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# Distributed optimization via dynamic event-triggered scheme with metric subregularity condition

Xin Yu<sup>1</sup>, Xi Chen<sup>1</sup>, Yuan Fan<sup>1</sup> and Songsong Cheng<sup>1\*</sup>

## Abstract

In this paper, we present a continuous-time algorithm with a dynamic event-triggered communication (DETC) mechanism for solving a class of distributed convex optimization problems that satisfy a metric subregularity condition. The proposed algorithm addresses the challenge of limited bandwidth in multi-agent systems by utilizing a continuous-time optimization approach with DETC. Furthermore, we prove that the distributed event-triggered algorithm converges exponentially to the optimal set, even without strong convexity conditions. Finally, we provide a comparison example to demonstrate the efficiency of our algorithm in communication resource-saving.

**Keywords:** Distributed optimization, Event-triggered, Metric subregularity, Exponential convergence

## 1 Introduction

In recent decades, network systems have become increasingly significant in numerous practical fields, including multi-robot [1], reinforcement learning [2], power system [3], and smart manufacturing [4]. Many tasks over network systems can be formulated as distributed optimization problems, which involve searching for the optimal solution of aggregating all the nodes' local objective functions. Various algorithms, such as continuous-time version [5–9] and discrete-time version [10–15], have been proposed to solve distributed optimization problems.

The convergence rate of optimization algorithms is a well-recognized topic in optimization theory and the corresponding algorithm design. The rate of the first-order gradient-based algorithm is generally expressed as  $O(1/k)$  [16] or  $O(1/k^2)$  (with an accelerated strategy) [17] for optimization problems with general convex and Lipschitz continuous objective functions. However, to attain a faster convergence performance (such as linear convergence), the objective function must satisfy the condition of strong

convexity [18]. Despite its theoretical appeal, strong convexity is not always a practical assumption in various applications. To address this issue, several methods have been developed to relax the strict assumption, including metric subregularity [19], restricted secant inequality [20], and Polyak-Łojasiewicz (PL) conditions [21]. For smooth constrained optimization problems, [22] proposed a relaxation of the strong convexity conditions as quasi-strong convexity, quadratic under-approximation, quadratic gradient growth, and quadratic functional growth. These relaxed conditions have been shown to enable linear convergence for several first-order methods.

Effective communication resource management has always been vital for evaluating distributed algorithms, as communication is energy-intensive and resources are often constrained. To alleviate the communication burden, researchers have proposed using event-triggered communication. Several studies have demonstrated that event-triggered communication is highly effective in reducing communication costs, including works by [23–33]. For optimization problems involving smooth cost functions, [34] developed a communication-efficient event-triggered first-order primal-dual algorithm, achieving an  $O(1/k)$

\*Correspondence: [sscheng@ahu.edu.cn](mailto:sscheng@ahu.edu.cn)

<sup>1</sup>School of Electrical Engineering and Automation, Anhui University, Hefei, 230601, Anhui, China

convergence rate. [11] combined the event-triggered mechanism with the mirror descent algorithm for distributed stochastic optimization problems. More recently, researchers have proposed a more advanced theory called the dynamic event-triggered communication (DETC) mechanisms, which employ additional internal dynamic variables. This mechanism has better communication resource saving than static event-triggered communication. For undirected and connected graphs, [35] proposed a distributed zero-gradient-sum algorithm that uses dynamic event-triggered communication, which converges exponentially to the global minimizer. In addition, [36] proposed a fully distributed algorithm with a dynamic event-triggered communication mechanism for the second-order continuous-time multi-agent systems, excluded the Zeno behavior, and showed an exponential convergence.

Given the non-strong convexity of the metric subregularity for linear convergence and the potential of the event-triggered communication mechanism to save communication resources, we solve the distributed optimization problem with a dynamic event-triggered communication mechanism and provide the corresponding convergence performance. Correspondingly, the main contributions of this work are summarized as follows.

- 1) We design a distributed dynamic event-triggered primal-dual algorithm for solving distributed optimization problems with metric subregularity. Compared with the results in [19], the proposed algorithm significantly reduces the communication burdens but does not sacrifice the convergence rate.
- 2) The designed algorithm linearly converges to the optimum set with an explicit convergence rate, which is faster than the asymptotic convergence in [37] with strong convexity. Moreover, compared with the restricted strong convexity in [38], we relax the condition as metric subregularity and achieve the same convergence rate of [38].
- 3) Compared to the seminal dynamic event-triggered mechanism, which needs the exchanging of the consensus error  $x_j(t_{k_j^j}^j) - x_i(t_{k_i^i}^i)$  at the triggered time, for distributed optimization in [35], our method is more simple by removing the term  $x_j(t_{k_j^j}^j) - x_i(t_{k_i^i}^i)$  and we provide a different method for convergence analysis.

The rest of the paper is organized as follows. Sect. 2 gives some basic knowledge of metric subregularity and graph theory. Sect. 3 introduces the distributed optimization problem and algorithm design. Sect. 4 analyzes the performance of the proposed algorithm and Sect. 5 gives a comparison example and Sect. 6 ends this paper with some concluding remarks.

## 2 Preliminaries

### 2.1 Notation

$A^\top$  is the transpose of  $A$ .  $\mathbb{R}$ ,  $\mathbb{R}^n$ , and  $\mathbb{R}^{m \times n}$  denote the sets of real numbers,  $n$ -dimensional vector, and  $m \times n$  real matrices, respectively.  $\|\cdot\|$  denotes the  $l_2$ -norm of a vector or the induced dual norm of a matrix.  $l_{ij}$  is the element at the  $j$ -th column and  $i$ -th row of  $L$ .  $\mathbf{1}_n \in \mathbb{R}^n$  stands for a column vector of  $n$  dimensions with all one.  $\text{col}\{z_i\}_{i=1}^N = [z_1^\top, z_2^\top, \dots, z_n^\top]^\top$  ( $\text{col}\{z_1, z_2\}$ ) denotes an augmented vector stacked by vectors  $z_1, z_2, \dots, z_n$  ( $z_1$  and  $z_2$ ).  $A \otimes B$  is the Kronecker product of the two matrices.  $d(x, \mathcal{X})$  is the distance between the point  $x$  and set  $\mathcal{X}$ .

### 2.2 Metric subregularity

Consider a map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and define

$$\text{gph } H := \{(x, y) | y = H(x), x, y \in \mathbb{R}^n\},$$

$$\text{rge } H := \{y | y = H(x), x, y \in \mathbb{R}^n\},$$

$$H^{-1}(y) := \{x | (x, y) \in \text{gph } H, x, y \in \mathbb{R}^n\}, \quad \forall y \in \text{rge } H.$$

By the above map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we give the definition of  $\kappa$ -metrically subregular.

**Definition 1** (see [19]) For a map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $(x_o, y_o) \in \text{gph } H$ , if there exist  $\kappa > 0$  and a neighborhood  $\mathcal{D}$  of  $x_o$  such that

$$d(x, H^{-1}(y_o)) \leq \kappa \|y_o - H(x)\|, \quad \forall x \in \mathcal{D},$$

then  $H$  is  $\kappa$ -metrically subregular at  $(x_o, y_o)$ .

### 2.3 Comparison lemma

For the convenience of the convergence analysis hereinafter, we provide the following comparison lemma on nonlinear systems.

**Lemma 1** (see [39]) Consider the following scalar differential equation

$$\dot{u} = f(t, u), \quad u(t_0) = u_0,$$

where  $f(t, u)$  is continuous in  $t$  and locally Lipschitz in  $u$ , for all  $t \geq 0$  and all  $u \in J \subset \mathbb{R}$ . Let  $[t_0, T)$  ( $T$  may be infinite) be the maximal interval of solution existence  $u(t)$ , and suppose  $u(t) \in J$  for all  $t \in [t_0, T)$ . Let  $v(t)$  be a continuous function with an upper right-hand derivative  $D^+v(t)$

satisfying the differential inequality

$$D^+v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0,$$

where  $v(t) \in J$  for all  $t \in [t_0, T)$ . Then,  $v(t) \leq u(t)$  for all  $t \in [t_0, T)$ .

### 2.4 Graph theory

For an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ ,  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  and  $\mathcal{V} = \{1, 2, \dots, n\}$  are the sets of edges and nodes, respectively.  $(i, j) \in \mathcal{E}$  means that node  $i$  exchanges information with node  $j$ .  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  is the adjacency matrix with  $a_{ij} = a_{ji} > 0$  if  $\{j, i\} \in \mathcal{E}$  and  $a_{ij} = 0$  otherwise. Defining  $D = \text{diag}\{d_1, \dots, d_n\}$  with  $d_i = \sum_{j=1}^n a_{ji}$ ,  $i \in \mathcal{V}$ ,  $L = D - A$  is the Laplacian matrix of  $\mathcal{G}$ . Specifically, if  $\mathcal{G}$  is directed and connected, then  $L = L^\top \geq 0$ .

### 3 Problem formulation and algorithm design

In this section, we consider a distributed convex optimization problem and propose an event-triggered primal-dual algorithm to solve this problem.

#### 3.1 Problem formulation

Consider the distributed optimization problem over networks as follows

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^{Nn}} \quad & \psi(\mathbf{x}) := \sum_{i=1}^N \psi_i(x_i) \\ \text{s.t.} \quad & x_i = x_j, \quad \forall i, j \in \mathcal{V}, \end{aligned} \quad (1)$$

where  $\psi_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathbf{x} = \text{col}\{x_i\}_{i=1}^N \in \mathbb{R}^{Nn}$ . In the meantime, we take the following mild assumption.

**Assumption 1** The local functions  $\psi_i(\cdot)$ ,  $i \in \mathcal{V}$  are convex and differentiable, and the corresponding gradients are local Lipschitz continuous.

To solve the above optimization problem in a distributed manner, a multi-agent system on communication graph  $\mathcal{G}$  must be employed. In that way, each agent updates its local variable  $x_i$  using local data and communication to reach the optimal solution over  $\mathcal{G}$ . We take the following mild assumption for  $\mathcal{G}$ .

**Assumption 2** The graph  $\mathcal{G}$  is connected and undirected.

Note that Assumption 2 implies that the Laplacian matrix  $L$  is symmetric and positive semi-definite.

#### 3.2 Algorithm design

We propose the following event-triggered-based algorithm from the primal-dual perspective to solve the above problem.

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#### Algorithm 1 Dynamic Event-triggered-based Algorithm Design

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**Initialization:** Choose the parameters  $a_i, b_i, c_i$ , and  $d_i > 0, \theta \in \mathbb{R}_0^+$ ,  $l_{ij}$  is the  $(i, j)$ -th element of  $L$ . Initialize  $x_i(0) \in \mathbb{R}^n, q_i(0) \in \mathbb{R}^n, t_k^i = 0, \forall i \in \mathcal{V}$ .

**Update flows:** For each  $i \in \mathcal{V}$ ,

**S. 1. for**  $t = 1, 2, \dots, T$  **do**

**S. 2.** Update the local variables:

$$\begin{cases} \dot{x}_i(t) = -\nabla \psi_i(x_i(t)) - \sum_{j=1}^N l_{ij}[x_j(t_{k_j^j}^j) + q_j(t_{k_j^j}^j)], \\ \dot{q}_i(t) = \sum_{j=1}^N l_{ij}x_j(t_{k_j^j}^j), \end{cases} \quad (2)$$

**S. 3.** Update the event time  $t_{k+1}^i$  by

$$t_{k+1}^i = \max_{t \geq t_k^i} \left\{ t \mid \begin{cases} \eta_i(t) + \theta \|x_i(t) - x_i(t_k^i)\| \leq a_i e^{-b_i t} \\ \eta_i(t) + \theta \|q_i(t) - q_i(t_k^i)\| \leq c_i e^{-d_i t} \end{cases} \right\}, \quad (3)$$

**S. 4. if** an event occurs at agent  $i$  **then**

**S. 5.** Update the latest event time  $t_{k_i(t)}^i = t_{k+1}^i$ ; Update and broadcast state estimates  $x_i(t_{k_i(t)}^i)$  and  $q_i(t_{k_i(t)}^i)$ ;

**S. 6. else**

**S. 7.** Keep local memory constant.

**S. 8. end if**

**S. 9. end for**

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In Algorithm 1,  $\eta_i(t)$  is an internal dynamic variable which satisfies

$$\dot{\eta}_i(t) = -\beta \eta_i(t) + \|x_i(t) - x_i(t_k^i)\| \quad (4)$$

with  $\beta > 0$  and  $\eta_i(0) \in \mathbb{R}_0^+$ .

*Remark 1* Algorithm 1 is introduced from the seminal work of [19] but with the event-triggered mechanism. Compared with event-triggered based algorithms proposed in [11, 28, 29, 34–36, 38], it relaxes of strong convexity as  $\kappa$ -metrically subregular. Furthermore, the proposed method modifies the static event-triggered communication scheme into a dynamic counterpart, resulting in a communication scheme that is more efficient compared to the approach presented in [11, 28, 34, 38].

*Remark 2* The event-triggered communication (ETC) in [11, 28, 34, 38] is denoted as static ETC, as it exclusively involves the current values of local variables. In contrast, the ETC in Algorithm 1 is termed dynamic ETC, as it incorporates an additional internal dynamic variable  $\eta_i$ . The main idea behind the dynamic event-triggered communication scheme involves introducing an internal nonnegative dynamic variable  $\eta_i$  based on the static event-triggered communication scheme. This modification ensures that  $\theta \|x_i(t) - x_i(t_k^i)\| - a_i e^{-b_i t}$  and  $\theta \|q_i(t) - q_i(t_k^i)\| - c_i e^{-d_i t}$  do not have to be strictly non-negative at all times but rather non-negative on average. Furthermore, in the limit where

$\theta \rightarrow +\infty$ , the static ETC can be considered a special case of the dynamic ETC.

For the designed DETC, we have the following basic conclusion.

**Lemma 2** *Let  $x, q$ , and  $\eta$  be generated by the proposed algorithm in (2)-(4). Then, for all  $t \in [0, t_\infty)$ ,  $\eta_i(t) + \theta \|x_i(t) - x_i(t_k^i)\| - a_i e^{-b_i t} \geq 0$  and  $\eta_i(t) \geq 0$ .*

*Proof* By (3), the communication condition triggers when

$$\eta_i(t) + \theta \|x_i(t) - x_i(t_k^i)\| - a_i e^{-b_i t} \geq 0, \quad t \in [0, t_\infty).$$

According to (3) and (4),

$$\begin{aligned} \dot{\eta}_i(t) &\geq -\frac{(1+\theta\beta)}{\theta} \eta_i(t) + \frac{a_i}{\theta} e^{-b_i t} \\ &\geq -\frac{(1+\theta\beta)}{\theta} \eta_i(t) + \frac{a_0}{\theta} e^{-b_0 t}, \end{aligned}$$

where  $a_0 = \min\{a_i, c_i\}_{i=1}^N$ ,  $b_0 = \max\{b_i, d_i\}_{i=1}^N$ . Then

$$\eta_i(t) \geq \eta_i(0) \left( C_1 e^{-\frac{1+\theta\beta}{\theta} t} + C_2 e^{-b_0 t} \right) > 0,$$

where  $C_1 = -\frac{a_0}{1+\theta(\beta-b_0)}$  and  $C_2 = -C_1$ .  $\square$

**Remark 3** The value of  $\eta_i$  depends on the values of  $\frac{(1+\theta\beta)}{\theta}$  and  $b_0$ . If  $b_0 > \frac{(1+\theta\beta)}{\theta}$ , then  $C_1 > 0$  and yields  $\eta_i(t) > 0$ , otherwise  $C_2 > 0$  and achieves  $\eta_i(t) > 0$  as well.

#### 4 Main results

In this section, we present the linear convergence of the proposed algorithm with the aid of the Lyapunov theory. Define

$$\begin{aligned} \hat{x}_i(t) &= x_i(t_k^i), & \hat{q}_i(t) &= q_i(t_k^i), \\ \mathbf{x}(t) &= \text{col}\{x_i(t)\}_{i=1}^N, & \hat{\mathbf{x}}(t) &= \text{col}\{\hat{x}_i(t)\}_{i=1}^N, \\ \mathbf{q}(t) &= \text{col}\{q_i(t)\}_{i=1}^N, & \hat{\mathbf{q}}(t) &= \text{col}\{\hat{q}_i(t)\}_{i=1}^N, \\ e_i^x(t) &= \hat{x}_i(t) - x_i(t), & \mathbf{e}^x &= \text{col}\{e_i^x\}_{i=1}^N, \\ e_i^q(t) &= \hat{q}_i(t) - q_i(t), & \mathbf{e}^q &= \text{col}\{e_i^q\}_{i=1}^N, \\ \mathbf{z}(t) &= \text{col}\{\mathbf{x}(t), \mathbf{q}(t)\}, & \hat{\mathbf{z}}(t) &= \text{col}\{\hat{\mathbf{x}}(t), \hat{\mathbf{q}}(t)\}, \\ \eta(t) &= \text{col}\{\eta_i(t)\}_{i=1}^N. \end{aligned}$$

We rewrite the algorithm in (2) into the following compact form

$$\dot{\mathbf{z}}(t) = -H(\mathbf{z}(t), \hat{\mathbf{z}}(t)), \tag{5}$$

where  $H(\mathbf{z}, \hat{\mathbf{z}}) = \text{col}\{\nabla\psi(\mathbf{x}) + \bar{L}\hat{\mathbf{x}} + \bar{L}\hat{\mathbf{q}}, -\bar{L}\hat{\mathbf{x}}\}$ .

Define  $[\mathbf{z}]^*$  as the projection from a point  $\mathbf{z}$  to the optimal solution set  $\mathcal{Z}^* = X^* \times Y^*$ , then

$$d(\mathbf{z}, \mathcal{Z}^*) = \|\mathbf{z} - [\mathbf{z}]^*\|.$$

Based on the optimization problem in (1) and the primal-dual event-triggered algorithm in (5), we take the following assumption, which is more general than the strongly convex and restrict strongly convex in [18, 38].

**Assumption 3**  $H$  is  $k_0$ -metrically subregular at any point  $(z^*, 0) \in \text{gph}H$ , namely, there exist a constant  $k_0 > 0$  and  $\mathcal{Z}^* \subset \mathcal{D}$  such that

$$\|H(\mathbf{z}, \hat{\mathbf{z}})\| \geq k_0^{-1} d(\mathbf{z}, \mathcal{Z}^*), \quad \forall \mathbf{z} \in \mathcal{D}. \tag{6}$$

For the proposed algorithm in (2), we design the following Lyapunov function

$$V(\mathbf{z}, \eta) = 6\sigma_0 V_1(\mathbf{z}, \eta) + \frac{4}{3} V_2(\mathbf{z}, \eta), \tag{7}$$

$$V_1(\mathbf{z}, \eta) = \frac{1}{2} d^2(\mathbf{z}, \mathcal{Z}^*) + A \|\eta\|^2, \tag{8}$$

$$V_2(\mathbf{z}, \eta) = \psi(\mathbf{x}) - \psi([\mathbf{x}]^*) + \frac{1}{2} \langle \mathbf{x} + 2\mathbf{q}, \bar{L}\mathbf{x} \rangle + B \|\eta\|^2, \tag{9}$$

where  $\bar{L} = L \otimes I_N$  and  $\sigma_0$  is the largest singular value of  $L$ .

**Lemma 3** *Under Assumptions 1-3, let  $(\mathbf{x}, \mathbf{q})$  be generated by the proposed algorithm in (2). Then, the Lyapunov function can be bounded as follows*

$$V(\mathbf{z}, \eta) \geq \frac{7\sigma_0}{3} \|\mathbf{z} - [\mathbf{z}]^*\|^2,$$

$$V(\mathbf{z}, \eta) \leq \frac{2l_0 + 13\sigma_0}{3} \|\mathbf{z} - [\mathbf{z}]^*\|^2 + \left( 6\sigma_0 A + \frac{4}{3} B \right) \|\eta\|^2,$$

where  $l_0$  is the Lipschitz constant of  $\nabla\psi(\cdot)$ .

*Proof* According to the optimal conditions  $\nabla\psi([\mathbf{x}]^*) = -\bar{L}[\mathbf{q}]^*$  and  $[\mathbf{x}^*]^\top \bar{L} = 0$ , we have

$$\begin{cases} \langle \mathbf{x} - [\mathbf{x}]^*, \nabla\psi([\mathbf{x}]^*) \rangle = -\langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}[\mathbf{q}]^* \rangle, \\ \langle \mathbf{x}, \bar{L}\mathbf{q} \rangle = \langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}\mathbf{q} \rangle. \end{cases}$$

Then we rewrite  $V_2(\mathbf{z})$  as

$$\begin{aligned} V_2(\mathbf{z}, \eta) &= \psi(\mathbf{x}) - \psi([\mathbf{x}]^*) - \langle \mathbf{x} - [\mathbf{x}]^*, \nabla\psi([\mathbf{x}]^*) \rangle \\ &\quad + \frac{1}{2} \bar{L} \langle \mathbf{x} - [\mathbf{x}]^* + 2\mathbf{q} - 2[\mathbf{q}]^* \rangle + B \|\eta\|^2. \end{aligned}$$

The convexity of  $\psi$  and  $\bar{L} \geq 0$  imply,

$$\psi(\mathbf{x}) \geq \psi([\mathbf{x}]^*) + \langle \mathbf{x} - [\mathbf{x}]^*, \nabla\psi([\mathbf{x}]^*) \rangle,$$

$$\langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}(\mathbf{x} - [\mathbf{x}]^*) \rangle \geq 0.$$

Therefore,

$$\begin{aligned} V_2(\mathbf{z}, \eta) &\geq \langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}(\mathbf{q} - [\mathbf{q}]^*) \rangle \\ &\geq -\frac{\sigma_0}{2} (\|\mathbf{x} - [\mathbf{x}]^*\|^2 + \|\mathbf{q} - [\mathbf{q}]^*\|^2) \\ &= -\frac{\sigma_0}{2} \|\mathbf{z} - [\mathbf{z}]^*\|^2 \end{aligned} \quad (10)$$

and

$$\begin{aligned} V_1(\mathbf{z}, \eta) &= \frac{1}{2} d^2(\mathbf{z}, \mathcal{Z}^*) + A\|\eta\|^2 \\ &\geq \frac{1}{2} d^2(\mathbf{z}, \mathcal{Z}^*). \end{aligned} \quad (11)$$

According to (10) and (11), the lower bound of  $V(\mathbf{z})$  is

$$V(\mathbf{z}, \eta) \geq \frac{7\sigma_0}{3} \|\mathbf{z} - [\mathbf{z}]^*\|^2. \quad (12)$$

By Lemma 1 and the  $l_0$ -Lipchitz continuity of  $\nabla\psi(\cdot)$ , we have

$$\begin{aligned} \psi(\mathbf{x}) - \psi([\mathbf{x}]^*) - \langle \mathbf{x} - [\mathbf{x}]^*, \nabla\psi([\mathbf{x}]^*) \rangle &\leq \frac{l_0}{2} \|\mathbf{x} - [\mathbf{x}]^*\|^2, \\ \langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}(\mathbf{x} - [\mathbf{x}]^*) \rangle &\geq \sigma_0 \|\mathbf{x} - [\mathbf{x}]^*\|^2. \end{aligned}$$

Moreover,

$$\begin{aligned} &\langle \mathbf{x} - [\mathbf{x}]^*, \bar{L}(\mathbf{q} - [\mathbf{q}]^*) \rangle \\ &\leq \sigma_0 \|\mathbf{x} - [\mathbf{x}]^*\| \|\mathbf{q} - [\mathbf{q}]^*\| \\ &\leq \frac{\sigma_0}{2} \left( \varepsilon \|\mathbf{x} - [\mathbf{x}]^*\|^2 + \frac{1}{\varepsilon} \|\mathbf{q} - [\mathbf{q}]^*\|^2 \right), \end{aligned}$$

for any  $\varepsilon > 0$ . By  $\varepsilon = \frac{\sigma_0}{l_0 + \sigma_0}$ , we have

$$\begin{aligned} V_2(\mathbf{z}, \eta) &\leq \frac{l_0 + \sigma_0}{2} (\|\mathbf{x} - [\mathbf{x}]^*\|^2 + \|\mathbf{q} - [\mathbf{q}]^*\|^2) \\ &\quad + \frac{\sigma_0^2}{2(l_0 + \sigma_0)} \|\mathbf{x} - [\mathbf{x}]^*\|^2 + B\|\eta\|^2 \\ &\leq \frac{l_0 + 2\sigma_0}{2} \|\mathbf{z} - [\mathbf{z}]^*\|^2 + B\|\eta\|^2. \end{aligned}$$

The upper bound is

$$V(\mathbf{z}, \eta) \leq \frac{2l_0 + 13\sigma_0}{3} \|\mathbf{z} - [\mathbf{z}]^*\|^2 + \left( 6\sigma_0 A + \frac{4}{3} B \right) \|\eta\|^2. \quad (13)$$

Therefore, we obtain the bounds of  $V(\mathbf{z})$ .  $\square$

**Theorem 1** Under Assumptions 1-3, let  $(\mathbf{x}, \mathbf{q})$  be generated by the proposed algorithm in (2). Then  $\mathbf{z}(t)$  linearly converges to the optimal set  $\mathcal{Z}^*$  as follows

$$\begin{aligned} &\|\mathbf{z}(t) - [\mathbf{z}(t)]^*\| \\ &\leq \sqrt{\frac{3V(\mathbf{z}(T_0))}{7\sigma_0}} \left[ C_3 e^{-\frac{3}{k_0^2(2\eta_0 + 13\sigma_0)}(t-T_0)} \right. \\ &\quad \left. + C_4 e^{-\frac{b}{2}(t-T_0)} \right], \quad t \geq T_0, \end{aligned} \quad (14)$$

where  $C_3 = \frac{2\gamma_3 N k_0^2 (2l_0 + 13\sigma_0)}{b k_0^2 (2l_0 + 13\sigma_0) - 6}$ ,  $C_4 = -C_3$ , and  $T_0 < \infty$ .

*Proof* For the designed Lyapunov function in (7), we calculate its first-order derivatives with respect to  $t$ ,

$$\begin{aligned} \dot{V}_1(\mathbf{z}, \eta) &= \langle \mathbf{x} - [\mathbf{x}]^*, -\bar{L}(\mathbf{x} + \mathbf{e}^x) - \bar{L}(\mathbf{q} + \mathbf{e}^q) - \nabla\psi(\mathbf{x}) \rangle \\ &\quad + \langle \mathbf{q} - [\mathbf{q}]^*, \bar{L}(\mathbf{x} + \mathbf{e}^x) \rangle + 2A\langle \eta, \dot{\eta} \rangle \\ &= \langle \mathbf{x} - [\mathbf{x}]^*, -\bar{L}(\mathbf{x} + \mathbf{e}^x) - \bar{L}(\mathbf{q} - [\mathbf{q}]^* + \mathbf{e}^q) \rangle \\ &\quad + \nabla\psi([\mathbf{x}]^*) - \nabla\psi(\mathbf{x}) + \langle \mathbf{q} - [\mathbf{q}]^*, \bar{L}(\mathbf{x} + \mathbf{e}^x) \rangle \\ &\quad + 2A\langle \eta, \dot{\eta} \rangle \\ &= -\langle \mathbf{x}, \bar{L}(\mathbf{x} + \mathbf{e}^x + \mathbf{e}^q) \rangle + \langle \mathbf{q} - [\mathbf{q}]^*, \bar{L}\mathbf{e}^x \rangle \\ &\quad - \langle [\mathbf{x}]^* - \mathbf{x}, \nabla\psi([\mathbf{x}]^*) - \nabla\psi(\mathbf{x}) \rangle \\ &\quad + 2A\langle \eta, \dot{\eta} \rangle \\ &\leq -\frac{1}{2} \mathbf{x}^\top \bar{L} \mathbf{x} + \sigma_0 (\|\mathbf{e}^x\|^2 + \|\mathbf{e}^q\|^2) \\ &\quad + \sigma_0 \|\mathbf{q} - [\mathbf{q}]^*\| \|\mathbf{e}^x\| \\ &\quad - \langle [\mathbf{x}]^* - \mathbf{x}, \nabla\psi([\mathbf{x}]^*) - \nabla\psi(\mathbf{x}) \rangle + 2A\langle \eta, \dot{\eta} \rangle \\ &\leq -\frac{1}{2} \mathbf{x}^\top \bar{L} \mathbf{x} + \sigma_0 (\|\mathbf{e}^x\|^2 + \|\mathbf{e}^q\|^2) \\ &\quad + \frac{b}{2} \|\mathbf{q} - [\mathbf{q}]^*\|^2 \\ &\quad + \frac{\sigma_0^2}{2b} \|\mathbf{e}^x\|^2 - \langle [\mathbf{x}]^* - \mathbf{x}, \nabla\psi([\mathbf{x}]^*) - \nabla\psi(\mathbf{x}) \rangle \\ &\quad + 2A\langle \eta, \dot{\eta} \rangle, \end{aligned} \quad (15)$$

where the first equation follows from  $\mathbf{e}^x = \hat{\mathbf{x}} - \mathbf{x}$  and  $\mathbf{e}^q = \hat{\mathbf{q}} - \mathbf{q}$ , the second equation is deduced based on  $\bar{L}[\mathbf{q}]^* = -\nabla\psi([\mathbf{x}]^*)$ , the first inequality is deduced based on Young's inequality and Cauchy-Schwarz inequality, and the last inequality is deduced by  $b = \min\{b_1, \dots, b_N, d_1, \dots, d_N\} \geq 0$ . Define

$$a = \max\{a_1, \dots, a_N, c_1, \dots, c_N\} \geq 0.$$

By (3), we have

$$\begin{cases} \|\mathbf{e}^x\|^2 \leq \sum_{i=1}^N \left\| \frac{a_i e^{-b_i t} - \eta_i}{\theta} \right\|^2 \leq \frac{2(Na^2 e^{-2bt} + \|\eta\|^2)}{\theta^2}, \\ \|\mathbf{e}^q\|^2 \leq \sum_{i=1}^N \left\| \frac{c_i e^{-d_i t} - \eta_i}{\theta} \right\|^2 \leq \frac{2(Nc^2 e^{-2dt} + \|\eta\|^2)}{\theta^2}. \end{cases} \quad (16)$$

From (16),

$$\begin{aligned} \langle \eta, \dot{\eta} \rangle &\leq \sum_{i=1}^N \left\langle \eta_i, -\beta \eta_i + \frac{a_i e^{-b_i t} - \eta_i}{\theta} \right\rangle \\ &= \sum_{i=1}^N \left( -\beta \eta_i^2 - \frac{1}{\theta} \eta_i^2 + \frac{1}{\theta} \eta_i a_i e^{-b_i t} \right) \\ &\leq -\beta \|\eta\|^2 + \frac{1}{4\theta} Na^2 e^{-2bt}, \end{aligned} \quad (17)$$

where the first inequality follows from (17) and the second inequality is deduced based on  $-\eta_i^2 + \eta_i a_i e^{-b_i t} \leq a_i^2 e^{-2b_i t} \leq a^2 e^{-2bt}$ . By (8),

$$\|\mathbf{q} - [\mathbf{q}]^*\|^2 \leq 2V_1(\mathbf{z}, \eta). \quad (18)$$

Based on (15), (16), and (18),

$$\begin{aligned} \dot{V}_1(\mathbf{z}, \eta) &\leq bV_1(\mathbf{z}, \eta) + \frac{4\sigma_0}{\theta^2} Na^2 e^{-2bt} + \frac{4\sigma_0}{\theta^2} \|\eta\|^2 \\ &\quad + \frac{\sigma_0^2}{b\theta^2} Na^2 e^{-2bt} + \frac{\sigma_0^2}{b\theta^2} \|\eta\|^2 \\ &\quad - 2A\beta \|\eta\|^2 + \frac{A}{2\theta} Na^2 e^{-2bt} \\ &\leq bV_1(\mathbf{z}, \eta) \\ &\quad + \left( \frac{4\sigma_0}{\theta^2} + \frac{\sigma_0^2}{b\theta^2} + \frac{A}{2\theta} \right) Na^2 e^{-2bt} \\ &\quad - \left( 2A\beta - \frac{4\sigma_0}{\theta^2} - \frac{\sigma_0^2}{b\theta^2} \right) \|\eta\|^2 \\ &\leq bV_1(\mathbf{z}, \eta) \\ &\quad + \left( \frac{4\sigma_0}{\theta^2} + \frac{\sigma_0^2}{b\theta^2} + \frac{A}{2\theta} \right) Na^2 e^{-2bt}, \end{aligned} \quad (19)$$

where the last inequality holds under  $2A\beta - \frac{4\sigma_0}{\theta^2} - \frac{\sigma_0^2}{b\theta^2} > 0$ . Therefore,

$$V_1(\mathbf{z}, \eta) \leq \gamma_1 e^{bt}, \quad (20)$$

where  $\gamma_1 = V_0(\mathbf{z}, \eta) + \left( \frac{4\sigma_0}{3b\theta^2} + \frac{\sigma_0^2}{3b\theta^2} + \frac{A}{6b\theta} \right) Na^2$ . By (18), we have

$$\|\mathbf{q} - [\mathbf{q}]^*\| \leq \sqrt{2\gamma_1} e^{\frac{b}{2}t}. \quad (21)$$

According to (15), (16) and (21),

$$\begin{aligned} \dot{V}_1(\mathbf{z}, \eta) &\leq -\frac{1}{2} \langle \mathbf{x}, \bar{\mathbf{L}}\mathbf{x} \rangle + \frac{4\sigma_0}{\theta^2} Na^2 e^{-2bt} \\ &\quad + \frac{4\sigma_0}{\theta^2} \|\eta\|^2 + \frac{\sqrt{2\gamma_1}\sigma_0}{\theta} Na e^{-\frac{b}{2}t} \\ &\quad - \frac{\sigma_0}{\theta} \sqrt{\gamma_1} \eta e^{\frac{b}{2}t} - 2A\beta \|\eta\|^2 \\ &\quad + \frac{A}{2\theta} Na^2 e^{-2bt} \\ &\leq -\frac{1}{2} \sigma_0^{-1} \|\bar{\mathbf{L}}\mathbf{x}\|^2 + \gamma_2 N e^{-\frac{b}{2}t} \\ &\quad - \left( 2A\beta - \frac{4\sigma_0}{\theta^2} \right) \|\eta\|^2, \end{aligned} \quad (22)$$

where  $\gamma_2 = a \left( \frac{4a}{\theta^2} + \frac{\sqrt{2\gamma_1}}{\theta} + \frac{Aa}{2\theta\sigma_0} \right) \sigma_0$ . Moreover,

$$\begin{aligned} \dot{V}_2(\mathbf{z}, \eta) &= \langle \nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}, \dot{\mathbf{x}} \rangle + \langle \bar{\mathbf{L}}\mathbf{x}, \dot{\mathbf{q}} \rangle - 2B \langle \eta, \dot{\eta} \rangle \\ &= \langle \nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}, \\ &\quad - (\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q} + \bar{\mathbf{L}}\mathbf{e}^x + \bar{\mathbf{L}}\mathbf{e}^q) \rangle \\ &\quad + \langle \bar{\mathbf{L}}\mathbf{x}, \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{e}^x \rangle - 2B \langle \eta, \dot{\eta} \rangle \\ &= -\|\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}\|^2 + \|\bar{\mathbf{L}}\mathbf{x}\|^2 \\ &\quad - \langle \nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}, \bar{\mathbf{L}}\mathbf{e}^x + \bar{\mathbf{L}}\mathbf{e}^q \rangle \\ &\quad + \langle \bar{\mathbf{L}}\mathbf{x}, \bar{\mathbf{L}}\mathbf{e}^x \rangle - 2B \langle \eta, \dot{\eta} \rangle, \end{aligned} \quad (23)$$

where the second equation follows from  $\mathbf{e}^x = \hat{\mathbf{x}} - \mathbf{x}$  and  $\mathbf{e}^q = \hat{\mathbf{q}} - \mathbf{q}$ . Due to  $-a^\top b \leq \frac{\epsilon}{4} \|a\|^2 + \frac{1}{\epsilon} \|b\|^2$ , we have

$$\begin{aligned} &-\langle \nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}, \bar{\mathbf{L}}\mathbf{e}^x + \bar{\mathbf{L}}\mathbf{e}^q \rangle \\ &\leq \frac{1}{4} \|\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}\|^2 + \|\bar{\mathbf{L}}\mathbf{e}^x + \bar{\mathbf{L}}\mathbf{e}^q\|^2 \end{aligned} \quad (24)$$

and

$$\langle \bar{\mathbf{L}}\mathbf{x}, \bar{\mathbf{L}}\mathbf{e}^x \rangle \leq \frac{1}{2} \|\bar{\mathbf{L}}\mathbf{x}\|^2 + \frac{1}{2} \|\bar{\mathbf{L}}\mathbf{e}^x\|^2. \quad (25)$$

Substituting (24) and (25) into (23), we have

$$\begin{aligned} \dot{V}_2(\mathbf{z}, \eta) &\leq -\|\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}\|^2 + \|\bar{\mathbf{L}}\mathbf{x}\|^2 \\ &\quad + \frac{1}{4} \|\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}\|^2 \\ &\quad + \|\bar{\mathbf{L}}\mathbf{e}^x + \bar{\mathbf{L}}\mathbf{e}^q\|^2 + \frac{1}{2} \|\bar{\mathbf{L}}\mathbf{x}\|^2 + \frac{1}{2} \|\bar{\mathbf{L}}\mathbf{e}^x\|^2 \\ &\quad + 2B \langle \eta, \dot{\eta} \rangle \\ &\leq -\frac{3}{4} \|\nabla \psi(\mathbf{x}) + \bar{\mathbf{L}}\mathbf{x} + \bar{\mathbf{L}}\mathbf{q}\|^2 + \frac{3}{2} \|\bar{\mathbf{L}}\mathbf{x}\|^2 \end{aligned}$$

$$\begin{aligned}
 & + \|\bar{L}\mathbf{e}^x + \bar{L}\mathbf{e}^q\|^2 + \frac{1}{2}\|\bar{L}\mathbf{e}^x\|^2 + 2B\langle \eta, \dot{\eta} \rangle \\
 \leq & -\frac{3}{4}\|\nabla\psi(\mathbf{x}) + \bar{L}\mathbf{x} + \bar{L}\mathbf{q}\|^2 + \frac{3}{2}\|\bar{L}\mathbf{x}\|^2 \\
 & + \frac{9N\sigma_0^2 a^2}{\theta^2}e^{-2bt} + \frac{9\sigma_0^2}{\theta^2}\|\eta\|^2 \\
 & - 2B\beta\|\eta\|^2 + \frac{BNa^2}{2\theta}e^{-2bt},
 \end{aligned}$$

where the last inequality follows from (16). As a result,

$$\begin{aligned}
 \dot{V}(\mathbf{z}, \eta) \leq & -\|\nabla\psi(\mathbf{x}) + \bar{L}\mathbf{x} + \bar{L}\mathbf{q}\|^2 - \|\bar{L}\mathbf{x}\|^2 \\
 & + 6\sigma_0\gamma_2Ne^{-\frac{b}{2}t} - 12\sigma_0\left(A\beta - \frac{2\sigma_0}{\theta^2}\right)\|\eta\|^2 \\
 & + \frac{12\sigma_0^2}{\theta^2}Na^2e^{-2bt} + \frac{12\sigma_0^2}{\theta^2}\|\eta\|^2 - \frac{8}{3}B\beta\|\eta\|^2 \\
 & + \frac{2B}{3\theta}Na^2e^{-2bt}
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 = & -\|H(\mathbf{z}, \dot{\mathbf{z}})\|^2 + \gamma_3Ne^{-\frac{b}{2}t} \\
 & - \left(12A\sigma_0\beta - \frac{36\sigma_0^2}{\theta^2} + \frac{8}{3}B\beta\right)\|\eta\|^2,
 \end{aligned}$$

where  $\gamma_3 = 6\sigma_0\gamma_2 + \left(\frac{12\sigma_0^2}{\theta^2} + \frac{2B}{3\theta}\right)e^{-\frac{3}{2}bt}$ . Based on (6), (13) and (26),

$$\begin{aligned}
 \dot{V}(\mathbf{z}, \eta) \leq & -\|H(\mathbf{z}, \dot{\mathbf{z}})\|^2 + \gamma_3Ne^{-\frac{b}{2}t} \\
 & - \left(12A\sigma_0\beta - \frac{36\sigma_0^2}{\theta^2} + \frac{8}{3}B\beta\right)\|\eta\|^2 \\
 \leq & -\frac{3}{k_0^2(2l_0 + 13\sigma_0)}V(\mathbf{z}, \eta) + \gamma_3Ne^{-\frac{b}{2}t} \\
 & + \frac{3}{k_0^2(2l_0 + 13\sigma_0)}\left(6\sigma_0A + \frac{4}{3}B\right)\|\eta\|^2 \\
 & - \left(12A\sigma_0\beta - \frac{36\sigma_0^2}{\theta^2} + \frac{8}{3}B\beta\right)\|\eta\|^2 \\
 \leq & -\frac{3}{k_0^2(2l_0 + 13\sigma_0)}V(\mathbf{z}, \eta) + \gamma_3Ne^{-\frac{b}{2}t},
 \end{aligned} \tag{27}$$

where  $\beta > \frac{3}{2k_0^2(2l_0 + 13\sigma_0)} + \frac{18}{\theta^2(6\sigma_0A + \frac{4}{3}B)}$ . Defining

$$\dot{U}(\mathbf{z}, \eta) = -\frac{3}{k_0^2(2l_0 + 13\sigma_0)}U(\mathbf{z}, \eta) + \gamma_3Ne^{-\frac{b}{2}t}. \tag{28}$$

By the Laplace transform of (28), we have

$$U(\mathbf{s}) = \frac{\gamma_3N}{\left(s + \frac{3}{k_0^2(2l_0 + 13\sigma_0)}\right)\left(s + \frac{b}{2}\right)}.$$

Then, there exists a finite constant  $T_0$  such that

$$\begin{aligned}
 U(\mathbf{z}(t), \eta(t)) = U(\mathbf{z}(T_0), \eta(T_0)) & \left[ C_3e^{-\frac{3}{k_0^2(2l_0 + 13\sigma_0)}(t-T_0)} \right. \\
 & \left. + C_4e^{-\frac{b}{2}(t-T_0)} \right], \quad \forall t \geq T_0,
 \end{aligned}$$

where

$$C_3 = \frac{2\gamma_3Nk_0^2(2l_0 + 13\sigma_0)}{bk_0^2(2l_0 + 13\sigma_0) - 6} \quad \text{and} \quad C_4 = -C_3.$$

According to Lemma 1, we have

$$\begin{aligned}
 V(\mathbf{z}(t), \eta(t)) \leq V(\mathbf{z}(T_0), \eta(T_0)) & \left[ C_3e^{-\frac{3}{k_0^2(2l_0 + 13\sigma_0)}(t-T_0)} \right. \\
 & \left. + C_4e^{-\frac{b}{2}(t-T_0)} \right], \quad \forall t \geq T_0. \tag{29}
 \end{aligned}$$

Combining (29) with (12),

$$\begin{aligned}
 \|\mathbf{z}(t) - [\mathbf{z}(t)]^*\| \leq \sqrt{\frac{3V(\mathbf{z}(T_0))}{7\sigma_0}} & \left[ C_3e^{-\frac{3}{k_0^2(2\eta_0 + 13\sigma_0)}(t-T_0)} \right. \\
 & \left. + C_4e^{-\frac{b}{2}(t-T_0)} \right]^{\frac{1}{2}}, \quad t \geq T_0.
 \end{aligned}$$

Consequently,  $\mathbf{z}(t)$  linearly converges to  $\mathcal{Z}^*$ .  $\square$

*Remark 4* By Theorem 1, the convergence rate depends on the values of  $\frac{3}{k_0^2(2\eta_0 + 13\sigma_0)}$  and  $\frac{b}{2}$ , where the optimization problem determines the first one and the other one depends on the event-triggered strategy. If  $\frac{b}{2} > \frac{3}{k_0^2(2\eta_0 + 13\sigma_0)}$ , then  $C_3 > 0$  and it yields  $\|\mathbf{x}(t) - [\mathbf{x}]^*\| \leq O(e^{-\frac{3}{2k_0^2(2\eta_0 + 13\sigma_0)}t})$ , otherwise  $C_4 > 0$  and we have  $\|\mathbf{x}(t) - [\mathbf{x}]^*\| \leq O(e^{-\frac{b}{4}t})$ .

*Remark 5* For continuous-time algorithms with event-triggered communication, Zeno behavior is an important topic. In this paper, Zeno Behavior is precluded in the algorithm (2) and the event-triggered scheme (3) is designed by referring to the work of [38]. The analysis of the Zeno behavior of the primal-dual algorithm is similar to Theorem 3 of [38] and we omit it in this paper.

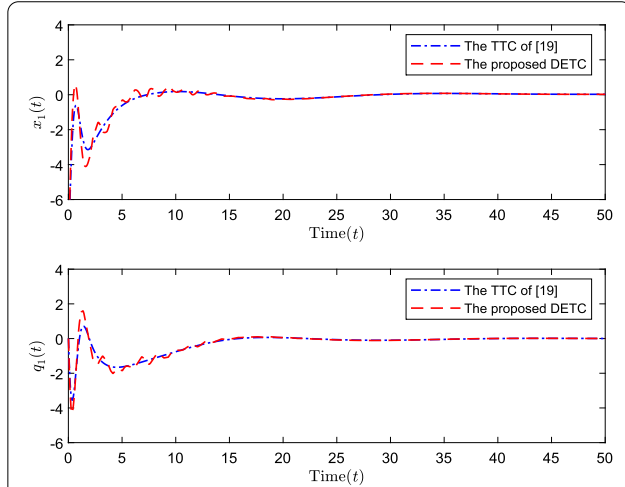
## 5 Simulation

To illustrate the effectiveness of the proposed method, we take the same example in [19], which considers a network containing 12 agents connected by a ring topology and equipped with the following local objective functions

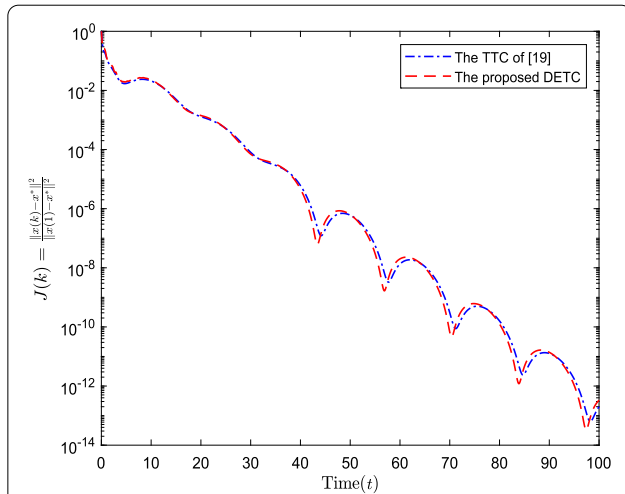
$$\psi_i(x) = \begin{cases} C_5^i(x - r_i)^2, & \text{if } x \geq r_i, \\ C_6^i(x + s_i)^2, & \text{if } x \leq -s_i, \\ 0, & \text{if } -s_i < x < r_i, \end{cases} \tag{30}$$

**Table 1** Parameters of the simulation

$i$	$C_5$	$C_6$	$r$	$s$	$x_i[0]$	$i$	$C_5$	$C_6$	$r$	$s$	$x_i[0]$
1	0.1	0.5	1	9	-9	7	0.5	0.4	1	5	6
2	0.3	0.2	3	3	4	8	0.6	1	7	5	6
3	0.8	0.5	3	7	-9	9	0.2	0	5	9	4
4	0	0.6	7	2	-9	10	0.5	0.9	8	6	-7
5	0.9	0.7	1	0	0	11	1	0.9	7	6	3
6	0.7	0.4	7	7	-8	12	0.5	0.8	9	9	0



**Figure 1** The trajectories of agent 1 in the methods of [19] and this paper



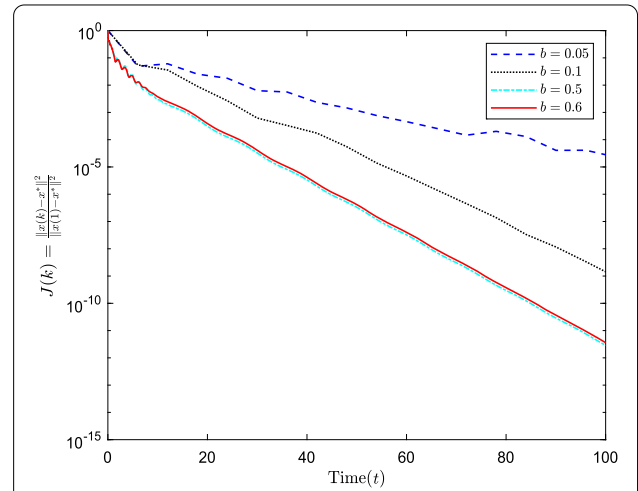
**Figure 2** The trajectories of  $J(k)$  in the methods of [19] and this paper

where  $C_5^i, C_6^i, r_i,$  and  $s_i > 0$  are given in Table 1. Then  $\psi_i(x)$  is convex and differentiable with

$$\nabla \psi_i(x) = \begin{cases} 2C_5^i(x - r_i), & \text{if } x \geq r_i, \\ 2C_6^i(x + s_i), & \text{if } x \leq -s_i, \\ 0, & \text{if } -s_i < x < r_i. \end{cases}$$

**Table 2** The times of triggering to achieve 0.01 convergence accuracy

$i$	TTC in [19]	Proposed DETC	Communication-saving
1	444	11	97.52%
2	500	12	97.60%
3	399	10	97.49%
4	262	5	98.10%
5	2766	34	98.77%
6	135	5	96.30%
7	135	3	97.78%
8	96	4	95.83%
9	200	3	98.50%
10	184	7	96.20%
11	1587	26	98.36%
12	2282	27	98.82%



**Figure 3** The trajectories of  $J(k)$  of Algorithm 1 about the event-triggered parameters  $b$

For the designed event-triggered scheme, we randomly select  $a_i$  and  $b_i$  in the triggering mechanism (10) and (15). We provide the trajectories of  $x_i(t)$  and  $q_i(t)$  of agent 1 generated by the algorithm of [19] and the proposed algorithm with DETC in Fig. 1. Moreover, we show the relative errors  $J(k) = \|x(k) - 1_N \otimes x^*\|^2 / \|x(1) - 1_N \otimes x^*\|^2$  of these algorithms in Fig. 2 and summarize the times of triggering to achieve  $J_i(k) = \|x_i(k) - x^*\|^2 / \|x_i(1) - x^*\|^2 = 0.01$  convergence accuracy in Table 2. Furthermore, we show the relative errors  $J(k)$  of Algorithm 1 when considering the event-triggered parameters  $b$  in Fig. 3. As we state in Remark 4, when  $b = \{0.5, 0.6\}$ , which implies  $\frac{b}{2} > \frac{3}{k_0^2(2\eta_0 + 13\sigma_0)}$ , the convergence rate is determined by the optimization problem, then  $\|\mathbf{x}(t) - [\mathbf{x}]^*\| \leq O(e^{-\frac{3}{2k_0^2(2\eta_0 + 13\sigma_0)}t})$ , otherwise the convergence rate is determined by the event-triggered strategy, then  $\|\mathbf{x}(t) - [\mathbf{x}]^*\| \leq O(e^{-\frac{b}{4}t})$ .

From Figs. 1-2 and Table 2, the DETC achieves almost the same convergence rate as the counterparts of the TTC



in [19] while the DETC significantly reduces communication burdens than TTC (98.36% for achieving 0.01 convergence accuracy).

## 6 Conclusion

In this paper, we designed a distributed continuous-time primal-dual algorithm for the distributed optimization problem with the metric subregularity condition. To alleviate the resource burden caused by continuous communication, we proposed a dynamic event-triggered communication mechanism without Zeno behavior. Moreover, we proved that the proposed algorithm achieved a linear convergence rate. Finally, we illustrated the effectiveness of the proposed algorithm with a comparison example.

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### Author contributions

The design and implementation of the research involved contributions from all authors. XY, XC, YF, and SC conducted material preparation and analysis. XY, XC, and SC were involved in problem formulation, discussion of ideas, mathematical derivation, and proof of results. Yuan Fan contributed to problem formulation and discussion of ideas. The final manuscript was reviewed and approved by all authors.

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### Data availability

Not applicable.

## Declarations

### Competing interests

The authors declare no competing interests.

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