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Enhanced parameter estimation by measurement of non-Hermitian operators

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Abstract

Quantum metrology aims at delivering new quantum-mechanical improvement to technologies of parameter estimations with precision bounded by the quantum Cramér-Rao bound. The currently used quantum Cramér-Rao bound was established with measurements of observables restricted to be Hermitian. This constrains the bound and limits the precision of parameter estimation. In this paper, we lift the constraint and derive a previously unknown quantum Cramér-Rao bound. We find that the new bound can reach arbitrary small value with mixed states and it breaks the Heisenberg limit in some cases. We construct a setup to measure non-Hermitian operators and discuss the saturation of the present bound. Two examples—the phase estimation with Greenberger-Horne-Zeilinger states of trapped ions and the adiabatic quantum parameter estimation with the nuclear magnetic resonance—are employed to demonstrate the theory. The present study might open a new research direction—non-Hermitian quantum metrology.

Keywords Parameter estimation, Quantum metrology, Non-Hermitian operators, Quantum Cramér-Rao bound

1 Introduction

One among the fields advanced by quantum mechanics is metrology, which concerns the estimation of unknown physical parameters [1–5] aiming at improving the estimation beyond classical precision bounds. Higher estimation precision demands more resources. The trade-off between the precision and the resources is constrained by the quantum Cramér-Rao bound (QCRB), which states that the variance of the estimation is at least as high as the inverse of the quantum Fisher information [6–13]. To attain the highest achievable precision, one needs to optimize over the input state and the encoding Hamiltonian as well as the measurements performed at the output.

In the past decades since Yuen and Lax [14] who first proposed an idea to estimate a complex parameter by measuring non-Hermitian observables, no progress has been made in the improvement of parameter estimation by optimizations of measurements invoking non-Hermitian operators. The reason is probably that in quantum mechanics observables are represented by Hermitian operators. In the last two decades, the study of non-Hermitian system and their unique properties have attracted fast growing interest [15–29], this motivates the examination of parameter estimation involving non-Hermitian operators. We should address that there are estimation protocols (or sensors) based on non-Hermitian system recently [30–38], but all analyses so far are based on either the quantum Fisher information and quantum Cramér-Rao bound with Hermitian operators or the properties of exceptional points. There is no study of quantum Cramér-Rao bound and its consequent estimation theory based on measurements of non-Hermitian operators.

In this paper, inspiring by the recent progress on the measurement of non-Hermitian operators [39–43], we present a scheme to enhance the precision of parameter

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estimation by measurement of non-Hermitian operators. To this goal, we first derive a previously unknown quantum Cramér-Rao bound defined by a quantum Fisher information for non-Hermitian operators. We find that the Fisher information is significantly increased due to the optimization over measurements including non-Hermitian operators, in particular for systems in mixed states. This is in contrary to the results of the quantum Fisher information defined by symmetric logarithmic derivatives. Saturation of the bound is analyzed and the optimal measurement to attain the bounds is derived. We demonstrate the present theory with the phase estimation setup in trapped ions [5, 44] and the adiabatic quantum parameter estimation setup.

The remainder of this paper is organized as follows. In Section 2, we first give the Robertson-Schrödinger uncertainty relation for non-Hermitian operators and extend the error propagation function from Hermitian systems [45, 46] to non-Hermitian systems. Then, we present a bound for parameter estimation based on the quantum Fisher information invoking measurements of non-Hermitian operators. An expression for the quantum Fisher information is derived and the optimal measurement to saturate the bound is given. In Section 3, we apply this theory to quantum parameter estimation with GHZ states of ions and show that mixed states can further enhance the sensitivity of phase estimation with measurements of non-Hermitian operators. The optimal measurement for this setup is derived in details. In Section 4, we apply this theory to adiabatic quantum parameter estimation, taking the estimation of magnetic field on a nuclear magnetic resonance as an example. The measurement of the non-Hermitian optimal operator is discussed in Section 5, and finally we conclude in Section 6.

2 Quantum Cramér-Rao bound and quantum Fisher information of non-Hermitian system

In this section, we first give the Robertson-Schrödinger uncertainty relation for non-Hermitian operators. Then, the error propagation is extended from Hermitian to non-Hermitian systems. Finally, we derive the non-Hermitian quantum Fisher information based on the new error propagation.

2.1 Robertson-Schrödinger uncertainty relation for non-Hermitian operators

In order to present a Robertson-Schrödinger uncertainty relation for non-Hermitian operators that will be used to

derive a non-Hermitian quantum Fisher information with non-Hermitian operators, let us consider two linear operators A and B , which can be either Hermitian or non-Hermitian. Defining $\Delta A = A - \langle A \rangle$, $\Delta B = B - \langle B \rangle$ and

$$\sigma_O^2 \equiv \langle \Delta O^\dagger \Delta O \rangle$$

as the variance for operator O ($O = A, B$), the Robertson-Schrödinger uncertainty relation follows from the Schwarz inequality $\langle F|F \rangle \langle G|G \rangle \geq |\langle F|G \rangle|^2$ with $|F\rangle = \Delta A|\Psi\rangle$, $|G\rangle = \Delta B|\Psi\rangle$ and $|\Psi\rangle$ being an arbitrary state of a system that,

$$\langle \Delta A^\dagger \Delta A \rangle \langle \Delta B^\dagger \Delta B \rangle \geq |\langle A^\dagger B \rangle - \langle A^\dagger \rangle \langle B \rangle|^2. \quad (1)$$

Though the proof is performed for pure states, it is easy to find that the uncertainty relation holds valid for mixed states (see Appendix A).

2.2 Error propagation for non-Hermitian systems

In order to derive a quantum Fisher information invoking measurements of non-Hermitian operators, here, we extend the error propagation from Hermitian to non-Hermitian systems. Take $\rho = \rho(\theta)$ as the encoding state of estimation parameter θ , the expectation value of operator A^\dagger and its conjugate A would depend on the parameter. Let us denote θ_j as i -th measurement result for the estimation parameter θ . The fluctuation due to the quantum uncertainty is then,

$$\langle \Delta A^\dagger \Delta A \rangle = \sum_j p_j [\langle A^\dagger(\theta_j) \rangle - \langle A^\dagger(\bar{\theta}) \rangle] [\langle A(\theta_j) \rangle - \langle A(\bar{\theta}) \rangle], \quad (2)$$

where $\bar{\theta} = \sum_j p_j \theta_j$, and p_j ($j = 1, 2, 3, \dots$) stand for probabilities of obtaining θ_j . Expanding $\langle A \rangle(\theta_j)$ around $\bar{\theta}$ as

$$\langle A \rangle(\theta_j) = \langle A \rangle(\bar{\theta}) + \frac{\partial \langle A \rangle}{\partial \theta} \Big|_{\bar{\theta}} (\theta_j - \bar{\theta}) + \dots$$

and keeping $\langle \Delta A^\dagger \Delta A \rangle$ up to the second order in $(\theta_j - \bar{\theta})$, we obtain

$$\langle \Delta A^\dagger \Delta A \rangle = \frac{\partial \langle A^\dagger \rangle}{\partial \theta} \Big|_{\bar{\theta}} \frac{\partial \langle A \rangle}{\partial \theta} \Big|_{\bar{\theta}} (\Delta \theta)^2, \quad (3)$$

where $(\Delta \theta)^2 = \sum_j p_j (\theta_j - \bar{\theta})^2$. This is the error propagation [47] invoking measurement of non-Hermitian operators.

2.3 Non-Hermitian quantum Fisher information

With the error propagation (3), we now derive the non-Hermitian quantum Fisher information where we concern the estimation of the unknown parameter θ encoded in $\rho(\theta)$. Introducing an operator L , which is so-called right logarithmic-derivative and defined by [14, 48]

$$\frac{\partial \rho}{\partial \theta} = L\rho = \rho L^\dagger, \tag{4}$$

we have

$$(\Delta\theta)^2 \frac{\partial \langle A^\dagger \rangle}{\partial \theta} \frac{\partial \langle A \rangle}{\partial \theta} \langle \Delta L^\dagger \Delta L \rangle = \langle \Delta A^\dagger \Delta A \rangle \langle \Delta L^\dagger \Delta L \rangle \geq |\langle A^\dagger L \rangle - \langle A^\dagger \rangle \langle L \rangle|^2 = \langle A^\dagger L \rangle \langle L^\dagger A \rangle. \tag{5}$$

where the second equality holds for $A = \gamma L$ with a constant γ (not necessary real), and $\langle L \rangle = 0$ has been used in the last equality implied by $\text{Tr}\left(\frac{\partial \rho}{\partial \theta}\right) = 0$. Substituting Eq. (4) into Eq. (5), we obtain

$$(\Delta\theta)^2 \geq \frac{1}{F_{nH}}, \quad F_{nH} \equiv \langle L^\dagger L \rangle. \tag{6}$$

Here, F_{nH} is defined as non-Hermitian quantum Fisher information.

The present derivation is based on the uncertainty relation, which has the following advantages. (i) It is easy to find an optimal measurement A^{opt} to saturate the bound,

$$A^{opt} = \gamma L, \text{ and } \langle L \rangle = \langle L^\dagger \rangle = 0, \tag{7}$$

with γ being a constant. And (ii) it provides an alternative understanding for the origin of the variance and bound. Namely, the variance and bound results from the uncertainty relation in contrast to its classical counterpart that is from statistics. In later discussions, we focus only on the saturation condition $A^{opt} = \gamma L$, since $\langle L \rangle = \langle L^\dagger \rangle = 0$.

In the following, we give the expression of the quantum Fisher information in terms of the eigenstates and eigenvalues of the encoding density matrix. We assume that the dimension of the Hilbert space is N and the state $\rho(\theta)$ encoding the unknown parameter θ may not be of full rank [49]. Its j -th eigenvalue is denoted by $p_j(\theta)$ and the corresponding eigenstate by $|\phi_j(\theta)\rangle$. Namely, the density matrix $\rho(\theta)$ can be decomposed as

$$\rho(\theta) = \sum_{j=1}^M p_j(\theta) |\phi_j(\theta)\rangle \langle \phi_j(\theta)|. \tag{8}$$

Here, we assume j running from 1 to M and $M \leq N$. With these notations, the quantum Fisher information defined in Eq. (6) takes

$$F_{nH} = \langle L^\dagger L \rangle = \sum_{j=1}^N \sum_{i=1}^M p_i L_{ij}^\dagger L_{ji}. \tag{9}$$

Recalling Eq. (4), we have

$$L_{ij}^\dagger = \frac{(\partial_\theta \rho)_{ij}}{p_i}, \quad L_{ji} = \frac{(\partial_\theta \rho)_{ji}}{p_i}, \quad p_i \neq 0. \tag{10}$$

Note that L should not be limited to the space spanned by the eigenstates of $\rho(\theta)$. This means that $L|\phi_i\rangle$ might be out of the space spanned by the eigenstates of $\rho(\theta)$. To compute L_{ij} , we need to know $(\partial_\theta \rho)_{ij}$. From Eq. (8) we have

$$\partial_\theta \rho = \sum_i \partial_\theta p_i |\phi_i\rangle \langle \phi_i| + \sum_i p_i |\partial_\theta \phi_i\rangle \langle \phi_i| + \sum_i p_i |\phi_i\rangle \langle \partial_\theta \phi_i|, \tag{11}$$

leading to

$$\begin{aligned} (\partial_\theta \rho)_{ij} &= \partial_\theta p_i \delta_{ij} + p_i \langle \partial_\theta \phi_i | \phi_j \rangle + p_j \langle \phi_i | \partial_\theta \phi_j \rangle \\ &= \partial_\theta p_i \delta_{ij} + (p_i - p_j) \langle \partial_\theta \phi_i | \phi_j \rangle. \end{aligned} \tag{12}$$

Here $\langle \partial_\theta \phi_i | \phi_j \rangle + \langle \phi_i | \partial_\theta \phi_j \rangle = 0$ has been applied. Substituting Eq. (12) and Eq. (10) into Eq. (9), we have

$$\begin{aligned} F_{nH} &= F_c + \sum_{i=1}^M p_i \langle \partial_\theta \phi_i | \partial_\theta \phi_i \rangle \\ &\quad + \sum_{i,j=1}^M \left(\frac{p_i^3 + p_j^3}{2p_i p_j} - (p_i + p_j) \right) |\langle \partial_\theta \phi_i | \phi_j \rangle|^2, \end{aligned} \tag{13}$$

where $F_c = \sum_{i=1}^M \frac{(\partial_\theta p_i)^2}{p_i}$ is the so-called classical Fisher information, which would be zero for θ -independent p_i . We will focus on this situation hereafter. The non-Hermitian quantum Fisher information F_{nH} for pure state $|\phi\rangle$ reduces to $F_{nH} = \langle \partial_\theta \phi | \partial_\theta \phi \rangle - \langle \partial_\theta \phi | \phi \rangle \langle \phi | \partial_\theta \phi \rangle$, which is 1/4 times smaller than the Hermitian quantum Fisher information $F_H^p = 4(\langle \partial_\theta \phi | \partial_\theta \phi \rangle - \langle \partial_\theta \phi | \phi \rangle \langle \phi | \partial_\theta \phi \rangle)$. The situation, however, is different for mixed states. Consider an encoding mixed state $\rho(\theta) = \sum_{j=1}^M p_j |\phi_j(\theta)\rangle \langle \phi_j(\theta)|$ with an eigenvalue, say, p_1 very close to zero and all p_j independent of θ . Equation (13) shows that the terms with $\frac{p_1^3 + p_j^3}{2p_1 p_j}$ dominate F_{nH} , and it could approach an arbitrary large value depending on how small p_1 might take in experiments. This makes the present quantum Fisher information different from the well-known one defined through the symmetric logarithmic derivative \mathcal{L} , which satisfies $\frac{\partial \rho}{\partial \theta} = \frac{1}{2}(\mathcal{L}\rho + \rho\mathcal{L})$. We note that the Hermitian quantum Fisher information F_H for a density matrix is always smaller than or equal to that for any eigenstate of the density matrix.

3 Quantum parameter estimation with Greenberger-Horne-Zeilinger (GHZ) states of several ions

Since the non-Hermitian quantum Fisher information F_{nH} benefits from mixed states, we will focus on encoding the unknown parameter into mixed states in the

undergoing example. Without loss of generality, we consider a type of simplest mixed states—there are only two eigenstates and the corresponding eigenvalues are p_1 and p_2 , respectively, where $p_1 + p_2 = 1$.

For mixed states of form $\rho = \sum_{i=1}^2 p_i |\phi_i\rangle\langle\phi_i|$ with θ -independent p_i , the Hermitian quantum Fisher information F_H [50–52] and non-Hermitian quantum Fisher information F_{nH} reduce to,

$$\begin{aligned}
 F_H &= \sum_{i=1}^2 \left(4p_i \langle \partial_\theta \phi_i | \partial_\theta \phi_i \rangle - 4p_i |\langle \phi_i | \partial_\theta \phi_i \rangle|^2 \right) - 16p_1 p_2 |\langle \phi_1 | \partial_\theta \phi_2 \rangle|^2, \\
 F_{nH} &= \sum_{i=1}^2 \left(p_i \langle \partial_\theta \phi_i | \partial_\theta \phi_i \rangle - p_i |\langle \phi_i | \partial_\theta \phi_i \rangle|^2 \right) \\
 &\quad + \left(\frac{p_2(1-3p_1)}{p_1} + \frac{p_1(1-3p_2)}{p_2} \right) |\langle \phi_1 | \partial_\theta \phi_2 \rangle|^2.
 \end{aligned}
 \tag{14}$$

Take GHZ states $|\phi_{1,2}\rangle = \frac{1}{\sqrt{2}}(|a\rangle^{\otimes N} \pm |b\rangle^{\otimes N})$ as the two eigenstates and suppose that the parameter θ is encoded into these states through a spin rotation defined by $U(\theta) = e^{-i\theta J_z}$ with Hermitian signal Hamiltonian $J_z = \sum_{j=1}^N s_z^{(j)}$, the encoding states follow,

$$|\phi_{1,2}(\theta)\rangle = \frac{1}{\sqrt{2}}(e^{i\frac{1}{2}N\theta} |a\rangle^{\otimes N} \pm e^{-i\frac{1}{2}N\theta} |b\rangle^{\otimes N}). \tag{15}$$

The rotation $U(\theta) = e^{-i\theta J_z}$ generates a relative phase $N\theta$ between states $|a\rangle^{\otimes N}$ and $|b\rangle^{\otimes N}$. The GHZ states have been created with up to $N = 6$ ${}^9\text{Be}^+$ ions [44] and $N = 14$ ${}^{40}\text{Ca}^+$ ions [53] in a linear Paul trap. With these states, the encoded mixed state reads

$$\rho(\theta) = \sum_{i=1}^2 p_i |\phi_i(\theta)\rangle\langle\phi_i(\theta)|,$$

and p_i ($i = 1, 2$) are independent of θ .

It is easy to find that

$$\begin{aligned}
 \langle \partial_\theta \phi_1 | \partial_\theta \phi_1 \rangle &= \frac{1}{4} N^2, \\
 \langle \partial_\theta \phi_2 | \partial_\theta \phi_2 \rangle &= \frac{1}{4} N^2, \\
 \langle \partial_\theta \phi_1 | \phi_1 \rangle &= \langle \partial_\theta \phi_2 | \phi_2 \rangle = 0, \\
 |\langle \partial_\theta \phi_1 | \phi_2 \rangle|^2 &= \frac{1}{4} N^2.
 \end{aligned}
 \tag{16}$$

Collecting these results, we obtain

$$\begin{aligned}
 F_H &= (2p - 1)^2 N^2, \\
 F_{nH} &= \frac{(2p - 1)^2}{4p(1 - p)} N^2.
 \end{aligned}
 \tag{17}$$

The dependance of F_H and F_{nH} are shown in Fig. 1. We find that for mixed states the non-Hermitian

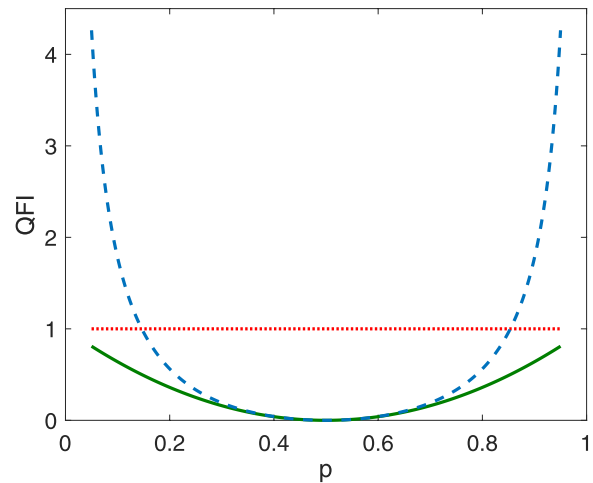


Fig. 1 F_H (green-solid line) and F_{nH} (blue-dashed line) are Hermitian and non-Hermitian quantum Fisher information, respectively, as a function of p . p is the eigenvalue of the density matrix. The quantum Fisher information F_H^p (red-dotted line) of pure state $|\phi_{1,2}(\theta)\rangle$ is also shown for comparison. All the quantum Fisher information is plotted in units of N^2

quantum Fisher information are always larger than the Hermitian quantum Fisher information, except the points $p = 0, 0.5, 1$. The quantum Fisher information of pure states F_H^p is larger than F_{nH} for $(2 - \sqrt{2})/4 < p < (2 + \sqrt{2})/4$. Especially, when the mixed state gets very close to pure states ($p \rightarrow 0$ or $p \rightarrow 1$), F_{nH} has a very large value manifesting itself as a witness of transition from mixed states to pure states. This feature might be used to design sensors. It is worth addressing that at points $p = 0, 1$, the state is pure. So, the quantum Fisher information should be calculated by the formula of pure states, which yield $F_H = N^2$ and $F_{nH} = 0.25N^2$. This suggests that a sensor working with mixed states very close to pure states is sensitive more than that working with pure states.

The variance $(\Delta\theta)^2$ of estimation can be given by the error propagation in Eq. (3), which is bounded by the quantum Cramér-Rao bound defined through the quantum Fisher information as $(\Delta\theta)^2 \geq \frac{1}{\nu F}$, $F = F_H, F_{nH}$, where ν is the number of repetitions of the experiment. Given a signal Hamiltonian and initial state, the bounds can be saturated by carefully chosen measurement described by operator A . Denoting A_1^{opt} as the optimal measurement and noticing Eqs. (4) and (8), we find that,

$$A_1^{opt} = \frac{1}{1 - p} |\phi_1\rangle\langle\phi_2| - \frac{1}{p} |\phi_2\rangle\langle\phi_1|, \tag{18}$$

where p , $(1 - p)$ and $|\phi_i\rangle$ are the eigenvalues and its corresponding eigenstates of $\rho(\theta)$, see Eqs. (15) and (17).

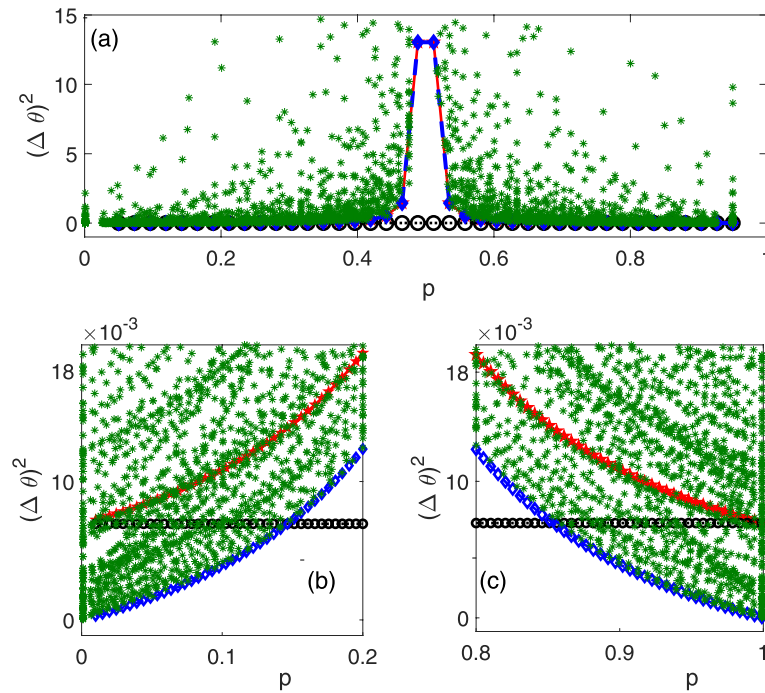


Fig. 2 Variance $(\Delta\theta)^2$ versus p , which is bounded by $1/F_H$ (red-solid lines with pentagams), $1/F_{nH}$ (blue-dashed lines with squares), and $1/F_H^P$ (black-dashed lines with circles). p is defined in Eq. (17). The variance (green-stars) is calculated by randomly generating operator A with Eq. (3). **a**, **b**, and **c** are for different range of p . The other parameter is chosen as $N = 12$

Applying this optimal measurement A_1^{opt} to $(\Delta\theta)^2 = \frac{\langle \Delta A_1^{opt\dagger} \Delta A_1^{opt} \rangle}{\partial \langle A_1^{opt\dagger} \rangle / \partial \theta \cdot \partial \langle A_1^{opt} \rangle / \partial \theta}$ (see Eq. (3)) and comparing it with numerical results, we find that the operator given in Eq. (18) indeed saturate the bound $1/F_{nH}$, see Fig. 2. With this setup, measurement of non-Hermitian operator would enhance the estimation precision for $p < (2 - \sqrt{2})/4$ or $p > (2 + \sqrt{2})/4$. The variance approaches to zero as the encoding states tend to pure states.

4 Parameter estimation on a nuclear magnetic resonance

Nuclear magnetic resonance (NMR) has long been at the forefront of precision tests of fundamental physics [54, 55], including the earliest efforts to search for a parity and time-reversal violating permanent electric dipole moment of the neutron [56, 57] and recent applications in battery and sensors [58, 59].

Most recently, an experimental adiabatic quantum parameter estimation on the NMR quantum processor has been performed [60]. The authors demonstrated that with the optimized adiabatic path the precision can achieve the Heisenberg scaling. The physics behind this

enhanced precision is the first order quantum phase transition and the speedup of the adiabatic evolution in the sensor. This scheme possesses the advantages of easy implementation, robustness against decays, and tunable energy gaps. Here, we propose the other scheme with measuring non-Hermitian operators and encoding unknown parameters in mixed states to enhance the precision of this setup.

The effective dimensionless Hamiltonian to describe the setup can be written as [60]

$$H = -B_z \sigma_0 + (1 - B_z) \sigma_z + \sqrt{2} B_x \sigma_x, \tag{19}$$

where $B_z > 0$ is the parameter to be estimated, B_x stands for the small transverse field assisting the estimation of B_z , σ_0 denotes the 2×2 identity operator, and $\sigma_{x,z}$ are Pauli operators. The eigenstates of the effective Hamiltonian H can be written as

$$\begin{aligned} |g(B_z)\rangle &= -\sin \frac{\theta}{2} |a\rangle + \cos \frac{\theta}{2} |b\rangle, \\ |e(B_z)\rangle &= \cos \frac{\theta}{2} |a\rangle + \sin \frac{\theta}{2} |b\rangle, \end{aligned}$$

where $|a\rangle = |11\rangle$, $|b\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ and $\tan \theta = \frac{\sqrt{2} B_x}{1 - B_z}$ [61].

The quantum Fisher information of eigenstates $|g\rangle \equiv |g(B_z)\rangle$ and $|e\rangle \equiv |e(B_z)\rangle$ is equal as $\langle \partial_{B_z} g | \partial_{B_z} g \rangle = \langle \partial_{B_z} e | \partial_{B_z} e \rangle$ and $\langle g | \partial_{B_z} g \rangle = \langle e | \partial_{B_z} e \rangle = 0$. With these considerations and the use of $F_H^p = 4(\langle \partial_{B_z} g | \partial_{B_z} g \rangle - |\langle g | \partial_{B_z} g \rangle|^2)$ [60], the quantum Fisher information of eigenstates $|g\rangle$ or $|e\rangle$ reads,

$$F_H^p = \frac{2B_x^2}{[(1 - B_z)^2 + 2B_x^2]^2}, \tag{20}$$

where superscript p in F_H^p stands for the quantum Fisher information of pure states. The key point of Ref. [60] to enhance the estimation precision is that F_H^p has a large value when $B_x \rightarrow 0$ near the critical point $B_z \sim 1$, and the quantum fisher information scales as T^2 , where T is the cost—the time required for the adiabatic evolution.

The time cost T required for the adiabatic path is determined by the adiabatic condition, which makes T long, although the shortcut to adiabaticity can solve this difficulty in some circumstances. We here propose to encode the estimation parameter B_z into mixed states,

$$\rho(B_z) = p|g(B_z)\rangle\langle g(B_z)| + (1 - p)|e(B_z)\rangle\langle e(B_z)| \tag{21}$$

with B_z -independent p . Considering $\langle g | \partial_{B_z} g \rangle = \langle e | \partial_{B_z} e \rangle = 0$, we have

$$F_{nH} = p\langle \partial_{B_z} g | \partial_{B_z} g \rangle + (1 - p)\langle \partial_{B_z} e | \partial_{B_z} e \rangle + \left(\frac{(1 - p)(1 - 3p)}{p} + \frac{p(3p - 2)}{1 - p} \right) |\langle g | \partial_{B_z} e \rangle|^2. \tag{22}$$

Straightforward calculation yields,

$$\langle \partial_{B_z} g | \partial_{B_z} g \rangle = \langle \partial_{B_z} e | \partial_{B_z} e \rangle = \frac{1}{4} \left(\frac{\partial \theta}{\partial B_z} \right)^2,$$

and

$$\langle \partial_{B_z} g | e \rangle = \frac{1}{2} \frac{\partial \theta}{\partial B_z}$$

with

$$\frac{\partial \theta}{\partial B_z} = \frac{\sqrt{2}B_x}{(1 - B_z)^2 + 2B_x^2}.$$

Substituting the last three equations into F_{nH} , we obtain

$$F_{nH} = \frac{(2p - 1)^2}{4p(1 - p)} F_H^p. \tag{23}$$

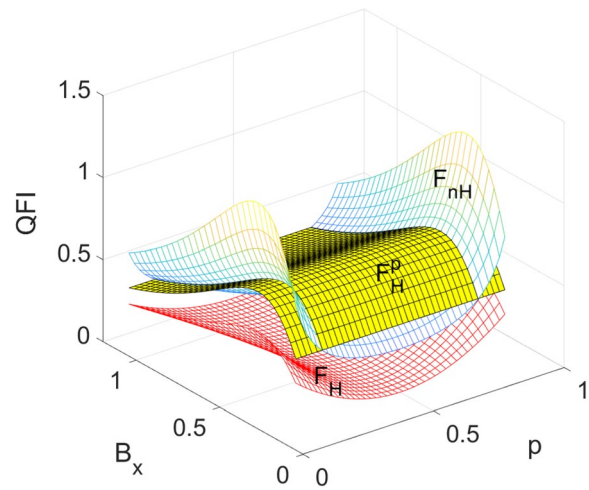


Fig. 3 F_H^p is Hermitian quantum Fisher information for pure state $|g(\theta)\rangle$, and F_H and F_{nH} are Hermitian and non-Hermitian quantum Fisher information for mixed state, respectively. p is the eigenvalue of the density matrix, B_x is the amplitude of the transverse field. $B_z = 0.4$ is chosen for this plot

It is always larger than the Hermitian quantum Fisher information of mixed state given by $F_H = (2p - 1)^2 F_H^p$. The dependence of quantum Fisher information on p and B_x is shown in Fig. 3. Note that $F_{nH} > F_H$ and $F_H^p > F_H$ holds for all p , while $F_{nH} > F_H^p$ for $p < \frac{1}{4}(2 - \sqrt{2})$ or $p > \frac{1}{4}(2 + \sqrt{2})$. In particular, F_{nH} has a very large value when the mixed states are very close to pure states. This feature provides a scheme for parameter estimation to enhance the precision. The estimation of B_z is bounded by $1/F_{nH}$. To saturate the bound, we need to optimize the measurement A . It is easy to find that

$$\begin{aligned} \langle g | L | g \rangle &= \langle e | L | e \rangle = 0, \\ \langle g | L | e \rangle &= \frac{1 - 2p}{2(1 - p)} \frac{\partial \theta}{\partial B_z}, \\ \langle e | L | g \rangle &= \frac{1 - 2p}{2p} \frac{\partial \theta}{\partial B_z}. \end{aligned} \tag{24}$$

Thus

$$A_2^{opt} = \frac{\gamma}{2} (1 - 2p) \frac{\partial \theta}{\partial B_z} \left(\frac{1}{1 - p} |g\rangle\langle e| + \frac{1}{p} |e\rangle\langle g| \right). \tag{25}$$

From the error propagation function Eq. (3), we can find that the factor $\frac{\gamma}{2} (1 - 2p) \frac{\partial \theta}{\partial B_z}$ can be dropped.

We perform numerically searching for the optimal measurement A . The result is shown in Fig. 4. We find that the variance of B_z becomes very small while p approaches to 0 as shown in Fig. 4 (a). This is reminiscent of sensors designed near quantum phase transition [60].

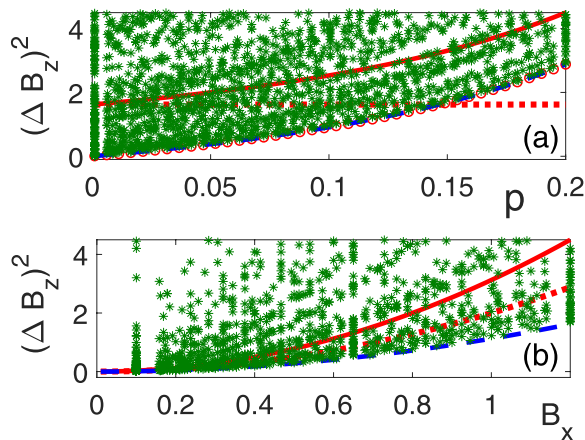


Fig. 4 **a, b** The variance $(\Delta B_z)^2$ (green-stars) versus ρ and B_x , respectively. The different lines are bounded by $1/F_H$ (red-solid) and $1/F_{nH}$ (blue-dashed) for mixed state, and $1/F_H^p$ (red-solid) is for pure state. ρ is defined in Eq. (17). The variance $(\Delta B_z)^2$ is calculated by Eq. (3) with random variable A

5 Measuring the non-Hermitian optimal operator

In the last sections, we have derived an unknown quantum Cramér-Rao bound defined by a quantum Fisher information for non-Hermitian operators and demonstrated the theory with the phase estimation of trapped ions and a setup of adiabatic quantum parameter estimation. One may wonder how to measure the non-Hermitian operators A_i^{opt} ($i = 1, 2$) to attain the bounds? Here, we follow the proposal in Refs. [39–42] to show it in experiment.

The expectation value of A_i^{opt} ($i = 1, 2$) in a quantum state $|\phi_{in}\rangle$ is given by $\langle \phi_{in} | A_i^{opt} | \phi_{in} \rangle$. It is complex in general and makes the non-Hermitian operator A_i^{opt} unobservable in experiment. Nevertheless, recent studies shown that this obstacle can be overcome with the help of polar decomposition [62]. Namely, $A_i^{opt} = UR$ with unitary operator U and Hermitian semidefinite operator R , $R = \sqrt{(A_i^{opt})^\dagger A_i^{opt}}$. This connects the average of non-Hermitian operator A_i^{opt} with the weak value of Hermitian operator R as follows,

$$\langle \phi_{in} | A_i^{opt} | \phi_{in} \rangle = \langle \phi_{in} | UR | \phi_{in} \rangle = \frac{\langle \phi | R | \phi_{in} \rangle}{\langle \phi | \phi_{in} \rangle} \langle \phi | \phi_{in} \rangle, \quad (26)$$

where $\langle \phi | \equiv \langle \phi_{in} | U$. It is well-known that $\frac{\langle \phi | R | \phi_{in} \rangle}{\langle \phi | \phi_{in} \rangle}$ is a weak value of the positive-semidefinite operator R , which can be measured directly in the weak measurement with successful probability depending on the post- and pre-selection states [63]. However, there are other approaches to obtain the weak value via the weak measurement [40, 41] without post-selection [42] in experiment. And the measurement of non-Hermitian operator

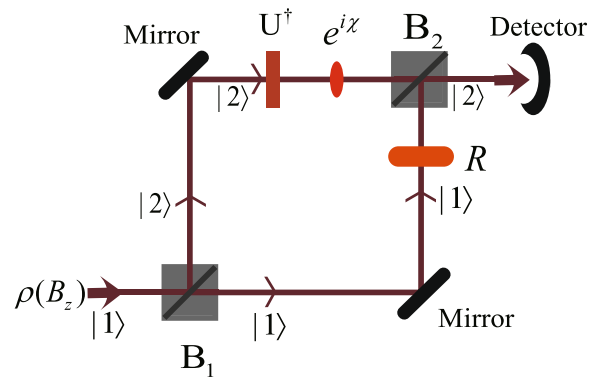


Fig. 5 Schematic setup to measure $A_2^{opt} = UR$ on state $\rho(B_z)$. B_1 and B_2 are 50:50 Hadamard-type beam splitters, which split the spatial modes representing by $|1\rangle$ and $|2\rangle$. $e^{i\chi}$ is a phase shifter that introduces a relative phase χ between the two arms. We measure the intensity at the detector as a function of χ and the value of $\text{Tr}(\rho(B_z)A_2^{opt})$ would be read out from the intensity of the output

can be implemented with an interferometric technique [41]. The average of the non-Hermitian operator can be determined through the intensity measured by a detector at the output without post-selection, so that there are no concerns of successful probability. Next, before going into details of such a technique, we first find R and U to decompose the non-Hermitian operator $A^{opt} = UR$.

To be specific, let us consider the optimal operator A_2^{opt} given in Eq. (25). Straightforward derivation yields,

$$R = \frac{1}{p} |g\rangle\langle g| + \frac{1}{1-p} |e\rangle\langle e|, \quad (27)$$

$$U = |e\rangle\langle g| + |g\rangle\langle e|.$$

R can be rewritten as a project operator, $R = \frac{1}{p}(\sigma_0 + \frac{2p-1}{1-p}|e\rangle\langle e|)$, and U can be treated as a beam splitter. Both Hermitian operator R and unitary operator U depend on B_z and can be realized in experiments with the established B_z based on prior data [60, 64]. σ_0 is the 2×2 identity operator. The schematic setup to measure the average of A_2^{opt} on state $\rho(B_z)$ is illustrate in Fig. 5.

To simplify the notation, we write $|g\rangle = |g(B_z)\rangle$ and $|e\rangle = |e(B_z)\rangle$. To measure A_2^{opt} , we first show that the average of A_2^{opt} in pure state $|g\rangle$, $\langle g | A_2^{opt} | g \rangle \equiv |A_2^{opt}| e^{i\zeta_g}$ can be readout from the intensity measured by the detector. Then, we extend this result to mixed state $\rho(B_z) = p|g\rangle\langle g| + (1-p)|e\rangle\langle e|$. Namely, the average of A_2^{opt} on state $\rho(B_z)$ can be read out from the intensity at the detector too.

In bases spanned by $|a, 1\rangle, |b, 1\rangle, |a, 2\rangle, |b, 2\rangle$ ($|x, n\rangle \equiv |x\rangle \otimes |n\rangle, x = a, b; n = 1, 2$) [65], these operation B_i ($i = 1, 2$), R , U^\dagger , $e^{i\chi}$ can be represented by [47],

$$\begin{aligned}
 B_{1,2} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} R_{aa} & R_{ab} & 0 & 0 \\ R_{ba} & R_{bb} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
 e^{i\chi} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\chi} & 0 \\ 0 & 0 & 0 & e^{i\chi} \end{pmatrix}, U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{aa}^\dagger & U_{ab}^\dagger \\ 0 & 0 & U_{ba}^\dagger & U_{bb}^\dagger \end{pmatrix},
 \end{aligned} \tag{28}$$

where $R_{ab} = \langle a|R|b\rangle$, and similar notations hold for R_{aa}, R_{bb}, R_{ba} and $U_{xy}^\dagger, x, y = a, b$. The input state that only occupies spatial state $|1\rangle$ can be written as,

$$|g\rangle = \begin{pmatrix} \phi_a \\ \phi_b \\ 0 \\ 0 \end{pmatrix}, \phi_a \equiv \langle a|g\rangle, \phi_b \equiv \langle b|g\rangle. \tag{29}$$

The output state reads

$$|\phi_{out}\rangle = B_2 e^{i\chi} U^\dagger R B_1 |g\rangle. \tag{30}$$

Simple algebra yields,

$$|\phi_{out}\rangle = \frac{1}{2} \begin{pmatrix} R_{aa}\phi_a + R_{ab}\phi_b - (U_{aa}^\dagger\phi_a + U_{ab}^\dagger\phi_b)e^{i\chi} \\ R_{ba}\phi_a + R_{bb}\phi_b - (U_{ba}^\dagger\phi_a + U_{bb}^\dagger\phi_b)e^{i\chi} \\ i[R_{aa}\phi_a + R_{ab}\phi_b + (U_{aa}^\dagger\phi_a + U_{ab}^\dagger\phi_b)e^{i\chi}] \\ i[R_{ba}\phi_a + R_{bb}\phi_b + (U_{ba}^\dagger\phi_a + U_{bb}^\dagger\phi_b)e^{i\chi}] \end{pmatrix}.$$

The intensity the detector measures can be represented by

$$I_g(\chi) = |\langle \phi_D | \phi_{out} \rangle|^2, \quad |\phi_D\rangle \equiv D|\phi_{out}\rangle, \tag{31}$$

with

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{32}$$

leading to

$$I_g(\chi) = \frac{1}{4} \left(1 + \langle g|R^2|g\rangle + 2|\langle g|A_2^{opt}|g\rangle| \cos(\chi - \zeta_g) \right), \tag{33}$$

where ζ_g is the argument of $\langle g|A_2^{opt}|g\rangle$, i.e., $\zeta_g = \arg \langle g|A_2^{opt}|g\rangle$.

To measure the average of A_2^{opt} on mixed state $\rho(B_z) = p|g\rangle\langle g| + (1-p)|e\rangle\langle e|$, we replace the input state $|g\rangle$ by $\rho(B_z)$. The output state is then,

$$\rho_{out} = pQ|g\rangle\langle g|Q^\dagger + (1-p)Q|e\rangle\langle e|Q^\dagger, \tag{34}$$

where $Q = DB_2 e^{i\chi} U^\dagger R B_1$. The intensity that the detector measures is

$$I(\chi) = |\text{Tr}(\rho_{out})|^2 = pI_g(\chi) + (1-p)I_e(\chi), \tag{35}$$

where $I_g(\chi) = |\langle g|Q^\dagger Q|g\rangle|^2$ and $I_e(\chi) = |\langle e|Q^\dagger Q|e\rangle|^2$. Substituting Q into $I_x(x = g, e)$, we find

$$I_x(\chi) = 1 + \langle x|R^2|x\rangle + 2|\langle x|A_2^{opt}|x\rangle| \cos(\chi - \zeta_x),$$

and $\langle x|A_2^{opt}|x\rangle = |\langle x|A_2^{opt}|x\rangle| e^{i\zeta_x}$. In experimental implementations, the intensity $I(\chi)$ together with $\langle x|R^2|x\rangle$ (the average of Hermitian operator R^2 on state $|x\rangle$) can determine the average of non-Hermitian operator A_2^{opt} , as both $|\langle x|A|x\rangle|$ and ζ_x can be inferred from the intensity $I(\chi)$.

6 Conclusion

The framework of quantum mechanics in which observables are associated only with Hermitian operators constrains the optimization in quantum parameter estimation. Considering the fact that in the past two decades the non-Hermitian physics has attracted fast growing interest in various research field, we have extended measurements from Hermitian operators to all operators including non-Hermitian ones. With this extension, a previously unknown expression for quantum Fisher information has been derived and a new quantum Cramér-Rao bound has been established. The saturation of the bound has been analyzed and the optimal measurement to attain the bounds have been given. The theory was elucidated with two experimentally feasible systems. In the example of phase estimation, we find that mixed states can further enhance the sensitivity of phase estimation with measurements of non-Hermitian operators. While in the example of adiabatic quantum parameter estimation for the estimation of magnetic fields on a nuclear magnetic resonance, the variance of the magnetic field under estimation can reach arbitrary small values depending on the encoding states. A setup to measure non-Hermitian operator was also proposed.

Appendix A: The uncertainty relation Eq. (1) for mixed states

Here, we show that the uncertainty relation Eq. (1) holds for mixed states. We prove this by introducing an ancilla a , such that a mixed state $\rho = \sum_j q_j |\psi_j\rangle\langle\psi_j|$ can be purified to be

$$|\Psi'\rangle = \sum_j \sqrt{q_j} |\psi_j\rangle \otimes |\phi_j\rangle_a, \tag{36}$$

and the state of the system is obtained by tracing $|\Psi'\rangle$ over the ancilla, $\rho = \text{Tr}_a |\Psi'\rangle\langle\Psi'|$. With this consideration, Eq. (1) can be straightforwardly extended to the composite system consisting of the system and the ancilla,

$$\begin{aligned} \langle \Psi' | \Delta A^\dagger \otimes I_a \Delta A \otimes I_a | \Psi' \rangle \langle \Psi' | \Delta B^\dagger \otimes I_a \Delta B \otimes I_a | \Psi' \rangle &\geq |\langle \Psi' | A^\dagger \otimes I_a B \otimes I_a | \Psi' \rangle \\ &- \langle \Psi' | A^\dagger \otimes I_a | \Psi' \rangle \langle \Psi' | B \otimes I_a | \Psi' \rangle|^2. \end{aligned} \quad (37)$$

Here, I_a is the identity operator of ancilla a . Noticing $\langle \Psi' | \Delta A^\dagger \otimes I_a \Delta A \otimes I_a | \Psi' \rangle = \text{Tr}(\rho \Delta A^\dagger \Delta A)$ and denoting $\text{Tr}(\rho \Delta A^\dagger \Delta A) = \langle \Delta A^\dagger \Delta A \rangle$ with Tr representing the trace over the system, we finish the proof of the weaker uncertainty relation in the main text.

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Authors' contributions

X. X. Yi developed the concept for the article. Jianning Li and X. X. Yi performed the specific calculation. Haodi Liu and Zhihai Wang checked the algebra and discussed the results. Finally, X. X. Yi and Jianning Li drafted the work and critically revised the work. All authors read and approved the final manuscript.

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Availability of data and materials

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare that they have no competing interests.

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