Probab. Theory Relat. Fields 118, 147–168 (2000)
Digital Object Identifier (DOI) 10.1007/s004400000076

Lorenzo Zambotti

An analytic approach to existence and uniqueness for martingale problems in infinite dimensions

Received: 18 May 1998 / Revised version: 27 September 1999 /
Published online: 5 September 2000 –

Abstract. We prove existence and uniqueness for a class of martingale problems in a Hilbert space. We solve the associated Kolmogorov equation and prove that the corresponding semigroup is determined by a kernel of measures if a Schauder-type regularity is satisfied.

1. Introduction

In this paper we propose a new analytic approach to the study of existence and uniqueness of solutions to martingale problems in infinite dimensions.

We consider a SPDE in a separable Hilbert space H

$$\begin{cases} dX = (AX + F(X))dt \; + \; \sqrt{B(X)}\, dW(t) \\ \\ X_0 = x \; \in \; H \end{cases} \tag{1}$$

where A generates a strongly continuous contraction semigroup in H, $F : H \mapsto H$ is bounded Hölder-continuous, and $B(\cdot)$ is obtained by a bounded Hölder-continuous trace-class perturbation of a constant operator.

Our method corresponds to the procedure presented by Stroock and Varadhan in the finite-dimensional case (see [17], chap. 3 and 6). First, we solve the parabolic Kolmogorov equation associated to (1), in the space $C_b(H)$ of bounded continuous functions on H. Secondly, we prove that the corresponding semigroup $(P_t)_{t\geq 0}$ is determined by a measurable kernel $\{\pi(t, x, \cdot)\}$ of probability measures on H.

The first step provides uniqueness of solutions to the martingale problem (1), identifying the law at fixed (t, x) of any solution X with the functional $C_b(H) \ni \varphi \mapsto P_t\varphi(x)$. The second step says that the transition function $\{\pi(t, x, \cdot)\}$ satisfies the Chapman-Kolmogorov equation. This allows to construct in a classical way a Markov process $X(t, x)$ with values in H, which turns out to be a solution to the martingale problem (1).

There are many results on existence for martingale problems in infinite dimensions, mainly based on compactness techniques or Girsanov's Theorem, while

L. Zambotti: Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy. e-mail: zambotti@cibs.sns.it

Mathematics Subject Classification (2000): 60H15, 35R15, 60J60

uniqueness is still largely open and has been proven only in particular cases: see [5], [8], [11], [14], [19], [22], [21]. For an analytic approach based on Dirichlet Forms see [13].

The main novelties of the present paper are contained in a generalization of the Riesz Representation Theorem to the Hilbert space case. More precisely, we discuss the following question: given a positive semigroup $(P_t)_{t\geq 0}$ in $C_b(H)$ satisfying a parabolic equation, does there exist a measurable kernel of finite positive measures $\{\pi(t, x, \cdot)\}$ on H such that

$$P_t\varphi(x) = \int_H \varphi(y)\, \pi(t, x, dy) \tag{2}$$

for all φ bounded and continuous? The answer in general is negative (see the beginning of Section 4 for a detailed discussion). However, we state the following general principle, giving a positive answer in many interesting cases: let $(\mathscr{L}, D(\mathscr{L}))$ be, at least formally, the infinitesimal generator in $C_b(H)$ of (1), and suppose that there exists an Ornstein-Uhlenbeck operator $(\mathscr{M}, D(\mathscr{M}))$ in $C_b(H)$ such that a uniform Schauder-type regularity holds for the operators $\alpha\mathscr{L} + (1-\alpha)\mathscr{M}$, $\alpha \in [0, 1]$. Then there exists a unique kernel of finite measures $\pi(t, x, \cdot)$ on H such that (2) holds for all $\varphi \in C_b(H)$, $t \geq 0$, $x \in H$. It is interesting that a regularity in the *backward* variable x, i.e. Schauder Estimates, produces a regularity in the *forward* variable y, i.e. the representation as a σ-additive measure, for the kernel $\{\pi(t, x, dy)\}$.

In sections 3–5 we prove the aforementioned general principle. We state Schauder Estimates and Positivity for the operators $(\lambda - \mathscr{L})^{-1}$, $\lambda > 0$, then we prove existence and regularity of the transition semigroup associated to (1), and both existence and uniqueness of solutions to the martingale problem (1). The technique presented here could be applied also to other class of coefficients, provided an analogous Schauder-type regularity holds: see for instance [3] and [21]. Notice also that our technique is entirely analytic and independent of tightness methods (see [8], chap. 8).

Our proof of Schauder Estimates, given in section 6, requires the hypothesis that the trace-class norm of the perturbation in the noise-term of (1) is bounded by a fixed constant: see (9) below. This hypothesis is due to technical difficulties arising from the unboundedness of the operator A in the drift term of (1), and we stress that it is not needed in the general procedure of sections 3–5. No assumptions are required on the bound of the Hölder seminorm of the perturbation: see (10) below.

In our opinion a full understanding of Schauder-type regularity in infinite dimension is still missing. We think that a better insight in infinite-dimensional analytic regularity will allow to apply our method with greater generality (see Remarks 9 and 10 below).

2. Notations and assumptions

In this paper we consider a separable real Hilbert space H, with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We will denote by $\mathscr{L}(H)$ the space of bounded linear operators on H with the usual sup-norm $\|\cdot\|$, by $\mathscr{L}_1(H)$ the space of trace-class operators on

H, endowed with the trace-norm $\|\cdot\|_{\mathscr{L}_1(H)}$, and by $\mathscr{L}_1^+(H)$ the space of positive symmetric operators in $\mathscr{L}_1(H)$.

We write equation (1) in the following way:

$$\begin{cases} dX = (AX + F(X))dt \ + \ (BB^* + G(X))^{1/2}\, dW(t) \\ X_0 = x \ \in \ H \end{cases} \tag{3}$$

with the following assumptions:

Hypothesis 1. *1. $A : D(A) \subset H \mapsto H$ is the generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ in H, such that*

$$\|e^{tA}\| \ \leq 1 \qquad\qquad \forall\, t \geq 0. \tag{4}$$

$(W_t)_{t\geq 0}$ is a cylindrical white-noise in H and $B \in \mathscr{L}(H)$.

2. The Ornstein-Uhlenbeck process $\{Z(t,x)\}_{t\geq 0, x}$, defined by

$$Z(t,x) \ := \ e^{tA}x \ + \ \int_0^t e^{(t-s)A} B\, dW_s, \tag{5}$$

takes values in H, i.e. for all $t > 0$

$$\mathrm{Tr}\, Q_t \ < \ \infty \qquad Q_t x \ := \ \int_0^t e^{sA} BB^* e^{sA^*}\, x\, ds. \tag{6}$$

3. The O.U. process is Strong Feller, i.e. $e^{tA}x \ \in \ Q_t^{1/2}(H)$ for all $t > 0$, $x \in H$. The operator $\Gamma(t) := Q_t^{-1/2} e^{tA}$, bounded in H by the Closed Graph Theorem, satisfies for some $\nu > 0$

$$\|\Gamma(t)\| \leq \frac{\nu}{\sqrt{t}} \quad \forall\, t > 0. \tag{7}$$

Hypothesis 2. $\theta \in]0,1[$, $\quad F : H \mapsto H$, $\quad G : H \mapsto \mathscr{L}_1^+(H)$ *and*

$$\sup_{x\in H} |F(x)| + \sup_{x\neq y} \frac{|F(x) - F(y)|}{|x-y|^\theta} =: \|F\|_{C_b^\theta(H;H)} < \infty \tag{8}$$

$$\sup_{x\in H} \|G(x)\|_{\mathscr{L}_1(H)} \ =: \ \varepsilon \ < \ \theta\, (120 \cdot 2^{\frac{1-\theta}{2}} \nu^{2-\theta})^{-1} \tag{9}$$

$$\sup_{x\neq y} \frac{\|G(x) - G(y)\|_{\mathscr{L}_1(H)}}{|x-y|^\theta} \ =: \ M \ < \ \infty. \tag{10}$$

Assumptions 1–3 in Hypothesis 1 are common in SPDEs literature, see [8] and [9]. Recently, regularity of invariant measures was studied in [1] for an infinite-dimensional Stochastic Differential Equation, with a diffusion term given by the Identity, plus a fixed trace-class operator times a $\mathscr{L}(H)$-valued Lipschitz function.

Example. Consider the following SPDE on $I = [0, 1]$:

$$\begin{cases} \partial_t X(t,\xi) = \left(\partial^2_{\xi\xi} X(t,\xi) + f(X(t,\xi))\right) dt \\ \qquad\qquad + \sum_{k=1}^{\infty} \left(\frac{1}{\nu^2} + \lambda_k \cdot g_k(X(t,\xi))\right)^{1/2} e_k(\xi)\, d\beta_k(t) \\ X(t,0) = X(t,1) = 0 \qquad \forall\, t > 0 \\ X(0,\xi) = x(\xi) \qquad \forall\, \xi \in [0,1] \end{cases} \tag{11}$$

where:

- $x \in L^2(0,1)$ and $\{e_k\}_k$ is a complete orthonormal system in $L^2(0,1)$.
- $\{\beta_k(t)\}_{t\geq 0, k}$ are independent linear Brownian Motions.
- $\theta \in]0,1[$, $f : \mathbb{R} \mapsto \mathbb{R}$ is bounded and θ-Hölder continuous.
- For all $k \in \mathbb{N}$, $g_k : \mathbb{R} \mapsto \mathbb{R}$ and

$$0 \leq g_k(x) \leq 1, \quad \forall\, x \in \mathbb{R}, \qquad \sup_{x \neq y} \frac{|g_k(x) - g_k(y)|}{|x-y|^\theta} \leq M < \infty$$

- $\nu > 0$, $\lambda_k > 0$ for all $k \in \mathbb{N}$, and $\sum_k \lambda_k =: \varepsilon < \theta(120 \cdot 2^{\frac{1-\theta}{2}} \nu^{2-\theta})^{-1}$.

Under these assumptions, (11) satisfies Hypothesis 1 and 2. □

We define $C_b(H)$ as the space of all bounded uniformly continuous real functions on H, endowed with the sup-norm $\|\cdot\|_0$; $C_b^\alpha(H)$, with $\alpha \in]0,1[$, as the space of functions in $C_b(H)$ which are α-Hölder continuous; $C_b^{2+\alpha}(H)$ as the space of functions in $C_b(H)$ which are twice Fréchet differentiable, with first Fréchet differential ∇u uniformly bounded and second Fréchet differential $D^2 u$ bounded and α-Hölder continuous as a map from H to $\mathscr{L}(H)$. We set:

$$\|u\|_0 := \sup_{x \in H} |u(x)|, \quad [u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \quad \|u\|_\alpha := \|u\|_0 + [u]_\alpha$$

$$\|u\|_{2+\alpha} := \|u\|_0 + \sup_x |\nabla u(x)| + \sup_x \|D^2 u(x)\| + \sup_{x \neq y} \frac{\|u(x) - u(y)\|}{|x-y|^\alpha}$$

If $(S_t)_{t \geq 0}$ is a semigroup acting on $C_b(H)$, we will say, following [4], that $(S_t)_{t\geq 0}$ is *weakly right-continuous* (respectively *weakly continuous*) if for all $\varphi \in C_b(H)$ and $x \in H$, the map $[0,\infty) \ni t \mapsto S_t\varphi(x)$ is right-continuous (resp. continuous).

The transition semigroup $(R_t)_{t\geq 0}$ in $C_b(H)$ of (5)

$$R_t\varphi(x) \;:=\; \int_H \varphi(y)\,\mathcal{N}(e^{tA}x, Q_t)(dy) \qquad x \in H,\; \varphi \in C_b(H) \tag{12}$$

is *not* strongly continuous in $C_b(H)$. However, $(R_t)_{t\geq 0}$ is weakly continuous. Therefore, one can define the continuous operators $F_\lambda : C_b(H) \mapsto C_b(H)$

$$F_\lambda\varphi(x) \;:=\; \int_0^\infty e^{-\lambda t} R_t\varphi(x)dt \qquad \lambda > 0,\; x \in H,\; \varphi \in C_b(H). \tag{13}$$

The family $\{F_\lambda\}_\lambda$ is a *Pseudo-resolvent* on $C_b(H)$ (see [20]): indeed, by the semigroup law of $(R_t)_{t\geq 0}$ it satisfies the Resolvent Formula

$$(\lambda - \mu)F_\mu F_\lambda \;=\; F_\mu - F_\lambda \qquad \forall\,\lambda, \mu > 0$$

and by the weak continuity of $(R_t)_{t\geq 0}$, F_λ is one-to-one for all $\lambda > 0$. Therefore, there exists a unique closed operator $\mathcal{M} : D(\mathcal{M}) \subset C_b(H) \mapsto C_b(H)$ such that $F_\lambda = (\lambda - \mathcal{M})^{-1} =: R(\lambda, \mathcal{M})$ for all $\lambda > 0$. The operator $(\mathcal{M}, D(\mathcal{M}))$ is said to be the *infinitesimal generator* of $(R_t)_{t\geq 0}$. Moreover, the semigroup $(R_t)_{t\geq 0}$ on $C_b(H)$, the resolvent family $\{R(\lambda, \mathcal{M})\}_\lambda$ and the operator $(\mathcal{M}, D(\mathcal{M}))$ determine uniquely each other. We will call $(\mathcal{M}, D(\mathcal{M}))$ the *Ornstein-Uhlenbeck* operator.

We introduce now the operator $(\mathcal{L}, D(\mathcal{L}))$ in $C_b(H)$, which will turn out to be the infinitesimal generator of (3). We set

$$D(\mathcal{Q}) := C_b^{2+\theta}(H), \qquad \mathcal{Q}u \;:=\; \frac{1}{2}\mathrm{Tr}[GD^2u] \;+\; \langle F, \nabla u\rangle,$$

$$D(\mathcal{L}) := D(\mathcal{M}) \cap D(\mathcal{Q}), \qquad \mathcal{L}u := \mathcal{M}u + \mathcal{Q}u, \tag{14}$$

where $\theta \in \,]0, 1[$ is the same as in Hypothesis 2.

We define the coordinate process on $H^{[0,T]}$ by

$$X_t : H^{[0,T]} \mapsto H, \qquad X_t(\omega) := \omega(t), \qquad t \in [0, T].$$

We can now give the following

Definition 1. *A solution to the martingale problem (3) on* $[0, T]$ *is a probability measure* $\mathbb{P}_x$ *on* $H^{[0,T]}$ *such that*

$$\mathbb{P}_x\,\{\,\omega : [0, T] \mapsto H,\; \omega(0) = x,\; \omega \text{ Borel function}\,\} \;=\; 1,$$

$$\textit{and the process} \qquad \left\{ f(X_t) - \int_0^t \mathcal{L}f(X_r)\,dr \right\}_{t\geq 0} \tag{15}$$

is a $\mathbb{P}_x$*-martingale for all* $f \in D(\mathcal{L})$*. If*

$$\lim_{t\to 0} \mathbb{P}_x\,\{\,\omega : |\omega(t) - x| \leq \delta\,\} = 1 \qquad \forall\,\delta > 0 \tag{16}$$

then the solution $\mathbb{P}_x$ *is said to be* stochastically continuous.

It is proven in [9], § 2.1, that a Markov process $\{\mathbb{P}_x\}_x$ with values in H satisfies (16) if and only if its transition semigroup $(P_t)_{t\geq 0}$ is weakly right-continuous.

We will often use the following important type of convergence, introduced in [15]: given a sequence $\{f_n : H \mapsto B,\ n \in \mathbb{N}\cup\{\infty\}\}$ of bounded continuous functions from H to a Banach space B, we write

$$f_n \xrightarrow{\pi} f_\infty \qquad \text{if and only if} \tag{17}$$

$$\forall\, x \in H \quad |f_\infty(x) - f_n(x)|_B \to 0 \quad \text{as } n \to \infty \quad \& \quad \sup_{n\in\mathbb{N}} \sup_{x\in H} |f_n(x)|_B < \infty.$$

Given a metric space K, we will denote by $\mathscr{B}(K)$ the Borel σ-algebra of K. A positive σ-additive measure on the measurable space $(K, \mathscr{B}(K))$ will be called shortly a measure on K. If K and J are two metric spaces, $T : K \mapsto J$ is a measurable map and μ is a measure on K, then we will denote by $T^*\mu$ the unique measure on J such that $T^*\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathscr{B}(J)$.

In the sequel, we will consider finite dimensional subspaces H_n of H. Splitting $H \cong H_n \oplus (H_n)^\perp$ and denoting by δ_0 the Dirac measure at 0 on $(H_n)^\perp$, we will extend canonically a finite measure μ^n on H_n to the finite measure $\mu^n \otimes \delta_0$, again denoted by μ^n, on H.

If $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H and $H_n := \text{Span}\langle e_1, \ldots, e_n\rangle$ for all $n \in \mathbb{N}$, then there exist canonical projections $\Pi_{n\to k} : H_n \mapsto H_k$, and $\Pi_k : \mathbb{R}^{\mathbb{N}} \mapsto H_k$, for all $n \geq k$. Given for all $n \in \mathbb{N}$ a finite measure μ^n on H_n, the system $\{\mu^n\}_{n\in\mathbb{N}}$ is said to be *projective* if $(\Pi_{n\to k})^*\mu^n = \mu^k$ for all $n \geq k$.

We will use Kolmogorov's Extension Theorem for countable products:

Theorem 1. *Given a projective system $\{\mu^n\}_{n\in\mathbb{N}}$ on $(\{H_n\},\{\Pi_{n\to k}\}_{n\geq k})$, there exists a unique finite measure $\overline{\mu}$ on $\mathbb{R}^{\mathbb{N}}$ such that $(\Pi_k)^*\overline{\mu} = \mu^k$ for all $k \in \mathbb{N}$.*

We will say that a set $\mathscr{F}$ of real Borel functions on H are a *determining class* if two probability measures on H, μ_1 and μ_2, are equal if and only if $\int_H f\, d\mu_1 = \int_H f\, d\mu_2$ for all $f \in \mathscr{F}$.

3. Kolmogorov Equation

In this section we study Kolmogorov Equations associated to the operator $\mathscr{L}$ defined in (14). In Theorems 2 and 3, whose proofs are postponed to Section 6, we state Positivity and Schauder-type regularity for the resolvent operators $(\lambda - \mathscr{L})^{-1} : C_b(H) \mapsto C_b(H)$, $\lambda > 0$. Then we prove the existence of a semigroup, acting on a subspace X of $C_b(H)$, naturally associated to (3). Notice that a direct resolution of the parabolic equation

$$u_t = \mathscr{L}u \qquad u(0) = \varphi \in C_b(H) \tag{18}$$

seems to be very hard (see Remark 2 below). On the other hand, solving the related elliptic equation $(\lambda - \mathscr{L})u = f$ for all $f \in C_b^\theta(H)$, yields:

1. There exists, if any, a unique weakly right-continuous semigroup in $C_b(H)$ whose infinitesimal generator is an extension of $\mathscr{L}$.

2. If X is the closure of $D(\mathscr{L})$ in $C_b(H)$, then there exists a strongly continuous semigroup $(P_t^X)_{t\geq 0}$ acting on X, whose infinitesimal generator is an extension of the part of $\mathscr{L}$ in X.

We consider the following elliptic equation, where $\mathscr{L}$ is defined as in (14):

$$\lambda u - \mathscr{L} u \;=\; f \tag{19}$$

with $f \in C_b^\theta(H)$, $\lambda > 0$, $u \in D(\mathscr{L})$. The proof of the following two Theorems is postponed to section 6.

Theorem 2 (Maximum Principle). *For any $f \in C_b^\theta(H)$ and $u \in D(\mathscr{L})$ such that equation (19) holds, we have*

$$\text{a. } \|u\|_0 \;\leq\; \frac{1}{\lambda}\|f\|_0, \qquad \textit{and} \qquad \text{b. } f \geq 0 \Longrightarrow u \geq 0 \tag{20}$$

Theorem 3 (Schauder Estimate). *There exists a constant $C = C(\lambda) > 0$ such that for all $f \in C_b^\theta(H)$ and $u \in D(\mathscr{L})$ such that equation (19) holds, we have*

$$\|\mathscr{M} u\|_\theta \;+\; \|u\|_{2+\theta} \;\leq\; C\,\|f\|_\theta \tag{21}$$

It is a standard fact that using the classical Continuity Method, Theorem 3 yields an existence Theorem for solutions to (19). We give here the proof, which is well known, because we will repeat it several times in the sequel, in order to prove several properties of the operator $R(\lambda, \mathscr{L}) := (\lambda - \mathscr{L})^{-1}$.

Theorem 4 (Continuity Method). *For all $f \in C_b^\theta(H)$ there exists one and only one $u \in D(\mathscr{L})$ such that* (19) *holds.*

Proof. For all $\alpha \in [0, 1]$ we introduce the operator

$$D(\mathscr{L}_\alpha) \;:=\; D(\mathscr{L}), \qquad \mathscr{L}_\alpha u \;:=\; \mathscr{M} u \;+\; \alpha \mathscr{Q} u \tag{22}$$

We introduce the set Λ of all $\alpha \in [0, 1]$ such that the equation

$$\lambda u - \mathscr{L}_\alpha u \;=\; f \tag{23}$$

has a unique solution $u = R(\lambda, \mathscr{L}_\alpha) f$ for all $f \in C_b^\theta(H)$. We will prove that there exists $\delta > 0$ such that if $\alpha_0 \in \Lambda$, $\alpha \in [0, 1]$ and $|\alpha - \alpha_0| \leq \delta$, then $\alpha \in \Lambda$. Since $0 \in \Lambda$, this will prove the thesis.

Let $\alpha_0 \in \Lambda$, and let u_0 be the corresponding solution to (23). We are going to show that equation (23) has a solution for all α close to α_0. We set

$$\gamma_f \,:=\; C_b^{2+\theta}(H) \mapsto C_b^{2+\theta}(H) \quad \gamma_f(v) := R(\lambda, \mathscr{L}_{\alpha_0})\,(f + (\alpha - \alpha_0)\mathscr{Q} v) \tag{24}$$

In virtue of Theorem 3 this map is well defined and continuous, and $u = \gamma_f(v)$ is the solution of $\lambda u - \mathscr{L}_{\alpha_0} u = f + (\alpha - \alpha_0)\mathscr{Q} v$.

Moreover, u is a solution to (23) if and only if u is a fixed point of γ_f. Let $u = \gamma_f(v)$ and $\overline{u} = \gamma_f(\overline{v})$. We set $\mathscr{K}_1 := \|G\|_{C_b^\theta(H;\mathscr{L}_1(H))} + \|F\|_{C_b^\theta(H;H)}$. Then by Theorem 3 there exists $C > 0$ such that

$$\|u - \overline{u}\|_{2+\theta} \leq\; C\mathscr{K}_1\,|\alpha_0 - \alpha|\;\|v - \overline{v}\|_{2+\theta}.$$

Setting $\delta_1 := 1/(2C\mathscr{K}_1)$, we obtain that if $|\alpha - \alpha_0| \le \delta_1$, γ_f is a contraction in $C_b^{2+\theta}(H)$. The conclusion follows, since δ_1 does not depend on α_0. □

We define now X := closure of $D(\mathscr{L})$ in $C_b(H)$. It is known that $X \neq C_b(H)$, since the semigroup $(R_t)_{t\ge0}$ is not strongly continuous: see [4]. We denote by $(\mathscr{L}_X, D(\mathscr{L}_X))$ the part of $\mathscr{L}$ in X: $D(\mathscr{L}_X) := \{u \in D(\mathscr{L}) : \mathscr{L}u \in X\}$, $\mathscr{L}_X u := \mathscr{L}u$ for all $u \in D(\mathscr{L}_X)$. Then we have the following

Proposition 1. *$D(\mathscr{L}_X)$ is dense in X.*

Proof. It is enough to prove that $D(\mathscr{L})$ is contained in the closure of $D(\mathscr{L}_X)$. Let $x \in D(\mathscr{L})$. Then for all $\lambda > 0$ there is $y_\lambda \in D(\mathscr{L})$ such that $\lambda y_\lambda - \mathscr{L}y_\lambda = x$, and moreover $y_\lambda \in D(\mathscr{L}_X)$. Then $y_\lambda = (\lambda - \mathscr{L}_X)^{-1}x$ and $\mathscr{L}_X y_\lambda = (\lambda - \mathscr{L})^{-1}\mathscr{L}x$. It follows, by (20.a),

$$\|\lambda y_\lambda - x\| = \|\mathscr{L}_X y_\lambda\| \le \frac{1}{\lambda}\|\mathscr{L}x\| \to 0 \text{ as } \lambda \to +\infty. \qquad \square$$

Corollary 1. *The operator $(\mathscr{L}_X, D(\mathscr{L}_X))$ is closable in X: we will denote by $\overline{\mathscr{L}_X}$ its closure. The unique continuous linear extension of $(\lambda - \mathscr{L}_X)^{-1}$ to X is one-to-one and coincides with $(\lambda - \overline{\mathscr{L}_X})^{-1}$ for all $\lambda > 0$.*

Proof. The Corollary follows from the following Closure Lemma (I.4.3 in [6]): if $(\mathscr{J}, D(\mathscr{J}))$ is a dissipative linear operator in a Banach space E, and if both $D(\mathscr{J})$ and $(I - \mathscr{J})D(\mathscr{J})$ are dense, then $\mathscr{J}$ is closable in E. □

Corollary 2. *By Hille-Yosida Theorem, $\overline{\mathscr{L}_X}$ generates a strongly continuous semigroup in X, that we call $(P_t^X)_{t\ge0}$.*

Remark 1. Notice that X is closed and strictly smaller than $C_b(H)$, so that we have a priori no way to extend $(P_t^X)_{t\ge0}$ to $C_b(H)$. This problem will be solved in Corollary 4 below.

Remark 2. Notice also that very little can be said about $D(\mathscr{L}_X)$ beyond its abstract definition: this shows why a direct study of the parabolic equation (18) seems so hard.

It is now a standard fact (see [5], [11], [17]) that Theorem 4 and the density of $C_b^\theta(H)$ in $C_b(H)$ imply the following

Theorem 5. *For all $x \in H$ and $T \ge 0$, there exists at most one stochastically continuous solution on $[0, T]$ to the martingale problem (3).*

However, we have proven more than probabilistic uniqueness:

Theorem 6. *There exists at most one weakly right-continuous semigroup in $C_b(H)$ (respectively in X) whose infinitesimal generator extends $\mathscr{L}$ (resp. $\mathscr{L}_X$).*

4. The kernel of measures for the Resolvent

This section is devoted to the proof of the following

Theorem 7. *For all* $\lambda > 0,\ x \in H$, *there exists a finite measure on* $H\ \ \rho(\lambda, x, \cdot)$ *such that*

$$R(\lambda, \mathscr{L})\varphi(x) = \int_H \varphi(y)\ \rho(\lambda, x, dy) \qquad\qquad \forall\, \varphi \in C_b^\theta(H) \tag{25}$$

Moreover, for all $A \in \mathscr{B}(H)$, *the map* $(\lambda, x) \mapsto \rho(\lambda, x, A)$ *is measurable.*

The main tools will be: 1. Feller Property: $R(\lambda, \mathscr{L})(C_b(H)) \subset C_b(H)$; 2. Positivity (20.b); 3. Schauder-type regularity (21) of $R(\lambda, \mathscr{L})$.

We fix a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ in H. For all $n \in \mathbb{N}$, let $H_n :=$ Span$\langle e_1,\ldots,e_n\rangle$. We will use the identification of $H \cong l^2 \subset \mathbb{R}^{\mathbb{N}}$ given by $\{e_k\}$. For all $\varphi \in C_b(\mathbb{R}^n)$ we set

$$\varphi^H \in C_b(H), \qquad \varphi^H(z) := \varphi(\langle z, e_1\rangle, \ldots, \langle z, e_n\rangle)$$

$$\mathscr{F}^\theta := \{\varphi^H : \varphi \in C_b^\theta(\mathbb{R}^n)\}, \quad \mathscr{F}^\infty := \{\varphi^H : \varphi \in C_0^\infty(\mathbb{R}^n)\} \tag{26}$$

Notice that $C_0^\infty(\mathbb{R}^n) \oplus \mathbb{R}$ can be identified with a dense subset of all C^∞ functions on the n-dimensional sphere S^n, which is compact. Therefore, by Riesz Representation Theorem for positive distributions on compact spaces, we have that for all $n \in \mathbb{N}$ there exists a finite measure $r^n(\lambda, x, \cdot)$ on $\mathbb{R}^n$ which is equal to the functional $C_0^\infty(\mathbb{R}^n) \ni \varphi \mapsto R(\lambda, \mathscr{L})\varphi^H(x)$.

Remark 3. By the positivity we know that $r^n(\lambda, x, \mathbb{R}^n) \le 1/\lambda$. However, since $1 \notin C_0^\infty(\mathbb{R}^n)$, it might happen that $r^n(\lambda, x, \mathbb{R}^n) < 1/\lambda$. Moreover, it is not evident that the system $\{r^n(\lambda, x, \cdot) : n \in \mathbb{N}\}$ is projective. Indeed, if $f \in C_0^\infty(\mathbb{R}^n)$, then setting $\overline{f} \in C_b(\mathbb{R}^{n+1})$, $\overline{f}(x_1, \ldots, x_{n+1}) := f(x_1, \ldots, x_n)$, we have that $\overline{f} \notin C_0^\infty(\mathbb{R}^{n+1})$, so that we can *not* say that

$$\int_{\mathbb{R}^n} f(y)\, r^n(\lambda, x, dy) \ = \ \int_{\mathbb{R}^{n+1}} \overline{f}(y)\, r^{n+1}(\lambda, x, dy) \qquad \Box$$

In order to obtain a measure $\rho(\lambda, x, \cdot)$ on H such that (25) holds, we have to prove that:

1. $r^n(\lambda, x, \mathbb{R}^n) = 1/\lambda$ for all $n \in \mathbb{N}$. The system $\{r^n(\lambda, x, \cdot)\}_{n\in\mathbb{N}}$ is projective and therefore determines a measure $\overline{r}(\lambda, x, \cdot)$ on $\mathbb{R}^{\mathbb{N}}$
2. The measure $\overline{r}(\lambda, x, \cdot)$ is concentrated on the space $l^2 \subset \mathbb{R}^{\mathbb{N}}$, which can be identified with H by means of the chosen orthonormal system
3. The measure $\rho(\lambda, x, \cdot)$ on H determined by the restriction of $\overline{r}(\lambda, x, \cdot)$ to $\mathscr{B}(H)$, satisfies (25).

We will prove that properties 1-3 hold in our case, using a unifying technique, namely the transfer of regularity properties from the resolvent $R(\lambda, \mathscr{M})$ of the Ornstein-Uhlenbeck operator to $R(\lambda, \mathscr{L})$, by means of the Continuity Method of Theorem 4.

However, we stress that the existence of the measures $r^n(\lambda, x, \cdot)$ follows only from the Positivity and the Feller property of the operator $R(\lambda, \mathscr{L})$, while each of the properties 1–3 could fail to hold for a generic positive operator in $C_b(H)$, as the following remarks show.

Remark 4. Recall the definition given in (26), and consider the closure $\mathscr{F}$ of the space $\mathscr{F}^\theta$ in $C_b(H)$. If we denote by $(K_t^n)_{t\geq 0}$ the Heat semigroup on $C_b(\mathbb{R}^n)$, then we can define $K_t^H \varphi^H := (K_t^n \varphi)^H$ for all $\varphi \in C_b^\theta(\mathbb{R}^n)$. $(K_t^H)_{t\geq 0}$ is a well defined contraction semigroup on $\mathscr{F}^\theta$, and therefore has a unique extension to a contraction semigroup on $\mathscr{F}$. For all $t > 0$ and $x \in H$, the map $C_b(\mathbb{R}^n) \ni \varphi \mapsto K_t^H \varphi^H(x)$ is equal to a Gaussian measure $\mu^n(t,x)$ on $\mathbb{R}^n$, and the system $\{\mu^n(t,x)\}_{n\in\mathbb{N}}$ is clearly projective, but it is well known that the measure $\mu(t,x)$ that it determines on $\mathbb{R}^\mathbb{N}$ is concentrated on $\mathbb{R}^\mathbb{N}\backslash l^2$, and therefore property 2 can not hold. □

Remark 5. Property 3 can fail to hold even if property 2 holds. Suppose that, for a given positive functional $T : C_b(H) \mapsto \mathbb{R}$, there exists a finite measure m on H which is equal to T on $\mathscr{F}^\infty$. Two such measures must be equal, since $\mathscr{F}^\infty$ is a determining class. But the same uniqueness does *not* hold for positive functionals on $C_b(H)$: indeed, notice that there exist functions $g \in C_b(H)$ such that

$$g \geq 0 \quad \& \quad g \neq 0 \quad \& \quad \sup\left\{\varphi(x) : \varphi \in \mathscr{F}^\infty \,\&\, \varphi \leq g\right\} = 0 \quad \forall\, x \in H,$$

for instance, any $g(x) = a(|x|)$ with $a : \mathbb{R} \mapsto \mathbb{R}$ continuous, $a \geq 0$, $a \equiv 0$ outside a bounded interval and $a(0) = 1$. By Hahn-Banach Theorem for positive functionals (see [2], chap. II, § 3, n. 4, Proposition 6) we can define a functional $\hat{T} = T$ on $\mathscr{F}^\infty$, $\hat{T} g = \alpha$ for any $\alpha \in [0, \sup g]$ and extend it to a *positive* linear functional on $C_b(H)$. This implies that there exist *infinitely many* positive functionals $\hat{T}$ which are equal to T and therefore to m on $\mathscr{F}^\infty$, but $\hat{T} \not\equiv T$ on $C_b(H)$. □

Remark 6. Suppose now that the functional T is of the form $T\varphi = T(\lambda, x)\varphi = R(\lambda, \mathscr{L})\varphi(x)$. The positivity of T could be combined with some algebraic relations linking the functionals $\{T(\lambda, x)\}_{\lambda>0,x}$, such as the Resolvent formula. However, this formula holds for the functionals $\{T(\lambda, x)\}$, but possibly *not* for the measures $\{m(\lambda, x)\}$, since the space $\mathscr{F}^\infty$, on which $T(\lambda, x) = m(\lambda, x)$, is not invariant for the operators $\{R(\lambda, \mathscr{L})\}_{\lambda>0}$. □

We come to the proof of Theorem 7. Recall the definition (22) of $\mathscr{L}_\alpha$: for all $\alpha \in [0, 1]$, set $R(\lambda, \mathscr{L}_\alpha) := (\lambda - \mathscr{L}_\alpha)^{-1} : C_b^\theta(H) \mapsto D(\mathscr{L})$.

Proposition 2. *For all $n \in \mathbb{N}$, there exists a set of finite positive measures on $\mathbb{R}^n$ $\{r_\alpha^n(\lambda, x, \cdot) : \lambda > 0, x \in H\}$, such that:*

1. For all $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$R(\lambda, \mathscr{L}_\alpha)\varphi^H(x) = \int_{\mathbb{R}^n} \varphi(y)\, r_\alpha^n(\lambda, x, dy) \tag{27}$$

2. $r_\alpha^n(\lambda, x, \mathbb{R}^n) \leq R(\lambda, \mathscr{L}_\alpha)1(x) = \dfrac{1}{\lambda}.$

Proof. The first assertion follows applying Riesz Representation Theorem to the positive functional $C_0^\infty(\mathbb{R}^n) \ni \varphi \mapsto T\varphi := R(\lambda, \mathscr{L}_\alpha)\varphi^H(x)$. The second assertion follows from the positivity of T and the fact that $T(1) = 1/\lambda$. □

We want now to prove that the system $\{r_\alpha^n(\lambda, x, \cdot) : n \in \mathbb{N}\}$ is projective. By Remark 3, this does not follow from Proposition 2.

We prove our first transfer result of properties from Ornstein-Uhlenbeck operator to operator $(\mathscr{L}, D(\mathscr{L}))$.

Lemma 1. *If* $\{f, f_n : n \in \mathbb{N}\} \subset C_b^\theta(H)$, $f_n \xrightarrow{\pi} f$ *and* $\sup_n \|f_n\|_\theta < \infty$, *then* $R(\lambda, \mathscr{L})f_n \xrightarrow{\pi} R(\lambda, \mathscr{L})f$.

Remark 7. Daniell's Theorem, Thm. 2.5.5. in [10], says that a positive linear functional $T : C_b(H) \mapsto \mathbb{R}$ can be extended to a positive finite measure on $\mathscr{B}(H)$ if and only if: $Tf_n \uparrow Tf$ for all sequences $f_n, f \in C_b(H)$ such that $f_n \uparrow f$, where $\uparrow$ denotes monotone non-decreasing pointwise convergence. However, (20) gives only continuity of T through uniform convergence. In Lemma 1 we show that the continuity of T with respect to an intermediate convergence, namely the π-convergence of *uniformly* Hölder-continuous functions, is a consequence of Schauder Estimates and the properties of the kernel of the O.U. semigroup (12).

Proof of Lemma 1. Consider the set Λ of all $\alpha \in [0, 1]$ such that

$$\{f, f_n : n \in \mathbb{N}\} \subset C_b^\theta(H),$$
$$f_n \xrightarrow{\pi} f \text{ and } \sup_n \|f_n\|_\theta < \infty \Longrightarrow R(\lambda, \mathscr{L}_\alpha)f_n \xrightarrow{\pi} R(\lambda, \mathscr{L}_\alpha)f,$$
$$\nabla R(\lambda, \mathscr{L}_\alpha)f_n \xrightarrow{\pi} \nabla R(\lambda, \mathscr{L}_\alpha)f, \text{ and } D^2R(\lambda, \mathscr{L}_\alpha)f_n \xrightarrow{\pi} D^2R(\lambda, \mathscr{L}_\alpha)f.$$

First, $0 \in \Lambda$: indeed, if $\sup_n \|f_n\|_\theta < \infty$, then for all $x \in H$ (see (47) and (48) below and [8], § 9.4.1):

$$\sup_{n,t} t^{1/2}|\nabla R_t f_n(x)| < \infty, \qquad \sup_{n,t} t^{1-\theta/2}\|D^2R_t f_n(x)\| < \infty$$

$$\nabla R(\lambda, \mathscr{M})f_n(x) = \int_0^\infty \frac{e^{-\lambda t}}{t^{1/2}} \left(t^{1/2}\nabla R_t f_n(x)\right) dt$$

$$D^2R(\lambda, \mathscr{M})f_n(x) = \int_0^\infty \frac{e^{-\lambda t}}{t^{1-\theta/2}} \left(t^{1-\theta/2}D^2R_t f_n(x)\right) dt$$

and the conclusion follows from the Dominated Convergence Theorem.

Moreover, Λ is open: arguing as in the proof of Lemma 3, we take $\alpha_0 \in \Lambda$ and $\alpha \in [0, 1]$ such that the mappings γ_{f_n} defined in (24) are contractions in $C_b^{2+\theta}(H)$. We write

$$R(\lambda, \mathscr{L}_\alpha)(f - f_n) = (R(\lambda, \mathscr{L}_\alpha)f - (\gamma_f)^k(0)) + ((\gamma_f)^k(0) - (\gamma_{f_n})^k(0))$$
$$+ ((\gamma_{f_n})^k(0) - R(\lambda, \mathscr{L}_\alpha)f_n) =: I_1 + I_2 + I_3$$

One can prove inductively on k that for all n, $\gamma_{f_n}(0) = R(\lambda, \mathscr{L}_{\alpha_0})f_n$ and

$$(\gamma_{f_n})^{k+1}(0) = R(\lambda, \mathscr{L}_{\alpha_0})f_n + (\alpha - \alpha_0)R(\lambda, \mathscr{L}_{\alpha_0})\mathscr{Q}\left[(\gamma_{f_n})^k(0)\right]$$

and, setting $\mathscr{K}_1 := \|G\|_{C_b^\theta(H;\mathscr{L}_1(H))} + \|F\|_{C_b^\theta(H;H)}$, by Theorem 3 there exists $C > 0$ such that

$$\|(\gamma_{f_n})^{k+1}(0))\|_{2+\theta} \le C\left(\|f_n\|_\theta + |\alpha-\alpha_0|\,\mathscr{K}\,\|(\gamma_{f_n})^k(0)\|_{2+\theta}\right)$$
$$\le \left(\sum_{i=0}^{k}\left[|\alpha-\alpha_0|C\,\mathscr{K}\right]^i\right)\|f_n\|_\theta$$

Since $\alpha_0 \in \Lambda$, one obtains by induction on k that for all $x \in H$ and $k \ge 1$,

$$(\gamma_{f_n})^{k+1}(0)(x) \xrightarrow{\pi} (\gamma_f)^{k+1}(0)(x)\,,\ \nabla(\gamma_{f_n})^{k+1}(0)(x) \xrightarrow{\pi} \nabla(\gamma_f)^{k+1}(0)(x),$$

$$\text{and}\quad D^2(\gamma_{f_n})^{k+1}(0)(x) \xrightarrow{\pi} D^2(\gamma_f)^{k+1}(0)(x),\quad \text{as } n\to\infty$$

By Theorem 4, if $|\alpha-\alpha_0| \le \delta_1 = 1/(2C\mathscr{K})$ the terms I_1 and I_3 defined above converge to 0 in $C_b^{2+\theta}(H)$ uniformly in n as $k\to\infty$. Therefore, for all $\eta > 0$ we can find k_0 such that $(|I_1|+|I_3|)(k_0,n) < \eta$ for all n. Fixing k_0 and letting $I_2(k_0,n)\to 0$ as $n\to\infty$, we obtain that $\alpha\in\Lambda$. Since δ does not depend on α_0 and $0\in\Lambda$, $\Lambda=[0,1]$. □

Corollary 3. *Formula (27) holds for all* $\varphi \in C_b^\theta(\mathbb{R}^n)$ *and for all* $\lambda > 0$, $x \in H$, $\alpha\in[0,1]$. *In particular,* $r_\alpha^n(\lambda,x,\mathbb{R}^n) = 1/\lambda$ *for all* $n\in\mathbb{N}$ *and the system* $\{r_\alpha^n(\lambda,x,\cdot) : n\in\mathbb{N}\}$ *is projective.*

Proof. Fix $n\in\mathbb{N}$, and let $f\in C_b^\theta(\mathbb{R}^n)$. Take any sequence $\{f_k\}\subset C_0^\infty(\mathbb{R}^n)$ such that $\sup_k \|f_k\|_\theta < \infty$ and $f_k \xrightarrow{\pi} f$. Then by the Dominated Convergence Theorem and Lemma 1

$$\int_{\mathbb{R}^n} f(y)\, r_\alpha^n(\lambda,x,dy) = \lim_k \int_{\mathbb{R}^n} f_k(y)\, r_\alpha^n(\lambda,x,dy)$$
$$= \lim_k R(\lambda,\mathscr{L}_\alpha) f_k^H(x) \;=\; R(\lambda,\mathscr{L}_\alpha) f^H(x) \qquad \square$$

By Kolmogorov's Extension Theorem and Corollary 3, we have that there exists a unique finite measure $\overline{r}_\alpha(\lambda,x,\cdot)$ on $\mathbb{R}^{\mathbb{N}}$ endowed with the product σ-algebra, having $r_\alpha^n(\lambda,x,\cdot)$ as marginal distributions. Notice that every $\varphi\in\mathscr{F}^\theta$ has a unique continuous extension $\overline{\varphi}$ on $\mathbb{R}^{\mathbb{N}}$. Then formula (27) becomes

$$R(\lambda,\mathscr{L}_\alpha)\varphi(x) \;=\; \int_{\mathbb{R}^{\mathbb{N}}} \overline{\varphi}(z)\ \overline{r}_\alpha(\lambda,x,dz) \qquad \forall\,\varphi\in\mathscr{F}^\theta \tag{28}$$

What we have to prove now, is that $\overline{r}_\alpha(\lambda,x,\cdot)$ is concentrated on H. The key observation is the following: if we are given a measure m on a measurable space (Y,Σ) and a measurable positive function $\psi : Y \mapsto [0,\infty]$, then

$$\int_Y \psi\, dm \;<\infty \implies \psi < \infty \quad m-\text{almost everywhere} \tag{29}$$

Therefore, if we prove that the function

$$\Psi : \mathbb{R}^{\mathbb{N}} \mapsto [0,\infty], \qquad z=\{z_k\} \mapsto \Psi(z) := \sum_{i=1}^{\infty} |z_i|^2$$

has finite $\overline{r}_\alpha(\lambda, x, \cdot)$-integral, then $\mathbb{R}^{\mathbb{N}} \backslash H = \{z \in \mathbb{R}^{\mathbb{N}} : \Psi(z) = \infty\}$ has $\overline{r}(\lambda, x, \cdot)$-measure equal to 0. We set:

$$\Psi_n : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R} \qquad\qquad z=\{z_k\} \mapsto \Psi_n(z) := \sum_{i=1}^{n} |z_i|^2 \wedge n \tag{30}$$

$$\psi : H \mapsto \mathbb{R} \qquad \psi := \Psi_{|H} = |\cdot|^2 \qquad\qquad \psi_n \in C_b(H) \qquad \psi_n := (\Psi_n)_{|H} \tag{31}$$

Lemma 2. *If there exists $C(\alpha) > 0$ and $\lambda > 0$ such that*

$$\sup_n R(\lambda, \mathscr{L}_\alpha)\psi_n(x) \le \frac{C(\alpha)}{\lambda}(1+\psi(x)) \qquad \forall x \in H \tag{32}$$

then $\overline{r}_\alpha(\lambda, x, \cdot)$ restricts to a finite measure $\rho_\alpha(\lambda, x, \cdot)$ on $(H, \mathscr{B}(H))$. Moreover

$$\int_H |z|^2 \, \rho_\alpha(\lambda, x, dz) \le \frac{C(\alpha)}{\lambda}(1+|x|^2) \tag{33}$$

Proof. We will prove that if (32) holds, then the integral of Ψ on $(\mathbb{R}^{\mathbb{N}}, \overline{r}_\alpha(\lambda, x, \cdot))$ is finite. Notice that $\{\Psi_n\}$ is an increasing sequence of bounded measurable functions on $\mathbb{R}^{\mathbb{N}}$ such that $\Psi = \sup_{n\in\mathbb{N}} \Psi_n$. Since $\psi_n \in \mathscr{F}$, by formula (28) we have for all n

$$R(\lambda, \mathscr{L}_\alpha)\psi_n(x) = \int_{H_n} \psi_n(y)\, r_\alpha^n(\lambda, x, dy) = \int_{\mathbb{R}^{\mathbb{N}}} \Psi_n(z)\, \overline{r}_\alpha(\lambda, x, dz).$$

By (32), we have that $\lambda R(\lambda, \mathscr{L}_\alpha)\psi_n \le C(\alpha)(1+\psi)$ for all $n \in \mathbb{N}$, so by Beppo-Levi Theorem

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{N}}} \Psi(z)\, \overline{r}_\alpha(\lambda, x, dz) &= \sup_n \int_{\mathbb{R}^{\mathbb{N}}} \Psi_n(z)\, \overline{r}_\alpha(\lambda, x, dz) \\ &= \sup_n \, R(\lambda, \mathscr{L}_\alpha)\psi_n(x) \le \frac{C(\alpha)}{\lambda}(1+\psi(x)) < +\infty\end{aligned}$$

and the assertion follows. Notice now that if Σ is the Borel σ-algebra of $\mathbb{R}^{\mathbb{N}}$, then the trace of Σ on H is the Borel σ-algebra of H, $\mathscr{B}(H)$. Therefore, $\bar{r}_\alpha(\lambda, x, \cdot)$ restricts to a well defined finite measure $\rho_\alpha(\lambda, x, \cdot)$ on $(H, \mathscr{B}(H))$. □

Lemma 3. *There exists $C \ge 1$ such that $\forall x \in H, \lambda > 0, \alpha \in [0,1]$*

$$\begin{aligned}&\sup_n R(\lambda, \mathscr{L}_\alpha)\psi_n(x) + \sup_n |\nabla R(\lambda, \mathscr{L}_\alpha)\psi_n(x)| \\ &\quad + \sup_n \|D^2 R(\lambda, \mathscr{L}_\alpha)\psi_n(x)\| \le \frac{C}{\lambda}(1+\psi(x))\end{aligned} \tag{34}$$

Proof. An explicit computation shows that there exists $C \geq 1$ such that (34) holds for $\alpha = 0$. Suppose that for $\alpha_0 \in [0, 1]$ there exists $C(\alpha_0) \geq 1$ such that $\forall x \in H$ and $\lambda > 0$

$$\begin{aligned} &\sup_n R(\lambda, \mathscr{L}_{\alpha_0})\psi_n(x) + \sup_n |\nabla R(\lambda, \mathscr{L}_{\alpha_0})\psi_n(x)| \\ &+ \sup_n \|D^2 R(\lambda, \mathscr{L}_{\alpha_0})\psi_n(x)\| \leq \frac{C(\alpha_0)}{\lambda}(1 + \psi(x)) \end{aligned} \tag{35}$$

Then Lemma 2 applies for α_0. We know that if $|\alpha - \alpha_0|$ is small enough, then $R(\lambda, \mathscr{L}_\alpha)\psi_n$ is the limit in $C_b^{2+\theta}(H)$ of $(\gamma_{\psi_n})^k(0)$ as $k \to \infty$, where γ_{ψ_n} is defined as in (24). By (35) and (33), we have by induction on $k \in \mathbb{N}$ that setting $\mathscr{K}_2 := \sup_x \|G(x)\|_{\mathscr{L}_1(H)} + \sup_x |F(x)|$

$$\begin{aligned} &(\gamma_{\psi_n})^k(0) + |\nabla(\gamma_{\psi_n})^k(0)| + \|D^2(\gamma_{\psi_n})^k(0)\| \\ &\leq \frac{C(\alpha_0)}{\lambda} \left\{ \left(\sum_{i=0}^k \left[|\alpha - \alpha_0| \mathscr{K}_2 \frac{C(\alpha_0)}{\lambda} \right]^i \right) (1 + \psi) + \sum_{j=1 \wedge k}^k \left[|\alpha - \alpha_0| \mathscr{K}_2 \right]^j \right\} \\ &\leq 3 \frac{C(\alpha_0)}{\lambda}(1 + \psi) \end{aligned}$$

provided $|\alpha - \alpha_0| \leq \delta_2 := 1/(2\mathscr{K}_2)$. We set $\delta := \delta_1 \wedge \delta_2$, where δ_1 was defined in the proof of Theorem 4. Now we can set in (35), $C(\alpha_0) := 3^{1/\delta} C(0)$ for all $\alpha_0 \in [0, 1]$, and the Lemma is proven. □

We can now prove that (27) holds *for all* $\varphi \in C_b^\theta(H)$: in fact, take $\varphi \in C_b^\theta(H)$, and set $\varphi_n \in \mathscr{F}_\theta$, $\varphi_n(x) := \varphi(x_n)$, where x_n is the projection of x onto H_n. Then $\sup_n \|\varphi_n\|_\theta \leq \|\varphi\|_\theta$, and $\varphi_n \xrightarrow{\pi} \varphi$, so by Lemma 1 and the Dominated Convergence Theorem,

$$\begin{aligned} \int_H \varphi(y)\, \rho_\alpha(\lambda, x, dy) &= \lim_k \int_H \varphi_k(y)\, \rho_\alpha(\lambda, x, dy) \\ &= \lim_k R(\lambda, \mathscr{L}_\alpha)\varphi_k(x) = R(\lambda, \mathscr{L}_\alpha)\varphi(x) \end{aligned}$$

The measurability of the map $(\lambda, x) \mapsto \rho_\alpha(\lambda, x, A)$ for all $A \in \mathscr{B}(H)$ follows as an application of the Monotone Class Theorem. If we set $\rho(\lambda, x, \cdot) := \rho_1(\lambda, x, \cdot)$, then Theorem 7 is proven. □

5. The kernel of measures for the Semigroup

In this section we prove the following

Theorem 8. *For all $t \geq 0$, $x \in H$, there exists a unique probability measure on H $\pi(t, x, \cdot)$ such that*

$$P_t^X \varphi(x) = \int_H \varphi(y)\, \pi(t, x, dy) \qquad \forall \varphi \in X = \overline{D(\mathscr{L})} \tag{36}$$

For all $A \in \mathscr{B}(H)$, the map $(t,x) \mapsto \pi(t,x,A)$ is measurable and the family $\{\pi(t,x,\cdot) : t, x\}$ satisfies the Chapman-Kolmogorov equation

$$\pi(t+s,x,B) \;=\; \int_H \pi(t,y,B)\,\pi(s,x,dy) \qquad \forall\, B \in \mathscr{B}(H),\ t,s \geq 0,\ x \in H. \tag{37}$$

Proof. We fix a complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$. Since $D(A^*)$ is dense in H, we can suppose that $\{e_k\}_{k\in\mathbb{N}} \subset D(A^*)$. Similarly to (26), we define the spaces

$$\mathscr{F}_0 := \{\varphi^H : \varphi \in C_b(\mathbb{R}^n),\, \exists \lim_{|x|\to 0} \varphi(x)\}, \quad \mathscr{F}^\infty := \{\varphi^H : \varphi \in C_0^\infty(\mathbb{R}^n)\}$$

$\{e_k\}_{k\in\mathbb{N}} \subset D(A^*)$ implies $\mathscr{F}^\infty \subset D(\mathscr{L})$ and therefore $\mathscr{F}_0 \subset X$. We also define functions $\psi_n \in C_b(H)$ as in (30), noting that now $\psi_n \in X$.

Two probability measures satisfying (36) must coincide: indeed, $\mathscr{F}_0$ is a determining class. This proves the uniqueness statement.

Since the operator $\overline{\mathscr{L}_X}$ generates the semigroup $(P_t^X)_{t\geq 0}$, then for all $f \in X$ we have by Hille-Yosida Theorem

$$P_t^X f \;=\; \lim_{m\to\infty}\left(I - \frac{t\,\overline{\mathscr{L}_X}}{m}\right)^{-m} f \;=\; \lim_{m\to\infty}\left[mR(m, t\,\overline{\mathscr{L}_X})\right]^m f \ \text{ in } X. \tag{38}$$

Formula (38) will be our basic tool to deduce properties of $(P_t^X)_{t\geq 0}$ from the properties of $\{R(\lambda,\mathscr{L})\}_{\lambda>0}$ proven in the previous section.

Arguing as section 4, we obtain that for all $n \in \mathbb{N}$, there exists a measure $p^n(t,x,\cdot)$ on $\mathbb{R}^n$ with total mass less or equal to 1, such that

$$P_t^X \varphi^H(x) \;=\; \int_{\mathbb{R}^n} \varphi(y)\; p^n(t,x,dy) \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n). \tag{39}$$

For any $f \in C_b(\mathbb{R}^n)$ such that $\exists \lim_{|x|\to\infty} f(x)$, consider a sequence $\{f_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that $f_k \xrightarrow{\pi} f$. Then $f^H,\, f_k^H \in \mathscr{F}_0$ and by (38)

$$P_t^X f^H - P_t^X f_k^H \;=\; (P_t^X f^H - \left[mR(m,t\,\overline{\mathscr{L}_X})\right]^m f^H)$$

$$+\; \left[mR(m,t\,\overline{\mathscr{L}_X})\right]^m (f^H - f_k^H) \;+\; (\left[mR(m,t\,\overline{\mathscr{L}_X})\right]^m f_k^H - P_t^X f_k^H)$$

Arguing as in the proof of Lemma 1, we obtain that $P_t^X f_k^H \to P_t^X f^H$ as $k\to\infty$. This implies that for all $t, x,\ p^n(t,x,H_n) = 1$ and the family $\{p^n(t,x,\cdot) : m \in \mathbb{N}\}$ is projective and determines a unique probability measure $\overline{p}(t,x,\cdot)$ on $\mathbb{R}^{\mathbb{N}}$. Moreover, by (38)

$$P_t^X \psi_n \;=\; (P_t^X \psi_n - \left[mR(m,t\,\overline{\mathscr{L}_X})\right]^m \psi_n) + \left[mR(m,t\,\overline{\mathscr{L}_X})\right]^m \psi_n \;=:\; I_1 + I_2 \tag{40}$$

By (34), $I_2 \leq C(1+\psi)$ uniformly in m, n, and I_1 tends to 0 as $m\to\infty$, so we get

$$\sup_n P_t^X \psi_n \;\leq\; C(1+\psi) \;<\; \infty$$

and arguing as in the proof of Lemma 2, $\overline{p}(t, x, \cdot)$ is concentrated on H and restricts to a well defined measure $\pi(t, x, \cdot)$ on $(H, \mathscr{B}(H))$.

Formula (39) says that (36) holds for all $\varphi \in \mathscr{F}^\infty$. We prove now that (36) holds for all $\varphi \in X$.

Consider, for all $k \in \mathbb{N}$, a cut-off function $\chi_k \in C_0^\infty(\mathbb{R}^k)$ such that

$$0 \le \chi_k \le 1, \quad \chi_k \equiv 1 \text{ on } \{|z|_{\mathbb{R}^k} \le k\}, \quad \text{and} \quad \chi \equiv 0 \text{ on } \{|z|_{\mathbb{R}^k} \ge k+1\}.$$

For any $\varphi \in X$, set $\varphi_k(x) := \varphi(x_k)\chi_k(x_k)$, where x_k is the projection of $x \in H$ onto H_k. Then $\varphi_k \in \mathscr{F}_0$ and $\varphi_k \xrightarrow{\pi} \varphi$. Now

$$\begin{aligned} P_t^X \varphi - P_t^X \varphi_k = (P_t^X \varphi - \left[mR(m, t\,\overline{\mathscr{L}_X})\right]^m \varphi) + \left[mR(m, t\,\overline{\mathscr{L}_X})\right]^m (\varphi - \varphi_k) \\ +(\left[mR(m, t\,\overline{\mathscr{L}_X})\right]^m \varphi_k - P_t^X \varphi_k) =: I_1 + I_2 + I_3 \end{aligned}$$

By (38), I_1 and I_3 converge to 0 uniformly in k as $m \to \infty$, while at fixed m I_2 converges to 0 as $k \to \infty$ by Theorem 7. The proof of (39) is complete. The measurability of the map $(t, x) \mapsto \pi(t, x, A)$, with $A \in \mathscr{B}(H)$, follows as an application of the Monotone Class Theorem, and Chapman Kolmogorov equation (37) is a consequence of the semigroup law of $(P_t^X)_{t \ge 0}$. Therefore, the proof of Theorem 8 is complete. □

We can now prove the existence result for the martingale problem (3). Recall that the coordinate process on $H^{[0,T]}$ is defined by

$$X_t : H^{[0,T]} \mapsto H, \qquad X_t(\omega) = \omega(t)$$

Theorem 9. *For all $x \in H$ and $T \ge 0$, there exists a solution $\mathbb{P}_x$ on $[0, T]$ to the martingale problem (3), such that for all $k \in D(A)$, the process $\{\langle \omega(t), k \rangle\}_t$ is cad-lag for all ω in a set $N \subset H^{[0,T]}$, with $\mathbb{P}_x(N) = 1$ for all $x \in H$. Moreover, the family $\{\mathbb{P}_x\}_x$ is measurable.*

Proof. It is standard from Theorem 8 that for all $T \ge 0$ there exists a unique Markov Process $\{\mathbb{P}_x\}_x$ on $H^{[0,T]}$ with transition function $\{\pi(t, x, \cdot)\}_{t,x}$. For the construction of the desired cad-lag modification of the coordinate process, we follow [16], Chapter III.

We take a countable dense subset of H, $Z = \{z_n\}_n$, and consider

$$h_\infty : H \mapsto \mathbb{R}, \qquad h_\infty(x) := |R(1, A)x|^2 + 1,$$

$$\mathscr{H}_1 := \left\{ h_{n,m,q} : H \mapsto \mathbb{R},\ h_{n,m,q}(x) := 1 + \sum_{k=1}^n |\langle z_m + qR(1, A)x, e_k \rangle|^2 \right\},$$

where $l, m, n \in \mathbb{N}$, $q \in \mathbb{Q}$. We set $\mathscr{H} := \mathscr{H}_1 \cup \{h_\infty\}$. Notice that $\mathscr{H}$ is contained in the set of continuous functions on H, but *not* in $C_b(H)$. However, we can define, by approximation with $C_b(H)$ functions, $\mathscr{L}h$ for every $h \in \mathscr{H}$. In particular, remember that in Lemma 3 it was proven that the measures $\rho(\lambda, x, \cdot)$ have integrable second moment.

We show now that for $\lambda > 0$ big enough, $\lambda h_\infty - \mathscr{L} h_\infty \geq 0$. Indeed,

$$\lambda h_\infty(x) - \mathscr{L} h_\infty(x) = \lambda h_\infty(x) - \mathrm{Tr}[R(1, A)(BB^* + G(x))R(1, A^*)]$$

$$-2\langle AR(1, A)x, R(1, A)x\rangle - 2\langle R(1, A)F(x), R(1, A)x\rangle.$$

Since $-\langle AR(1, A)x, R(1, A)x\rangle \geq 0$ by (4), we find, setting $w = |R(1, A)x|^2$,

$$\lambda h_\infty(x) - \mathscr{L} h_\infty(x) \geq \lambda w^2 - c_1 w + \lambda - c_2$$

with c_1, c_2 positive and independent of x. Therefore, for λ_0 big enough, $\lambda_0 h_\infty - \mathscr{L} h_\infty =: g_\infty \geq 0$. Since g_∞ is continuous and with quadratic growth, we can define by approximation $R(\lambda_0, \mathscr{L})g_\infty$ and prove that it coincides with h_∞. Then, the process $\{e^{-\lambda_0 t} h_\infty(X_t)\}_{t\geq 0}$ is a supermartingale and therefore has right limits along $\mathbb{Q}^+$ on a set $N_1 \subset H^{[0,T]}$ with $\mathbb{P}_x(N_1) = 1$ for all $x \in H$.

Arguing similarly for all functions $h \in \mathscr{H}_1$, we find a set $N \subset H^{[0,T]}$ with $\mathbb{P}_x(N) = 1$ for all $x \in H$, such that for all $\omega \in N$, $h \in \mathscr{H}$, the map $t \mapsto h(\omega(t))$ has right limits along $\mathbb{Q}^+$.

For $t \in \mathbb{R}^+$, denote by $\mathscr{S}_t$ the set of all sequences $\{t_n\}_n \subset \mathbb{Q}^+$, with $t_n \downarrow t$.

Suppose that $\omega \in N$ and $|R(1, A)\omega(t_n)| \to +\infty$ for $\{t_n\}_n \in \mathscr{S}_t$. Then, for any sequence $\{s_n\}_n \in \mathscr{S}_t$, we have

$$\lim_n |R(1, A)\omega(s_n)| = \lim_n |R(1, A)\omega(t_n)| = +\infty$$

and therefore, $\lim_{s\downarrow t, s\in\mathbb{Q}^+} |R(1, A)\omega(s)| = +\infty$. We denote by $M_1(t)$ the set of all such ω.

Take now $\omega \in N - M_1(t) =: M_2(t)$. Therefore, for some sequence $\{t_n\}_n \in \mathscr{S}_t$, and hence for all, $|R(1, A)\omega(t_n)|$ is bounded. Then, there exists, along some $\{s_n\}_n \in \mathscr{S}_t$, a limit $y \in H$ of $R(1, A)\omega(s_n)$ in the weak topology of H. Suppose that for $\{r_n\}_n \in \mathscr{S}_t$, $R(1, A)\omega(r_n) \rightharpoonup z \neq y$. Then, we can find $n, m \in \mathbb{N}$, $q \in \mathbb{Q}$ such that

$$\frac{3}{2} \leq 1 + \sum_{k=1}^{n} |\langle z_m + q \cdot y, e_k\rangle|^2, \qquad 1 + \sum_{k=1}^{n} |\langle z_m + q \cdot z, e_k\rangle|^2 \leq \frac{5}{4}.$$

We denote by h the corresponding $h_{n,m,q} \in \mathscr{H}_1$. By construction, we have

$$\frac{3}{2} \leq \lim_n h(\omega(s_n)) = \lim_n h(\omega(r_n)) \leq \frac{5}{4}$$

which is absurd. Hence, $\lim_{s\downarrow t, s\in\mathbb{Q}^+} R(1, A)\omega(t) = y$ in the weak topology.

Therefore, for all $\omega \in N$ and $t \in \mathbb{R}^+$, either $\lim_{s\downarrow t, s\in\mathbb{Q}^+} |R(1, A)\omega(s)| = +\infty$, or $\lim_{s\downarrow t, s\in\mathbb{Q}^+} R(1, A)\omega(s)$ exists in the weak topology. In other words, $\omega(s)$ has right weak limits with values in $D(A^{-1}) \cup \{\infty\}$, where $D(A^{-1})$ is the completion of H with respect to the norm $|R(1, A) \cdot|$. We define $\tilde{X}_t := \lim_{s\downarrow t, s\in\mathbb{Q}^+} X_s$, when such limit exists in the sense specified above, and $\tilde{X}_t := 0$ otherwise. We claim that for all t, $\tilde{X}_t = X_t$ a.s., which will also imply that $\tilde{X}_t$ takes values in H a.s.

Notice that the function $v_n(x) := |x| \wedge n$, $n \in \mathbb{N}$, is lower semicontinuous with respect to the weak topology of H, and the function $u_n(x) := v_n(R(1, A)x)$ belongs to X. Then we have for all $n \in \mathbb{N}$

$$\mathbb{E}[|R(1,A)\tilde{X}_t| \wedge n] \leq \liminf_{s\downarrow t, s\in\mathbb{Q}^+} \mathbb{E}[|R(1,A)X_s| \wedge n]$$

$$= \lim_{s\downarrow t, s\in\mathbb{Q}^+} \mathbb{E}[P^X_{t-s} u_n(X_t)] = \mathbb{E}[|R(1,A)X_t| \wedge n]$$

which implies $\mathbb{E}[|R(1,A)\tilde{X}_t|] \leq \mathbb{E}[|R(1,A)X_t|]$. Since the right hand side is finite, then $\mathbb{P}_x(M_1(t)) = 0$ for all $x \in H$ and $t \geq 0$.

Let u, v be functions in $C_b(H)$, v being cylindrical with respect to an orthonormal basis in $D(A)$, so that in particular: first $v \in X$, and secondly $R(1, A)x_n \rightharpoonup R(1, A)x$ in H implies $v(x_n) \to v(x)$. Then

$$\mathbb{E}[u(X_t)v(\tilde{X}_t)] = \lim_{s\downarrow t, s\in\mathbb{Q}^+} \mathbb{E}[u(X_t)v(X_s)]$$

$$= \lim_{s\downarrow t, s\in\mathbb{Q}^+} \mathbb{E}[u(X_t)P^X_{s-t}v(X_t)] = \mathbb{E}[u(X_t)v(X_t)]$$

By the Monotone Class Theorem and the fact that the indicator function of the diagonal in $H \times H$ is Borel, the claim follows. Therefore, $\{\tilde{X}_t\}_t$ is a right-continuous modification of $\{X_t\}_t$.

Now, since right-continuous real-valued supermartingales have a.s. left-limits along $\mathbb{R}^+$, proceeding as before we find that $\{\tilde{X}_t\}_t$ is the desired modification of the coordinate process.

Notice now that for all $\varphi \in D(\mathscr{L}_X)$ the process

$$f(\tilde{X}_t) - \int_0^t \mathscr{L}\varphi(\tilde{X}_r)\,dr \qquad t \geq 0$$

is a martingale by construction. For all $u \in D(\mathscr{L})$ we set $f := u - \mathscr{L}u$. Then there exists a sequence $f_n \in X \cap C_b^\theta(H)$ such that $f_n \xrightarrow{\pi} f$ and $u_n := R(\lambda, \mathscr{L})f_n \xrightarrow{\pi} u$. Then $\{X_u(t)\}_t$ is a martingale for all $u \in D(\mathscr{L})$. The measurability of $\{\mathbb{P}_x\}_x$ follows from the Monotone Class Theorem. □

In Remark 1 we noticed that a priori there is no way to extend a positive semigroup on X to a semigroup on $C_b(H)$. However, Theorem 8 has the following corollary:

Corollary 4. *For all $\varphi \in C_b(H)$ set*

$$P_t\varphi(x) := \int_H \varphi(y)\,\pi(t, x, dy) \qquad t \geq 0,\ x \in H$$

Then $(P_t)_{t\geq 0}$ is a semigroup acting on $C_b(H)$. In particular, the process $\{\mathbb{P}_x\}_x$ is Feller.

Proof. If $\varphi \in C_b^\theta(H)$, then we have shown that there exists a sequence $\varphi_n \in X \cap C_b^\theta(H)$, such that $\varphi_n \xrightarrow{\pi} \varphi$ and $\|\varphi_n\|_\theta \le \|\varphi\|_\theta$ for all n. One can easily prove by means of the transfer principle, that P_t^X is a bounded operator in $X \cap C_b^\theta(H)$ for all $t \ge 0$. Then

$$\varphi_n \xrightarrow{\pi} \varphi \;\;\&\;\; \sup_n \|\varphi_n\|_\theta \le \|\varphi\|_\theta \implies P_t^X \varphi_n \xrightarrow{\pi} P_t\varphi \;\;\&\;\; \|P_t\varphi\|_\theta \le C_t \|\varphi\|_\theta$$

By the density of $C_b^\theta(H)$ in $C_b(H)$, we have that $P_t(C_b(H)) \subset C_b(H)$. □

Remark 8. If $G(\cdot)$ is constant, then the results of the previous sections can be repeated also in the case of F being *bounded and continuous*, and *without assumptions on* ε in (9). In particular Lemma 1 can be improved, with similar proof, in the following way: *If* $f_n, f \in C_b(H)$ *is a sequence such that* $f_n \xrightarrow{\pi} f$, *then* $R(\lambda, \mathscr{L}) f_n \xrightarrow{\pi} R(\lambda, \mathscr{L}) f$. By Daniell's Theorem, this implies directly Theorem 7.

6. Schauder Estimate and Maximum Principle

In this section we prove positivity and Schauder-type regularity for the equation

$$\lambda u - \mathscr{L} u = f \tag{41}$$

We assume that Hypothesis 1 and 2 hold.

Proof of Theorem 2. We have to prove that for any $f \in C_b^\theta(H)$ and $u \in D(\mathscr{L})$ such that (41) holds, we have

$$\inf_H f \le \lambda u(x) \le \sup_H f \qquad \forall\, x \in H. \tag{42}$$

Valentine's Theorem says that if E is a Hilbert space, then the space $\mathrm{Lip}_b(H; E)$ of bounded Lipschitz continuous maps from H to E is dense in the space of bounded uniformly continuous maps (see [18]). Therefore there exists a sequence $\{F_n\} \subset \mathrm{Lip}_b(H; H)$ approximating F uniformly in H. Moreover, approximating G by means of functions taking values into finite-rank symmetric operators, we can find a sequence $\{G_n\} \subset \mathrm{Lip}_b(H; \mathscr{L}_1^+(H))$ such that $\|G(x) - G_n(x)\|_{\mathscr{L}_1(H)} \to 0$ uniformly in $x \in H$. Then we can write

$$D(\mathscr{L}_n) := D(\mathscr{L}) \qquad \mathscr{L}_n u := \mathscr{M} u + \frac{1}{2}\mathrm{Tr}[G_n D^2 u] + \langle F_n, \nabla u\rangle$$

$$\lambda u - \mathscr{L}_n u = f + \frac{1}{2}\mathrm{Tr}[(G_n - G) D^2 u] + \langle (F_n - F), \nabla u\rangle =: f + g_n \tag{43}$$

By Chapters 7 and 9 of [8], there exist H-valued processes $\{X_n(t,x)\}_{t\ge 0, x}$ such that, if $\varphi, g \in C_b(H)$ satisfy

$$\lambda\varphi - \mathscr{L}_n \varphi = g, \qquad \text{then} \qquad \varphi = \int_0^\infty e^{-\lambda t}\,\mathbb{E}[g(X_n(t,x))]dt. \tag{44}$$

Therefore (43) and (44) imply

$$\inf_H (f + g_n) \le \lambda u(x) \le \sup_H (f + g_n) \qquad \forall\, x \in H$$

Since $\|g_n\|_0 \to 0$ as $n \to \infty$, the Theorem is proven. □

Proof of Theorem 3. We have to prove that there exists a constant $C = C(\lambda) > 0$ such that for all $f \in C_b^\theta(H)$ and $u \in D(\mathscr{L})$ satisfying equation (41), we have

$$\|\mathscr{M} u\|_\theta \;+\; \|u\|_{2+\theta} \;\le\; C\,\|f\|_\theta \tag{45}$$

We follow [3] and [7]: see also [12] for another approach. Recall the definition (12) of the Ornstein-Uhlenbeck semigroup $(R_t)_{t\ge 0}$. Setting $\|f\|_1 := \|f\|_0 + \sup_{x\in H} |\nabla f(x)|$ for $f \in C_b^1(H)$, we have

$$f \in C_b(H) \implies \|D^2 R_t f(x)\| \le \frac{\sqrt{2}\nu^2}{t}\|f\|_0 \quad \forall\, t > 0,\; x \in H \tag{46}$$

$$f \in C_b^1(H) \implies \|D^2 R_t f(x)\| \le \frac{\nu}{\sqrt{t}}\|f\|_1 \quad \forall\, t > 0,\; x \in H \tag{47}$$

where ν is as in (7): see also [8], § 9.4.1. By interpolation we obtain

$$f \in C_b^\theta(H) \implies \|D^2 R_t f(x)\| \le 3\frac{2^{\frac{1-\theta}{2}}\,\nu^{2-\theta}}{t^{1-\theta/2}}\|f\|_\theta \tag{48}$$

which implies $D^2 R(\lambda, \mathscr{M}) f(x) \in \mathscr{L}(H)$. For $v \in C_b^2(H)$ we set

$$[D^2 v]_{\theta,\mathscr{M}} := \sup_{|h|\le 1}\left\{ \sup_{t\in]0,1]} \; t^{-\theta}\|R_t(\langle D^2 v\cdot h, h\rangle) - \langle D^2 v \cdot h, h\rangle\|_0 \right\}$$

Then a computation shows that (see respectively [7] and [3] for details):

$$[D^2 v]_\theta \le 2(1 + 3e\Gamma(1/2)\theta)\left(\|D^2 v\|_0 + [D^2 v]_{\theta,\mathscr{M}}\right), \tag{49}$$

$$[D^2 R(\lambda, \mathscr{M}) f]_{\theta,\mathscr{M}} \;\le\; 6\frac{2^{\frac{1-\theta}{2}}\,\nu^{2-\theta}}{\theta}\|f\|_\theta. \tag{50}$$

We write equation (41) in the following way:

$$\lambda u - \mathscr{M} u = f + \frac{1}{2}\mathrm{Tr}[G D^2 u] + \langle F, \nabla u\rangle$$

Then by (50)

$$[D^2 u]_{\theta,\mathscr{M}} \le 6\frac{2^{\frac{1-\theta}{2}}\,\nu^{2-\theta}}{\theta}\left(\|f\|_\theta + \|F\|_{C_b^\theta(H;H)}\|\nabla u\|_\theta + \varepsilon[D^2 u]_\theta + M\|D^2 u\|_0\right)$$

Using the interpolatory inequalities:

$$\|D^2 u\|_0 \;\le\; C_\theta\,\|u\|_0^{\frac{\theta}{2+\theta}}[D^2 u]_\theta^{\frac{2}{2+\theta}} \qquad \|\nabla u\|_\theta \;\le\; C_\theta\,\|u\|_0^{\frac{1+\theta}{2+\theta}}[D^2 u]_\theta^{\frac{1}{2+\theta}}$$

and Young's inequality, we obtain by (49)

$$[D^2u]_\theta \leq 120\frac{2^{\frac{1-\theta}{2}}\nu^{2-\theta}}{\theta}\left(\|f\|_\theta + (\varepsilon + r)[D^2u]_\theta + C(r)\|u\|_0\right)$$

By (9), we can choose now $r > 0$ such that $(\varepsilon + r)2^{\frac{1-\theta}{2}}\nu^{2-\theta}120/\theta < 1$ and using the Maximum Principle, we obtain

$$\|D^2u\|_\theta \;\leq\; C\,\|f\|_\theta$$

□

Remark 9. Using a Localization technique analogous to the one in [3], we could avoid the hypothesis (9) on ε. On the other hand this would require, with the techniques presently available, strong assumptions on G such as uniform continuity with respect to the norm on H $x \mapsto |R(1, A)x|$.

Remark 10. In this paper we restrict to differential operators of the form (14). The reason is that, up to now, Schauder Estimates for the Ornstein-Uhlenbeck operator $(\mathscr{M}, D(\mathscr{M}))$ can be proven only for the $\mathscr{L}(H)$-norm of the second derivatives: see [7]. However, notice that the operator $2\mathscr{M}$ is not of the form (14), nonetheless it satisfies Schauder-type regularity and its semigroup is determined by a kernel of probability measures. A detailed characterization of $D(\mathscr{M})$ could allow to treat more general operators than (14).

References

[1] Bogachev, V.I., Krylov, N., Röckner, M.: Regularity of Invariant Measures: The Case of Non-constant Diffusion Part, J. Funct. Anal., **138**, No. 1, 223–242 (1996)

[2] Bourbaki, N.: Espaces Vectoriels Topologiques, Livre **V**, Hermann & C. Éditeurs, Paris (1953)

[3] Cannarsa, P., Da Prato, G.: Infinite-dimensional elliptic equations with hölder-continuous coefficients, Advances in Differential Equations, **1**, No. 3, 425–452 (1996)

[4] Cerrai, S.: A Hille-Yosida Theorem for weakly continuous semigroups, Semigroup Forum, **49**, 349–367 (1994)

[5] Chojnowska-Michalik, A., Goldys, B.: Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, Prob. Theory and Rel. Fields, **102**, 331–356 (1995)

[6] Da Prato, G.: Applications croissantes et équations d'évolution dans les espaces de Banach, Academic Press (1976)

[7] Da Prato, G.: Some results on elliptic and parabolic equations in Hilbert spaces, Rend. Mat. Acc. Lincei, Serie IX, **VII** 3, 181–199 (1996)

[8] Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, Cambridge University Press (1992)

[9] Da Prato, G., Zabczyk, J.: Ergodicity for Infinite Dimensional Systems, Cambridge University Press (1996)

[10] Federer, H.: Geometric Measure Theory, Springer-Verlag, Berlin (1969)

[11] Gatarek, D., Goldys, B.: On uniqueness in law of solutions to stochastic evolution equations in Hilbert spaces, Stoch. Anal. and Appl. **12**, 192–203 (1994)

[12] Lunardi, A.: An interpolation method to characterize domains of generators of semigroups, Semigroup Forum, **53** (1996)

[13] Ma, Z.M., Rockner, M.: Introduction to the Theory of (Non Symmetric) Dirichlet Forms, Springer-Verlag (1992)
[14] Metivier, M.: Stochastic Partial Differential Equations in Infinite Dimensional Spaces, Quaderni Scuola Normale Superiore di Pisa (1988)
[15] Priola, E.: π-semigroups and applications, Preprint Scuola Normale Superiore di Pisa (1998)
[16] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Springer-Verlag, Berlin (1991)
[17] Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes, Springer-Verlag, New York (1979)
[18] Valentine, F.A.: A Lipschitz condition preserving extension for a vector function, Amer. J. Math. , **67**, n. 1, 83–93 (1945)
[19] Viot, M.: Solutions faibles d'équations aux dérivées partielles non linéaires. Thèse Université Pierre et Marie Curie, Paris (1976)
[20] Yosida, K.: Functional Analysis, Springer-Verlag, Berlin (1970)
[21] Zambotti, L.: Infinite dimensional elliptic and stochastic equations with Hölder-continuous coefficients, Stoch. Anal. and Appl., **17**(3), 487–508 (1999)
[22] Eberle, A.: Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators, LNM 1718, Springer-Verlag (1999)