \\
&= \frac{1}{1/2 - \kappa} \|f\|^2_{L^1(\mathbb{R})} + 2\pi \|f\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where as usual $\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(u)| du$ and $\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(u)|^2 du\right)^{1/2}$. □

**Remarks.**

1. Observe that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a condition on $f$, whereas $f \in \widetilde{\Lambda}^{\kappa}$ involves a condition on $\widehat{f}$.

2. Proposition 3.1 does not hold when $-1/2 < \kappa < 0$. As a counterexample, consider the function $f(u) = \mathrm{sgn}(u)e^{-|u|}/|u|^p$ with $\kappa + 1/2 < p < 1/2$. Clearly, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Its Fourier transform equals (see formula 3.945,(1) in Gradshteyn and Ryzhik [7], p. 491)

$$2\Gamma(1-p)(x^2+1)^{\frac{p-1}{2}} \sin[(1-p)\mathrm{Arctan}(x)]$$

and hence it behaves (up to a constant) like $|x|^{p-1}$ at $x = \infty$. Since $2p - 2\kappa - 2 + 1 = 2(p - \kappa - 1/2) > 0$, the function $|x|^{2(p-1)}|x|^{-2\kappa} = |x|^{2p-2\kappa-2}$ is not integrable around $x = \infty$. This implies that $f \notin \widetilde{\Lambda}^\kappa$ for $-1/2 < \kappa < 0$.

### *3.2. Class of integrands in the "time domain" for* $0 < \kappa < 1/2$

Instead of working in the "spectral domain" as in Section 3.1, we now work in the "time domain". Recall that a standard fBm $\{B^\kappa(t)\}_{t\in\mathbb{R}}$ with index $-1/2 < \kappa < 1/2$ has the moving average representation (see Samorodnitsky and Taqqu [15], p. 320)

$$\{B^\kappa(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \frac{1}{c_1(\kappa)} \int_{\mathbb{R}} \left((t-s)_+^\kappa - (-s)_+^\kappa\right) dB(s) \right\}_{t\in\mathbb{R}}, \tag{3.7}$$

where $B$ is a standard Brownian motion,

$$c_1(\kappa) = \left( \int_0^\infty \left((1+s)^\kappa - s^\kappa\right)^2 ds + \frac{1}{2\kappa+1} \right)^{1/2}, \tag{3.8}$$

$a_+^\kappa = 0$, for $a \le 0$, and $a_+^\kappa = a^\kappa$, for $a > 0$. Observe that, when $0 < \kappa < 1/2$,

$$(t-s)_+^\kappa - (-s)_+^\kappa = \kappa \int_{\mathbb{R}} 1_{[0,t)}(u)(u-s)_+^{\kappa-1} du, \tag{3.9}$$

where $1_{[0,t)}$ is interpreted as $(-1_{[t,0)})$, if $t < 0$. Let $I_-^\alpha \phi$ denote a fractional integral of order $\alpha > 0$ of a function $\phi$ defined by

$$(I_-^\alpha \phi)(s) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)(s-u)_-^{\alpha-1} du, \tag{3.10}$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)(u-s)_+^{\alpha-1} du, \;\; s \in \mathbb{R} \tag{3.11}$$

(see Chapter 2 in Samko et al. [14] and, in particular, (5.2) on p. 94. While the right-hand side of (3.10) indicates that we are dealing with the convolution of $\phi$ with $u_-^{\alpha-1}$, relation (3.11) is more convenient to manipulate.) Then the relation (3.9) can be expressed as

$$(t-s)_+^\kappa - (-s)_+^\kappa = \Gamma(\kappa+1)(I_-^\kappa 1_{[0,t)})(s). \tag{3.12}$$

It follows from (3.7) and (3.12) that, for $f \in \mathscr{E}$ and $0 < \kappa < 1/2$,

$$\int_{\mathbb{R}} f(u) dB^\kappa(u) \stackrel{d}{=} \frac{\Gamma(\kappa+1)}{c_1(\kappa)} \int_{\mathbb{R}} (I_-^\kappa f)(s) dB(s) \tag{3.13}$$

and hence that, for $f, g \in \mathscr{E}$ and $0 < \kappa < 1/2$,

$$E(\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g)) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}}(I_-^\kappa f)(s)(I_-^\kappa g)(s)ds. \tag{3.14}$$

This leads us to introduce the following class of functions

**Definition 3.2.**

$$\begin{aligned}\Lambda^\kappa &= \left\{f : \int_{\mathbb{R}}\left[(I_-^\kappa f)(s)\right]^2 ds < \infty\right\} \\ &= \left\{f : \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1}du\right]^2 ds < \infty\right\},\end{aligned} \tag{3.15}$$

for $0 < \kappa < 1/2$.

The inner integral in (3.15) is understood in the Lebesgue sense, which implies in particular that

$$\int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1}du < \infty \ \text{ a.e. } ds. \tag{3.16}$$

The following theorem is the "time domain" analogue of Theorem 3.1 and is proved in Section 5.2.

**Theorem 3.2.** *For* $0 < \kappa < 1/2$, *the class of functions* $\Lambda^\kappa$, *defined by* (3.15), *is a linear space with the inner product*

$$\left(f, g\right)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}}(I_-^\kappa f)(s)(I_-^\kappa g)(s)ds. \tag{3.17}$$

*The set of elementary functions* $\mathscr{E}$ *is dense in the space* $\Lambda^\kappa$. *The space* $\Lambda^\kappa$ *is not complete.*

Theorem 3.2 and the relation (3.14) show that $\Lambda^\kappa$ satisfies the conditions of Proposition 2.1. Hence, by Proposition 2.1, $\Lambda^\kappa$ is isometric to a strict linear subspace of $\overline{Sp}(B^\kappa)$. We denote this isometry map by $\mathscr{I}^\kappa(f) = \int_{\mathbb{R}} f(u)dB^\kappa(u)$, for $f \in \Lambda^\kappa$, as in the "spectral domain". The use of the same notation is justified in Section 3.4. Observe also that the relations (3.13) and (3.14) hold for $f \in \Lambda^\kappa$.

**Remarks.**

1. In the case $0 < \kappa < 1/2$, by (3.15),

$$\Lambda^\kappa = \left\{f : \phi_f = I_-^\kappa f \in L^2(\mathbb{R})\right\}, \tag{3.18}$$

and hence (3.17) states that

$$\left(f, g\right)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\left(\phi_f, \phi_g\right)_{L^2(\mathbb{R})}. \tag{3.19}$$

2. The fractional integral notation provides a convenient way of interpreting (3.13). For $\alpha > 0$ and "sufficiently good" functions $\psi$ and $\phi$, the fractional integration by parts formula (see (5.17) in Samko et al. [14])

$$\int_{\mathbb{R}} (I_-^\alpha \psi)(s)\phi(s)ds = \int_{\mathbb{R}} \psi(u)(I_+^\alpha \phi)(u)du \tag{3.20}$$

holds, where $I_+^\alpha \phi$, $\alpha > 0$, is the fractional integral defined by

$$(I_+^\alpha \phi)(u) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{u} \frac{\phi(s)ds}{(u-s)^{1-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(s)(u-s)_+^{\alpha-1} ds, \ u \in \mathbb{R} \tag{3.21}$$

(see Chapter 2 in Samko et al. [14] and, in particular, (5.3) on p. 94). Then the relation (3.13) with $f \in \Lambda^\kappa$ can be informally expressed as

$$\int_{\mathbb{R}} f(u) \frac{dB^\kappa}{du}(u)du = \frac{\Gamma(\kappa+1)}{c_1(\kappa)} \int_{\mathbb{R}} f(u) \left( I_+^\kappa \frac{dB}{du} \right)(u)du$$

or

$$\frac{dB^\kappa}{du}(u) = \frac{\Gamma(\kappa+1)}{c_1(\kappa)} \left( I_+^\kappa \frac{dB}{du} \right)(u).$$

Thus, when $0 < \kappa < 1/2$ the "fractional noise" $dB^\kappa/du$ is a $\kappa$–fractional integral of the "white noise" $dB/du$. This fact could be formalized by using generalized processes.

We now establish an analogue to Proposition 3.1.

**Proposition 3.2.** *For* $0 < \kappa < 1/2$,

$$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset L^{2/2\kappa+1}(\mathbb{R}) \subset \Lambda^\kappa. \tag{3.22}$$

*Proof.* The first inclusion in (3.22) follows from the inequality

$$|f(u)|^{2/2\kappa+1} \leq |f(u)| + |f(u)|^2,$$

which holds for $0 < \kappa < 1/2$. On the other hand, the operator $I_-^\kappa$ maps $L^{2/2\kappa+1}(\mathbb{R})$ into $L^2(\mathbb{R})$ for $0 < \kappa < 1/2$ (Theorem 5.3 in Samko et al. [14], p. 103). This implies the second inclusion in (3.22). □

**Remark.** It is easy to verify directly that, for $0 < \kappa < 1/2$, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ implies $f \in \Lambda^\kappa$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1} du \right]^2 ds \\
&\quad \leq 2 \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1} 1_{\{u-s>1\}}(u)du \right]^2 ds \\
&\qquad + 2 \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(u+s)| \, u_+^{\kappa-1} 1_{\{u<1\}}(u)du \right]^2 ds < \infty,
\end{aligned}$$

after applying (3.37) below.

### *3.3. Class of integrands in the "time domain" for $-1/2 < \kappa < 0$*

To construct the class of integrands $\Lambda^\kappa$ in the "time domain" when $0 < \kappa < 1/2$, we used the moving average representation (3.7) of fBm and the relation (3.12). Although (3.12) holds for $0 < \kappa < 1/2$ only, there is an analogous relation in the case $-1/2 < \kappa < 0$. Recall (see Samko et al. [14], p. 111) that Marchaud fractional derivatives $\mathbf{D}^\alpha_\pm \phi$ of order $\alpha \in (0,1)$ of a function $\phi$ are defined by

$$\mathbf{D}^\alpha_\pm \phi = \lim_{\epsilon \to 0} \mathbf{D}^\alpha_{\pm,\epsilon} \phi, \tag{3.23}$$

where

$$(\mathbf{D}^\alpha_{\pm,\epsilon}\phi)(s) = \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty \frac{\phi(s) - \phi(s \mp u)}{u^{1+\alpha}} du. \tag{3.24}$$

We then have

**Lemma 3.1.** *If $-1/2 < \kappa < 0$ and $t > 0$, then, for all $s \in \mathbb{R}$,*

$$(t-s)^\kappa_+ - (-s)^\kappa_+ = \Gamma(\kappa+1)(\mathbf{D}^{-\kappa}_- 1_{[0,t)})(s). \tag{3.25}$$

*The formula* (3.25) *also holds for $t < 0$, if $1_{[0,t)}$ is interpreted as* $(-1_{[-t,0)})$.

*Proof.* Let $t > 0$. By (3.24), we have

$$\begin{aligned}\Gamma(\kappa+1)(\mathbf{D}^{-\kappa}_{-,\epsilon} 1_{[0,t)})(s) &= -\kappa \int_\epsilon^\infty \big(1_{[0,t)}(s) - 1_{[0,t)}(s+u)\big) u^{\kappa-1} du \\ &= \kappa \int_\mathbb{R} 1_{[0,t)}(v)(v-s)^{\kappa-1}_+ 1_{\{\epsilon < v-s\}}(v) dv + 1_{[0,t)}(s)\epsilon^\kappa.\end{aligned} \tag{3.26}$$

One can verify that, if $\epsilon < t$, the function (3.26) is equal to $(t-s)^\kappa_+ - (-s)^\kappa_+$, $s \in \mathbb{R}$, on the set $A_\epsilon = \{s < -\epsilon\} \cup \{0 \le s < t - \epsilon\} \cup \{s \ge t\}$. The relation (3.25) follows by letting $\epsilon \to 0$. When $t < 0$, the proof of (3.25) is similar. □

By using the moving average representation (3.7) of fBm and (3.25), we obtain that, for $f \in \mathcal{E}$ and $-1/2 < \kappa < 0$,

$$\int_\mathbb{R} f(u) dB^\kappa(u) \overset{d}{=} \frac{\Gamma(\kappa+1)}{c_1(\kappa)} \int_\mathbb{R} (\mathbf{D}^{-\kappa}_- f)(s) dB(s) \tag{3.27}$$

and hence that, for $f, g \in \mathcal{E}$ and $-1/2 < \kappa < 0$,

$$E(\mathcal{I}^\kappa(f)\mathcal{I}^\kappa(g)) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_\mathbb{R} (\mathbf{D}^{-\kappa}_- f)(s)(\mathbf{D}^{-\kappa}_- g)(s) ds. \tag{3.28}$$

Observe now that, if $f = I^{-\kappa}_- \phi$, for some $\phi \in L^2(\mathbb{R})$, then Theorem 6.1 in Samko et al. [14], p. 125, implies that

$$\mathbf{D}^{-\kappa}_- f = \mathbf{D}^{-\kappa}_- (I^{-\kappa}_- \phi) = \phi \tag{3.29}$$

and hence that

$$\int_\mathbb{R} \big[(\mathbf{D}^{-\kappa}_- f)(s)\big]^2 ds = \int_\mathbb{R} \phi^2(u) du < \infty.$$

Based on this observation and the relation (3.28), we introduce the next class of functions.

**Definition 3.3.** Let

$$\Lambda^\kappa = \left\{ f : \exists\, \phi \in L^2(\mathbb{R}) \text{ such that } f = I_-^{-\kappa}\phi \right\}, \tag{3.30}$$

for $-1/2 < \kappa < 0$.

(Note that the class $\Lambda^\kappa$ coincides with the space of functions $I_-^{-\kappa}(L_2)$ from Samko et al. [14], Chapter 6, p. 122.) Compare this definition of $\Lambda^\kappa$ with the corresponding definition for $0 < \kappa < 1/2$ (see (3.18)). The definition of $\Lambda^\kappa$ in the cases $0 < \kappa < 1/2$ and $-1/2 < \kappa < 0$ can be viewed as dual to each other.

Theorem 3.2 concerned $\Lambda^\kappa$ when $0 < \kappa < 1/2$. The following theorem deals with $\Lambda^\kappa$ in the case $-1/2 < \kappa < 0$ and is proved in Section 5.3.

**Theorem 3.3.** *For* $-1/2 < \kappa < 0$, *the class of functions* $\Lambda^\kappa$, *defined by* (3.30), *is a linear space with the inner product*

$$\left(f, g\right)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (\mathbf{D}_-^{-\kappa} f)(s)(\mathbf{D}_-^{-\kappa} g)(s)\,ds. \tag{3.31}$$

*The set of elementary functions* $\mathscr{E}$ *is dense in the space* $\Lambda^\kappa$. *The space* $\Lambda^\kappa$ *is complete and hence it is isometric to* $\overline{Sp}(B^\kappa)$.

**Remarks.**

1. The definition (3.30) of $\Lambda^\kappa$ states that to each $f \in \Lambda^\kappa$ there corresponds a function $\phi_f \in L^2(\mathbb{R})$ such that $f = I_-^{-\kappa}\phi_f$. By (3.29), the inner product (3.31) can be expressed as

$$\left(f, g\right)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \left(\phi_f, \phi_g\right)_{L^2(\mathbb{R})}. \tag{3.32}$$

   While (3.32) also holds when $0 < \kappa < 1/2$ (see (3.19)), $\phi_f$ plays a different role (see (3.18)).
2. Theorem 3.3 states that, for $-1/2 < \kappa < 0$, $\Lambda^\kappa$ is in fact isometric to the space $\overline{Sp}(B^\kappa)$ itself because $\Lambda^\kappa$ is complete. This is where the case $-1/2 < \kappa < 0$ is different from the case $0 < \kappa < 1/2$ considered in Section 3.2: by Theorem 3.2, $\Lambda^\kappa$ is *not complete* for $0 < \kappa < 1/2$. The proofs of Theorems 3.2 and 3.3 show that this difference in completeness is a consequence of the following two facts:
   (a) if $-1/2 < \kappa < 0$, then the equation

$$\mathbf{D}_-^{-\kappa} f = \phi$$

   has a solution $f = I_-^{-\kappa}\phi$ for every $\phi \in L^2(\mathbb{R})$,
   (b) when $0 < \kappa < 1/2$, however, there are functions $\phi \in L^2(\mathbb{R})$ for which the equation

$$I_-^{\kappa} f = \phi$$

   is not solvable.

3. Observe that the relations (3.27) and (3.28) hold for $f, g \in \Lambda^\kappa$ as well. As in the case $0 < \kappa < 1/2$, we may express the relation (3.27) informally as

$$\frac{dB^\kappa}{du}(u) = \frac{\Gamma(\kappa+1)}{c_1(\kappa)} \left(\mathbf{D}_+^{-\kappa} \frac{dB}{du}\right)(u).$$

Thus, when $-1/2 < \kappa < 0$ the "fractional noise" $dB^\kappa/du$ is a $(-\kappa)$–fractional derivative of the "white noise" $dB/du$.

*3.4. "Time domain" versus "spectral domain"*

In this section we compare the classes of integrands $\widetilde{\Lambda}^\kappa$ and $\Lambda^\kappa$, as well as the corresponding isometry maps $\mathscr{I}^\kappa$'s or, equivalently, the definitions of the integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ in the "spectral domain" and the "time domain".

**Proposition 3.3.** *Suppose either* $-1/2 < \kappa < 0$ *or* $0 < \kappa < 1/2$. *Then:*
*(1) The inclusion*

$$\widetilde{\Lambda}^\kappa \subset \Lambda^\kappa \tag{3.33}$$

*holds.*
*(2) For* $f, g \in \widetilde{\Lambda}^\kappa$,

$$\left(f, g\right)_{\widetilde{\Lambda}^\kappa} = \left(f, g\right)_{\Lambda^\kappa}. \tag{3.34}$$

*(3) For* $f \in \widetilde{\Lambda}^\kappa$, *the Fourier transform of* $I_-^\kappa f$ *or* $\mathbf{D}_-^{-\kappa} f$ *is*

$$C^\kappa(x)\widehat{f}(x)|x|^{-\kappa}, \tag{3.35}$$

*where*

$$C^\kappa(x) = e^{-i\pi\kappa/2} 1_{\{x>0\}} + e^{i\pi\kappa/2} 1_{\{x<0\}},\ x \in \mathbb{R}. \tag{3.36}$$

*Proof.* The proof is in two parts: (1) $0 < \kappa < 1/2$, and (2) $-1/2 < \kappa < 0$.

(1) We first assume that $0 < \kappa < 1/2$. In the proof given below we make use of the following two facts:

$(i)$ If $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$ are two functions, then their convolution $f_1 * f_2 \in L^2(\mathbb{R})$ because

$$\begin{aligned} &\left[\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f_1(u)|\ |f_2(s-u)|du\right]^2 ds\right]^{1/2} \\ &\quad \le \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f_1(u)|^2\ |f_2(s-u)|^2 ds\right]^{1/2} du \\ &\quad = \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}. \end{aligned} \tag{3.37}$$

This last relation follows from the generalized Minkowsky inequality

$$\left\{\int_{\mathbb{R}} \left|\int_{\mathbb{R}} g(u,s)du\right|^p ds\right\}^{1/p} \le \int_{\mathbb{R}} \left\{\int_{\mathbb{R}} |g(u,s)|^p ds\right\}^{1/p} du, \tag{3.38}$$

which holds for $p \geq 1$ (see (1.33) in Samko et al. [14], p. 9). Moreover, by approximating $f_2$ with functions $f_{2,n} = f_2 1_{\{|\cdot| \leq n\}}$ which are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and converge to $f_2$ in $L^2(\mathbb{R})$, one has

$$\widehat{f_1 * f_2}(x) = \widehat{f_1}(x)\widehat{f_2}(x).$$

$(ii)$ If $f \in L^2(\mathbb{R})$, then the function

$$\int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} du = \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} 1_{\{u-s \leq 1\}} du + \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} 1_{\{u-s>1\}} du$$

is well-defined because the first term is well-defined by (3.37) and the second term by the Cauchy-Schwarz inequality.

Now let $f \in \widetilde{\Lambda}^\kappa$ and set, for $n \in \mathbb{N}$,

$$h(s) = (I_-^\kappa f)(s) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} du,$$

$$h_n(s) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} 1_{\{u-s<n\}} du.$$

Since $f \in L^2(\mathbb{R})$, $h$ and $h_n$ are well-defined by part $(ii)$. The functions $h_n$ converge to $h$ a.e. $ds$. Let us show that they converge to $h$ in $L^2(\mathbb{R})$ as well. Since $h_n$ is the convolution (up to a constant) of the $L^2(\mathbb{R})$-function $f(u)$ and the $L^1(\mathbb{R})$-function $u_-^{\kappa-1} 1_{\{u_- < n\}}$, it belongs to $L^2(\mathbb{R})$ by part $(i)$ and

$$\widehat{h}_n(x) = \frac{1}{\Gamma(\kappa)} \widehat{f}(x) \int_{\mathbb{R}} e^{ixu} u_-^{\kappa-1} 1_{\{u_- < n\}} du = C_n^\kappa(x) \widehat{f}(x) |x|^{-\kappa},$$

where

$$C_n^\kappa(x) = \frac{1}{\Gamma(\kappa)} \left( \int_0^{n|x|} e^{-iv} v^{\kappa-1} dv 1_{\{x>0\}} + \int_0^{n|x|} e^{iv} v^{\kappa-1} dv 1_{\{x<0\}} \right).$$

Since the improper integral $\int_0^\infty e^{\pm iv} v^{\gamma-1} dv$ equals $e^{\pm i\pi\gamma/2}\Gamma(\gamma)$, for $\gamma > 0$, the functions $C_n^\kappa(x)$ converge to $C^\kappa(x)$ given by (3.36). Observe also that $C_n^\kappa(x)$'s are uniformly bounded in $n$ and $x$. We then have

$$\begin{aligned} \|h_n - h_m\|_{L^2(\mathbb{R})}^2 &= (2\pi)^{-1} \|\widehat{h}_n - \widehat{h}_m\|_{L^2(\mathbb{R})}^2 \\ &= (2\pi)^{-1} \int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\kappa} |C_n^\kappa(x) - C_m^\kappa(x)|^2 dx. \end{aligned}$$

Since by assumption $\int_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{-2\kappa} dx < \infty$, the dominated convergence theorem implies that $\|h_n - h_m\|_{L^2(\mathbb{R})}^2 \to 0$, as $n, m \to \infty$. Therefore, there is $\widetilde{h} \in L^2(\mathbb{R})$ such that $\|h_n - \widetilde{h}\|_{L^2(\mathbb{R})}^2 \to 0$, as $n \to \infty$. By taking an a.e. convergent subsequence, we can deduce that $\widetilde{h}(s) = h(s)$ a.e. $ds$. In particular, $h \in L^2(\mathbb{R})$ or, equivalently, $f \in \Lambda^\kappa$. This establishes the inclusion (3.33) when $0 < \kappa < 1/2$.

Observe that $\widehat{h}_n(x)$ converges to the function given by (3.35) for all $x \in \mathbb{R}$, $x \neq 0$. On the other hand, since $h_n$ converges to $h$ in $L^2(\mathbb{R})$, $\widehat{h}_n$ converges to $\widehat{h}$ in $L^2(\mathbb{R})$ as well. Hence, the Fourier transform of $h = I_-^\kappa f$, for $f \in \widetilde{\Lambda}_\kappa$, is given by (3.35). Since $|C^\kappa(x)| = 1$, if $x \neq 0$, the Parseval's equality implies that, for $f, g \in \widetilde{\Lambda}^\kappa$,

$$\int_{\mathbb{R}} (I_-^\kappa f)(s)(I_-^\kappa g)(s)ds = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(x)\overline{\widehat{g}(x)}|x|^{-2\kappa}dx,$$

or, by (3.17) and (3.6), that

$$\big(f, g\big)_{\Lambda^\kappa} = c(\kappa)\ \big(f, g\big)_{\widetilde{\Lambda}^\kappa},$$

for some constant $c(\kappa)$. Since both $\big(f, g\big)_{\Lambda^\kappa}$ and $\big(f, g\big)_{\widetilde{\Lambda}^\kappa}$ are equal to $E(\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g))$, for $f, g \in \mathscr{E}$, we get $c(\kappa) = 1$. Thus, the relation (3.34) holds when $0 < \kappa < 1/2$.

(2) We now consider the case $-1/2 < \kappa < 0$. Let us show that, if $f \in \widetilde{\Lambda}^\kappa$, then $f \in \Lambda^\kappa$ as well, that is, there is $\phi \in L^2(\mathbb{R})$ such that

$$f = I_-^{-\kappa}\phi. \tag{3.39}$$

Since $\widehat{f}(x)|x|^{-\kappa}$ is in $L^2(\mathbb{R})$, $|C^{-\kappa}(x)| = 1$, if $x \neq 0$, and $\overline{C^{-\kappa}(x)} = C^{-\kappa}(-x)$, the function

$$\widehat{\phi}(x) = (C^{-\kappa}(x))^{-1}\widehat{f}(x)|x|^{-\kappa} \tag{3.40}$$

is in $L^2(\mathbb{R})$, satisfies $\overline{\widehat{\phi}(x)} = \widehat{\phi}(-x)$ and hence is the Fourier transform of some function $\phi \in L^2(\mathbb{R})$. Let us show that this function satisfies (3.39). From (3.40) we get

$$\widehat{f}(x) = C^{-\kappa}(x)\widehat{\phi}(x)|x|^{\kappa} = C^{-\kappa}(x)\widehat{\phi}(x)|x|^{-(-\kappa)}. \tag{3.41}$$

This implies that

$$|\widehat{\phi}(x)|^2|x|^{-2(-\kappa)} = |\widehat{f}(x)|^2$$

and, since $\widehat{f} \in L^2(\mathbb{R})$, that $\phi \in \widetilde{\Lambda}_{-\kappa}$. Since $0 < -\kappa < 1/2$, we can apply (3.35) to evaluate the Fourier transform of $I_-^{-\kappa}\phi$. Since it is identical to (3.41), we conclude that

$$f = I_-^{-\kappa}\phi,$$

that is, (3.39) holds and hence $f \in \Lambda^\kappa$.

To verify (3.35), we must evaluate the Fourier transform of $\mathbf{D}_-^{-\kappa} f$. Observe that our $\phi$ satisfies (3.29) and hence the Fourier transform of $\mathbf{D}_-^{-\kappa} f$ is the Fourier transform of $\phi$, that is (3.40). Since $(C^{-\kappa}(x))^{-1} = C^\kappa(x)$, we obtain (3.35). The relation (3.34) for $-1/2 < \kappa < 0$ follows as in the case $0 < \kappa < 1/2$. □

**Corollary 3.1.** *Let $f \in \widetilde{\Lambda}^\kappa$, for $-1/2 < \kappa < 0$ or $0 < \kappa < 1/2$. Then the integrals $\int_{\mathbb{R}} f(u)dB^\kappa(u)$ defined in the "spectral domain" and the "time domain" are equal almost surely.*

*Proof.* The integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ in the "spectral domain" is defined as an $L^2(\Omega)$-limit of the integrals $\int_{\mathbb{R}} f_n(u)dB^{\kappa}(u)$, where $(f_n)$ is a sequence of elementary functions such that $f_n \to f$ in $\widetilde{\Lambda}^{\kappa}$. By (3.34), $f_n \to f$ also in $\Lambda^{\kappa}$. Therefore, the integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ in the "time domain" is also an $L^2(\Omega)$-limit of $\int f_n(u)dB^{\kappa}(u)$. Since an $L^2(\Omega)$-limit of a sequence of random variables is almost surely unique, we obtain the required result. □

**Proposition 3.4.** *The inclusion* $\widetilde{\Lambda}^{\kappa} \subset \Lambda^{\kappa}$ *is strict when* $-1/2 < \kappa < 0$ *or* $0 < \kappa < 1/2$.

*Proof.* Consider the case $0 < \kappa < 1/2$ first. It is enough to find a function $f$ such that $f \in \Lambda^{\kappa}$ but $f \notin L^2(\mathbb{R})$ (so that its $L^2$–Fourier transform is not defined and hence $f \notin \widetilde{\Lambda}^{\kappa}$). By Theorem 5.3 in Samko et al. [14], p. 103, the operator $I_{-}^{\kappa}$ maps $L^{2/2\kappa+1}(\mathbb{R})$ into $L^2(\mathbb{R})$. Therefore, for $f$ we can take any function from $L^{2/2\kappa+1}(\mathbb{R})$ which is not in $L^2(\mathbb{R})$. Suppose now that $-1/2 < \kappa < 0$. If $\widetilde{\Lambda}^{\kappa} = \Lambda^{\kappa}$, then the inner product spaces $\widetilde{\Lambda}^{\kappa}$ and $\Lambda^{\kappa}$ are identical because, by (3.34), the corresponding inner products are equal. However, this is not possible, since $\widetilde{\Lambda}^{\kappa}$ is not a complete inner product space (Theorem 3.1) whereas $\Lambda^{\kappa}$ is a complete inner product space (Theorem 3.3). Therefore, the inclusion $\widetilde{\Lambda}^{\kappa} \subset \Lambda^{\kappa}$ is strict in the case $-1/2 < \kappa < 0$ as well. □

## 4. An alternative class of integrands in the "time domain"

If $\Gamma^{\kappa}$ is the covariance function of a standard fBm $B^{\kappa}$ with $-1/2 < \kappa < 1/2$ given by (1.1), then, for $f, g \in \mathscr{E}$,

$$E(\mathscr{I}^{\kappa}(f)\mathscr{I}^{\kappa}(g)) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)d^2\Gamma^{\kappa}(u,v), \tag{4.1}$$

where the double integral is defined to be linear and to satisfy

$$\begin{aligned}\int_{[a,b]}\int_{[c,d]} d^2\Gamma^{\kappa}(u,v) &= \Delta^2\Gamma^{\kappa}\Big|_{[a,b]\times[c,d]}\\ &= \Gamma^{\kappa}(d,b) - \Gamma^{\kappa}(d,a) - \big(\Gamma^{\kappa}(c,b) - \Gamma^{\kappa}(c,a)\big),\end{aligned}$$

for any real numbers $a < b$ and $c < d$. This may suggest that one can define the integral $\int_{\mathbb{R}} f(u)dB^{\kappa}(u)$ for functions $f$ from the space

$$|\Lambda|^{\kappa} = \left\{f : \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)|d^2|\Gamma^{\kappa}|(u,v) < \infty\right\},$$

where $|\Gamma^{\kappa}|$ is the total variation measure of $\Gamma^{\kappa}$. Observe, however, that when $-1/2 < \kappa < 0$ the function $\Gamma^{\kappa}$ is not of bounded variation (around the diagonal $u = v$) and hence the measure $|\Gamma^{\kappa}|$ is not defined. But in the case $0 < \kappa < 1/2$, we have

$$d^2\Gamma^{\kappa}(u,v) = \kappa(2\kappa+1)|u-v|^{2\kappa-1}du\,dv, \tag{4.2}$$

and hence, we have

**Definition 4.1.**

$$|\Lambda|^\kappa = \left\{ f : \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du\, dv < \infty \right\},$$

for $0 < \kappa < 1/2$.

Although the class $|\Lambda|^\kappa$ arises in the "time domain" as naturally as the class $\Lambda^\kappa$, interestingly, it turns out that $|\Lambda|^\kappa$ is a *strict* subset of $\Lambda^\kappa$.

**Proposition 4.1.** *Let* $0 < \kappa < 1/2$. *Then the inclusion*

$$|\Lambda|^\kappa \subset \Lambda^\kappa$$

*holds and it is strict.*

*Proof.* The inclusion $|\Lambda|^\kappa \subset \Lambda^\kappa$ follows from the relation

$$\begin{aligned} &\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1} du \right]^2 ds \\ &\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)| \left[ \int_{\mathbb{R}} (u-s)_+^{\kappa-1} (v-s)_+^{\kappa-1} ds \right] du\, dv \\ &\quad = \mathrm{B}\,(\kappa, 1-2\kappa) \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du\, dv, \end{aligned} \tag{4.3}$$

which can be obtained by making the change of variables

$$s = \min(u,v) - |v-u| \left( \frac{1}{z} - 1 \right)$$

above ($\mathrm{B}(p,q) = \int_0^1 (1-v)^{p-1} v^{q-1} dv,\ p,q > 0$, is the beta function). To show that the inclusion $|\Lambda|^\kappa \subset \Lambda^\kappa$ is strict, we have to find $f \in \Lambda^\kappa$ for which $\int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} dudv = \infty$. By Proposition 3.3, it is enough to provide an example of a function $f$ which is in $\widetilde{\Lambda}^\kappa$ but not in $|\Lambda|^\kappa$. Take $f(u) = \mathrm{sign}(u)|u|^{-p}\sin(u)$ with $1/2 < p < 1$, which is in $L^2(\mathbb{R})$. To calculate its Fourier transform, consider a sequence of functions $f_n(u) = \mathrm{sign}(u)|u|^{-p}\sin(u)1_{\{|u|<n\}}$, which converges to $f$ in $L^2(\mathbb{R})$. The Fourier transform of $f_n$ is

$$\begin{aligned} \widehat{f_n}(x) &= 2 \int_0^n \cos(xu)|u|^{-p} \sin(u) du \\ &= \int_0^n u^{-p} \sin[(x+1)u] du - \int_0^n u^{-p} \sin[(x-1)u] du \\ &= \mathrm{sign}(x+1)|x+1|^{p-1} \int_0^{n|x+1|} v^{-p} \sin(v) dv \\ &\quad - \mathrm{sign}(x-1)|x-1|^{p-1} \int_0^{n|x-1|} v^{-p} \sin(v) dv. \end{aligned}$$

It converges to the limit

$$\Big(\operatorname{sign}(x+1)|x+1|^{p-1} - \operatorname{sign}(x-1)|x-1|^{p-1}\Big)\int_0^\infty v^{-p}\sin(v)dv,$$

which is also $\widehat{f}$. Since $1/2 < p < 1$, we have $\int_{\mathbb{R}} |\widehat{f}(x)|^2|x|^{-2\kappa}dx < \infty$ and hence $f \in \widetilde{\Lambda}^\kappa$. Let us show that, for $1/2 < p < \kappa + 1/2$,

$$I = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1}du\right]^2 ds = \infty,$$

that is, by (4.3), $f \notin |\Lambda|^\kappa$. We have

$$\begin{aligned}
I &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} |u|^{-p}|\sin u|(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\geq \int_{\mathbb{R}}\left[\int_{\{u>\pi/4\}} u^{-p}|\sin u|(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\geq \frac{1}{2}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}\int_{\pi k+\pi/4}^{\pi k+3\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\geq \frac{1}{4}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}\int_{\pi k+\pi/4}^{\pi k+3\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\quad + \frac{1}{4}\int_{\mathbb{R}}\left[\sum_{k=1}^{\infty}\int_{\pi k-\pi/4}^{\pi k+\pi/4}\left(u+\frac{\pi}{2}\right)^{-p}\left(u+\frac{\pi}{2}-s\right)_+^{\kappa-1}du\right]^2 ds \\
&\geq \frac{1}{4}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}\int_{\pi k+\pi/4}^{\pi k+3\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\quad + \frac{2^{-2p}}{4}\int_{\mathbb{R}}\left[\sum_{k=1}^{\infty}\int_{\pi k-\pi/4}^{\pi k+\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\geq \frac{2^{-2p}}{8}\int_{\mathbb{R}}\left[\sum_{k=0}^{\infty}\int_{\pi k+\pi/4}^{\pi k+3\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right. \\
&\quad \left. + \sum_{k=1}^{\infty}\int_{\pi k-\pi/4}^{\pi k+\pi/4} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&= c_p\int_{\mathbb{R}}\left[\int_{\{u>\pi/4\}} u^{-p}(u-s)_+^{\kappa-1}du\right]^2 ds \\
&\geq c_p\int_0^\infty\left[\int_{\{v\geq\pi/4s\}} v^{-p}(v-1)_+^{\kappa-1}dv\right]^2 s^{2\kappa-2p}ds.
\end{aligned} \tag{4.4}$$

(We have used the inequalities $|\sin u| > 1/2$, for $u \in (\pi k + \pi/4, \pi k + 3\pi/4)$, $a^2 + b^2 \geq (a+b)^2/2$ and $(u + \frac{\pi}{2})^{-p} \geq (2u)^{-p}$, for $u > 3\pi/4$.) Since the lower bound (4.4) diverges around $s = \infty$ for $1/2 < p < \kappa + 1/2$, we obtain that $I = \infty$. □

Observe that by (4.3) we also have

$$|\Lambda|^\kappa = \left\{ f : \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(u)|(u-s)_+^{\kappa-1} du \right]^2 ds < \infty \right\} \tag{4.5}$$

and comparing (4.5) with (3.15), we observe that the difference between $|\Lambda|^\kappa$ and $\Lambda^\kappa$ is that the absolute value of $f$ appears in (4.5). Since $|\Lambda|^\kappa \subset \Lambda^\kappa$, one can define an inner product on $|\Lambda|^\kappa$ in the same way as for $\Lambda^\kappa$, that is, $\left(f, g\right)_{|\Lambda|^\kappa} = \left(f, g\right)_{\Lambda^\kappa}$, for $f, g \in |\Lambda|^\kappa$, and then $|\Lambda|^\kappa$ becomes an inner product space. Since the set $\mathscr{E}$ is dense in $\Lambda^\kappa$ (Theorem 3.2), it is also dense in $|\Lambda|^\kappa$. Then, by Proposition 2.1, there is an isometry map between the space $|\Lambda|^\kappa$ and a linear subspace of $\overline{Sp}(B^\kappa)$. If we denote this isometry map by $|\mathscr{I}|^\kappa$, then clearly $|\mathscr{I}|^\kappa(f) = \mathscr{I}^\kappa(f)$, for $f \in |\Lambda|^\kappa$, where the isometry map $\mathscr{I}^\kappa$ corresponds to the space $\Lambda^\kappa$. The inner product on $|\Lambda|^\kappa$, $\left(f, g\right)_{|\Lambda|^\kappa} = \left(f, g\right)_{\Lambda^\kappa}$, can be expressed as in (3.17), or alternatively as

$$\left(f, g\right)_{|\Lambda|^\kappa} = \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1} du\, dv. \tag{4.6}$$

Indeed, by writing as in (4.3), $\mathrm{B}\,(\kappa, 1-2\kappa)\,|u-v|^{2\kappa-1} = \int_{\mathbb{R}} (u-s)_+^{\kappa-1}(v-s)_+^{\kappa-1} ds$, and applying the Fubini's theorem, for $f, g \in |\Lambda|^\kappa$, we get

$$\begin{aligned} &\mathrm{B}(\kappa, 1-2\kappa) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1} du\, dv \\ &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(u)(u-s)_+^{\kappa-1} du \right] \left[ \int_{\mathbb{R}} g(u)(u-s)_+^{\kappa-1} du \right] ds, \end{aligned}$$

which is $\left(f, g\right)_{\Lambda^\kappa}$ up to a multiplicative constant which depends on $\kappa$ only. Thus the relation (4.6) holds up to a multiplicative constant which also depends on $\kappa$ only. This constant, however, equals 1. This can be verified by direct computations (it is complicated!) or, more simply, by noting that both the left-hand side and the right-hand side of (4.6), the latter by the relations (4.1) and (4.2), are equal to $E(\mathscr{I}^\kappa(f)\mathscr{I}^\kappa(g))$, for $f, g \in \mathscr{E}$.

Since $|\Lambda|^\kappa$ is dense in $\Lambda^\kappa$ (Lemma 5.5) and $\Lambda^\kappa$ is not complete (Theorem 3.2), the inner product space $|\Lambda|^\kappa$ is not complete either. This is equivalent to saying that the normed space $|\Lambda|^\kappa$, with the norm

$$\|f\|_{|\Lambda|^\kappa} = \left( \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)|u-v|^{2\kappa-1} du\, dv \right)^{1/2}$$

induced by the inner product, is not complete. However, if we introduce a new norm on $|\Lambda|^\kappa$,

$$\|f\| = \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1} du\, dv \right)^{1/2},$$

we show in the theorem below that $(|\Lambda|^\kappa, \|\cdot\|)$ becomes a *complete* normed space. The norm $\|f\|$ is also easier to work with than the norm $\|f\|_{|\Lambda|^\kappa}$.

**Theorem 4.1.** *Let* $0 < \kappa < 1/2$. *Then the function* $\|\cdot\|$ *defines a norm on* $|\Lambda|^\kappa$ *and the space* $(|\Lambda|^\kappa, \|\cdot\|)$ *is a complete normed space.*

*Proof.* Using (4.3), it is easy to verify that $\|\cdot\|$ is a norm. To prove completeness, let $(f_n)$ be a Cauchy sequence in $(|\Lambda|^\kappa, \|\cdot\|)$, that is, $\|f_n - f_m\| \to 0$, as $n, m \to \infty$. We then have to find $f \in |\Lambda|^\kappa$ such that $\|f - f_n\| \to 0$, as $n \to \infty$. This is done in two steps:

(i) there is a subsequence $(n_l)$ and a function $f$ such that $f_{n_l} \to f$ a.e., as $l \to \infty$, and

(ii) $f$ is indeed the required function.

Let us first prove the step (i). Since, for any $k > 0$,

$$\|g\|^2 \geq (2k)^{2\kappa-1} \int_{|u|\leq k} \int_{|v|\leq k} |g(u)||g(v)| du\, dv = (2k)^{2\kappa-1} \|g\|^2_{L^1(-k,k)},$$

the convergence $\|f_n - f_m\| \to 0$ implies that $\|f_n - f_m\|_{L^1(-k,k)} \to 0, \forall\, k > 0$. Since the spaces $L^1(-k,k)$ are complete for all $k > 0$, it is easy to see that there is a function $f$, defined on the real line, such that $\|f_n - f\|_{L^1(-k,k)} \to 0, \forall\, k > 0$. To choose an almost everywhere convergent subsequence, we use the well-known diagonal argument. For $k = 1$, since $\|f_n - f\|_{L^1(-1,1)} \to 0$, we can take a subsequence $(n_l(1))$ such that $f_{n_l(1)} \to f$ a.e. on $(-1,1)$. By mathematical induction, we can thus construct subsequences $(n_l(k+1)) \subset (n_l(k)), k \in \mathbb{N}$, such that $f_{n_l(k)} \to f$ a.e. on $(-k,k)$. Now, by taking the indices $(n_l) = (n_l(l))$ in the diagonal, we obtain that $f_{n_l} \to f$ a.e. on the whole real line.

The step (ii) follows easily from the step (i) because, by Fatou's lemma, $\|f\| = \|\underline{\lim}_l f_{n_l}\| \leq \underline{\lim}_l \|f_{n_l}\| < \infty$ and $\|f - f_n\| \leq \|\underline{\lim}_l f_{n_l} - f_n\| \leq \underline{\lim}_l \|f_{n_l} - f_n\| \overset{n\to\infty}{\longrightarrow} 0$, since $(f_n)$ is Cauchy in the norm $\|\cdot\|$. □

**Proposition 4.2.** *For* $0 < \kappa < 1/2$,

$$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset L^{2/2\kappa+1}(\mathbb{R}) \subset |\Lambda|^\kappa \subset \Lambda^\kappa. \tag{4.7}$$

*Proof.* After using (3.22), it remains to show that $L^{2/2\kappa+1}(\mathbb{R}) \subset |\Lambda|^\kappa$. As noted in the proof of Proposition 3.2, the operator $I^\kappa_-$ is bounded from $L^{2/2\kappa+1}(\mathbb{R})$ into $L^2(\mathbb{R})$ for $0 < \kappa < 1/2$, implying that $\|I^\kappa_- f\|_{L^2(\mathbb{R})} \leq c_1(\kappa)\|f\|_{L^{2/2\kappa+1}(\mathbb{R})}$ and therefore

$$\|f\| = c_2(\kappa)\|I^\kappa_-|f|\|_{L^2(\mathbb{R})} \leq c_3(\kappa)\|f\|_{L^{2/2\kappa+1}(\mathbb{R})} \tag{4.8}$$

for some positive constants $c_i(\kappa)$, $i = 1, 2, 3$. □

## 5. Proof of Theorems 3.1, 3.2 and 3.3

### *5.1. The proof of Theorem 3.1*

We let $\lambda^\kappa$ be the measure on the real line defined by $\lambda^\kappa(dx) = |x|^{-2\kappa}dx$, for $-1/2 < \kappa < 1/2$. In the proof of Theorem 3.1 we will use the following result.

**Lemma 5.1.** *Suppose that $\widehat{f}$ is the Fourier transform of a function $f \in L^2(\mathbb{R})$. If, for some $-1/2 < \kappa < 1/2$, $\widehat{f} \in L^2(\lambda^\kappa)$, then there is a sequence of elementary functions $l_n$ such that*

$$\|\widehat{f} - \widehat{l}_n\|^2_{L^2(\lambda^\kappa)} = \int_{\mathbb{R}} |\widehat{f}(x) - \widehat{l}_n(x)|^2 |x|^{-2\kappa} dx \overset{n\to\infty}{\longrightarrow} 0. \tag{5.1}$$

*Proof.* Since $f(u) = (1/2)(f(u) + f(-u)) + (1/2)(f(u) - f(-u))$, for $u \in \mathbb{R}$, we may prove the lemma in two cases: (1) $f$ is an even function and, (2) $f$ is an odd function.

*Case 1:* If $f$ is an even function, then $\widehat{f}$ is real-valued and $\widehat{f}(x) = \widehat{f}(-x)$. To prove (5.1), we may show that, for arbitrary small $\epsilon > 0$, there is an elementary function $l$ such that $\|\widehat{f} - \widehat{l}\|_{L^2(\lambda^\kappa)} < \epsilon$. We will provide this approximation in several steps. As a first step, we approximate $\widehat{f}$ by simple functions. For any $\epsilon > 0$, there is a simple function $\widehat{g}(x) = \sum_{l=1}^k g_l 1_{G_l}(x),\ g_l \in \mathbb{R},\ G_l \in \mathscr{B}(\mathbb{R})$, such that $\|\widehat{f} - \widehat{g}\|_{L^2(\lambda^\kappa)} < \epsilon$. Since $\widehat{f}(x) = \widehat{f}(-x)$, we can take the sets $G_l$ to be symmetric around the origin $x = 0$. As a second step, observe that, for a symmetric around the origin set $G$ and any $\epsilon > 0$, there is a function $\widehat{h}(x) = \sum_{n=1}^m h_n 1_{[-H_n, H_n]}(x),\ h_n \in \mathbb{R},\ H_n > 0$, such that $\|1_G - \widehat{h}\|_{L^2(\lambda^\kappa)} < \epsilon$. It is therefore enough to show, for example, that the function $1_{[-1,1]}(x)$ can be approximated in $L^2(\lambda^\kappa)$ by the Fourier transform of an elementary function. In other words, that, for any $\epsilon > 0$, there is an elementary function $l$ such that $\|1_{[-1,1]} - \widehat{l}\|_{L^2(\lambda^\kappa)} < \epsilon$.

To construct $l$, observe first that

$$\int_{\mathbb{R}} |1_{[-1,1]}(x) - \widehat{l}(x)|^2 |x|^{-2\kappa} dx = \int_{\mathbb{R}} |x 1_{[-1,1]}(x) - x\widehat{l}(x)|^2 |x|^{-2\kappa-2} dx$$

and the measure $\lambda^{\kappa+1}(dx) = |x|^{-2\kappa-2}dx$ is finite around $x = \infty$. The idea then is to truncate the range of $x1_{[-1,1]}(x)$, perform a periodic extension and observe that its truncated Fourier series is of the form $x\widehat{l}(x)$, where $\widehat{l}$ is the (continuous) Fourier transform of an elementary function. We thus construct $l$ as follows.

First, choose $k > 1$ such that

$$\int_{\mathbb{R}} |x|^{-2\kappa-2} 1_{\{|x|>k\}} dx < \epsilon^2/2.$$

Now, let $U$ be the function which equals $x1_{[-1,1]}(x)$ on $[-k, k]$ and is periodically extended to $x \in \mathbb{R}$. It has the Fourier series $\sum_{n=-\infty}^{\infty} u_n e^{i\pi n x/k}$ which converges to $U$ everywhere on $[-k, k]$ except at the points $x = \pm 1$ where $U$ is discontinuous. Moreover, the partial sum $U_m(x) = \sum_{n=-m}^{m} u_n e^{i\pi n x/k}$ can be expressed as

$$U_m(x) = \frac{1}{k} \int_{-k}^{k} U(x-y) D_m\left(\frac{\pi y}{k}\right) dy,$$

where

$$D_m(y) = \sin(m + \frac{1}{2})y/(2\sin\frac{1}{2}y),\ \ y \in \mathbb{R},$$

is the well-known Dirichlet kernel. Observe that $U_m(0) = 0$. Let us show that the following two properties of the partial sums $U_m$ hold:

(i) $\sup_m \sup_x |U_m(x)| \leq \text{const}$,

(ii) $\sup_m |U_m(x)| \leq \text{const}|x|$, for small enough $x$.

These properties will be used in the sequel to apply the dominated convergence theorem.

We first prove the property (i). Since $U_m$ is periodic with period $2k$, we may assume that $x \in (-k, k)$. Then

$$\begin{aligned} U_m(x) &= \frac{1}{k}\int_{-k}^{k}(x-y)1_{\{|x-y|\leq 1\}}(y)D_m\left(\frac{\pi y}{k}\right)dy \\ &+\frac{1}{k}\int_{-k}^{k}(x-y-2k)1_{\{|x-y-2k|\leq 1\}}(y)D_m\left(\frac{\pi y}{k}\right)dy \\ &+\frac{1}{k}\int_{-k}^{k}(x-y+2k)1_{\{|x-y+2k|\leq 1\}}(y)D_m\left(\frac{\pi y}{k}\right)dy. \end{aligned}$$

Let us show that the second term is uniformly bounded in $m$ and $x \in (-k, k)$ by a constant. (The proof for the other two terms is similar.) The second term equals

$$\frac{(x-2k)}{\pi}\int_{-\pi}^{\pi}1_{\{|x-\frac{kv}{\pi}-2k|\leq 1\}}(v)D_m(v)dv - \frac{k}{\pi^2}\int_{-\pi}^{\pi}v1_{\{|x-\frac{kv}{\pi}-2k|\leq 1\}}(v)D_m(v)dv.$$

Since the kernel $D_m$ is an even function, it is enough to show that the integrals $\int_0^\xi D_m(v)dv$ and $\int_0^\xi vD_m(v)dv$ are uniformly bounded in $m$ and $\xi \in [0, \pi]$. This fact for the first integral is proved in Zygmund [18], (8·2) Lemma, p. 57. It holds for the second integral as well because the function $vD_m(v)$ can be bounded uniformly in $m$ and $v \in [0, \pi]$ by a constant.

Let us now prove the property (ii). Observe first that, for any $1/2 < |y| < k$, there is a constant $c_1$ such that $\sup_m |D_m(\pi y/k)| \leq 1/2|\sin(\pi y/2k)| \leq c_1$. Moreover, by the same (8·2) Lemma in Zygmund [18], the integral $(1/k)\int_{-k}^{k} 1_{\{|y-x|\leq 1\}\cap\{|y|\leq 1\}}(y)D_m(\pi y/k)dy$ can be bounded uniformly in $m$ and $x$ by a constant $c_2$. Observe also that, for $y \in (-k, k)$ and $x - y \in (-2k + 1, 2k - 1)$,

$$\begin{aligned} U(x-y) - U(-y) &= (x-y)1_{\{|y-x|\leq 1\}}(y) + y1_{\{|y|\leq 1\}} \\ &= x1_{\{|y-x|\leq 1\}\cap\{|y|\leq 1\}}(y) + y1_{\{|y-x|>1\}\cap\{|y|\leq 1\}}(y) \\ &\quad +(x-y)1_{\{|y-x|\leq 1\}\cap\{|y|>1\}}(y). \end{aligned}$$

Then, for small enough $x$,

$$\begin{aligned} |U_m(x)| = |U_m(x) - U_m(0)| &= \left|\frac{1}{k}\int_{-k}^{k}\left(U(x-y) - U(-y)\right)D_m\left(\frac{\pi y}{k}\right)dy\right| \\ &\leq \frac{|x|}{k}\left|\int_{-k}^{k}1_{\{|y-x|\leq 1\}\cap\{|y|\leq 1\}}(y)D_m\left(\frac{\pi y}{k}\right)dy\right| \\ &\quad +\frac{1}{k}\int_{-k}^{k}1_{\{|y-x|>1\}\cap\{|y|\leq 1\}}(y)\left|D_m\left(\frac{\pi y}{k}\right)\right|dy \end{aligned}$$

$$+\frac{1}{k}\int_{-k}^{k} 1_{\{|y-x|\le 1\}\cap\{|y|>1\}}(y)\left|D_m\left(\frac{\pi y}{k}\right)\right| dy$$
$$\le c_2|x| + \frac{2c_1}{k}\int_{-k}^{k} 1_{\{|y-1|\le |x|\}}(y)dy \le \text{const}|x|,$$

which is the property (ii).

By (i) and (ii), the dominated convergence theorem implies that

$$\int_{\{|x|\le k\}} |x1_{[-1,1]}(x) - U_m(x)|^2|x|^{-2\kappa-2}dx \overset{m\to\infty}{\longrightarrow} 0.$$

In particular, there is an integer $M$ such that

$$\int_{\{|x|\le k\}} |x1_{[-1,1]}(x) - U_M(x)|^2|x|^{-2\kappa-2}dx < \epsilon^2/2.$$

Since $U(0) = 0$ and $U(-x) = -U(x)$, we have that

$$\begin{aligned} u_n &= \frac{1}{k}\int_{-k}^{k} U(x)e^{i\pi nx/k}dx = \frac{2i}{k}\int_0^k U(x)\sin(\pi nx/k)dx \\ &= -ia_n,\ a_n \in \mathbb{R},\ n \ge 1, \end{aligned}$$

$u_0 = 0$ and $u_n = ia_n$, for $n \le -1$. Hence, $U_M(x) = \sum_{n=1}^{M}(-ia_n)(e^{i\pi nx/k} - e^{-i\pi nx/k})$. Since

$$\widehat{1}_{[-\pi n/k,\pi n/k)}(x) = \int_{\mathbb{R}} e^{ixu}1_{[-\pi n/k,\pi n/k)}(u)du = (-ix^{-1})(e^{i\pi nx/k} - e^{-i\pi nx/k}),$$

$x^{-1}U_M(x)$ is the Fourier transform of the elementary function $l = \sum_{n=1}^{M} a_n 1_{[-\pi n/k,\pi n/k)}$. We thus obtain the required approximation because

$$\begin{aligned} \|1_{[-1,1]} - \widehat{l}\|^2_{L^2(\lambda^\kappa)} &= \int_{\mathbb{R}} |x1_{[-1,1]}(x) - U_M(x)|^2|x|^{-2\kappa-2}dx \\ &\le \int_{\{|x|\le k\}} |x1_{[-1,1]}(x) - U_M(x)|^2|x|^{-2\kappa-2}dx \\ &\quad + \int_{\{|x|>k\}} |x|^{-2\kappa-2}dx < \epsilon^2. \end{aligned}$$

*Case 2:* If $f$ is an odd function, then $\widehat{f} = i\Im\widehat{f}$ and $\Im\widehat{f}(-x) = -\Im\widehat{f}(x)$. By the same arguments as in the previous case, it is enough to show that, for example, the function $i(1_{[0,1]}(x) - 1_{[-1,0]}(x))$ can be approximated by the Fourier transform of an elementary function. Equivalently, for arbitrary small $\epsilon > 0$, we have to find an elementary function $l$ such that $\|(1_{[0,1]} - 1_{[-1,0]}) - i\widehat{l}\|_{L^2(\lambda^\kappa)} < \epsilon$. The proof is similar to the previous case and we only outline it. Fix $k$ as in Case 1 and let $V$ be the function which equals $x(1_{[0,1]}(x) - 1_{[-1,0]}(x)) = |x|1_{[-1,1]}(x)$ on $[-k, k]$ and is periodically extended to $x \in \mathbb{R}$. Its truncated Fourier series $V_m(x) = \sum_{n=-m}^{m} v_n e^{i\pi nx/k}$ converges to $V$ everywhere on $[-k, k]$ except at the points $x = \pm 1$. It is not enough here to focus on $V_m(x)$ for small $x$ because $V_m(0)$ is not zero. Therefore, instead of dealing with $V_m(x)$, we will consider $V_m(x)-V_m(0)$. The function $V_m(x)-V_m(0)$ also converges to $V(x)$ a.e. $dx$ and one can show that $\sup_m \sup_x |V_m(x)-V_m(0)| \le \text{const}$ and $\sup_m |V_m(x)-V_m(0)| \le \text{const}|x|$, for small enough $x$. Moreover, $V_m(x)-V_m(0)=\sum_{n=1}^m b_n(e^{i\pi nx/k}+e^{-i\pi nx/k}-2)$, for some $b_n \in \mathbb{R}$, and hence $(V_m(x)-V_m(0))/x = i\widehat{l_m}$, where $l_m$ is the elementary function given by $l_m=\sum_{n=1}^m b_n(1_{[0,\pi n/k)}-1_{[-\pi n/k,0)})$. The conclusion follows as in Case 1. □

**Remarks.**

1. In the case $0<\kappa<1/2$, a second way to see that the function $1_{[-1,1]}$ can be approximated by the Fourier transform of an elementary function is as follows. Observe first that $1_{[-1,1]}(x)=\widehat{f_0}(x)$, where, by taking the inverse Fourier transform, $f_0(u)=(1/2\pi)\int_{\mathbb{R}} e^{-iux}1_{[-1,1]}(x)dx=(\sin u/\pi u)$. Since $f_0 \notin L^1(\mathbb{R})$, the function $f_0$ is not easy to deal with. Consider instead the function $f_a(u)=e^{-au}\sin u/\pi u$, for $a>0$, which is in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Since functions in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ can be approximated by simple and hence elementary functions (in the two norms simultaneously), by using the estimate obtained in the proof of Proposition 3.1, $\widehat{f_a}$ can be approximated in $L^2(\lambda^\kappa)$ by an elementary function. It is then enough to show that $\|1_{[-1,1]}-\widehat{f_a}\|_{L^2(\lambda^\kappa)} \to 0$, as $a \to 0$. By formula 3.947,(3) in Gradshteyn and Ryzhik [7],

$$\widehat{f_a}(x)=\frac{2}{\pi}\int_0^\infty e^{-au}\cos xu \sin u \frac{du}{u}=\frac{1}{\pi}\text{Arctan}\frac{2a}{x^2+a^2-1}+1_{\{x^2\le 1-a^2\}},$$

and hence $\widehat{f_a}(x)\to 1_{[-1,1]}(x)$ a.e. $dx$, as $a\to 0$. Moreover, since $|\text{Arctan}(y)|\le \min(\frac{\pi}{2},|y|)$,

$$\begin{aligned}\left|\text{Arctan}\frac{2a}{x^2+a^2-1}\right| &\le \frac{\pi}{2}1_{\{|x|\le 1\}}(x)+\frac{2a}{x^2+a^2-1}1_{\{|x|>1\}}(x)\\ &\le \frac{\pi}{2}1_{\{|x|\le 1\}}(x)+\frac{2}{x^2-1}1_{\{|x|>1\}}(x),\end{aligned}$$

if $0<a<1$, and the bound is in $L^2(\lambda^\kappa)$. The required convergence then follows from the dominated convergence theorem.
2. In the case $0<\kappa<1/2$, a third way of proving the lemma is to use Proposition 3.3 and Theorem 3.2.

*Proof of Theorem 3.1.* To show that $\widetilde{\Lambda}^\kappa$ is an inner product space, we check the least obvious condition. If $\left(f,f\right)_{\widetilde{\Lambda}^\kappa}=0$, for $f\in\widetilde{\Lambda}^\kappa$, then $\widehat{f}(x)=0$ a.e. $dx$ and hence $f(u)=0$ a.e. $du$, since $\|f\|^2_{L^2(\mathbb{R})}=(2\pi)^{-1}\|\widehat{f}\|^2_{L^2(\mathbb{R})}=0$.

The set of elementary functions $\mathscr{E}$ is dense in $\widetilde{\Lambda}^\kappa$, if $f\in\widetilde{\Lambda}^\kappa$ can be approximated in $L^2(\lambda^\kappa)$ by the Fourier transform of an elementary function. This follows from Lemma 5.1.

Finally, we show that $\widetilde{\Lambda}^\kappa$ is not complete if $\kappa \neq 0$. Suppose first that $0 < \kappa < 1/2$. The functions

$$\widehat{f_n}(x) = |x|^{-p} 1_{\{1<|x|<n\}}(x),\ \ p > 0,$$

are in $L^2(\mathbb{R})$, $\overline{\widehat{f_n}(x)} = \widehat{f_n}(-x)$, and hence they are the Fourier transforms of function $f_n \in L^2(\mathbb{R})$. It is clear that $f_n \in \widetilde{\Lambda}^\kappa$ and $f_n - f_m \to 0$ in $\widetilde{\Lambda}^\kappa$, as $n, m \to \infty$, if $-2p-2\kappa+1 < 0$ or $1/2-\kappa < p$. Let $1/2-\kappa < p < 1/2$ and suppose that there is $f \in \widetilde{\Lambda}^\kappa$ such that $f_n \to f$ in $\widetilde{\Lambda}^\kappa$. It is then necessary that $\widehat{f}(x) = |x|^{-p} 1_{\{1<|x|\}}(x)$, $x \in \mathbb{R}$, but this $\widehat{f}$ is not in $L^2(\mathbb{R})$ which is a contradiction. When $-1/2 < \kappa < 0$, an example of a Cauchy sequence in $\widetilde{\Lambda}^\kappa$ which does not converge is $(f_n) \subset L^2(\mathbb{R})$ with

$$\widehat{f_n}(x) = |x|^{-p} 1_{\{1/n<|x|<1\}}(x),\ \ 1/2 < p < 1/2 - \kappa.$$

□

### *5.2. The proof of Theorem 3.2*

We assume throughout this section that $0 < \kappa < 1/2$. The fractional integrals $I^\alpha_\pm$, $\alpha > 0$, and the inner product spaces $\Lambda^\kappa$ and $|\Lambda|^\kappa$ are defined in Sections 3.2 and 4. The fractional Marchaud derivatives $\mathbf{D}^\alpha_\pm$, $0 < \alpha < 1$, were introduced in Section 3.3. In the proof of Theorem 3.2 we will use the following results.

**Lemma 5.2.** *Let $a < b$ be real numbers. Then the function*

$$f_{a,b}(u) = (\mathbf{D}^\kappa_- 1_{[a,b)})(u) \tag{5.2}$$

$$= (\Gamma(1-\kappa))^{-1} \left((b-u)_+^{-\kappa} - (a-u)_+^{-\kappa}\right) \tag{5.3}$$

*satisfies the equation*

$$(I^\kappa_- f_{a,b})(s) = 1_{[a,b)}(s),\ \ \forall s \in \mathbb{R}. \tag{5.4}$$

*Proof.* The functions (5.2) and (5.3) are equal by Lemma 3.1 (where $\kappa$ is replaced by $-\kappa$). To show that the function (5.3) satisfies the equation (5.4), we have to verify that

$$\begin{aligned} J_1(s) :&= \int_{\mathbb{R}} \left((b-u)_+^{-\kappa} - (a-u)_+^{-\kappa}\right)(u-s)_+^{\kappa-1} du \\ &= \Gamma(\kappa)\Gamma(1-\kappa) 1_{[a,b)}(s) =: J_2(s). \end{aligned}$$

If $s \geq b$, then $J_1(s) = J_2(s) = 0$. If $s < b$, we use the following identity, valid for $t > 0$:

$$\begin{aligned} \int_0^t (t-v)_+^{-\kappa} v_+^{\kappa-1} dv &= t^{-\kappa+\kappa-1+1} \int_0^1 (1-s)^{-\kappa} s^{\kappa-1} ds = B(\kappa, 1-\kappa) \\ &= \frac{\Gamma(\kappa)\Gamma(1-\kappa)}{\Gamma(\kappa+1-\kappa)} = \Gamma(\kappa)\Gamma(1-\kappa). \end{aligned}$$

If $a \le s < b$, then

$$J_1(s) = \int_s^b (b-u)_+^{-\kappa}(u-s)_+^{\kappa-1}du = \int_0^{b-s} (b-s-v)_+^{-\kappa}v_+^{\kappa-1}dv = \Gamma(\kappa)\Gamma(1-\kappa),$$

which is also $J_2(s)$. In the case when $s < a$, we show in a similar way that

$$J_1(s) = \int_s^b (b-u)_+^{-\kappa}(u-s)_+^{\kappa-1}du - \int_s^a (a-u)_+^{-\kappa}(u-s)_+^{\kappa-1}du = 0 = J_2(s).$$

□

**Lemma 5.3.** *Let $a < b$ be real numbers. Then the function*

$$\psi_{a,b}(s) = (\mathbf{D}_+^\kappa 1_{(a,b]})(s) \tag{5.5}$$

$$= (\Gamma(1-\kappa))^{-1}\left((s-a)_+^{-\kappa} - (s-b)_+^{-\kappa}\right) \tag{5.6}$$

*satisfies the equation*

$$(I_+^\kappa \psi_{a,b})(u) = 1_{(a,b]}(u), \ \forall u \in \mathbb{R}. \tag{5.7}$$

*Proof.* The equality of the functions (5.5) and (5.6) follows from Lemma 3.1 by using the following property of fractional derivatives $\mathbf{D}_\pm^\alpha$: $Q\mathbf{D}_\pm^\alpha\psi = \mathbf{D}_\mp^\alpha Q\psi$, where $(Q\psi)(x) = \psi(-x)$, $x \in \mathbb{R}$ (see (5.61) in Samko et al. [14], p. 111, after correcting the typo in the reference). Indeed, by applying Lemma 3.1 (where $\kappa$ is replaced by $-\kappa$, $s$ by $(-s)$ and $1_{[0,t)}$ by $1_{[-b,-a)}$), we get

$$(\Gamma(1-\kappa))^{-1}\left((s-a)_+^{-\kappa} - (s-b)_+^{-\kappa}\right) = (\mathbf{D}_-^\kappa 1_{[-b,-a)})(-s)$$

$$= (\mathbf{D}_-^\kappa Q 1_{(a,b]})(-s) = (\mathbf{D}_+^\kappa 1_{(a,b]})(s).$$

The relation (5.7) follows from (5.4) because for fractional integrals $I_\pm^\alpha$ we similarly have $QI_\pm^\alpha f = I_\mp^\alpha Qf$ (see (5.9) in Samko et al. [14], p. 95). □

The following lemma shows that the fractional integration by parts formula holds for functions in $\Lambda^\kappa$, when associated with $\psi_{a,b}$.

**Lemma 5.4.** *If $f \in \Lambda^\kappa$, then, for any $a < b$,*

$$\int_{\mathbb{R}} \psi_{a,b}(s)(I_-^\kappa f)(s)ds = \int_{\mathbb{R}} (I_+^\kappa \psi_{a,b})(u)f(u)du \tag{5.8}$$

*or, equivalently,*

$$\int_{\mathbb{R}} \psi_{a,b}(s)(I_-^\kappa f)(s)ds = \int_{\mathbb{R}} 1_{(a,b]}(u)f(u)du. \tag{5.9}$$

*Proof.* The equivalence of the relations (5.8) and (5.9) follows from Lemma 5.3. To establish (5.8), it is enough to show by Fubini's theorem that, for $f \in \Lambda^\kappa$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||\psi_{a,b}(s)|(u-s)_+^{\kappa-1} ds du < \infty. \tag{5.10}$$

Since the function $\psi_{a,b}$ is zero on $(-\infty, a)$, positive on $(a, b)$ and negative on $(b, \infty)$, we have that

$$|\psi_{a,b}| = \psi_{a,b} - 2\psi_{a,b}1_{(b,\infty)}.$$

Hence, by Lemma 5.3,

$$\begin{aligned}
&(\Gamma(\kappa))^{-1}\int_{\mathbb{R}} |\psi_{a,b}(s)|(u-s)_+^{\kappa-1} ds \\
&= (I_+^\kappa|\psi_{a,b}|)(u) = (I_+^\kappa\psi_{a,b})(u) - 2(I_+^\kappa\psi_{a,b}1_{(b,\infty)})(u) \\
&= (I_+^\kappa\psi_{a,b})(u) - 2(I_+^\kappa\psi_{a,b}1_{(b,\infty)})(u)1_{(b,\infty)}(u) \\
&= (I_+^\kappa\psi_{a,b})(u) - 2(I_+^\kappa\psi_{a,b})(u)1_{(b,\infty)}(u) \\
&\quad +2(I_+^\kappa\psi_{a,b}1_{(a,b)})(u)1_{(b,\infty)}(u) \\
&= 1_{(a,b]}(u) - 21_{(a,b]}(u)1_{(b,\infty)}(u) \\
&\quad +2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_a^b (s-a)_+^{-\kappa}(u-s)_+^{\kappa-1} ds\ 1_{(b,\infty)}(u) \\
&\le 1_{(a,b]}(u) + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_a^b (s-a)_+^{-\kappa} ds\ (u-b)_+^{\kappa-1}.
\end{aligned}$$

We would complete the verification of (5.10), if we can show that

$$\int_{\mathbb{R}} |f(u)|1_{(a,b]}(u) du < \infty \tag{5.11}$$

and

$$\int_{\mathbb{R}} |f(u)|(u-b)_+^{\kappa-1} du < \infty. \tag{5.12}$$

By (3.16), $f \in \Lambda^\kappa$ implies that $f$ is integrable on any finite interval and hence (5.11) holds. Relation (5.12) may hold as well, but if it does not, by (3.16), we can take $(b_n)$ such that $b_n \to b$ and $\int_{\mathbb{R}} |f(u)|(u-b_n)_+^{\kappa-1} du < \infty$. Then the relation (5.9) holds with $b_n$ substituted for $b$. By letting $n$ tend to infinity, we obtain the R.H.S. of (5.9) with $b$ because $f$ is integrable on any finite interval. We also obtain the L.H.S. of (5.9) with $b$ because $I_-^\kappa f \in L^2(\mathbb{R})$ and $\|\psi_{a,b} - \psi_{a,b_n}\|_{L^2(\mathbb{R})} \to 0$, as $n \to \infty$. □

**Lemma 5.5.** *The set $|\Lambda|^\kappa$ is dense in the inner product space $\Lambda^\kappa$.*

*Proof.* If $f \in \Lambda^\kappa$, then, by (3.18), the function $g = I_-^\kappa f$ belongs to $L^2(\mathbb{R})$. Hence, there is a sequence $(g_n)$ of elementary functions such that $\|g - g_n\|_{L^2(\mathbb{R})} \to 0$. By Lemma 5.2, the elementary functions $g_n$ can be expressed as

$$g_n = I_-^\kappa f_n,$$

where $f_n$ is a linear combination of functions $f_{a,b}$, for some $a < b$, and the above equality holds almost everywhere. The functions $f_{a,b}$ are given in (5.3). Since, by (3.19),

$$\|f - f_n\|_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)}{c_1(\kappa)}\|g - g_n\|_{L^2(\mathbb{R})} \to 0,$$

$|\Lambda|^\kappa$ is dense in $\Lambda^\kappa$, if $f_n \in |\Lambda|^\kappa$. Since $f_n$ is a linear combination of functions $f_{a,b}$, it is in $|\Lambda|^\kappa$, if $f_{a,b} \in |\Lambda|^\kappa$ for every $a < b$, or, by (4.5), if

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f_{a,b}(u)|(u-s)_+^{\kappa-1} du\right]^2 ds < \infty. \tag{5.13}$$

To establish (5.13), we follow the idea of the proof of Lemma 5.4. Since $|f_{a,b}| = f_{a,b} - 2f_{a,b}1_{(-\infty,a)}$ (see (5.3)), we obtain by Lemma 5.2 that

$$\begin{aligned}
&(\Gamma(\kappa))^{-1}\int_{\mathbb{R}} |f_{a,b}(u)|(u-s)_+^{\kappa-1} du \\
&\quad = I_-^\kappa(|f_{a,b}|)(s) = (I_-^\kappa f_{a,b})(s) - 2(I_-^\kappa f_{a,b}1_{(-\infty,a)})(s)1_{(-\infty,a)}(s) \\
&\quad = (I_-^\kappa f_{a,b})(s) - 2(I_-^\kappa f_{a,b})(s)1_{(-\infty,a)}(s) + 2(I_-^\kappa f_{a,b}1_{(a,b)})(s)1_{(-\infty,a)}(s) \\
&\quad = 1_{[a,b)}(s) + 21_{[a,b)}(s)1_{(-\infty,a)}(s) \\
&\qquad + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_a^b (b-u)_+^{-\kappa}(u-s)_+^{\kappa-1} du\ 1_{(-\infty,a]}(s) \\
&\quad \le 1_{[a,b)}(s) + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_a^b (b-u)_+^{-\kappa} du\ (a-s)_+^{\kappa-1}1_{(-\infty,a-1]}(s) \\
&\qquad + 2(\Gamma(\kappa)\Gamma(1-\kappa))^{-1}\int_a^b (b-u)_+^{-\kappa}(u-a)_+^{\kappa-1} du\ 1_{(a-1,a]}(s).
\end{aligned} \tag{5.14}$$

Since the upper bound (5.14) is in $L^2(\mathbb{R})$, the function $f_{a,b}$ satisfies (5.13) and hence $|\Lambda|^\kappa$ is dense in $\Lambda^\kappa$. □

The following two lemmas will be used to show that the inner product space $\Lambda^\kappa$ is not complete.

**Lemma 5.6.** *The inner product space $\Lambda^\kappa$ is complete if and only if, for every $\phi \in L^2(\mathbb{R})$, there is a function $f_\phi \in \Lambda^\kappa$ such that*

$$\phi = I_-^\kappa f_\phi. \tag{5.15}$$

*Proof.* Suppose that $\Lambda^\kappa$ is complete and let $\phi \in L^2(\mathbb{R})$. There is a sequence $(\phi_n)$ of elementary functions such that $\phi_n \to \phi$ in $L^2(\mathbb{R})$. By Lemma 5.2, the elementary functions $(\phi_n)$ can be expressed as $\phi_n = I_-^\kappa f_{\phi_n}$, for some $f_{\phi_n} \in \Lambda^\kappa$. The sequence $(f_{\phi_n})$ is then Cauchy in $\Lambda^\kappa$ by (3.17). Since $\Lambda^\kappa$ is complete by assumption, there is $f \in \Lambda^\kappa$ such that $f_{\phi_n} \to f$ in $\Lambda^\kappa$. In other words, the convergence $\phi_n = I_-^\kappa f_{\phi_n} \to I_-^\kappa f$ holds in $L^2(\mathbb{R})$. Since $\phi_n \to \phi$ in $L^2(\mathbb{R})$ as well, we obtain (5.15) with $f = f_\phi$.

Conversely, suppose that (5.15) holds and let $(f_n)$ be a Cauchy sequence in $\Lambda^\kappa$. Then the sequence $(\phi_n)$, given by $\phi_n = I_-^\kappa f_n$, is Cauchy in $L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is complete, there is $\phi \in L^2(\mathbb{R})$ such that $\phi_n \to \phi$ in $L^2(\mathbb{R})$. By assumption, there is $f_\phi \in \Lambda^\kappa$ such that (5.15) holds. Since $\phi_n \to \phi$ in $L^2(\mathbb{R})$ implies $f_n \to f_\phi$ in $\Lambda^\kappa$, $\Lambda^\kappa$ is complete. □

**Lemma 5.7.** *Let* $0 < \kappa < 1/2$. *There is* $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *such that the function*

$$U_\phi(a) = \int_{\mathbb{R}} u_+^{-\kappa} \phi(u+a) du, \ a \in \mathbb{R},$$

*is not differentiable on a set of positive Lebesgue measure.*

*Proof.* Let $b > 1$ and $0 < p < \kappa$. Consider the complex-valued function

$$\phi^*(u) = c_0 \left[ \sum_{n=1}^{\infty} b^{-pn} e^{ib^n u} \right] 1_{[0,1]}(u),$$

where $c_0 = \left( \int_0^\infty e^{iv} v_+^{-\kappa} dv \right)^{-1} = \left( \Gamma(1-\kappa) e^{i\pi(1-\kappa)/2} \right)^{-1}$. Since $\phi^*$ is bounded with compact support, it belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Let us show that $U_{\phi^*}(a)$ is not differentiable on $[0, 1/2]$ (that is, its real and imaginary parts are not differentiable). Since, for $0 < a < 1$, we have

$$\begin{aligned} U_{\phi^*}(a) &= c_0 \int_0^{1-a} u_+^{-\kappa} \left[ \sum_{n=1}^{\infty} b^{-pn} e^{ib^n u} e^{ib^n a} \right] du \\ &= c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_0^{1-a} e^{ib^n u} u_+^{-\kappa} du \right] e^{ib^n a} \\ &= c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_0^{\infty} e^{ib^n u} u_+^{-\kappa} du \right] e^{ib^n a} \\ &\quad - c_0 \sum_{n=1}^{\infty} b^{-pn} \left[ \int_{1-a}^{\infty} e^{ib^n u} u_+^{-\kappa} du \right] e^{ib^n a} \\ &=: y_1(a) + y_2(a), \end{aligned}$$

it is enough to show that the function $y_1$ is not differentiable on $[0, 1]$ and that the function $y_2$ is differentiable on $[0, 1/2]$. We deal with $y_1$ first. By making a change of variables $b^n u = v$, we obtain that

$$y_1(a) = \sum_{n=1}^{\infty} b^{-(p-\kappa+1)n} e^{ib^n a}.$$

Since $b^{-(p-\kappa+1)} b = b^{\kappa-p} > 1$, the function $y_1$ is a particular case of the well-known Weierstrass function whose real and imaginary parts are nowhere differentiable functions (see, for example, Hardy [10] or Zygmund [18] for more details on the nowhere differentiable Weierstrass function). We now turn to the function $y_2$, which is well-defined because, by Lemma 5.8 below,

$$\sum_{n=1}^{\infty} \left| b^{-pn} \int_{1-a}^{\infty} e^{ib^n u} u_+^{-\kappa} du \ e^{ib^n a} \right| \le 4(1-a)^{-\kappa} \sum_{n=1}^{\infty} b^{-(p+1)n} < \infty.$$

It is differentiable on $[0, 1/2]$, since, by using Lemma 5.8,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{d}{da}\left(b^{-pn}\int_{1-a}^{\infty}e^{ib^n u}u_+^{-\kappa}du\; e^{ib^n a}\right)\right| \\ \le \sum_{n=1}^{\infty}\left|b^{-pn}e^{ib^n}(1-a)^{-\kappa}\right| + \sum_{n=1}^{\infty}\left|ib^n b^{-pn}\int_{1-a}^{\infty}e^{ib^n u}u_+^{-\kappa}du\; e^{ib^n a}\right| \\ \le (1-a)^{-\kappa}\sum_{n=1}^{\infty}b^{-pn} + 4\sum_{n=1}^{\infty}b^n b^{-pn}b^{-n}(1-a)^{-\kappa} \\ \le C\sum_{n=1}^{\infty}b^{-pn} < \infty,\end{aligned}$$

where the constant $C$ does not depend on $a \in [0, 1/2]$. Finally, for a function $\phi$ of the lemma, we can take $\phi = \Re\phi^*$ or $\phi = \Im\phi^*$. □

**Lemma 5.8.** *Let $a < b$ be two real numbers. Suppose that $f$ is a positive non-increasing function on the interval $[a, b]$. Then, for any $x \in \mathbb{R}$, $x \neq 0$,*

$$\left|\int_a^b e^{ixu}f(u)du\right| \le \frac{4f(a)}{x}. \tag{5.16}$$

*Proof.* Relation (5.16) follows from the inequalities

$$\left|\int_a^b \sin(xu)f(u)du\right| \le f(a)\sup_{a<y<b}\left|\int_a^y \sin(xu)du\right|$$

and

$$\left|\int_a^b \cos(xu)f(u)du\right| \le f(a)\sup_{a<y<b}\left|\int_a^y \cos(xu)du\right|,$$

valid for any $x \in \mathbb{R}$. □

*Proof of Theorem 3.2.* To show that $\Lambda^\kappa$ is an inner product space, we verify that $\left(f, f\right)_{\Lambda^\kappa} = 0$ implies $f(u) = 0$ a.e. $du$. If $\left(f, f\right)_{\Lambda^\kappa} = 0$, then, by (3.17), $(I_-^\kappa f)(s) = 0$ a.e. $ds$, and hence, by (5.9), $\int_{\mathbb{R}} f(u)1_{(a,b]}(u)du = 0$, for any real numbers $a < b$. By an approximation argument, we get that $\int_{\mathbb{R}} f(u)1_{\{f(u)>0\}\cap\{|u|\le n\}}(u)du = 0$, for any $n \in \mathbb{N}$. Consequently, $f(u) \le 0$ a.e. $du$ and, by symmetry, $f(u) = 0$ a.e. $du$.

Let us prove that the set of elementary functions $\mathscr{E}$ is dense in $\Lambda^\kappa$. By Lemma 5.5, it is enough to show that $\mathscr{E}$ is dense in $|\Lambda|^\kappa$. If $f \in |\Lambda|^\kappa$, then, by (4.6),

$$\|f\|_{\Lambda^\kappa}^2 = \kappa(2\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)f(v)|u-v|^{2\kappa-1}du\,dv.$$

Now, choose a sequence of simple functions $f_n$ which converges to $f$ a.e. and such that $|f_n| \le |f|$. Then, by the dominated convergence theorem, $\left\|f - f_n\right\|_{\Lambda^\kappa} \to 0$, as $n \to \infty$. On the other hand, for fixed $n$, there is a sequence of elementary functions

$g_{n,m}$ such that $g_{n,m} \to f_n$ a.e. and $\sup_m |g_{n,m}| \le c_n 1_{[-k_n,k_n)}$. The dominated convergence theorem implies again that $\left\| f_n - g_{n,m} \right\|_{\Lambda^\kappa} \to 0$, as $m \to \infty$. It follows that the set $\mathscr{E}$ is dense in $|\Lambda|^\kappa$.

We will show that the inner product space $\Lambda^\kappa$ is not complete by contradiction. Suppose that $\Lambda^\kappa$ is complete. Then, by Lemma 5.6, to every $\phi \in L^2(\mathbb{R})$ there corresponds a function $f_\phi \in \Lambda^\kappa$ such that

$$\phi = I_-^\kappa f_\phi.$$

It follows from (5.9) that

$$\int_{\mathbb{R}} \phi(s)\psi_{0,a}(s)ds = \int_{\mathbb{R}} f_\phi(u) 1_{(0,a]}(u)du, \tag{5.17}$$

for any $\phi \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$. Since the function on the right-hand side of (5.17) is differentiable a.e. $da$, the function

$$\int_{\mathbb{R}} \phi(s) \left( s_+^{-\kappa} - (s-a)_+^{-\kappa} \right) ds,$$

which is the left-hand side of (5.17) up to a multiplicative constant, is differentiable a.e. $da$ as well. If, in addition, $\phi \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} \phi(s) s_+^{-\kappa} ds \le \|\phi\|_{L^2(\mathbb{R})} \left( \int_0^1 s_+^{-2\kappa} ds \right)^{1/2} + \|\phi\|_{L^1(\mathbb{R})} < \infty$$

and the function

$$U_\phi(a) = \int_{\mathbb{R}} \phi(u)(u-a)_+^{-\kappa} du = \int_{\mathbb{R}} u_+^{-\kappa} \phi(u+a) du$$

is also differentiable a.e. $da$. In Lemma 5.7 above we provide an example of a function $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for which $U_\phi$ is not differentiable on a set of positive Lebesgue measure. We thus obtain a contradiction which shows that the inner product space $\Lambda^\kappa$ is not complete. □

### *5.3. The proof of Theorem 3.3*

In the proof of Theorem 3.3 we will use the following result which is a consequence of Lemma 5.1.

**Lemma 5.9.** *Let* $-1/2 < \kappa < 1/2$ *and suppose that* $\widehat{g}$ *is the Fourier transform of a function* $g \in L^2(\mathbb{R})$. *Then there is a sequence of elementary functions* $l_n$ *such that*

$$\int_{\mathbb{R}} \left| \widehat{g}(x) - \widehat{l_n}(x)|x|^{-\kappa} \right|^2 dx \overset{n\to\infty}{\longrightarrow} 0.$$

*Proof.* Suppose first that $-1/2 < \kappa < 0$. If $\widehat{g}_\epsilon(x) = \widehat{g}(x)1_{\{|x|>\epsilon\}}(x)$, then $\int_{\mathbb{R}} |\widehat{g}(x) - \widehat{g}_\epsilon(x)|^2 dx \to 0$, as $\epsilon \to 0$. Since

$$\widehat{g}_\epsilon(x) = \left(\widehat{g}(x)|x|^\kappa 1_{\{|x|>\epsilon\}}(x)\right)|x|^{-\kappa} = \widehat{f_\epsilon}(x)|x|^{-\kappa},$$

where $\widehat{f_\epsilon}$ is the Fourier transform of a function $f_\epsilon \in L^2(\mathbb{R})$, the result follows from Lemma 5.1. The proof in the case $0 \le \kappa < 1/2$ is similar. □

*Proof of Theorem 3.3.* Recall that $-1/2 < \kappa < 0$ in Theorem 3.3. It is obvious that the space $\Lambda^\kappa$, defined by (3.30), is an inner product space with the inner product (3.31). Let us show that $\mathscr{E}$ is dense in $\Lambda^\kappa$. If $f \in \Lambda^\kappa$, then

$$f = I_-^{-\kappa}\phi_f, \tag{5.18}$$

for some function $\phi_f \in L^2(\mathbb{R})$. When $f$ is an elementary function, Lemma 5.2 implies that $\phi_f$ equals to a linear combination of $\phi_{a,b}$, $a < b$, where the functions

$$\phi_{a,b}(s) = \Big(\Gamma(1-\kappa)\Big)^{-1}\Big((b-s)_+^\kappa - (a-s)_+^\kappa\Big) \tag{5.19}$$

satisfy the equation

$$I_-^{-\kappa}\phi_{a,b} = 1_{[a,b)}. \tag{5.20}$$

Therefore, by (3.32), the set $\mathscr{E}$ of elementary functions is dense in $\Lambda^\kappa$, if any $\phi \in L^2(\mathbb{R})$ can be approximated in $L^2(\mathbb{R})$ by a linear combination of functions $\phi_{a,b}$ from (5.19). To prove this fact, let us work with Fourier transforms. We have to show that any $\widehat{\phi} \in L^2(\mathbb{R})$ can be approximated in $L^2(\mathbb{R})$ by a linear combination of the functions $\widehat{\phi}_{a,b}$. Since, by Lemma 3.1,

$$\phi_{a,b} = \mathbf{D}_-^{-\kappa} 1_{[a,b)}$$

and, clearly, $1_{[a,b)} \in \widetilde{\Lambda}^\kappa$, Proposition 3.3, (3), implies that

$$\widehat{\phi}_{a,b}(x) = C^\kappa(x)\widehat{1}_{[a,b]}(x)|x|^{-\kappa}.$$

Then, applying Lemma 5.9 with $\widehat{g}(x) = C^\kappa(x)\widehat{\phi}(x)$, the required approximation of $\widehat{\phi}$ by a linear combination of $\widehat{\phi}_{a,b}$ follows.

To show that $\Lambda^\kappa$ is complete, let $(f_n)$ be a Cauchy sequence in $\Lambda^\kappa$ and use the facts that, by (3.32), the corresponding sequence $(\phi_{f_n})$ is Cauchy in $L^2(\mathbb{R})$ and that $L^2(\mathbb{R})$ is complete. Finally, the space $\Lambda^\kappa$ is isometric to $\overline{Sp}(B^\kappa)$ by Proposition 2.1. □

## 6. Connections to the RKHS of fractional Brownian motion

In this section we look at our results from the standpoint of the theory of reproducing kernel Hilbert spaces (RKHS). The RKHS of fBm also characterizes the Hilbert space $\overline{Sp}(B^\kappa)$: it is isometric to the space $\overline{Sp}(B^\kappa)$ itself, not only for $-1/2 < \kappa < 0$ but also for $0 < \kappa < 1/2$. The RKHS can be viewed as representing "the space of integrals", since the "simple" integral $B^\kappa(u_2) - B^\kappa(u_1)$ belonging to $\overline{Sp}(B^\kappa)$ is represented by the element of the RKHS, $\Gamma^\kappa(\cdot, u_2) - \Gamma^\kappa(\cdot, u_1)$, a function which depends on $\kappa$ ($\Gamma^\kappa$ is the covariance function of fBm). From this perspective, the spaces which associate to $B^\kappa(u_2) - B^\kappa(u_1)$ the indicator function $1_{[u_1,u_2)}$ can be regarded as "spaces of integrands". It is therefore interesting to see how the RKHS characterization compares with the inner product spaces introduced earlier.

We start by recalling the definition of a RKHS. Suppose that $X = \{X(t)\}_{t\in\mathbb{R}}$ is a second order mean zero stochastic process with the covariance function $\Gamma(s,t)$, $s,t \in \mathbb{R}$, and let $\overline{Sp}(X)$ be the closure in $L^2(\Omega)$ of all linear combinations $\sum_{i=1}^k a_i X(t_i)$, $a_i, t_i \in \mathbb{R}$. Then with the Hilbert space $\overline{Sp}(X)$ one can always associate an isometric Hilbert space $\mathbb{H}(\Gamma)$ of deterministic functions. The space $\mathbb{H}(\Gamma)$ is called the Reproducing Kernel Hilbert Space (RKHS) and is characterized by the following two properties: (1) $\Gamma(\cdot,t) \in \mathbb{H}(\Gamma)$, for all $t \in \mathbb{R}$, and (2) $(g, \Gamma(\cdot,t))_{\mathbb{H}(\Gamma)} = g(t)$, for all $t \in \mathbb{R}$ and $g \in \mathbb{H}(\Gamma)$. It consists of all functions of the form $f(s) = \sum_{i=1}^k a_i \Gamma(s,t_i)$, $s \in \mathbb{R}$, and their limits under the norm $\|f\|^2 = \sum_{i,j=1}^k a_i a_j \Gamma(t_i,t_j)$. (See Grenander [8], p. 93, for a quick reference to RKHS's, or Weinert [16], for a thorough introduction.) The isometry map $\mathscr{I}$ between the Hilbert spaces $\overline{Sp}(X)$ and $\mathbb{H}(\Gamma)$ satisfies

$$\mathscr{I} : \sum_{i=1}^k a_i \Gamma(\cdot, t_i) \longmapsto \sum_{i=1}^k a_i X(t_i)$$

and $(\mathscr{I}(g), \mathscr{I}(h))_{L^2(\Omega)} = (g,h)_{\mathbb{H}(\Gamma)}$. If the covariance function $\Gamma$ can be expressed as

$$\Gamma(s,t) = \int_U f_s(u) f_t(u) d\nu(u), \tag{6.1}$$

where $(U, \mathscr{U}, \nu)$ is a measure space and $\{f_t : t \in \mathbb{R}\} \subset L^2(\nu)$, then the RKHS $\mathbb{H}(\Gamma)$ is characterized by

$$\mathbb{H}(\Gamma) = \left\{ g : g(t) = \int_U g^*(u) f_t(u) d\nu(u), \text{ for some } g^* \in \overline{\text{span}}\{f_t, t \in \mathbb{R}\} \right\}, \tag{6.2}$$

$$(g,h)_{\mathbb{H}(\Gamma)} = \int_U g^*(u) h^*(u) d\nu(u), \tag{6.3}$$

where $\overline{\text{span}}\{f_t, t \in \mathbb{R}\}$ is the closure in $L^2(\nu)$ of all linear combinations of $f_t$, $t \in \mathbb{R}$ (see Grenander [8]).

*6.1. The case $-1/2 < \kappa < 0$*

We characterize, in the following proposition, the RKHS of fBm with index $-1/2 < \kappa < 0$.

**Proposition 6.1.** *Let $-1/2 < \kappa < 0$ and $B^\kappa$ be a standard fBm. Then the* RKHS $\mathbb{H}(\Gamma^\kappa)$ *of $B^\kappa$ is*

$$\mathbb{H}(\Gamma^\kappa) = \left\{ g : g(t) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} g^*(u)(\mathbf{D}_-^{-\kappa} 1_{[0,t)})(u)du, \right. \\ \left. \text{for some } g^* \in L^2(\mathbb{R}) \right\} \tag{6.4}$$

*with the inner product*

$$(g,h)_{\mathbb{H}(\Gamma^\kappa)} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} g^*(u)h^*(u)du. \tag{6.5}$$

*Proof.* By taking $f$ and $g$ in (3.28) to be indicator functions, one gets

$$\Gamma^\kappa(s,t) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (\mathbf{D}_-^{-\kappa} 1_{[0,s)})(u)(\mathbf{D}_-^{-\kappa} 1_{[0,t)})(u)du.$$

Thus, the relation (6.1) holds with $U = \mathbb{R}$, the Lebesgue measure $d\nu(u)$ (up to a constant) and

$$f_t(u) = (\mathbf{D}_-^{-\kappa} 1_{[0,t)})(u).$$

Since, by Proposition 3.3, (3),

$$\widehat{f_t}(x) = C^\kappa(x)\widehat{1}_{[0,t)}(x)|x|^{-\kappa}$$

and $|C^\kappa(x)| = 1$, Lemma 5.9 implies that $\{f_t, t \in \mathbb{R}\}$ spans $L^2(\mathbb{R})$. Hence, by (6.2) and (6.3), the RKHS $\mathbb{H}(\Gamma^\kappa)$ of fBm $B^\kappa$ with $-1/2 < \kappa < 0$ is characterized by (6.4) and (6.5). □

Let us denote the isometry map between the Hilbert spaces $\mathbb{H}(\Gamma^\kappa)$ and $\overline{Sp}(B^\kappa)$ by $\mathscr{J}^\kappa$. Then to every $X \in \overline{Sp}(B^\kappa)$ there corresponds a unique $g_X \in \mathbb{H}(\Gamma^\kappa)$ such that $X = \mathscr{J}^\kappa(g_X)$, that is, every element of $\overline{Sp}(B^\kappa)$ can be characterized by a deterministic function from $\mathbb{H}(\Gamma^\kappa)$.

**Example.** The random variable $B^\kappa(u_2) - B^\kappa(u_1)$ in $\overline{Sp}(B^\kappa)$ is represented by the function $g(\cdot) = \Gamma^\kappa(\cdot, u_2) - \Gamma^\kappa(\cdot, u_1)$ in $\mathbb{H}(\Gamma^\kappa)$. The corresponding function $g^*$ in (6.4) is $g^* = \mathbf{D}_-^{-\kappa} 1_{[u_1,u_2)}$.

To compare the spaces of functions $\mathbb{H}(\Gamma^\kappa)$ and $\Lambda^\kappa$ when $-1/2 < \kappa < 0$, observe that there is a natural map from the space of elementary functions $\mathscr{E} \subset \Lambda^\kappa$ into $\mathbb{H}(\Gamma^\kappa)$ defined by

$$\gamma^\kappa : \sum_{k=1}^n f_k 1_{[u_k,u_{k+1})} \longmapsto \sum_{k=1}^n f_k \left(\Gamma^\kappa(\cdot, u_{k+1}) - \Gamma^\kappa(\cdot, u_k)\right). \tag{6.6}$$

To see the connection between our integral $\mathscr{I}^\kappa$ and the RKHS integral $\mathscr{J}^\kappa$, note that, for all $f, g \in \mathscr{E}$,

$$\mathscr{I}^\kappa(f) = \mathscr{J}^\kappa(\gamma^\kappa(f)), \tag{6.7}$$

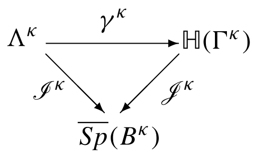
where

$$(\gamma^\kappa(f))(t) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (\mathbf{D}_-^{-\kappa} f)(u)(\mathbf{D}_-^{-\kappa} 1_{[0,t)})(u)du, \tag{6.8}$$

so that

$$\begin{aligned}(f,g)_{\Lambda^\kappa} &= (\mathscr{I}^\kappa(f), \mathscr{I}^\kappa(g))_{L^2(\Omega)} = (\mathscr{J}^\kappa(\gamma^\kappa(f)), \mathscr{J}^\kappa(\gamma^\kappa(g))_{L^2(\Omega)} \\ &= (\gamma^\kappa(f), \gamma^\kappa(g))_{\mathbb{H}(\Gamma^\kappa)}. \end{aligned} \tag{6.9}$$

To see that the definition (6.6) of $\gamma^\kappa$ can be extended to an isometry between $\Lambda^\kappa$ and $\mathbb{H}(\Gamma^\kappa)$, observe that, for every $g^* \in L^2(\mathbb{R})$, there is a function $f$ such that $g^* = \mathbf{D}_-^{-\kappa} f$. Indeed, by Theorem 6.1 in Samko et al. [14], p. 125, we can take $f = I_-^{-\kappa} g^*$. This means that the map $\gamma^\kappa$ in (6.6) can be extended by the relation (6.8) involving elementary functions to the map from the space $\Lambda^\kappa$ onto the space $\mathbb{H}(\Gamma^\kappa)$. Now $\mathscr{E}$ is dense in the space $\Lambda^\kappa$ (Theorem 3.3) and $\gamma^\kappa(\mathscr{E})$ is dense in the space $\mathbb{H}(\Gamma^\kappa)$ (by construction). This means that the relations (6.7) and (6.9) extend to $f, g \in \Lambda^\kappa$. Hence, when $-1/2 < \kappa < 0$, the Hilbert spaces $\Lambda^\kappa$, $\mathbb{H}(\Gamma^\kappa)$ and $\overline{Sp}(B^\kappa)$ are isometric. This is illustrated in the following diagram:




where the maps $\mathscr{I}^\kappa$, $\mathscr{J}^\kappa$ and $\gamma^\kappa$ are all isometries between the corresponding spaces.

**Remarks.**

1. The isometry between $\mathbb{H}(\Gamma^\kappa)$ and $\Lambda^\kappa$ also follows from the facts that $\mathbb{H}(\Gamma^\kappa)$ is isometric to $\overline{Sp}(B^\kappa)$ and $\Lambda^\kappa$ is isometric to $\overline{Sp}(B^\kappa)$ (Theorem 3.3). The difficult step in the proof of either Theorem 3.3 or Proposition 6.1 (which characterizes $\mathbb{H}(\Gamma^\kappa)$) is Lemma 5.9 (or Lemma 5.1 on which Lemma 5.9 is based).

2. The argument proceeding these remarks provides an explicit identification of the map $\gamma^\kappa$.

3. Since the space of functions $\widetilde{\Lambda}^\kappa$ with $-1/2 < \kappa < 0$ is a proper subspace of $\Lambda^\kappa$, it is isometric to a subspace of $\mathbb{H}(\Gamma^\kappa)$. We can identify this subspace as a subset of functions $g \in \mathbb{H}(\Gamma^\kappa)$ such that the function $\widehat{g^*}(x)|x|^\kappa$ is in $L^2(\mathbb{R})$. (The functions $g$ and $g^*$ are related by (6.4).) Indeed, if the latter condition holds, there is a function $f^* \in L^2(\mathbb{R})$ such that

$$\widehat{g^*}(x)|x|^\kappa = C^\kappa(x)\widehat{f^*}(x).$$

In particular, the function $\widehat{f^*}(x)|x|^\kappa$ is in $L^2(\mathbb{R})$ and, by Proposition 3.3, (3),

$$g^* = \mathbf{D}_-^{-\kappa} f^*,$$

since both sides of the equation have the same Fourier transforms. This shows that $g = \gamma^\kappa(f^*)$ with $f^* \in \widetilde{\Lambda}^\kappa$. On the other hand, one can similarly show that, if $f^* \in \widetilde{\Lambda}^\kappa$, the function $g^*$ which corresponds to $g = \gamma^\kappa(f^*)$ satisfies $\widehat{g^*}(x)|x|^\kappa \in L^2(\mathbb{R})$.

### 6.2. *The case* $0 < \kappa < 1/2$

Proceeding as in the case $-1/2 < \kappa < 0$ considered above, one can show that the following proposition holds.

**Proposition 6.2.** *Let* $0 < \kappa < 1/2$ *and* $B^\kappa$ *be a standard fBm. Then the* RKHS $\mathbb{H}(\Gamma^\kappa)$ *of* $B^\kappa$ *is*

$$\mathbb{H}(\Gamma^\kappa) = \left\{ g : g(t) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} g^*(u)(I_-^\kappa 1_{[0,t)})(u)du, \right. \\ \left. \text{for some } g^* \in L^2(\mathbb{R}) \right\} \tag{6.10}$$

*with the inner product* (6.5).

The result (6.10) coincides with the characterization of Barton and Poor [2], who studied the case $0 < \kappa < 1/2$ and expressed $\mathbb{H}(\Gamma^\kappa)$ as

$$\mathbb{H}(\Gamma^\kappa) = \left\{ g : g(t) = \frac{1}{\Gamma(\kappa)} \int_0^t \int_{-\infty}^s (s-u)^{\kappa-1} \widetilde{g}(u)duds, \text{ for some } \widetilde{g} \in L^2(\mathbb{R}) \right\}. \tag{6.11}$$

Indeed, by using (3.21) and the fractional integration by parts formula (3.20), we can write the function $g$ in (6.11) as

$$g(t) = \int_{\mathbb{R}} (I_+^\kappa \widetilde{g})(s) 1_{[0,t)}(s)ds = \int_{\mathbb{R}} \widetilde{g}(u)(I_-^\kappa 1_{[0,t)})(u)du$$

and, hence, recover the representation in (6.10).

As in the case $-1/2 < \kappa < 0$, let us denote the isometry map between the Hilbert spaces $\mathbb{H}(\Gamma^\kappa)$ and $\overline{Sp}(B^\kappa)$ by $\mathscr{J}^\kappa$ and let also $\gamma^\kappa$ be the map defined by (6.6). Then the relations (6.7) and (6.9) hold also for $f, g \in \mathscr{E} \subset \Lambda^\kappa$ and $0 < \kappa < 1/2$, while (6.8) becomes

$$(\gamma^\kappa(f))(t) = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2} \int_{\mathbb{R}} (I_-^\kappa f)(u)(I_-^\kappa 1_{[0,t)})(u)du. \tag{6.12}$$

Using the relation (6.12), we can extend the map $\gamma^\kappa$ to the map from the space $\Lambda^\kappa$ into the space $\mathbb{H}(\Gamma^\kappa)$. Since the sets of functions $\mathscr{E}$ and $\gamma(\mathscr{E})$ are dense in the spaces $\Lambda^\kappa$ and $\mathbb{H}(\Gamma^\kappa)$, respectively, the extended map $\gamma^\kappa$ satisfies (6.7) and (6.9) for any $f, g \in \Lambda^\kappa$ and $0 < \kappa < 1/2$. However, since the inner product space $\Lambda^\kappa$ is not complete (Theorem 3.2), $\Lambda^\kappa$ is isometric only to a subspace of the Hilbert space $\mathbb{H}(\Gamma^\kappa)$. The case $0 < \kappa < 1/2$ is therefore different from the case $-1/2 < \kappa < 0$

considered earlier because when $-1/2 < \kappa < 0$, there is an isometry between $\mathbb{H}(\Gamma^\kappa)$ and $\Lambda^\kappa$.

A slightly different perspective on the relationship between the spaces $\mathbb{H}(\Gamma^\kappa)$ and $\Lambda^\kappa$ when $0 < \kappa < 1/2$ is as follows. Let $(\gamma^\kappa)^{-1}$ be the isometry map defined from a linear subspace of $\mathbb{H}(\Gamma^\kappa)$ into the space of elementary functions $\mathcal{E}$ as the inverse map of (6.6). We can then ask whether we can extend the map $(\gamma^\kappa)^{-1}$ to the isometry between $\mathbb{H}(\Gamma^\kappa)$ and some space of deterministic functions. If such an isometry exists, then the class of functions $(\gamma^\kappa)^{-1}(\mathbb{H}(\Gamma^\kappa))$ is isometric to the space $\overline{Sp}(B^\kappa)$ itself (since $\mathbb{H}(\Gamma^\kappa)$ is isometric to $\overline{Sp}(B^\kappa)$) and is also a class of integrands for which we can extend the natural definition $\mathcal{I}^\kappa(f) = \int_{\mathbb{R}} f(u)dB^\kappa(u)$, $f \in \mathcal{E}$, of the integral with respect to fBm. To obtain this isometry, one would naturally try to do the following: in view of (6.10) and (6.12), express the function $g^*$ in (6.10) as

$$g^* = I^\kappa_- f \tag{6.13}$$

for some function $f$, and then define $(\gamma^\kappa)^{-1}(g) = f$. However, as we have shown in Section 5.2 there are functions $g^* \in L^2(\mathbb{R})$ which do not admit the representation (6.13). This is why $\Lambda^\kappa = \{f : g^* = I^\kappa_- f \in L^2(\mathbb{R})\}$ is isometric only to a subspace of $\mathbb{H}(\Gamma^\kappa)$. Since by the definition of the RKHS, any function in $\mathbb{H}(\Gamma^\kappa)$ can be approximated by functions in $\gamma^\kappa(\mathcal{E})$, $\mathbb{H}(\Gamma^\kappa)$ is isometric to the space $\mathcal{C}$ of all *Cauchy sequences* in $\mathcal{E}$. But is $\mathcal{C}$ a set of functions? We do not know the answer. Therefore we do not know whether the map $\gamma^\kappa : \mathcal{E} \mapsto \mathbb{H}(\Gamma^\kappa)$ can be extended to an isometry between some space of *functions* $\mathcal{C}$ and the space $\mathbb{H}(\Gamma^\kappa)$.

**Remark.** When $0 < \kappa < 1/2$, we introduced in Sections 3.1 and 4 two other classes of integrands, namely, the spaces $\widetilde{\Lambda}^\kappa$ and $|\Lambda|^\kappa$ which are strict subsets of the class $\Lambda^\kappa$. Their images under the map $\gamma^\kappa$ are

$$\gamma^\kappa\left(\widetilde{\Lambda}^\kappa\right) = \left\{g \in \mathbb{H}(\Gamma^\kappa) \ : \ \widehat{g^*}(x)|x|^\kappa \in L^2(\mathbb{R})\right\},$$

$$\gamma^\kappa\left(|\Lambda|^\kappa\right) = \left\{g \in \mathbb{H}(\Gamma^\kappa) \ : \ g^* = I^\kappa_- f, \text{ for some function } f \in |\Lambda|^\kappa\right\},$$

where $g^*$ is as in (6.10). These relations follow easily from the relation (6.12) applied to $f \in \Lambda^\kappa$ and the fact that $\widetilde{\Lambda}^\kappa$ and $|\Lambda|^\kappa$ are subsets of $\Lambda^\kappa$.

## 7. Conclusion

In this work we have introduced a number of different inner product spaces, namely

$$\widetilde{\Lambda}^\kappa = \left\{f : f \in L^2(\mathbb{R}), \ \int_{\mathbb{R}} |\widehat{f}(x)|^2|x|^{-2\kappa}dx < \infty\right\}, \text{ for } -1/2 < \kappa < 1/2,$$

$$\Lambda^\kappa = \left\{f : \int_{\mathbb{R}} \left[(I^\kappa_- f)(s)\right]^2 ds < \infty\right\}, \text{ for } 0 < \kappa < 1/2,$$

$$\Lambda^\kappa = \left\{f : \exists\, \phi_f \in L^2(\mathbb{R}) \text{ such that } f = I^{-\kappa}_- \phi_f\right\}, \text{ for } -1/2 < \kappa < 0,$$

$$|\Lambda|^\kappa = \left\{f : \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\kappa-1}dudv < \infty\right\}, \text{ for } 0 < \kappa < 1/2,$$

with the inner products given by

$$(f,g)_{\widetilde{\Lambda}^\kappa} = \frac{1}{c_2(\kappa)^2}\int_{\mathbb{R}} \widehat{f}(x)\overline{\widehat{g}(x)}|x|^{-2\kappa}dx,$$
$$\big(f,g\big)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}}(I_-^\kappa f)(s)(I_-^\kappa g)(s)\,ds,$$
$$\big(f,g\big)_{\Lambda^\kappa} = \frac{\Gamma(\kappa+1)^2}{c_1(\kappa)^2}\int_{\mathbb{R}}\phi_f(s)\phi_g(s)ds,$$
$$\big(f,g\big)_{|\Lambda|^\kappa} = \kappa(2\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1}dudv,$$

respectively. (The constants $c_1(\kappa)$ and $c_2(\kappa)$ are given in (3.8) and (3.2).)[1] The definition of $\Lambda^\kappa$ uses the integral fractional operator $I_-^\alpha$, $\alpha>0$, defined in (3.10). We have shown that

$$\widetilde{\Lambda}^\kappa \subset \Lambda^\kappa,\ -1/2<\kappa<1/2,\ \kappa\neq 0, \quad \text{and} \quad |\Lambda|^\kappa \subset \Lambda^\kappa,\ 0<\kappa<1/2,$$

where all the inclusions are strict. In fact, for $0<\kappa<1/2$, one has (see (4.7)),

$$L^1(\mathbb{R})\cap L^2(\mathbb{R}) \subset L^{2/2\kappa+1}(\mathbb{R}) \subset |\Lambda|^\kappa \subset \Lambda^\kappa.$$

Each of the inner product spaces $\widetilde{\Lambda}^\kappa$, $\Lambda^\kappa$ and $|\Lambda|^\kappa$ is isometric to a linear subspace of the Hilbert space $\overline{Sp}(B^\kappa)$, defined by (1.5), with the usual $L^2(\Omega)$ inner product.

[1] For convenience to the reader, we also display the spaces and the inner products in the commonly used parametrization $H=\kappa+1/2$:

$$\widetilde{\Lambda}_H = \left\{ f : f\in L^2(\mathbb{R}),\ \int_{\mathbb{R}}|\widehat{f}(x)|^2|x|^{-2H+1}dx<\infty\right\},\ \text{for } 0<H<1,$$
$$\Lambda_H = \left\{ f : \int_{\mathbb{R}}\Big[(I_-^{H-\frac12}f)(s)\Big]^2ds<\infty\right\},\ \text{for } 1/2<H<1,$$
$$\Lambda_H = \left\{ f : \exists\,\phi_f\in L^2(\mathbb{R}) \text{ such that } f=I_-^{\frac12-H}\phi_f\right\},\ \text{for } 0<H<1/2,$$
$$|\Lambda|_H = \left\{ f : \int_{\mathbb{R}}\int_{\mathbb{R}}|f(u)||f(v)||u-v|^{2H-2}dudv<\infty\right\},\ \text{for } 1/2<H<1,$$

and, respectively,

$$(f,g)_{\widetilde{\Lambda}_H} = \frac{1}{C_2(H)^2}\int_{\mathbb{R}}\widehat{f}(x)\overline{\widehat{g}(x)}|x|^{-2H+1}dx,$$
$$\Big(f,g\Big)_{\Lambda_H} = \frac{\Gamma(H+1/2)^2}{C_1(H)^2}\int_{\mathbb{R}}(I_-^{H-\frac12}f)(s)(I_-^{H-\frac12}g)(s)\,ds,$$
$$\Big(f,g\Big)_{\Lambda_H} = \frac{\Gamma(H+1/2)^2}{C_1(H)^2}\int_{\mathbb{R}}\phi_f(s)\phi_g(s)ds,$$
$$\Big(f,g\Big)_{|\Lambda|_H} = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}}f(u)g(v)|u-v|^{2H-2}dudv,$$

where $C_1(H)=c_1(H-1/2)$ and $C_2(H)=c_2(H-1/2)$.

The isometry map $\mathscr{I}^\kappa$ is an extension of the natural definition of the integral with respect to fBm

$$\int_{\mathbb{R}} f(u)dB^\kappa(u) = \sum_{k=1}^{n} f_k \left(B^\kappa(u_{k+1}) - B^\kappa(u_k)\right), \tag{7.1}$$

where $f$ is an elementary (step) function $f(u) = \sum_{k=1}^n f_k 1_{[u_k,u_{k+1})}(u)$, $u \in \mathbb{R}$, and, therefore, it is still denoted by

$$\mathscr{I}^\kappa(f) = \int_{\mathbb{R}} f(u)dB^\kappa(u) \tag{7.2}$$

and called the integral on the real line of a function $f$ with respect to fBm $B^\kappa$. The inner product spaces $\widetilde{\Lambda}^\kappa$, $\Lambda^\kappa$ and $|\Lambda|^\kappa$ can thus also be viewed as classes of integrands. The completeness of an inner product space (a class of integrands) is a desirable property because the space is then (and only then) isometric to the space $\overline{Sp}(B^\kappa)$ itself and hence every element of the space $\overline{Sp}(B^\kappa)$ can, in this case, be expressed as an integral of a function with respect to fBm $B^\kappa$. We have shown that the inner product space $\widetilde{\Lambda}^\kappa$, when either $-1/2 < \kappa < 0$ or $0 < \kappa < 1/2$, and the spaces $|\Lambda|^\kappa$, $\Lambda^\kappa$, when $0 < \kappa < 1/2$, are not complete, whereas the space $\Lambda^\kappa$, when $-1/2 < \kappa < 0$, is complete. We do not know whether there is an inner product space of functions isometric to the space $\overline{Sp}(B^\kappa)$ itself when $0 < \kappa < 1/2$, where the isometry extends the natural definition (7.1) of the integral with respect to fBm for elementary functions.

We compared the classes of *integrands* $\widetilde{\Lambda}^\kappa$, $\Lambda^\kappa$ and $|\Lambda|^\kappa$ to the reproducing kernel Hilbert space $\mathbb{H}(\Gamma^\kappa)$ of fBm, which can be regarded as representing the space of *integrals*. We obtained an explicit characterization of the isometries between these classes of integrands and either $\mathbb{H}(\Gamma^\kappa)$ or subspaces of $\mathbb{H}(\Gamma^\kappa)$.

The space $\widetilde{\Lambda}^\kappa$ is referred to as the class of integrands in the "spectral domain" because it is constructed by using the spectral representation (3.1) of fBm $B^\kappa$ and involves a condition on its Fourier transform $\widehat{f}$. On the other hand, the space $\Lambda^\kappa$ is called the class of integrands in the "time domain" because it is constructed by using the time domain representation (3.7) of fBm. The integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$, defined in the "spectral domain", is equal almost surely to the integral $\int_{\mathbb{R}} f(u)dB^\kappa(u)$, defined in the "time domain". The space $|\Lambda|^\kappa$ is a practical, alternative class of integrands in the "time domain" when $0 < \kappa < 1/2$.

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