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Bohdan Maslowski · Jan Seidler

Probabilistic approach to the strong Feller property*

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Abstract. A new probabilistic method, based on the Girsanov theorem, for establishing the strong Feller property of diffusion processes in both finite and infinite dimensional spaces is proposed. Applications to second order stochastic differential equations, stochastic delay equations and stochastic partial differential equations of parabolic type are discussed, with a twofold aim: both to extend some older results, usually by weakening the assumptions on the drift term, and to obtain simpler proofs of them.

0. Introduction

Consider a homogeneous Markov process in a state space S with a strong Feller irreducible transition probability P. In many cases, a rather complete description of its long time behaviour is available by relating in an almost one-to-one way recurrence properties of the process to existence and uniqueness of a (σ-finite) invariant measure. In particular, if an invariant probability measure ν exists, then the process is recurrent and the measures $P_t(x,\cdot)$ converge to ν in the total variation norm as $t \to \infty$ for all starting points $x \in S$. For locally compact spaces S, these results were obtained in late fifties by G. Maruyama and H. Tanaka [31] and R. Z. Khas'minskiĭ [27], for recent extensions to a wider class of state spaces (including separable Banach spaces) see e.g. [41], [42] and references therein.

Let us recall that the transition probability P (or the corresponding transition semigroup (P_t)) is strong Feller if $P_t\varphi$ is a continuous function on S for each $t > 0$ and every bounded Borel function φ on S, and irreducible if $P_t\mathbf{1}_U > 0$ on S for each $t > 0$ and all open sets $U \neq \emptyset$. Of these two properties, the former is usually more difficult to verify. A Markov process defined by a stochastic differential equation in $\mathbb{R}^n$ with sufficiently regular coefficients such that the diffusion matrix is uniformly positive definite possesses a transition probability having a density which is a fundamental solution of the backward Kolmogorov equation, therefore, the strong Feller property and irreducibility follow by the properties of fundamental solutions (see e.g. [18], Theorem 6.5.4). A more refined result in this direction,

B. Maslowski, J. Seidler: Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic. e-mail: {maslow; seidler}@math.cas.cz

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also based on PDE techniques, may be found in [44], Theorem 7.2.4. In the degenerate case, K. Ichihara and H. Kunita found a characterization of hypoelliptic differential operators $L = \frac{1}{2}\sum_{i=1}^{r} X_i^2 + Y$, where $X_1, \dots, X_r, Y$ are $\mathscr{C}^\infty$-vector fields on a d-dimensional $\mathscr{C}^\infty$-manifold M, $d \geq r$, that generate a strong Feller process (see [23], Lemma 5.1). Recently, in the paper [39] the strong Feller property was established for diffusions in $\mathbb{R}^n$ with drifts merely integrable using Dirichlet forms.

If we turn to solutions to stochastic partial differential equations (SPDE's, for brevity) which are Markov processes in infinite dimensional spaces the above mentioned tools cease to be easily applicable. Specially, the theory of Kolmogorov equations in infinitely many variables has been developed only recently, more or less parallelly to the study of the strong Feller property.

The first proofs of the strong Feller property for SPDE's were based either on finite dimensional approximations (e.g. in [32]) or on smoothing properties of the infinite dimensional Kolmogorov equations ([34], [10]). Another approach, based on the Bismut-Elworthy formula for directional derivatives of the transition semigroup, was initiated in the paper [11], extended to equations with a multiplicative noise and Lipschitz continuous nonlinear terms in [38], and subsequently applied to various particular models, as stochastic reaction-diffusion equations with a polynomial drift and stochastic Burgers and Navier-Stokes equations, see Chapters 7 and 14 of the monograph [14] for discussion and references, or the more recent papers [17], [6], [7], [8], [22]. Tools from the Malliavin calculus were employed in [19], [20].

All these proofs are rather complicated from the technical point of view and, as in the finite dimensional case, they depend more on analytical than on probabilistic methods. It was our intention to find a different argument, more straightforward and of a probabilistic nature, yielding the strong Feller property of solution of some classes of SPDE's. The procedure we propose is based on two ingredients: First, we characterize the strong Feller property of a transition probability P in terms of equicontinuity of measures $P_t(x, \cdot)$, this is done in Section 1 below. Second, we show that, roughly speaking, an application of the Girsanov theorem to a strongly Feller equation leads to an equation defining again a strong Feller process. Precise formulations and an easy proof may be found at the beginning of Section 2. It should be mentioned here that this paper having been essentially completed we learned that a similar measure-theoretic description of the strong Feller property was used in a different context by Ł. Stettner in [15]. The Girsanov theorem was employed to prove the (weak) Feller property of weak solutions to stochastic differential equations in $\mathbb{R}^d$ in a related but different manner in the paper [43].

The rest of Section 2 is devoted to illustrative examples. The emphasis is laid upon the argument, not on reaching maximal possible generality. There are two types of applications of the proposed method. Either one starts with a linear equation, whose solutions are Gaussian processes (and hence necessary and sufficient conditions for the strong Feller property are available) and obtains a self-contained proof of the strong Feller property of a solution to a semilinear stochastic differential equation with an additive noise. Or one starts with an equation that can be treated with some of the analytical methods and uses the probabilistic procedure to relax assumptions on the drift or to simplify considerably earlier proofs. Moreover, the Girsanov theorem based procedure often yields irreducibility as an immediate consequence of the proof of the strong Feller property.

Although we have aimed primarily at applications to SPDE's, it turns out that new results may be obtained even in the finite dimensional case. In Example 2.1 we consider a second order stochastic differential equation written symbolically as

$$\ddot{x} + F(x, \dot{x}) = \Sigma \dot{w} \tag{0.1}$$

and show that its solution is a strong Feller process if Σ is an invertible matrix and F is a bounded continuous function. Let us note that the Kolmogorov equation corresponding to (0.1) strongly degenerates, so the theory of parabolic partial differential equations cannot be applied directly.

A stochastic delay equation

$$\mathrm{d}x = \left(\int_{-r}^{0} x(t+s)\, \mathrm{d}\eta(s) + F(x(t)) \right) \mathrm{d}t + \Sigma\, \mathrm{d}w$$

is dealt with in Example 2.2. We provide sufficient conditions for the process x to be strong Feller and irreducible for $t > r$, answering in this way a question posed by G. Da Prato and J. Zabczyk (see [14], §10.3).

A stochastic parabolic equation

$$\mathrm{d}X = \big(AX + f(X)\big)\, \mathrm{d}t + \sigma(X) Q^{1/2}\, \mathrm{d}W \tag{0.2}$$

with a bounded continuous drift f in a Hilbert space is treated in Examples 2.3, 2.4 in the cases $\sigma = I$ and σ boundedly invertible, $Q = I$, respectively. We relate here our results to those obtained in [10], [19], and [38]. Finally, a one-dimensional heat equation with a nonlinear nonhomogeneous white noise boundary condition is discussed in Example 2.5. Notwithstanding the simplicity of the considered problem, the result does not seem to be provable by the available analytic methods.

In all these examples, the use of Girsanov's theorem may be justified without difficulties due to the boundedness (or the linear growth) of the drift. If the drift is not bounded, some approximation procedures may be invoked, and we give two different examples in this spirit. First, using Lyapunov functions techniques, we extend results concerning the equation (0.1) to drifts of the form $F(x, \dot{x}) = b(x, \dot{x}) + \nabla G(x)$, where b is a locally Lipschitz function obeying some one-sided growth conditions and the potential G is bounded from below. Equations of this type and their long time behaviour have been investigated recently e.g. in [30], [2], [1]. Especially, in the paper [1], §2, it was noted that the strong Feller property may be established using the results from [23] if $F \in \mathscr{C}^\infty(\mathbb{R}^{2n})$. Our method does not require any smoothness of the drift.

Second, in Section 3 we return to the problem (0.2) assuming that f is defined and continuous only on some Banach subspace of the state space, but satisfies a suitable dissipativity hypothesis.

In the last Section 4 we sketch an alternative approach to the result proven by M. Fuhrman in [20]. He considered semilinear stochastic equations with an additive noise in a Hilbert space, whose drifts belong to a certain special class, introduced in the finite dimensional case by G. Jona-Lasinio and R. Sénéor [24]. Drifts in this class, nevertheless, may be of considerable interest as they are subject only to weak growth restrictions and they need not be dissipative. We will reprove the main result of [20] on the strong Feller property replacing a difficult step using the Malliavin calculus by a straightforward application of our Theorem 2.1 and showing that the most stringent assumption of [20] may be omitted.

We close this section with introducing some notation to be used in what follows. By $\mathbf{1}_\Lambda$ we denote the indicator of a set Λ, by $\mathbb{M}_{m\times n}$ the space of all $m\times n$ matrices with real entries. If X, Y are metric spaces, $\mathscr{C}(X;Y)$ stands for the space of all continuous mapping from X to Y, $\mathscr{C}(X)=\mathscr{C}(X;\mathbb{R})$, and $\mathscr{C}_b(X)$ for the subspace of bounded functions. If X, Y are Banach spaces, then $\mathscr{C}^k(X)$ denotes the space of all functions having continuous Fréchet derivatives up to order k, $\mathscr{L}(X,Y)$ the space of all bounded linear operators from X to Y, $\mathscr{L}(X)=\mathscr{L}(X,X)$; $I\in\mathscr{L}(X)$ is the identity operator. If (Ω,μ) is a measure space, then $L^p(\Omega;X)$ denotes the space of all Bochner p-integrable functions form Ω to X, and $\xrightarrow{\mu}$ denotes the convergence in measure μ. Finally, if X and Y are Hilbert spaces, then $\|B\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of an operator $B\in\mathscr{L}(X,Y)$.

1. Preliminaries on convergence of measures

Let E be a Polish space, $\mathscr{B}$ the σ-algebra of Borel sets in E. Denote by τ_s the topology of pointwise convergence in the space of finite signed measures on $\mathscr{B}$; that is, a net $\{\mu_\gamma\}_{\gamma\in\Gamma}$ converges to μ in τ_s if and only if $\lim_{\gamma\in\Gamma}\mu_\gamma(A)=\mu(A)$ for every $A\in\mathscr{B}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of Borel probability measures on E, then the set $\{\mu_n;\ n\in\mathbb{N}\}$ is conditionally sequentially compact for τ_s iff it is equicontinuous, i.e.

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\mu_n(A_k)=0\quad\text{for all }\{A_k\}\subseteq\mathscr{B},\ A_k\downarrow\emptyset;\tag{1.1}$$

see [21], Theorem 2.6, cf. also [16], Theorems IV.9.1, IV.9.5. Therefore, Lemma 3.15 in [21] yields that $\mu_n\longrightarrow\mu$ in τ_s provided (1.1) holds and $\mu_n\longrightarrow\mu$ weakly in the probabilistic sense, that is

$$\lim_{n\to\infty}\int_E f\,\mathrm{d}\mu_n=\int_E f\,\mathrm{d}\mu\quad\text{for every } f\in\mathscr{C}_b(E).\tag{1.2}$$

Further, consider a Markov kernel $P=P(x,\cdot)$ on $(E,\mathscr{B})$. P is called strong Feller if $P(\cdot,A)\in\mathscr{C}_b(E)$ for any $A\in\mathscr{B}$. As a real valued function on E is continuous iff it is sequentially continuous, the strong Feller property is equivalent to the assertion that $P(x_n,A)\longrightarrow P(x_0,A)$ whenever $A\in\mathscr{B}$ and $x_n,x_0\in E$ are such that $x_n\to x_0$ in E. In other words, P is strong Feller iff $P(x_n,\cdot)\longrightarrow P(x_0,\cdot)$ in τ_s for all $x_n,x_0\in E$ such that $x_n\to x_0$.

Hence we have arrived at the following result:

Lemma 1.1. *Let E be a Polish space, $\mathscr{B}$ the σ-algebra of its Borel sets and P a Markov kernel on $(E,\mathscr{B})$. Then P is strong Feller if and only if P is Feller and the measures $\{P(x_n,\cdot);\ n\in\mathbb{N}\}$ are equicontinuous for any convergent sequence $\{x_n\}$ in E.*

Proof. It remains to note that P is Feller (that is, maps the space $\mathscr{C}_b(E)$ into itself) iff (1.2) holds for the measures $\mu_n = P(x_n, \cdot)$, $\mu = P(x_0, \cdot)$, whenever $\{x_n\}$ is a convergent sequence in E, $x_n \to x_0$. Q.E.D.

2. The strong Feller property for stochastic evolution equations

Let H be a separable Hilbert space and $A : \mathrm{Dom}(A) \longrightarrow H$ an infinitesimal generator of a strongly continuous semigroup on H, let W denote a standard cylindrical Wiener process in a real separable Hilbert space Υ. We shall denote the norm and the inner product in both H and Υ by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. Let us consider equations

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + \sigma(X)Q^{1/2}\,\mathrm{d}W, \tag{2.1}$$

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \sigma(Z)Q^{1/2}\,\mathrm{d}W, \tag{2.2}$$

where $Q \in \mathscr{L}(\Upsilon)$ is nonnegative and self-adjoint (but not necessarily nuclear), $f : H \longrightarrow H$ and $\sigma Q^{1/2} : H \longrightarrow \mathscr{L}(\Upsilon, H)$ are Borel mappings such that

(A) *1) There exists a probability space* $(\Omega, \mathscr{F}, \mathsf{P})$ *carrying a standard cylindrical Wiener process W and, for any $y \in H$, a mild solution Z^y to* (2.2) *satisfying* $Z^y(0) = y$.
2) For any $y \in H$ there exists a martingale solution $\big((\Theta_y, \mathscr{G}^y, p^y), (\mathscr{G}^y_t), W^y, (X^y(t))\big)$ *of* (2.1) *with* $X^y(0) = y$.
3) Uniqueness in law holds for both (2.1) *and* (2.2).

(Martingale solutions are defined as in [12], Chapter 8.) The assumption (A) implies that (2.1), (2.2) define Markov processes; let us denote by $P = P(t, y, \cdot)$, $R(t, y, \cdot)$ the transition probabilities corresponding to the equations (2.1), (2.2), respectively, that is

$$P(t, y, A) = \int_{\Theta_y} 1_A(X^y(t))\;\mathrm{d}p^y,$$
$$R(t, y, A) = \mathsf{E}1_A(Z^y(t)), \quad t \geq 0, \quad y \in H, \quad A \in \mathscr{B},$$

$\mathscr{B}$ denoting the σ-algebra of Borel sets in H.

We aim at proving the following

Theorem 2.1. *Suppose that* (A) *holds and let there exist a Borel function* $u : H \longrightarrow \Upsilon$ *satisfying* $f(\cdot) = \sigma(\cdot)Q^{1/2}u(\cdot)$. *Assume that*

(i) the transition probability R defined by (2.2) *is strong Feller,*

(ii) for all $t \geq 0$ and $y \in H$ we have $\mathsf{E}U(y, t) = 1$, *where*

$$U(y, t) = \exp\left(\int_0^t \big\langle u(Z^y(s)), \cdot\big\rangle\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \big|u(Z^y(s))\big|^2\,\mathrm{d}s\right),$$

and either

(iii) the set $\{U(y_n, t);\ n \in \mathbb{N}\}$ is uniformly integrable for any fixed $t \geq 0$ and any convergent sequence $\{y_n\}$ in H,

(iv) P is Feller,

or

(v) we have

$$U(y_n, t) \xrightarrow[n\to\infty]{\mathsf{P}} U(y, t) \tag{2.3}$$

and

$$\left|Z^{y_n}(t) - Z^{y}(t)\right| \xrightarrow[n\to\infty]{\mathsf{P}} 0 \tag{2.4}$$

for any convergent sequence $\{y_n\}$ in H, $y_n \longrightarrow y$, and for all $t > 0$.

Then P is strong Feller as well.

Remark 2.1. Note that if (2.3) and (ii) hold then due to the nonnegativity of U the hypothesis (iii) follows, in fact, we have $U(y_n, t) \longrightarrow U(y, t)$ in $L^1(\mathsf{P})$ (see e.g. [35], Theorem II.21). Hence the relevant information is that (ii) and (v) imply also (iv). The assumption (iv), the Feller property of P, can often be easily checked if the nonlinearities f, σ are (locally) Lipschitz continuous and of linear growth, cf. e.g. [12], §9.2.1. (In the case $\dim H < \infty$, another proof requiring only continuity and boundedness of f and σ is proposed in [44], Corollary 6.3.3.) In general, it may be helpful to know that we can obtain the Feller property of P using a procedure based on the Girsanov theorem.

Remark 2.2. The assumption (2.3) may be often verified in a straightforward way. For example, assume that the function

$$H \longrightarrow L^2([0,T] \times \Omega; \Upsilon), \quad y \longmapsto u(Z^y(\cdot))$$

is continuous for every $T > 0$. (Sufficient conditions for that can be easily given in terms of σ, f and Q.) Let $y, y_n \in H$ be such that $y_n \to y$; set for brevity

$$\mathfrak{u}(v) = \int_0^t \left\langle u(Z^v(s)), \cdot\right\rangle \mathrm{d}W(s) - \frac{1}{2}\int_0^t \left|u(Z^v(s))\right|^2 \mathrm{d}s.$$

Obviously, $\mathfrak{u}(y_n) \longrightarrow \mathfrak{u}(y)$ in probability as $n \to \infty$, hence also

$$U(y_n, t) = \exp \mathfrak{u}(y_n) \xrightarrow[n\to\infty]{\mathsf{P}} \exp \mathfrak{u}(y) = U(y, t).$$

Remark 2.3. Fix a $t > 0$ and a subset $M \subseteq H$. Tracing the proof of the Novikov condition as it is presented e.g. in [29], Theorem IV.3.5(a) or in [26], Theorem 1.5, it is possible to check easily that if

$$\sup_{y\in M} \mathsf{E} \exp\left(\left(\frac{1}{2} + \varepsilon\right)\int_0^t \left|u(Z_s^y)\right|^2 \mathrm{d}s\right) < \infty$$

for some $\varepsilon > 0$ then there exists a $p > 1$ such that

$$\sup_{y\in M} \mathsf{E} U^p(y, t) < \infty. \tag{2.5}$$

It is well known that (2.5) implies uniform integrability of the set $\{U(y,t);\ y \in M\}$. (We are indebted to M. Röckner for this remark.)

Remark 2.4. Let us note that, in fact, the assumption (A2) is superfluous since under (ii) the existence of martingale solutions of (2.1) follows from the Girsanov theorem.

Proof. First, assume that (i)–(iv) are satisfied. Let us fix $t > 0$ and a convergent sequence $\{y_n\}$ in H arbitrarily; we have to prove the equicontinuity of the measures $\{P(t, y_n, \cdot);\ n \in \mathbb{N}\}$. Take an arbitrary sequence $A_k \in \mathscr{B}$ such that $A_k \downarrow \emptyset$. By the assumption (i) we know that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} R(t, y_n, A_k) = \lim_{k\to\infty} \sup_{n\in\mathbb{N}} \mathsf{E}\mathbf{1}_{A_k}(Z^{y_n}(t)) = 0. \tag{2.6}$$

Set for simplicity $U(y_n) = U(y_n, t)$ and define a probability $\widetilde{\mathsf{P}}_n$ on Ω by $\mathrm{d}\widetilde{\mathsf{P}}_n = U(y_n)\,\mathrm{d}\mathsf{P}$. By the Girsanov theorem the process Z^{y_n} considered on the probability space $(\Omega, \mathscr{F}, \widetilde{\mathsf{P}}_n)$ solves the equation (2.1), therefore

$$P(t, y_n, A_k) = \widetilde{\mathsf{P}}_n\{Z^{y_n}(t) \in A_k\}$$

(see e.g. the proof of Theorem 10.18 in [12]). It follows

$$\begin{aligned}
\sup_{n\in\mathbb{N}} P(t, y_n, A_k) &= \sup_{n\in\mathbb{N}} \mathsf{E}\Big(U(y_n)\mathbf{1}_{A_k}(Z^{y_n}(t))\Big)\\
&= \sup_{n\in\mathbb{N}}\Big\{\mathsf{E}\Big(\mathbf{1}_{\{U(y_n)\le K\}}U(y_n)\mathbf{1}_{A_k}(Z^{y_n}(t))\Big)\\
&\qquad + \mathsf{E}\Big(\mathbf{1}_{\{U(y_n)>K\}}U(y_n)\mathbf{1}_{A_k}(Z^{y_n}(t))\Big)\Big\}\\
&\le K \sup_{n\in\mathbb{N}} \mathsf{E}\mathbf{1}_{A_k}(Z^{y_n}(t)) + \sup_{n\in\mathbb{N}} \mathsf{E}\Big(\mathbf{1}_{\{U(y_n)>K\}}U(y_n)\Big).
\end{aligned}$$

Fix an arbitrary $\varepsilon > 0$, by the uniform integrability we can choose $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathsf{E}\Big(\mathbf{1}_{\{U(y_n)>K\}}U(y_n)\Big) < \frac{\varepsilon}{2}.$$

According to (2.6),

$$\sup_{n\in\mathbb{N}} \mathsf{E}\mathbf{1}_{A_k}(Z^{y_n}(t)) < \frac{\varepsilon}{2K}$$

for all $k \in \mathbb{N}$ sufficiently large, which completes the first part of the proof.

Now assume (ii) and (v), it remains to prove that P is Feller. Denote by $\widetilde{\mathsf{P}}$ the probability with the density $U(y)$, $\mathrm{d}\widetilde{\mathsf{P}} = U(y)\,\mathrm{d}\mathsf{P}$, and choose $\varphi \in \mathscr{C}_b(H)$ arbitrarily. We aim at showing

$$\lim_{n\to\infty} \int_H \varphi(z)P(t, y_n, \mathrm{d}z) = \int_H \varphi(z)P(t, y, \mathrm{d}z).$$

By (2.4) we have $\varphi(Z^{y_n}(t)) \longrightarrow \varphi(Z^y(t))$ in probability, hence in $L^1(\mathsf{P})$. It follows

$$\begin{aligned}
&\left|\int_H \varphi(z)P(t, y_n, \mathrm{d}z) - \int_H \varphi(z)P(t, y, \mathrm{d}z)\right|\\
&\quad = \left|\widetilde{\mathsf{E}}_n\varphi(Z^{y_n}(t)) - \widetilde{\mathsf{E}}\varphi(Z^y(t))\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \mathsf{E}\Big|\{U(y_n) - U(y)\}\varphi(Z^{y_n}(t))\Big| + \mathsf{E}U(y)\Big|\varphi(Z^{y_n}(t)) - \varphi(Z^{y}(t))\Big| \\
&\le \sup_H |\varphi|\,\mathsf{E}\Big|U(y_n) - U(y)\Big| + K\mathsf{E}\Big|\varphi(Z^{y_n}(t)) - \varphi(Z^{y}(t))\Big| \\
&\qquad + 2\sup_H |\varphi|\,\mathsf{E}\Big(\mathbf{1}_{\{U(y)>K\}}U(y)\Big).
\end{aligned}
$$

Thus first fixing a K sufficiently large and then using (2.3), (2.4) we obtain the desired conclusion. Q.E.D.

Example 2.1. Stochastic nonlinear oscillators. Let us consider a second order differential equation in $\mathbb{R}^n$ perturbed with an additive noise, written symbolically as

$$\ddot{x} + F(x, \dot{x}) = \Sigma\dot{w}, \tag{2.7}$$

where $F \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, $\Sigma \in \mathbb{M}_{n\times n}$ is a nonsingular matrix, and $\dot{w}$ denotes the (distributional) derivative of an n-dimensional standard Wiener process w. We rewrite (2.7) as a first order system

$$\mathrm{d}\mathfrak{x} = \Big\{A\mathfrak{x} + f(\mathfrak{x})\Big\}\,\mathrm{d}t + \sigma\,\mathrm{d}w, \tag{2.8}$$

setting

$$
\begin{aligned}
&A \in \mathbb{M}_{2n\times 2n}, \qquad A = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \\
&f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \qquad f(X) = \begin{pmatrix} 0 \\ -F(x, v) \end{pmatrix}, \qquad X = (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \\
&\sigma \in \mathbb{M}_{2n\times n}, \qquad \sigma = \begin{pmatrix} 0 \\ \Sigma \end{pmatrix}.
\end{aligned}
$$

That is, componentwise (2.8) reads as

$$
\begin{aligned}
\mathrm{d}x &= v\,\mathrm{d}t, \\
\mathrm{d}v &= -F(x, v)\,\mathrm{d}t + \Sigma\,\mathrm{d}w.
\end{aligned}
$$

First, let us consider the linear problem corresponding to (2.8), namely

$$\mathrm{d}\mathfrak{z} = A\mathfrak{z} + \sigma\,\mathrm{d}w. \tag{2.9}$$

The solution to (2.9) is a Gaussian Markov process with a transition probability $Q(t, z, \cdot) = \mathscr{N}(e^{At}z, Q_t)$, $t \ge 0$, $z \in \mathbb{R}^{2n}$, where

$$Q_t = \int_0^t e^{A(t-s)}\sigma\sigma^* e^{A^*(t-s)}\,\mathrm{d}s$$

and $\mathscr{N}(h, R)$ stands for the Gaussian measure on $\mathbb{R}^{2n}$ with mean h and covariance matrix R. The process $\mathfrak{z}$ is strong Feller irreducible, provided the matrix Q_t is nonsingular for any $t > 0$. According to the Kalman rank condition (see e.g. [45], Theorem 1.2), the matrix Q_t is nonsingular for an arbitrary $t > 0$ iff

$$\operatorname{rank} \underbrace{\Big[\sigma, A\sigma, \cdots, A^{2n-1}\sigma\Big]}_{\in \mathbb{M}_{2n\times 2n^2}} = 2n. \tag{2.10}$$

An easy calculation shows that

$$\left[\sigma, A\sigma, \cdots, A^{2n-1}\sigma\right] = \begin{pmatrix} 0 & \Sigma & \cdots \\ \Sigma & 0 & \cdots \end{pmatrix},$$

hence (2.10) is a consequence of the invertibility of Σ.

Let us turn to the nonlinear equation (2.8) supposing that F is of a linear growth:

$$\exists K > 0 \quad \forall (x, v) \in \mathbb{R}^{2n} \quad |F(x, v)| \le K\big(1 + |x| + |v|\big). \tag{2.11}$$

Set $u = \Sigma^{-1}F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$; we are going to check the assumptions (ii) and (v) of Theorem 2.1. Let us denote by $\mathfrak{z}^{y_0}$ the solution to (2.9) satisfying $\mathfrak{z}^{y_0}(0) = y_0 \in \mathbb{R}^{2n}$, fix $T > 0$ and take an arbitrary sequence $\{y_n\}$ in R^{2n}, $y_n = (y_n^1, y_n^2)$, $y^n \longrightarrow y$. We claim that

$$\sup_{n\ge 0} \left|u(\mathfrak{z}^{y_n}(t))\right| \le c\Big(1 + \sup_{0\le t\le T} |w(t)|\Big), \quad 0 \le t \le T, \tag{2.12}$$

for a constant $c < \infty$. The estimate (2.12) implies (ii) by Corollary 3.5.16 in [25]. Further,

$$\mathfrak{z}^{y_n} \xrightarrow[n\to\infty]{} \mathfrak{z}^{y} \quad \text{in } L^2([0, T] \times \Omega; \mathbb{R}^{2n}),$$

and the continuity of F yields

$$F(\mathfrak{z}^{y_n}(\cdot)) \xrightarrow[n\to\infty]{} F(\mathfrak{z}^{y}(\cdot)) \quad \text{in measure on } [0, T] \times \Omega,$$

hence the function $y \longmapsto u(\mathfrak{z}^y(\cdot))$, $\mathbb{R}^{2n} \longrightarrow L^2([0, T] \times \Omega; \mathbb{R}^n)$ is continuous by (2.12) and the dominated convergence theorem. Consequently, there exists a weak solution to (2.8) for any nonrandom initial condition $\mathfrak{x}(0) = y \in \mathbb{R}^{2n}$. Assume that uniqueness in law holds for (2.8), then Theorem 2.1 together with the positivity of U imply that the Markov process solving (2.8) is strong Feller and irreducible. (If we suppose that F is even bounded, then weak uniqueness follows, see [25], Proposition 5.3.10.) It remains to prove (2.12), but using (2.11) we obtain

$$\begin{aligned}
\left|u(\mathfrak{z}^{y_n}(t))\right| &\le \|\Sigma^{-1}\| \left| F\left(y_n^1 + y_n^2 t + \int_0^t \Sigma w(s)\,\mathrm{d}s,\ y_n^2 + \Sigma w(t)\right)\right| \\
&\le c_1\left(1 + |y_n| + \|\Sigma\| \left\{ \int_0^t |w(s)|\,\mathrm{d}s + |w(t)|\right\}\right) \\
&\le c_2\big(1 + |y_n| + \sup_{0\le t\le T} |w(t)|\big) \\
&\le c_3\big(1 + \sup_{0\le t\le T} |w(t)|\big),
\end{aligned}$$

the last estimate holds since $\sup_{n\ge 1} |y_n| < \infty$.

Further, let us turn to equations whose drift F does not satisfy the linear growth condition (2.11). We content ourselves with equations of the type

$$\ddot{x} + b(x, \dot{x}) + \nabla G(x) = \Sigma \dot{w}, \tag{2.13}$$

under the hypothesis that $b : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a locally Lipschitz function and ∇G is the Fréchet derivative of a function $G \in \mathscr{C}^2(\mathbb{R}^n)$ such that $G(y) \geq G_0$ for a constant $G_0 \in \mathbb{R}$ and each $y \in \mathbb{R}^n$. We rewrite (2.13) in the form (2.8) as before. Following [30] we suppose that

$$\langle b(x, v), v\rangle \geq -\big(k_1 + k_2|x|^r + k_3|v|^2\big)$$

for some constants $k_1, k_2, k_3 \geq 0$, $r \in [0, 2[$ and all $(x, v) \in \mathbb{R}^{2n}$. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$ be an arbitrary fixed stochastic basis carrying an n-dimensional Wiener process w, then for every $z \in \mathbb{R}^{2n}$ there exists a unique solution $\mathfrak{x}^z = (x^z, v^z)$ to (2.13) satisfying $\mathfrak{x}^z(0) = z$ according to [30], Theorem 2.1. Set

$$\tau_m^z = \inf\{s \geq 0;\ |v^z(s)| \geq m\}, \quad m \in \mathbb{N}.$$

Let us fix a ball B in $\mathbb{R}^{2n}$ and $t > 0$ arbitrarily. The proof of Theorem 2.1 in [30] yields

$$\sup_{m\geq 1}\sup_{z\in B} \mathsf{E}|v^z(t \wedge \tau_m^z)|^2 < \infty,$$

which in turn implies

$$\lim_{m\to\infty}\sup_{z\in B} \mathsf{P}\{\tau_m^z \leq t\} = 0. \tag{2.14}$$

Let N be such that $|x_0| \leq N$ whenever $(x_0, v_0) \in B$, let us consider an equation

$$\mathrm{d}\mathfrak{x}_n = \big\{A\mathfrak{x}_n + f_n(\mathfrak{x}_n)\big\}\,\mathrm{d}t + \sigma\,\mathrm{d}w, \tag{2.15}$$

with a bounded Lipschitz continuous function $f_n : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ satisfying

$$f_n(x, v) = \begin{pmatrix} 0 \\ -b(x, v) - \nabla G(x) \end{pmatrix} \quad \text{for all } |v| \leq n,\ |x| \leq N + nt.$$

Denoting by $\mathfrak{x}_n^z$ the solution of (2.15) with $\mathfrak{x}_n^z(0) = z \in \mathbb{R}^{2n}$ (defined on the same stochastic basis as the process $\mathfrak{x}^z$) we obtain

$$\mathfrak{x}^z(t \wedge \tau_n^z) = \mathfrak{x}_n^z(t \wedge \tau_n^z) \quad \mathsf{P}\text{-almost surely} \tag{2.16}$$

for each $z \in B$ by a standard local uniqueness theorem. If P, P_n are the transition probabilities corresponding to (2.13), (2.15), respectively, then

$$\begin{aligned}\sup_{z\in B}\big|P_n(t, z, \Gamma) - P(t, z, \Gamma)\big| &= \sup_{z\in B}\big|\mathsf{P}\{\mathfrak{x}_n^z(t) \in \Gamma\} - \mathsf{P}\{\mathfrak{x}^z(t) \in \Gamma\}\big| \\ &\leq 2\sup_{z\in B}\mathsf{P}\{\tau_n^z \leq t\} \xrightarrow[n\to\infty]{} 0\end{aligned}$$

for any Borel set $\Gamma \subseteq \mathbb{R}^{2n}$ by (2.14) and (2.16). Since the ball B was arbitrary, we have

$$P_n(t, \cdot, \Gamma) \xrightarrow[n\to\infty]{} P(t, \cdot, \Gamma) \quad \text{locally uniformly on } \mathbb{R}^{2n}.$$

Therefore, $P(t, \cdot, \Gamma)$ is a continuous function for every Borel set Γ and $t > 0$, in other words, P is strongly Feller. We cannot claim, however, that P is irreducible, as we have only $P(t, z, \cdot) \ll Q(t, z, \cdot)$ for all $t > 0$, $z \in \mathbb{R}^{2n}$.

As a simple particular one-dimensional case of (2.13) satisfying the above assumptions we may consider an equation

$$\ddot{x} + \mu \dot{x}^2 \operatorname{sign} \dot{x} = \Sigma \dot{w}$$

with $\mu > 0$. (Compare [28], Example 22.2, where the physical relevance of the deterministic counterpart to this equation is discussed.)

Example 2.2. Stochastic delay equations. We shall be concerned with stochastic delay equations of the form

$$\mathrm{d}x(t) = \left(\int_{-r}^{0} x(t+s)\,\mathrm{d}\eta(s) + F(x(t)) \right) \mathrm{d}t + \Sigma\,\mathrm{d}w(t), \tag{2.17}$$

where w is a d-dimensional Brownian motion, $\Sigma \in \mathbb{M}_{d\times d}$ an invertible matrix, η is an $\mathbb{M}_{d\times d}$-valued Borel measure on $[-r, 0]$ and $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel function. We interpret the equation (2.17) as an infinite dimensional problem (see [9] or [14], Chapter 10)

$$\mathrm{d}X = \left\{ AX + f(X) \right\} \mathrm{d}t + \sigma\,\mathrm{d}w \tag{2.18}$$

in the Hilbert space $M^2 = \mathbb{R}^d \times L^2((-r, 0); \mathbb{R}^d)$, setting

$$X(t) = \begin{pmatrix} x(t) \\ x_t(\cdot) \end{pmatrix}, \quad f\left(\begin{pmatrix} \varkappa \\ \lambda \end{pmatrix} \right) = \begin{pmatrix} F(\varkappa) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \varkappa \\ \lambda \end{pmatrix} \in M^2,$$

$$\sigma\varkappa = \begin{pmatrix} \Sigma\varkappa \\ 0 \end{pmatrix}, \quad \varkappa \in \mathbb{R}^d,$$

$$\mathrm{Dom}(A) = \left\{ \begin{pmatrix} \varphi(0) \\ \varphi \end{pmatrix};\ \varphi \in W^{1,2}((-r, 0); \mathbb{R}^d) \right\},$$

$$A \begin{pmatrix} \varphi(0) \\ \varphi \end{pmatrix} = \begin{pmatrix} \int_{-r}^{0} \varphi(s)\,\mathrm{d}\eta(s) \\ \frac{\mathrm{d}\varphi}{\mathrm{d}s}(\cdot) \end{pmatrix}$$

defining, as usual, the function $x_t : [-r, 0] \longrightarrow \mathbb{R}^d$ by $x_t(\cdot) = x(t + \cdot)$.
First, we have to consider the corresponding linear problem

$$\mathrm{d}z(t) = \left(\int_{-r}^{0} z(t+s)\,\mathrm{d}\eta(s) \right) \mathrm{d}t + \Sigma\,\mathrm{d}w(t),$$

that is

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \sigma\,\mathrm{d}w. \tag{2.19}$$

We assume that

(2.20) *the operator A generates a C_0-semigroup on M^2,*

(2.21) *the Markov process Z defined by* (2.19) *is strong Feller for all $t > r$.*

For example, it is known (see e.g. [3], Proposition 2.1) that (2.20) holds true if the measure η is of the form

$$\eta = \sum_{i=0}^{N} a_i \delta_{s_i} + b\,\mathrm{d}s, \quad a_i \in \mathbb{M}_{d\times d},\ b \in L^{\infty}((-r,0);\mathbb{M}_{d\times d}),$$
$$0 = s_0 > \cdots > s_N = -r.$$

If, moreover, $b = 0$ then (2.21) is fulfilled as well and the process Z is also irreducible for $t > r$ ([14], Theorem 10.2.3).

Now, suppose that $F \in \mathscr{C}(\mathbb{R}^d;\mathbb{R}^d)$ is a continuous function of a linear growth, i.e., there exists a constant $K < \infty$ such that $|F(\varkappa)| \le K(1+|\varkappa|)$ for every $\varkappa \in \mathbb{R}^d$. Set

$$u : M^2 \longrightarrow \mathbb{R}^d, \quad \begin{pmatrix} \varkappa \\ \lambda \end{pmatrix} \longmapsto \Sigma^{-1} F(\varkappa),$$

then u is also a continuous function of a linear growth and $f = \sigma u$. Let $T > 0$ and $y \in M^2$ be arbitrary but fixed. According to [12], Proposition 10.17, for the assumption (ii) of Theorem 2.1 to hold it suffices to find $\delta > 0$ such that

$$\sup_{t\in[0,T]} \mathsf{E} \exp\left(\delta |u(Z^y(t))|^2\right) < \infty, \tag{2.22}$$

where, as above, we denote by Z^y the solution of (2.19) with $Z^y(0) = y$. However, the random variable $Z^y(t)$ is Gaussian and

$$\sup_{t\in[0,T]} \mathsf{E} \exp\left(\varrho |Z^y(t)|^2\right) < \infty$$

for all $\varrho > 0$ sufficiently small, hence (2.22) easily follows. Obviously, $Z^{y_n} \longrightarrow Z^y$ in $L^2([0,T]\times\Omega; M^2)$ whenever $y_n \in M^2$, $y_n \to y$, so the assumption (v) can be checked using Remark 2.2, continuity and linear growth of u. Therefore, assuming that uniqueness in law holds for (2.18) we see that the Markov process X defined by the equation (2.18) is strong Feller (and irreducible provided Z is) for all $t > r$. This settles in the affirmative a conjecture posed by G. Da Prato and J. Zabczyk in [14], §10.3.

Let us note that the same proof applies to a more general equation

$$\mathrm{d}x(t) = \left(\int_{-r}^{0} x(t+s)\,\mathrm{d}\eta(s) + F(x(t), x_t) \right) \mathrm{d}t + \Sigma\,\mathrm{d}w(t),$$

provided F is continuous and of a linear growth as a function from M^2 to $\mathbb{R}^d$.

Remark 2.5. M. Scheutzow [40] studied the long-time behaviour of the Markov process solving the equation

$$\mathrm{d}x(t) = f(x_t)\,\mathrm{d}t + \mathrm{d}w(t), \tag{2.23}$$

assuming only that weak uniqueness holds for (2.23) and $f : \mathscr{C}([-r,0];\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ is Borel and locally bounded. Constructing directly an embedded Markov chain he established results on the ergodic behaviour of (2.23) which are standard consequences of the strong Feller property and irreducibility. In [40], solutions of (2.23) are considered as $\mathscr{C}([-r,0];\mathbb{R}^d)$-valued processes; in this space they need not be even Feller (see [40], Remark 5 in Section 3). Nevertheless, it may be of some interest to know whether the approach adopted in our paper can be extended to equations of the type (2.23) under assumptions similar to those in [40].

Example 2.3. An SPDE with an additive noise. Let us consider the equation (2.1) taking $H = \Upsilon$ and $\sigma = I$, i.e.

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W \tag{2.24}$$

and with a Borel function $f : H \longrightarrow H$ such that

$$\operatorname{Rng} f \subseteq \operatorname{Rng} Q^{1/2}, \quad Q^{-1/2} f \in \mathscr{C}(H; H) \text{ and with a linear growth}, \tag{2.25}$$

where $Q^{-1/2}$ denotes the pseudo-inverse to $Q^{1/2}$. (The most important particular case is, of course, $Q = I$, when we suppose simply that $f : H \longrightarrow H$ is a continuous function with a linear growth.) Let us denote by e^{At} the semigroup generated by A on H. We assume that

$$\int_0^t \|e^{As}Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s < \infty \quad \text{for all } t > 0 \tag{2.26}$$

(then the stochastic integral in the formula for mild solutions of (2.24) is well defined) and

$$\operatorname{Rng} e^{At} \subseteq \operatorname{Rng}\left(\int_0^t e^{As}Qe^{A^*s}\,\mathrm{d}s\right)^{1/2} \equiv \operatorname{Rng}(Q_t^{1/2}) \quad \text{for all } t > 0. \tag{2.27}$$

The hypothesis (2.27) is necessary and sufficient for the Ornstein-Uhlenbeck process Z defined by the linear counterpart

$$\mathrm{d}Z = AZ\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W$$

to (2.24) to be strong Feller (see [33], Proposition 1B; cf. also [12], §9.4.1); obviously, under (2.27) the process Z is also irreducible. We set $u = Q^{-1/2}f$, then the assumptions (ii) and (v) of Theorem 2.1 follow by the same argument as employed in the preceding example. So, if uniqueness in law holds for (2.24) and (2.25)–(2.27) are satisfied then the Markov process X is strong Feller and irreducible.

Let us compare this assertion with some related results. In the paper [19] the problem (2.24) is investigated under the assumptions (2.26) in a bit strengthened form, (2.27), $\operatorname{Rng} f \subseteq \operatorname{Rng} Q^{1/2}$ and with f and $Q^{-1/2}f$ Lipschitz continuous. Using the Malliavin calculus the author proves that X is strong Feller and irreducible ([19], Theorem 2.6). A. Chojnowska-Michalik and B. Goldys in [10] considered (2.24) with a bounded Borel mapping f such that $\langle f(\cdot), h\rangle \in \mathscr{C}_b(H)$ for any $h \in H$. They suppose (2.26), (2.27) and

$$\operatorname{Ker} Q_t = \{0\}, \qquad \int_0^T \|Q_t^{-1/2}e^{At}\|\,\mathrm{d}t < \infty,\ T > 0,$$

and they prove, by investigating the associated Kolmogorov equation, that there exists a weakly unique martingale solution of (2.24) which is a strong Feller irreducible process (see [10], Theorems 3, 4 and Propositions 3, 4).

Our proof of the strong Feller property for (2.24) is more straightforward than the ones in [19], [10], but it ought to be emphasized that much stronger regularity properties of the transition semigroup than the strong Feller property are established in the cited papers.

Example 2.4. An SPDE with a multiplicative noise. In this example we shall indicate that assumptions on the drift adopted in Peszat's and Zabczyk's paper [38] (cf. also [14], §7.1) may be relaxed. Consider the equation (2.1) with $H = \Upsilon$, $Q = I$, that is

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W, \tag{2.28}$$

assuming that

$$\int_0^t \|e^{As}\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty \quad \text{for all } t > 0,$$

$\sigma : H \longrightarrow \mathscr{L}(H)$ is a Lipschitz continuous mapping, $\sigma(z)$ is invertible for any $z \in H$ and

$$\sup_{z\in H} \|\sigma^{-1}(z)\| < \infty.$$

In [38] it is shown that the Markov process defined by (2.28) is strong Feller provided that $f : H \longrightarrow H$ is Lipschitz ([38], Corollary 1.1). If, moreover, either f or σ is bounded, then this Markov process is also irreducible ([38], Theorem 1.3). In particular, the Markov process defined by

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \sigma(Z)\,\mathrm{d}W$$

is strong Feller and irreducible under the above hypotheses on A and σ. Therefore, a straightforward application of Theorem 2.1 yields that the Markov process associated with (2.28) is strong Feller and irreducible whenever $f \in \mathscr{C}(H; H)$ is a bounded mapping. Indeed, set $u = \sigma^{-1} f$, then the assumptions (ii) and (v) of Theorem 2.1 follow easily by boundedness and continuity of u and Lipschitz continuity of σ.

Example 2.5. A stochastic heat equation with a white noise boundary condition. In this example we shall treat a one-dimensional heat equation with a nonhomogeneous nonlinear boundary condition containing a white noise term, written symbolically

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in [0,1],\\ &\Big(\frac{\partial u}{\partial x}(t,0), \frac{\partial u}{\partial x}(t,1)\Big) = f(u(t)) + \dot{w}_t, \quad t \geq 0, \end{aligned}\right\} \tag{2.29}$$

where w is a two-dimensional Brownian motion and f is a bounded continuous function from $W^{\alpha,2}((0,1))$ to $\mathbb{R}^2$ for a $\alpha \in [0, \frac{1}{2}[$. (We denote by $W^{\alpha,2}$ the usual Slobodeckiĭ spaces.) As well known (see e.g. [13]), the problem (2.29) can be reformulated as an equation

$$X_t = e^{At}X_0 + \int_0^t (A-I)e^{A(t-s)}Nf(X_s)\,\mathrm{d}s + \int_0^t (A-I)e^{A(t-s)}N\,\mathrm{d}w_s \tag{2.30}$$

in the Hilbert space $H = L^2((0,1))$, where

$$\mathrm{Dom}(A) = \Big\{ v \in W^{2,2}((0,1));\ \frac{\mathrm{d}v}{\mathrm{d}x}(0) = \frac{\mathrm{d}v}{\mathrm{d}x}(1) = 0 \Big\}, \quad A = \frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

and $N : \mathbb{R}^2 \longrightarrow H$ is the Neumann map, i.e., for any $\varrho = (\varrho_1, \varrho_2) \in \mathbb{R}^2$, $N\varrho$ is the (unique) solution of the problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - u = 0 \quad \text{on } [0,1], \quad \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \varrho_1, \quad \frac{\mathrm{d}u}{\mathrm{d}x}(1) = \varrho_2.$$

The Ornstein-Uhlenbeck process Z defined by

$$Z_t = e^{At} Z_0 + \int_0^t (A - I) e^{A(t-s)} N \,\mathrm{d}w_s$$

has paths continuous in $W^{\alpha,2}((0,1))$ and is strong Feller in H ([13], Proposition 3.3). Hence applying Theorem 2.1 in the same manner as above we see that the process X defined by (2.30) is strong Feller, provided uniqueness in law holds for (2.30).

3. The strong Feller property for stochastic evolution equations: the dissipative case

In Section 2, we applied Theorem 2.1 to stochastic partial differential equations whose drift contained nonlinear terms either bounded or of a linear growth, in which case the Girsanov transform may be used in a rather straightforward way. Now we shall discuss stochastic evolutions equations with unbounded nonlinearities in the drift. To handle this case, we need more detailed information about the behaviour of solutions than that provided by the hypothesis (A). Therefore, we shall study a more particular model, which, nonetheless, covers stochastic reaction-diffusion equations with polynomial nonlinearities (compare e.g. [12], §7.2). As in the previous section, we consider a pair of equations

$$\mathrm{d}X = \big(AX + f(X)\big)\,\mathrm{d}t + \sigma(X) Q^{1/2}\,\mathrm{d}W, \tag{3.1}$$

$$\mathrm{d}Z = AZ\,\mathrm{d}t + \sigma(Z) Q^{1/2}\,\mathrm{d}W, \tag{3.2}$$

in a separable Hilbert space H, assuming henceforth that W is a standard cylindrical Wiener process in a real separable Hilbert space Υ and $Q \in \mathscr{L}(\Upsilon)$ is a nonnegative self-adjoint operator. Let $(B, \|\cdot\|)$ be a separable Banach space embedded continuously into H. Suppose

(C1) $A : \mathrm{Dom}(A) \longrightarrow H$ *generates a* C_0*-semigroup* (e^{At}) *on* H *and*

$$\int_0^T t^{-\alpha} \left\| e^{At} \right\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty$$

for some $T > 0$ *and* $\alpha > 0$. *The part of* A *in* B, *denoted by* A_B, $\mathrm{Dom}(A_B) = \{x \in \mathrm{Dom}(A) \cap B;\ Ax \in B\}$, *generates a* C_0*-semigroup on* B.

We shall write $A_B = A$, $e^{A_B t} = e^{At}$ if there is no danger of confusion.

(C2) *The mapping* $\sigma : H \longrightarrow \mathscr{L}(\Upsilon, H)$ *satisfies*

$$\|\sigma(x)\|_{\mathscr{L}(\Upsilon,H)} \le k\big(1+|x|\big),$$
$$\|\sigma(x)-\sigma(y)\|_{\mathscr{L}(\Upsilon,H)} \le k|x-y|$$

for a constant $k < \infty$ *and all* $x, y \in H$.

Fix a standard cylindrical Wiener process W in Υ, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$. Given an arbitrary $x \in H$ there exists a unique $(\mathscr{F}_t)$-adapted mild solution Z^x to (3.2) with the initial condition $Z^x(0) = x$; this solution satisfies $Z^x \in \mathscr{C}(\mathbb{R}_+; H)$ almost surely.

Further we list hypotheses on the function f. We denote by $\langle \cdot, \cdot \rangle_{B,B^*}$ the duality between B and its dual space B^* and by $\partial\|x\|$ the subdifferential of the norm $\|\cdot\|$ at a point $x \in B$.

(C3) *Let the mapping* $f : B \longrightarrow B$ *be continuous and let there exist a nondecreasing function* $a : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$\big\langle Ax + f(x+y), x^* \big\rangle_{B,B^*} \le a(\|y\|)\big(1+\|x\|\big)$$

for each $x \in \mathrm{Dom}(A_B)$, $y \in B$ *and a certain* $x^* \in \partial\|x\|$. *Assume further that there exists a function* $u \in \mathscr{C}(B; \Upsilon)$ *bounded on bounded sets and satisfying* $f = \sigma Q^{1/2} u$ *on* B.

The last two assumptions are needed to guarantee that the process Z^x behaves well also in the state space B.

(C4) *Given* $T > 0$, *there exist* $p \in [2, \infty[$ *and* $C < \infty$ *such that for each standard cylindrical Wiener process* $\tilde{W}$ *defined on a filtered probability space* $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t), \tilde{\mathsf{P}})$ *and every* $(\tilde{\mathscr{F}}_t)$*-adapted process* $\xi \in L^p(\tilde{\Omega}; \mathscr{C}([0,T]; B))$ *we have*

$$\int_{\tilde{\Omega}} \sup_{0\le t\le T} \left\| \int_0^t e^{A(t-s)} \sigma(\xi_s) Q^{1/2}\, \mathrm{d}\tilde{W}_s \right\|^p \mathrm{d}\tilde{\mathsf{P}} \le C,$$

and the paths

$$t \longmapsto \int_0^t e^{A(t-s)} \sigma(\xi_s) Q^{1/2}\, \mathrm{d}\tilde{W}_s$$

belong to $\mathscr{C}([0,T]; B)$ $\tilde{\mathsf{P}}$*-almost surely.*

(C5) *Given* $T > 0$, *let* p *be the same as in* (C4). *Let there exist a constant* $\hat{C} < \infty$ *such that for any* $(\mathscr{F}_t)$*-adapted processes* $\xi, \zeta \in L^p(\Omega; \mathscr{C}([0,T]; B))$ *we have*

$$\mathsf{E} \sup_{0\le t\le T} \left\| \int_0^t e^{A(t-s)} \big[\sigma(\xi_s) - \sigma(\zeta_s)\big] Q^{1/2}\, \mathrm{d}W_s \right\|^p \le \hat{C}\mathsf{E} \int_0^T \|\xi_t - \zeta_t\|^p\, \mathrm{d}t.$$

We refer the reader to the papers [4], [5], [36], [37] for results on maximal inequalities yielding (C4), (C5) under various particular hypotheses. Let us note that (C4) usually requires boundedness of σ whilst (C5) its Lipschitz continuity (in suitable norms), so these assumptions are closely related to the hypothesis (C1). Proceeding in a standard manner, it is possible to check that the process Z^x is B-valued and $Z^x \in L^p(\Omega; \mathscr{C}([0,T]; B))$ for each $T > 0$, provided $x \in B$.

Now we can state the main result of this section. The equation (3.2) is well-posed both in H and in B, let us denote by R the transition probability of the associated Markov process. We say that R is *strong Feller in* B, if $R_t(\cdot, M)$ is continuous on B for any $t > 0$ and any Borel set M in B. Clearly, the strong Feller property of R in H, as considered in Section 2, implies that in B, since the embedding $B \hookrightarrow H$ is continuous. We shall see below that the equation (3.1) has a martingale solution for each initial datum $x \in B$ (cf. Corollary 3.4). If uniqueness in law holds as well then (3.1) defines a Markov process in B. Denote by P its transition probability, the strong Feller property of P in B is defined in an obvious way.

Theorem 3.1. *Assume* (C1)–(C5)*, let the equation* (3.1) *be well-posed in* B*. Let the transition probability* R *defined by the equation* (3.2) *be strong Feller in* B*, then the transition probability* P *defined by* (3.1) *is also strong Feller in* B*.*

Let us note that sufficient conditions for the strong Feller property of R are recalled in Example 2.4. Towards the proof of Theorem 3.1, we have to establish several technical lemmas. In what follows, we shall always suppose that the assumptions (C1)–(C5) are satisfied. First, note that (C4), (C5) and the Gronwall lemma yield

Lemma 3.2. *For any* $T > 0$*,* $R > 0$*, and* $x_n, x \in B$ *such that* $x_n \longrightarrow x$ *in* B *as* $n \to \infty$ *we have*

$$\sup_{\|x\| \le R} \mathsf{E} \sup_{0 \le t \le T} \|Z^x(t)\|^p < \infty,$$

$$\lim_{n\to\infty} \mathsf{E} \sup_{0 \le t \le T} \|Z^{x_n}(t) - Z^x(t)\|^p = 0.$$

As in Section 2 we set

$$U(x,t) = \exp\left(\int_0^t \langle u(Z_s^x), \cdot \rangle \, \mathrm{d}W_s - \frac{1}{2} \int_0^t |u(Z_s^x)|^2 \, \mathrm{d}s \right), \quad x \in B, \ t > 0.$$

Since the paths of Z^x are continuous in B, the random variable $U(x,t)$ is well defined. Consider stopping times $\tau_n^x \equiv \tau(x,n)$ defined by

$$\tau_n^x = \inf\left\{ t \ge 0; \ \|Z^x(t)\| \ge n \right\}.$$

As u is bounded on bounded set in B the process $u(Z_s^x)$ is bounded for $s \le \tau_n^x$, consequently $\mathsf{E}U(x, t \wedge \tau_n^x) = 1$ for every $n \in \mathbb{N}$, $x \in B$, $t \ge 0$. For a fixed $T > 0$ we obtain by the Girsanov theorem that

$$W_t^n = W_t - \int_0^{t \wedge \tau(x,n)} u(Z^x(s)) \, \mathrm{d}s, \quad 0 \le t \le T,$$

is a standard cylindrical Wiener process on the probability space $(\Omega, \mathscr{F}, \mathsf{P}_{x,n})$, the measure $\mathsf{P}_{x,n}$ being defined by $\mathrm{d}\mathsf{P}_{x,n} = U(x, T \wedge \tau_n^x)\,\mathrm{d}\mathsf{P}$.

The following lemma corresponds to Lemma 2 in [22], unfortunately, the proof in [22] is flawed, hence we present here a slightly modified one.

Lemma 3.3. *For any $R > 0$ and $T > 0$*

$$\lim_{n\to\infty} \sup_{\|x\|\le R} \mathsf{P}_{x,n}\big\{\tau_n^x \le T\big\} = 0$$

holds true.

Proof. Fix an arbitrary $T > 0$ and for $t \in [0, T]$ set

$$Z_n^x(t) = e^{At}x + \int_0^t e^{A(t-s)}\sigma(Z^x(s))Q^{1/2}\,\mathrm{d}W_s^n,$$
$$Y_n^x(t) = Z^x(t) - Z_n^x(t).$$

Then

$$Y_n^x(t) = \int_0^{t\wedge\tau(x,n)} e^{A(t-s)} f(Z^x(s))\,\mathrm{d}s$$

and hence the function $Y_n^x(\cdot,\omega)$ is a mild solution of the equation

$$\dot{Y}_n^x(t) = AY_n^x(t) + f(Z^x(t)), \quad Y_n^x(0) = 0, \quad 0 \le t \le T \wedge \tau_n^x$$

for almost all $\omega \in \Omega$. For $\lambda > 0$ in the resolvent set of A let $R(\lambda) = \lambda(\lambda I - A)^{-1}$ and let $Y_{\lambda,n}^x$ be the Yosida approximation to Y_n^x,

$$Y_{\lambda,n}^x(t) = R(\lambda)Y_n^x(t) = \int_0^{t\wedge\tau(x,n)} e^{A(t-s)}R(\lambda)f(Z^x(s))\,\mathrm{d}s.$$

Thus $Y_{\lambda,n}^x$ solves the equation

$$\dot{Y}_{\lambda,n}^x(t) = AY_{\lambda,n}^x(t) + R(\lambda)f(Z^x(t)), \quad Y_{\lambda,n}^x(0) = 0, \quad 0 \le t \le T \wedge \tau_n^x$$

in the strong sense. Put

$$\begin{aligned}\delta_\lambda(t) &= R(\lambda)f(Z^x(t)) - f(Y_{\lambda,n}^x(t) + Z_n^x(t))\\ &= R(\lambda)f(Y_n^x(t) + Z_n^x(t)) - f(Y_{\lambda,n}^x(t) + Z_n^x(t)),\end{aligned}$$

so

$$\dot{Y}_{\lambda,n}^x(t) = AY_{\lambda,n}^x(t) + f(Y_{\lambda,n}^x(t) + Z_x^n(t)) + \delta_\lambda(t)$$

and a standard application of the assumption (C3) (see e.g. [12], Appendix D) yields

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\big\|Y_{\lambda,n}^x(t)\big\| \le a(\|Z_n^x(t)\|)\big(1 + \|Y_{\lambda,n}^x(t)\|\big) + \delta_\lambda(t).$$

By the Gronwall lemma we get

$$\|Y_{\lambda,n}^x(t)\| \le \int_0^t \exp\left(\int_s^t a(\|Z_n^x(r)\|)\,\mathrm{d}r\right)\Big\{a(\|Z_n^x(s)\|) + \|\delta_\lambda(s)\|\Big\}\,\mathrm{d}s \qquad (3.3)$$

for $0 \le t \le T \wedge \tau_n^x$. Since $R(\lambda) \longrightarrow I$ as $\lambda \to \infty$ in the strong operator topology and $\|R(\lambda)\| \le M$ for a constant M and all $\lambda > 0$ sufficiently large, we have

$$\lim_{\lambda\to\infty} \int_0^{T\wedge\tau(x,n)} \|\delta_\lambda(s)\| \, \mathrm{d}s = 0$$

and the estimate

$$\|Y_n^x(t)\| \le \int_0^t \exp\left(\int_s^t a(\|Z_n^x(r)\|)\, \mathrm{d}r\right) a(\|Z_n^x(s)\|)\, \mathrm{d}s, \quad 0 \le t \le T \wedge \tau_n^x,$$

follows from (3.3) by passing $\lambda \to \infty$. Obviously, $Y_n^x(t) = e^{A(t-\tau(x,n))} Y_n^x(\tau_n^x)$ for $\tau_n^x < t \le T$, so

$$\|Y_n^x(t)\| \le \tilde{K} \int_0^t \exp\left(\int_s^t a(\|Z_n^x(r)\|)\, \mathrm{d}r\right) a(\|Z_n^x(s)\|)\, \mathrm{d}s$$

holds for a constant $\tilde{K} < \infty$ and all $t \in [0, T]$. Hence

$$\begin{aligned}
\sup_{0\le t\le T} \|Z^x(t)\| &= \sup_{0\le t\le T} \|Y_n^x(t) + Z_n^x(t)\| \\
&\le \sup_{0\le t\le T} \|Z_n^x(t)\| \\
&\quad + \tilde{K} \sup_{0\le t\le T} \int_0^t \exp\left(\int_s^t a(\|Z_n^x(r)\|)\, \mathrm{d}r\right) a(\|Z_n^x(s)\|)\, \mathrm{d}s \\
&\le \sup_{0\le t\le T} \|Z_n^x(t)\| + \tilde{K} \int_0^T \exp\left(\int_0^T a(\|Z_n^x(r)\|)\, \mathrm{d}r\right) a(\|Z_n^x(s)\|)\, \mathrm{d}s.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathsf{P}_{x,n}\left\{\tau_n^x \le T\right\} &= \mathsf{P}_{x,n}\left\{\sup_{0\le t\le T} \|Z^x(t)\| \ge n\right\} \\
&\le \mathsf{P}_{x,n}\left\{\sup_{0\le t\le T} \|Z_n^x(t)\| + \Phi(Z_n^x) \ge n\right\},
\end{aligned}$$

where we set

$$\Phi : \mathscr{C}([0, T]; B) \longrightarrow \mathbb{R}; \ \varphi \longmapsto \tilde{K} \exp\left(\int_0^T a(\|\varphi(s)\|)\, \mathrm{d}s\right) \int_0^T a(\|\varphi(s)\|)\, \mathrm{d}s.$$

By the hypothesis (C4) we can find a constant $C_1 = C_1(R, T) < \infty$ independent of n such that

$$\sup_{\|x\|\le R} \int_\Omega \sup_{0\le t\le T} \|Z_n^x(t)\|^p \, \mathrm{d}\mathsf{P}_{x,n} \le C_1.$$

Whence

$$\sup_{\|x\|\le R} \mathsf{P}_{x,n}\left\{\sup_{0\le t\le T} \|Z_n^x(t)\| \ge h\right\} \le \frac{C_1}{h^p} \underset{h\to\infty}{\longrightarrow} 0 \tag{3.4}$$

by the Chebyschev inequality, in particular

$$\lim_{n\to\infty} \sup_{\|x\|\le R} \mathsf{P}_{x,n}\Big\{ \sup_{0\le t\le T} \|Z_n^x(t)\| \ge n \Big\} = 0.$$

As a is a nondecreasing function, the mapping Φ is bounded on bounded sets, namely, if $\sup_{0\le t\le T} \|\psi(t)\| \le h$ then $|\Phi(\psi)| \le \tilde{K} T a(h) \exp(T a(h))$. Given $\varepsilon > 0$, use (3.4) to find $\varkappa > 0$ such that $\mathsf{P}_{x,n}\{\sup_{0\le t\le T} \|Z_n^x(t)\| \le \varkappa\} \ge 1-\varepsilon$. Let $n_0 \in \mathbb{N}$ be such that $\tilde{K} T a(\varkappa) \exp(T a(\varkappa)) < n_0$, then $\mathsf{P}_{x,n}\{\Phi(Z_n^x) \ge n\} < \varepsilon$ for all $n \ge n_0$, which completes the proof. Q.E.D.

A standard argument shows that Lemma 3.3 yields

Corollary 3.4. *For every $x \in B$ and $t \ge 0$ we have* $\mathsf{E}U(x,t) = 1$.

Proof. Obviously,

$$1 \ge \mathsf{E}U(x,t) \ge \mathsf{E}\mathbf{1}_{\{\tau(x,n)\ge t\}} U(x, t\wedge \tau_n^x) = \mathsf{P}_{x,n}\{\tau_n^x \ge t\} \underset{n\to\infty}{\longrightarrow} 1.$$

Q.E.D.

Lemma 3.5. *For each $t \ge 0$ and any convergent sequence $\{x_n\}$ in B, $x_n \longrightarrow x_0$, we have*

$$U(x_n,t) \xrightarrow[n\to\infty]{\mathsf{P}} U(x_0,t).$$

Proof. Fix $t > 0$, $\varepsilon > 0$ and a convergent sequence $\{x_n\}$, $x_n \longrightarrow x_0$ in B, arbitrarily. By Lemma 3.2 we can find $k \in \mathbb{N}$ such that

$$\sup_{n\ge 0} \mathsf{P}\big\{\tau_k^{x_n} < t\big\} < \frac{\varepsilon}{2}.$$

Set $A_n = \{\tau_k^{x_n} < t\} \cup \{\tau_k^{x_0} < t\}$ and choose a bounded function $\tilde{u} \in \mathscr{C}(B, \Upsilon)$ such that $u = \tilde{u}$ on the ball $\{x \in B;\ \|x\| \le k\}$. If we define

$$\tilde{U}(y,t) = \exp\left(\int_0^t \big\langle \tilde{u}(Z^y(s)), \cdot \big\rangle \, \mathrm{d}W(s) - \frac{1}{2}\int_0^t \big|\tilde{u}(Z^y(s))\big|^2 \, \mathrm{d}s \right), \quad y \in B,$$

then $\tilde{U}(x_n,t) = U(x_n,t)$, $\tilde{U}(x_0,t) = U(x_0,t)$ almost surely on $\Omega \setminus A_n$ for each $n \ge 1$, since $u(Z^y(s)) = \tilde{u}(Z^y(s))$ for $0 \le s \le \tau_k^y \wedge t$, $\|y\| \le k$. Hence

$$\begin{aligned} \mathsf{P}\big\{|U(x_n,t) - U(x_0,t)| > \varepsilon\big\} &\le \mathsf{P}(A_n) + \mathsf{P}\big\{|\tilde{U}(x_n,t) - \tilde{U}(x_0,t)| > \varepsilon\big\} \\ &\le \varepsilon + \mathsf{P}\big\{|\tilde{U}(x_n,t) - \tilde{U}(x_0,t)| > \varepsilon\big\}. \end{aligned}$$

To complete the proof we have to check that

$$\tilde{U}(x_n,t) \xrightarrow[n\to\infty]{\mathsf{P}} \tilde{U}(x_0,t).$$

By Remark 2.2 we know that it suffices to show that

$$\lim_{n\to\infty} \int_0^t \mathsf{E}\big|\tilde{u}(Z^{x_n}(s)) - \tilde{u}(Z^{x_0}(s))\big|^2 \, \mathrm{d}s = 0. \tag{3.5}$$

However, $\|Z^{x_n} - Z^{x_0}\| \longrightarrow 0$ in measure on $[0,T]\times\Omega$ by Lemma 3.2, thus (3.5) follows immediately by boundedness and continuity of $\tilde{u}$. Q.E.D.

Proof of Theorem 3.1. We can repeat the proof of Theorem 2.1 literally, taking into account that the crucial assumptions (ii) and (v) of Theorem 2.1 are satisfied according to Corollary 3.4. and Lemmas 3.5, 3.2, respectively. Q.E.D.

4. A remark on M. Fuhrman's result

Let H be a real separable Hilbert space, W a standard cylindrical Wiener process in H, defined on a fixed filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$, $Q \in \mathscr{L}(H)$ a nonnegative self-adjoin operator. Let $V \in \mathscr{C}^2(H)$, denote by V', V'' the first and second Fréchet derivative of V, respectively. We will consider the equation

$$\mathrm{d}X = \big(AX - QV'(X)\big)\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W \tag{4.1}$$

under the following assumptions: $A : \mathrm{Dom}(A) \longrightarrow H$ is an infinitesimal generator of a C_0-semigroup (e^{At}) on H such that

$$\int_0^T t^{-\alpha} \big\|e^{At} Q^{1/2}\big\|_{\mathrm{HS}}^2 \,\mathrm{d}t < \infty \tag{4.2}$$

for a $\alpha > 0$ and any $T \in \mathbb{R}_+$. Denote by $\mathscr{L}_1(H)$ the Banach space of all nuclear operators on H equipped with the nuclear norm. Suppose that the functions

$$V : H \longrightarrow \mathbb{R}, \quad V' : H \longrightarrow H, \quad V'' : H \longrightarrow \mathscr{L}(H)$$

are uniformly continuous and bounded on bounded sets in H, $\mathrm{Rng}(V') \subseteq \mathrm{Dom}\,(A^*)$, $\mathrm{Rng}(Q^{1/2}V''(\cdot)Q^{1/2}) \subseteq \mathscr{L}_1(H)$, and the functions

$$A^*V' : H \longrightarrow H, \quad Q^{1/2}V''Q^{1/2} : H \longrightarrow \mathscr{L}_1(H)$$

are continuous on H and bounded on bounded subsets of H. Set

$$LV(x) = \frac{1}{2}\mathrm{Tr}\big(Q^{1/2}V''(x)Q^{1/2}\big) + \big\langle A^*V'(x), x\big\rangle, \quad x \in H,$$

and assume that

$$V(x) \ge k_1, \quad LV(x) - \frac{1}{2}\big|Q^{1/2}V'(x)\big|^2 \le k_2 \tag{4.3}$$

for some constants $k_1, k_2 \in \mathbb{R}$ and each $x \in H$.

Then, by Theorem 2.5 of [20], there exists a unique mild solution to (4.1) for any initial condition $X(0) = x \in H$; denote by P the transition probability defined by (4.1). (The reader may consult Section 5 of [20], where nontrivial examples of equations satisfying the above hypotheses are presented.)

Assume further that (2.27) holds, that is, that the Ornstein-Uhlenbeck process

$$\mathrm{d}Z = AZ\,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W$$

is strong Feller, we aim at checking the strong Feller property of P using Theorem 2.1. Let us define $U(y,t)$, $y \in H$, $t \ge 0$, as in Theorem 2.1 with the choice $u = -Q^{1/2}V'$. We have

$$\int_0^t \big\langle Q^{1/2}V'(Z^y(s)), \cdot\big\rangle\,\mathrm{d}W(s) = V(Z^y(t)) - V(y) - \int_0^t LV(Z^y(s))\,\mathrm{d}s$$

by [20], Lemma 2.1, hence

$$U(y,t) = \exp\left(V(y) - V(Z^y(t)) + \int_0^t \left\{ LV(Z^y(s)) - \frac{1}{2}\left|Q^{1/2}V'(Z^y(s))\right|^2 \right\} \mathrm{d}s \right)$$

for any $y \in H$, $t \geq 0$, and so $\mathsf{E}U(y,t) = 1$ is an easy consequence of (4.3). Fix $t > 0$ and a convergent sequence $\{x_n\}$ in H, $x_n \longrightarrow x_0$ as $n \to \infty$. It remains to prove that

$$U(x_n, t) \xrightarrow[n\to\infty]{\mathsf{P}} U(x_0, t). \tag{4.4}$$

Obviously, $\sup_{0\leq s\leq t} |Z^{x_n}(s) - Z^{x_0}(s)| \longrightarrow 0$ almost surely, from which we obtain

$$V(x_n) \longrightarrow V(x_0), \quad V(Z^{x_n}(t)) \longrightarrow V(Z^{x_0}(t)) \quad \mathsf{P}-\text{almost surely}$$

by continuity of V. Due to (4.2), the assumptions of Theorem 5.9 of [12] are met and

$$\lim_{N\to\infty} \sup_{n\geq 0} \mathsf{P}\left\{ \sup_{0\leq s\leq t} |Z^{x_n}(s)| \geq N \right\} = 0$$

holds. Since the function $LV - \frac{1}{2}|Q^{1/2}V'|$ is continuous and bounded on bounded subsets of H, we may proceed as in the proof of Lemma 3.5 to show

$$\int_0^t \left\{ LV(Z^{x_n}(s)) - \frac{1}{2}\left|Q^{1/2}V'(Z^{x_n}(s))\right|^2 \right\} \mathrm{d}s$$
$$\xrightarrow[n\to\infty]{\mathsf{P}} \int_0^t \left\{ LV(Z^{x_0}(s)) - \frac{1}{2}\left|Q^{1/2}V'(Z^{x_0}(s))\right|^2 \right\} \mathrm{d}s$$

and the claim (4.4) follows. Therefore, the transition function P is strong Feller, obviously, it is also irreducible.

In [20], an additional assumption that $LV \in \mathscr{C}^1(H)$ and

$$|V'(x)| + |(LV)'(x)| + |V''(x)^* QV'(x)| \leq k_3 e^{k_4|x|^\gamma}$$

for some $k_3, k_4 \in \mathbb{R}_+$ and $\gamma < 2$ is adopted; we do not need this estimate. On the other hand, Theorem 3.4 in [20] asserts more than we can prove: for φ bounded Borel on H, $P_t\varphi$ is not only continuous on H but also Lipschitz on bounded subsets of H.

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