Furutani and Salasnich *AAPPS Bulletin* (2023) 33:19
https://doi.org/10.1007/s43673-023-00087-2

AAPPS Bulletin

REVIEW ARTICLE

Open Access

Fokker-Planck equations for a trapped particle in a quantum-thermal Ohmic bath: general theory and applications to Josephson junctions

Koichiro Furutani[1*] and Luca Salasnich[1,2]

Abstract

We consider a particle trapped by a generic external potential and under the influence of a quantum-thermal Ohmic bath. Starting from the Langevin equation, we derive the corresponding Schwinger-Keldysh action. Then, within the path-integral formalism, we obtain both the semiclassical Fokker-Planck equation and the quantum Fokker-Planck equation for this out-of-equilibrium system. In the case of an external harmonic potential and in the underdamped regime, we find that our Fokker-Planck equations contain an effective temperature T_{eff}, which crucially depends on the interplay between quantum and thermal fluctuations in contrast to the classical Fokker-Planck equation. In the regime of high temperatures, one recovers the classical Fokker-Planck equation. As an application of our result, we also provide the stationary solution of the semiclassical Fokker-Planck equations for a superconducting Josephson circuit and for a Bose Josephson junction, which are experimentally accessible.

1 Introduction

Inspired by the works of Einstein [1] and Smoluchowski [2] about the Brownian motion of a mesoscopic particle in a fluid, in 1908 Langevin introduced its stochastic equation [3]. Some years later, Johnson [4] and Nyquist [5] observed that, in addition to thermal effects, also the quantum mechanical noise plays a relevant role in the electric current of conductors. In 1951, the quantum version of the fluctuation-dissipation theorem of Callen and Welton [6] paved the way to the quantum Langevin equation [7–9]. In the first part of this paper, we explicitly show that, from the Langevin equation of a confined particle in a quantum-thermal Ohmic bath [10], one derives the corresponding semiclassical Schwinger-Keldysh action [11, 12]. The Schwinger-Keldysh action is usually obtained by adopting a quite different approach, which involves a closed time contour with forward and backward branches in time and where the dynamical variables of the system are doubled to take into account the two branches [10–12]. Moreover, it is well known that from the Schwinger-Keldysh action one obtains in the high-temperature regime the classical Martin-Siggia-Rose action [13]. Here, we follow a quite different path: for a particle in contact with a quantum-thermal bath, its semiclassical Schwinger-Keldysh action is obtained directly from the Langevin equation through a Martin-Siggia-Rose action, which is indeed the semiclassical Schwinger-Keldysh action functional.

*Correspondence:
Koichiro Furutani
koichiro.furutani@phd.unipd.it
[1] Dipartimento di Fisica e Astronomia "Galileo Galilei" and QTech Center, Università di Padova and INFN Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy
[2] Istituto Nazionale di Ottica del Consiglio Nazionale delle Ricerche, Via Nello Carrara 2, 50019 Sesto Fiorentino, Italy

Springer

In the second part, we derive the main results of the paper: the Fokker-Planck equations of our specific system. These Fokker-Planck equations are partial-differential equations describing the probability density $P(q, v, t)$ of finding a particle with position q and velocity v at time t [14–20] in a quantum-thermal Ohmic bath and trapped by a deterministic external potential $V(q)$. From the short-time propagator of the transition probability associated with the Schwinger-Keldysh action, we obtain both the semiclassical and the quantum Fokker-Planck equation for a confined particle under the effect of a quantum-thermal Ohmic bath. Remarkably, our Fokker-Planck equations are fully analytical and contain an effective temperature $T_{\rm eff}$. This effective temperature crucially depends on the interplay between quantum fluctuations, characterized by $\hbar\Omega$ with $\hbar$ the reduced Planck constant and Ω the frequency of harmonic potential, and thermal fluctuations, characterized by the thermal energy $k_B T$ with k_B the Boltzmann constant and T the temperature. Following our approach, we also obtain quantum Fokker-Planck equations for the superconducting phase in a Josephson circuit and for the population imbalance in an atomic Bose Josephson junction, which are described by generalized Langevin equations including quantum and thermal fluctuations.

2 Langevin equation for a particle in a quantum-thermal Ohmic bath

Let us consider a particle of mass m and coordinate $q(t)$ under the action of a deterministic potential $V(q(t))$ but also of a thermal bath which induces a dissipative force $-\gamma\dot{q}(t)$ with damping coefficient γ and a Gaussian stochastic force $\xi(t)$. The stochastic coordinate $q(t)$ of the particle satisfies the equation of motion

$$m\,\ddot{q}(t) + \gamma\,\dot{q}(t) + \frac{\partial V(q(t))}{\partial q} = \xi(t)\ . \tag{1}$$

Here, we assume the Markovian dynamics such that the damping term $\gamma\dot{q}(t)$ does not include any memory effect and the equation of motion (1) involves only one time variable t. Given a generic observable O which depends explicitly on the Gaussian random variable $\xi(t)$, the stochastic average has the following path integral representation

$$\langle O\rangle = \frac{\int D[\xi(t)]\, O[\xi(t)]\, e^{-\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \xi(t)C^{-1}(t-t')\xi(t')\,dt\,dt'}}{\int D[\xi(t)]\, e^{-\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \xi(t)C^{-1}(t-t')\xi(t')\,dt\,dt'}}, \tag{2}$$

which crucially depends on the choice of the correlation function $C(t)$, which, in general, is that $C(t-t') = \langle\xi(t)\xi(t')\rangle$.

Equation (1) is called semiclassical quantum Langevin equation [7–9] provided that the correlation function $C(t)$ is given by

$$C(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\gamma\,\hbar\omega\,\coth\left(\frac{\hbar\omega}{2k_B T}\right) e^{i\omega t}\ , \tag{3}$$

where T is the absolute temperature, k_B is the Boltzmann constant, and $\hbar$ is the reduced Planck constant. The correlation function of Eq. (3) is the one of a stochastic quantum-thermal Ohmic bath [7–9]. The presence of γ both in Eqs. (1) and (3) is a consequence of the fluctuation-dissipation theorem (FDT) [10]. Moreover, in Eq. (3) the term $e^{i\omega t}$ can be substituted by $\cos(\omega t)$ because the imaginary part is odd and its integral gives zero. Equation (1) with Eq. (3) is called semiclassical because the dynamical variable $q(t)$ is not a quantum operator. The quantum nature of Eq. (1) is however encoded in the correlator $C(t)$ through Eq. (3). Note that in the high-temperature regime $k_B T \gg \hbar\omega$, where $\coth(\hbar\omega/(2k_B T)) \to 2k_B T/(\hbar\omega)$, Eq. (3) gives $C(t) = 2\gamma\,k_B T\,\delta(t)$ with $\delta(t)$ the Dirac delta function and Eq. (1) becomes the familiar classical Langevin equation [3].

3 Martin-Siggia-Rose action

To derive the Martin-Siggia-Rose action [13] (see also Refs. [21–26]) from the semiclassical quantum Langevin equation (1), we observe that the expectation value of the generic observable O can also be written as

$$\begin{aligned}\langle O\rangle &= \int D[q(t)]O[q(t)]\langle\delta[q-q(t)]\rangle \\ &= \int D[q(t)]O[q(t)]\left\langle\delta\left[m\ddot{q}(t)+\gamma\dot{q}(t)+\frac{\partial V(q(t))}{\partial q}-\xi(t)\right]\right\rangle,\end{aligned} \tag{4}$$

where the Dirac delta function $\delta(x)$ appears because one considers the path integral over all possible $q(t)$ but with the constraint that $q(t)$ satisfies Eq. (1). This constraint ensures the correct implicit dependence of $q(t)$ with respect to $\xi(t)$. Taking into account the path integral representation of $\delta(x)$, we have

$$\langle O\rangle = \int D[q(t),\tilde{q}(t)]O[q(t)]\left\langle \exp\left[i\int_{-\infty}^{\infty} dt\tilde{q}(t)\left[m\ddot{q}(t)+\gamma\dot{q}(t)+\frac{\partial V(q(t))}{\partial q}-\xi(t)\right]\right]\right\rangle, \tag{5}$$

where $\tilde{q}(t)$ is an auxiliary response field. We now use Eq. (2) and the properties of Gaussian integrals obtaining

$$\langle O\rangle = \int D[q(t),\tilde{q}(t)]O[q(t)]e^{iS[q(t),\tilde{q}(t)]/\hbar}, \tag{6}$$

where

$$S[q(t),\tilde{q}(t)]/\hbar = -\int_{-\infty}^{+\infty} \tilde{q}(t)\left[m\ddot{q}(t) + \gamma\,\dot{q}(t) + \frac{\partial V(q(t))}{\partial q}\right] dt + \frac{i}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{q}(t)C(t-t')\tilde{q}(t')\,dt\,dt', \tag{7}$$

remembering that the stochastic noise $\xi(t)$ is time-translation invariant, namely $C(t-t') = \langle\xi(t)\xi(t')\rangle = \langle\xi(t-t')\xi(0)\rangle$. The functional of Eq. (7), with $C(t) = \langle\xi(t)\xi(0)\rangle$ given by Eq. (3), is our Martin-Siggia-Rose action [13].

We have seen that in the high-temperature regime $C(t) = 2\gamma k_B T\delta(t)$. In this regime, Eq. (7) is much simpler and the action is known as the classical Martin-Siggia-Rose action [13]. Quite remarkably, in this classical high-temperature regime, one can easily perform the path integral over the dual variable $\tilde{q}(t)$, obtaining

$$\langle O\rangle = \int D[q(t)]\; O[q(t)]\; e^{-S_{\rm OM}[q(t)]}, \tag{8}$$

with the real effective action

$$S_{\rm OM}[q(t)] = \frac{1}{4\gamma k_B T}\int_{-\infty}^{+\infty}\left[m\ddot{q}(t) + \gamma\,\dot{q}(t) + \frac{\partial V(q(t))}{\partial q}\right]^2 dt, \tag{9}$$

that is called Onsager-Machlup [27] action according to Olender and Elber [28].

4 Semiclassical Schwinger-Keldysh action

Let us restrict ourselves to a harmonic potential $V(q) = m\Omega^2 q^2/2$. In the underdamped regime $\gamma \ll m\Omega^2/\omega_{\rm cut}$ with $\omega_{\rm cut}$ being the ultraviolet cutoff frequency associated with the zero-point fluctuations in Eq. (3), we apply the approximation [29, 30]

$$C(t) \simeq \gamma\hbar\Omega\coth\left(\frac{\hbar\Omega}{2k_B T}\right)\delta(t) = 2\gamma k_B T_{\rm eff}\delta(t), \tag{10}$$

which is white noise including quantum fluctuations. The effective temperature is defined by

$$T_{\rm eff} = \frac{\hbar\Omega}{2k_B}\coth\left(\frac{\hbar\Omega}{2k_B T}\right). \tag{11}$$

This approximation is justified for the following reason. The Langevin equation (1) in the long-time limit gives [31]

$$\tilde{q}(\omega) = \int_{-\infty}^{\infty} dt\, q(t)e^{-i\omega t} = \tilde{\chi}(\omega)\tilde{\xi}(\omega), \tag{12}$$

where $\tilde{\xi}(\omega) = \int_{-\infty}^{\infty} dt\xi(t)e^{-i\omega t}$ and

$$\tilde{\chi}(\omega) = \frac{1}{-m\omega^2 - i\gamma\omega + m\Omega^2}, \tag{13}$$

is the response function in the frequency domain. Equation (12) leads to the correlation function

$$\langle q(t\to\infty)^2\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{C}(\omega)\|\tilde{\chi}(\omega)\|^2, \tag{14}$$

with $\tilde{C}(\omega) = \int_{-\infty}^{\infty} dte^{-i\omega t}C(t) = \gamma\hbar\omega\coth\left[\hbar\omega/(2k_B T)\right]$. The autocorrelation function of the conjugate momentum also involves the combination of $\tilde{C}(\omega)\|\tilde{\chi}(\omega)\|^2$. In the underdamped limit $\gamma \ll m\Omega^2/\omega_{\rm cut}$, the response function (13) is dominant only around $\omega = \pm\Omega$ as

$$\gamma\|\omega\|\|\tilde{\chi}(\omega)\|^2 \to \frac{\pi}{2m^2\Omega}\left[\delta(\omega-\Omega) + \delta(\omega+\Omega)\right]. \tag{15}$$

Consequently, in Eq. (14), we can safely use approximation that $\tilde{C}(\omega) \simeq \tilde{C}(\Omega)$, which justifies the white noise approximation in Eq. (10) in the underdamped limit $\gamma \ll m\Omega^2/\omega_{\rm cut}$. In the classical limit $\hbar\Omega \ll k_B T$, we recover $T_{\rm eff} = T$. In the low-temperature regime $\hbar\Omega \gg k_B T$, on the other hand, Eq. (10) provides $T_{\rm eff} = \hbar\Omega/(2k_B)$.

By using Eq. (10), the action (7) with the kernel reads

$$S[q(t),\tilde{q}(t)]/\hbar = -\int_{-\infty}^{+\infty}\tilde{q}(t)\left[m\ddot{q}(t) + \gamma\,\dot{q}(t) + m\Omega^2 q\right] dt + \frac{i\gamma}{2}\int_{-\infty}^{+\infty} 2k_B T_{\rm eff}\tilde{q}(t)^2\, dt. \tag{16}$$

Remarkably, Eq. (16) is very similar to the Schwinger-Keldysh action [11, 12] of a quantum particle in contact with an Ohmic bath. The only difference is due to the fact that instead of $\hbar\tilde{q}\,\partial V(q)/\partial q$ in the exact Schwinger-Keldysh action there is $[V(q+\hbar\tilde{q}) - V(q-\hbar\tilde{q})]/2$. See, for instance, page. 33 of Ref. [10]. Clearly, $[V(q+\hbar\tilde{q}) - V(q-\hbar\tilde{q})]/2 \simeq \hbar\tilde{q}\,\partial V(q)/\partial q$ under the assumption of a small $\hbar\tilde{q}$. It is important to stress that the Echern-Schon-Ambegaokar action [32] used for superconducting Josephson junctions is nothing else than the exact Schwinger-Keldysh action. At zero temperature, the Schwinger-Keldysh action has been used to study the effect of quantum noise in the quantum phase transition of a Josephson junction [33]. In the classical limit $T_{\rm eff} \to T$, one can readily find that Eq. (16) coincides with the classical dissipative action [10].

We call Eq. (16) semiclassical Schwinger-Keldysh action. Indeed, the classical Schwinger-Keldysh action, which is nothing else than the classical Martin-Siggia-Rose action, is obtained with $T_{\rm eff} \to T$.

Then, by functional integrating over $\tilde{q}(t)$ one recovers again the Onsager-Machlup action (9).

5 Fokker-Planck equations

In this section, we derive the corresponding Fokker-Planck equation for the MSR action in Eq. (7) or the Schwinger-Keldysh action in Eq. (16).

5.1 Semiclassical Fokker-Planck equation

Let us rewrite the action of Eq. (7) as

$$e^{iS[q(t),\tilde{q}(t)]/\hbar} \equiv \int D[v(t),\lambda(t)] e^{iS_{\rm e}[q(t),v(t),\tilde{q}(t),\lambda(t)]/\hbar}, \tag{17}$$

$$\begin{aligned} S_{\rm e}[q(t),v(t),\tilde{q}(t),\lambda(t)] &= \int_{-\infty}^{\infty} dt L_{\rm e}[q(t),v(t),\tilde{q}(t),\lambda(t)] \\ &= \int_{-\infty}^{\infty} dt\hbar \Bigg[-m\tilde{q}(t)\dot{v}(t) - m\tilde{q}(t)\frac{\partial F(q,v)}{\partial v} \\ &\quad + \frac{i}{2}\int_{-\infty}^{\infty} dt'\tilde{q}(t)C(t-t')\tilde{q}(t') + \lambda(t)[\dot{q}(t)-v(t)] \Bigg], \end{aligned} \tag{18}$$

where $S_{\rm e}[q(t),v(t),\tilde{q}(t),\lambda(t)]$ is a new effective action with a velocity field $v(t)$ and an auxiliary field $\lambda(t)$ as a Lagrange multiplier that guarantees $v(t)=\dot{q}(t)$ by $\delta S_{\rm e}/\delta\lambda = 0$ [34], and

$$F(q,v) \equiv \frac{\gamma}{2m}v^2 + \frac{V'(q)}{m}v. \tag{19}$$

The effective Lagrangian $L_{\rm e}[q(\bar{t}),v(\bar{t}),\tilde{q}(\bar{t}),\lambda(\bar{t})]$ can be used to introduce the propagator [34]

$$\begin{aligned} K(q',v',t'|q,v,t) \equiv & \int_{q(t)=q}^{q(t')=q'} D[q(\bar{t})] \int_{v(t)=v}^{v(t')=v'} D[v(\bar{t})] \\ & \times \int D[\tilde{q}(\bar{t}),\lambda(\bar{t})] e^{i\int_t^{t'} d\bar{t} L_{\rm e}[q(\bar{t}),v(\bar{t}),\tilde{q}(\bar{t}),\lambda(\bar{t})]/\hbar}, \end{aligned} \tag{20}$$

which gives the transition probability from the initial configuration (q, v, t) to the final configuration (q', v', t'). It follows that the probability $\mathcal{P}(q,v,t)$ of finding the system in the configuration (q, v, t) satisfies the convolution equation

$$\mathcal{P}(q',v',t') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dv\, K(q',v',t'|q,v,t)\, \mathcal{P}(q,v,t). \tag{21}$$

In Appendix 1, we show how to derive the semiclassical Fokker-Planck equation from this expression taking into account Eq. (20). The final result of this derivation is [34, 35]

$$\partial_t \mathcal{P}(q,v,t) = \left[-v\partial_q + \frac{\gamma}{m}\partial_v v + \frac{V'(q)}{m}\partial_v + \frac{\gamma k_{\rm B} T_{\rm eff}}{m^2}\partial_v^2 \right] \mathcal{P}(q,v,t), \tag{22}$$

under Eq. (10) where $V(q)$ is the harmonic potential of frequency Ω. In the classical limit $\hbar \to 0$, or equivalently $k_{\rm B}T \gg \hbar\Omega$, Eq. (22) reduces to the familiar classical one [10, 36, 37]

$$\partial_t \mathcal{P}(q,v,t) = \left[-v\partial_q + \frac{\gamma}{m}\partial_v v + \frac{V'(q)}{m}\partial_v + \frac{\gamma k_{\rm B} T}{m^2}\partial_v^2 \right] \mathcal{P}(q,v,t). \tag{23}$$

On the other hand, at $T = 0$, the effective temperature has a finite minimum $T_{\rm eff} = \hbar\Omega/(2k_{\rm B})$ as shown in Fig. 1. Figure 1 illustrates that the difference from the original temperature is significant in the low-temperature regime due to the quantum effects.

Quite remarkably, the semiclassical Fokker-Planck equation (22) has the following stationary analytical solution

$$\mathcal{P}_{\rm stat}(q,v) = Z^{-1} e^{-[mv^2/2+V(q)]/(k_{\rm B}T_{\rm eff})}, \tag{24}$$

where

$$Z \equiv \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv\, e^{-[mv^2/2+V(q)]/(k_{\rm B}T_{\rm eff})}. \tag{25}$$

With a harmonic potential $V(q) = m\Omega^2 q^2/2$, the stationary solution gives the second moments of the coordinate and velocity as

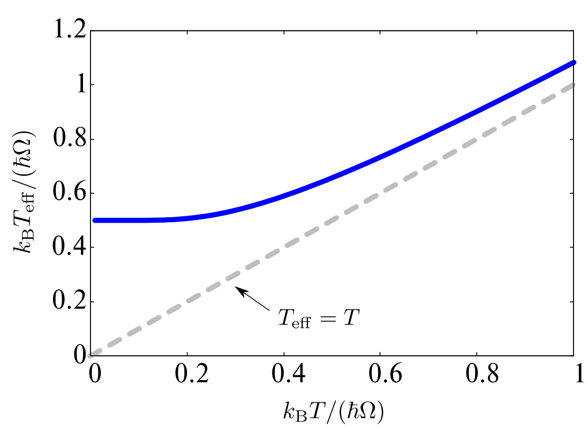

Fig. 1 Effective temperature in Eq. (11). The dashed line stands for $T_{\rm eff} = T$

$$\langle q^2 \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv q^2 \mathcal{P}_{\rm stat}(q,v) = \frac{k_{\rm B} T_{\rm eff}}{m\Omega^2}, \quad (26)$$

and

$$\langle v^2 \rangle = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv v^2 \mathcal{P}_{\rm stat}(q,v) = \frac{k_{\rm B} T_{\rm eff}}{m}. \quad (27)$$

These second moments are exactly consistent with a Langevin analysis in the underdamped limit [31, 38]. In the classical limit $T_{\rm eff} \to T$, Eqs. (26) and (27) recover the equipartition of energy. Due to the quantum correction in Eq. (11), both of Eqs. (26) and (27) are proportional to $\hbar\Omega$ in the low-temperature regime.

Note that Eqs. (26) and (27) are the underdamped results which are, strictly speaking, justified in the underdamped limit $\gamma \ll m\Omega^2/\omega_{\rm cut}$. With finite damping constant $\gamma > m\Omega^2/\omega_{\rm cut}$, the second moments of the velocity should have a logarithmic ultraviolet divergence [38]. We stress that, however, we are restricting ourselves to the underdamped limit and do not consider such cases throughout this paper. Only in this underdamped limit, do the second moments of the coordinate or the momentum have clear physical interpretation as they recover the equipartition of energy in the classical limit [31, 38].

5.2 Quantum Fokker-Planck equation

As mentioned in Section 4, the dissipative Schwinger-Keldysh action is different from the MSR action in terms of the potential term in the higher order of $\tilde{q}$. Let us call the corresponding Fokker-Planck equation derived from the Schwinger-Keldysh action the quantum Fokker-Planck equation since it includes higher order of the quantum component $\tilde{q}$, which should coincide with the semiclassical one to the first order of $\tilde{q}$.

To obtain the quantum Fokker-Planck equation, we can simply substitute the potential term in Eq. (18) as

$$\hbar\tilde{q}V'(q) \to \frac{V(q+\hbar\tilde{q}) - V(q-\hbar\tilde{q})}{2} = \sum_{n=0}^{\infty} (\hbar\tilde{q})^{2n+1} \frac{V^{(2n+1)}(q)}{(2n+1)!}, \quad (28)$$

where $V^{(k)}(q)$ denotes the k-derivative of $V(q)$ in q. Since it is time-local, one can proceed to the Fokker-Planck equation in a similar manner as in the last section. Eventually, one obtains the quantum Fokker-Planck equation

$$\partial_t \mathcal{P}(q,v,t) = \left[-v\partial_q + \frac{\gamma}{m}\partial_v v + \frac{\gamma k_{\rm B} T_{\rm eff}}{m^2}\partial_v^2 + \partial_v\left(\partial_v \tilde{F}\right)\right]\mathcal{P}(q,v,t), \quad (29)$$

where $\tilde{F}$, that gives the difference from semiclassical equation (22), is defined as

$$\begin{aligned}\tilde{F}(q,v) &\equiv \frac{v}{m}\sum_{n=0}^{\infty}\left(\frac{\hbar\partial_v}{im}\right)^{2n}\frac{V^{(2n+1)}(q)}{(2n+1)!} \\ &= \frac{V'(q)}{m}v + \frac{v}{m}\sum_{n=1}^{\infty}\left(\frac{\hbar\partial_v}{im}\right)^{2n}\frac{V^{(2n+1)}(q)}{(2n+1)!} \\ &= \Omega^2 qv.\end{aligned} \quad (30)$$

For a harmonic potential $V(q) = m\Omega^2 q^2/2$, as in the last equality in Eq. (30), the higher-order derivatives in Eq. (30) vanish. With a generic potential, the presence of infinite derivatives makes it quite difficult to solve it in full generality. While a truncation of the series with $n = 1$ or $n = 1, 2$ could be used to find reliable corrections to the semiclassical result, it may lead to negative probability distribution [39].

6 Fokker-Planck equation for Josephson junctions

In a resistively and capacitively shunted Josephson (RCSJ) junction, the superconducting phase $\phi(t)$ obeys the generalized Langevin equation [31, 40]

$$\frac{\hbar^2}{2E_C}\ddot{\phi} + \frac{\hbar\alpha}{2\pi}\dot{\phi} + E_J \sin\phi - \frac{\Phi_0 I_{\rm ext}}{2\pi} = \xi, \quad (31)$$

where $E_C = (2e)^2/(2C)$ is the charging energy with a capacitance C, $\alpha = R_Q/R$ is the ratio between the critical resistance $R_Q = h/(2e)^2$ and the resistance R, E_J is the Josephson energy, $I_{\rm ext}$ is the external current, and $\Phi_0 = h/(2e)$ is the magnetic flux quantum. The current noise $\xi(t)$ originates from the shunted resistor and is assumed to satisfy the FDT in Eq. (3). It has been theoretically predicted that the system is superconducting below the critical resistance $\alpha > 1$ while it is insulating above the resistance $\alpha < 1$ [41, 42]. The ultraviolet cutoff frequency can be chosen as $\omega_{\rm cut} = \Delta/\hbar$ with Δ being the superconducting gap. The approximation (10) is valid within the underdamped limit $\alpha \ll 2\pi E_J/\Delta$ [31].

We can identify this Langevin equation as Eq. (1) by replacing $q(t) \to \phi(t)$, $v \to \dot{\phi} = 2\pi V/\Phi_0$, $m \to \hbar^2/(2E_C)$, $\gamma \to \hbar\alpha/(2\pi)$, and $V(q) \to U_{\rm wash}[\phi] \equiv -E_J\cos\phi - \Phi_0 I_{\rm ext}/(2\pi)\cdot\phi$. Consequently, following the procedures in the last sections, we obtain the quantum Fokker-Planck equation for the RCSJ junction as

$$\begin{aligned}\partial_t \mathcal{P}(\phi,V,t) = \Bigg[&-\frac{2\pi V}{\Phi_0}\partial_\phi + \frac{\alpha E_C}{\pi\hbar}\partial_V V + \frac{\alpha\Phi_0^2 E_C^2}{2\pi^2\hbar^3}k_{\rm B}T_{\rm eff}\partial_V^2 \\ &+ \left(\frac{\Phi_0}{2\pi}\right)^2\partial_V\left(\partial_V \tilde{F}[\phi,V]\right)\Bigg]\mathcal{P}(\phi,V,t),\end{aligned} \quad (32)$$

with $V(t) = \Phi_0\dot{\phi}/(2\pi)$ the voltage. The function $\tilde{F}[\phi, V]$ is given by

$$\begin{aligned}\tilde{F}[\phi, V] &= \frac{4\pi E_C}{\Phi_0\hbar^2}\left[\frac{\partial U_{\mathrm{wash}}[\phi]}{\partial\phi} + \sum_{n=1}^{\infty}\left(\frac{\Phi_0 E_C}{i\pi\hbar}\partial_V\right)^{2n}\frac{U_{\mathrm{wash}}^{(2n+1)}[\phi]}{(2n+1)!}\right]V \\ &= \frac{4\pi E_C}{\Phi_0\hbar^2}\left(-\frac{\Phi_0 I_{\mathrm{ext}}}{2\pi} + E_J\phi\right)V,\end{aligned} \tag{33}$$

where, in the last row, we used $U_{\mathrm{wash}}[\phi] \simeq -E_J - \Phi_0 I_{\mathrm{ext}}/(2\pi)\cdot\phi + E_J\phi^2/2$. We can find the semiclassical stationary solution as

$$\mathcal{P}^{\mathrm{cl}}_{\mathrm{stat}}(\phi, V) = Z^{-1}\exp\left[-\frac{E_C}{\pi k_{\mathrm{B}}T_{\mathrm{eff}}}\left[\left(\frac{eV}{E_C}\right)^2 + \frac{U_{\mathrm{wash}}[\phi]}{E_C}\right]\right], \tag{34}$$

which is illustrated in Fig. 2(a) for $E_J/E_C = 1$, $k_{\mathrm{B}}T/E_C = 0.1$, and $\Phi_0 I_{\mathrm{ext}}/(2\pi E_J) = 0.3$. Also in Fig. 2(a), we use $U_{\mathrm{wash}}[\phi] \simeq -E_J - \Phi_0 I_{\mathrm{ext}}/(2\pi)\cdot\phi + E_J\phi^2/2$. The effective temperature is given by Eq. (11) with $\Omega = (2E_C E_J)^{1/2}/\hbar$. Equation (34) indicates that, in the stationary configuration, the voltage V is more localized as one increases $E_C/(k_{\mathrm{B}}T_{\mathrm{eff}})$, and the superconducting phase ϕ is localized with a large E_J/E_C. Experimentally, one can observe $E_J/E_C \simeq 3.8\times 10^5$ in a RCSJ circuit [43], which reflects the highly localized phase in the superconducting circuit.

In a one-dimensional Bose Josephson junction (BJJ) in a head-to-tail configuration, we have two one-dimensional Bose gasses in contact through a tunnel coupling $J(x) = J_0 L\delta(x)$ at a point $x = 0$ where J_0 is the strength of the Josephson coupling and L is the system size. The zero-mode of the population imbalance ζ_0 also obeys the Langevin equation [44, 45]

$$\ddot{\zeta}_0 + \gamma\dot{\zeta}_0 + \Omega^2\zeta_0 = \frac{\sqrt{M}\Omega}{\hbar\bar{\rho}}\xi, \tag{35}$$

in the linear regime $\|\zeta_0(t)\| \ll 1$ with Ω the Josephson frequency, γ the damping constant associated with the Josephson coupling J_0, M the effective mass related to the interparticle interaction strength g, and $\bar{\rho}$ the average atomic density. The stochastic noise $\xi(t)$ satisfies the FDT in Eq. (3). The ultraviolet cutoff can be chosen as $\omega_{\mathrm{cut}} = 2\pi c\bar{\rho} \simeq 10^4\Omega$ where $c = (g\bar{\rho}/m)^{1/2}$ is the speed of sound and m is the atomic mass [45, 46]. Then, the approximation (10) is valid if $\bar{N}J_0/Mc^2 \ll 10^{-8}$ with $\bar{N} \equiv \bar{\rho}L$ being the average number of atoms, which is the Josephson regime in which the tunneling energy $\bar{N}J_0$ is much smaller than the kinetic energy Mc^2. For this one-dimensional BJJ, we find the quantum Fokker-Planck equation as

$$\partial_t\mathcal{P}(\zeta_0, \phi_0, t) = \left[\Omega_{\mathrm{R}}\phi_0\partial_{\zeta_0} + \gamma\partial_{\phi_0}\phi_0 + \frac{2g}{\hbar^2 c}k_{\mathrm{B}}T_{\mathrm{eff}}\partial^2_{\phi_0} + \frac{1}{\Omega_{\mathrm{R}}^2}\partial_{\phi_0}\left(\partial_{\phi_0}\tilde{F}\right)\right]\mathcal{P}(\zeta_0, \phi_0, t), \tag{36}$$

with $\Omega_{\mathrm{R}} \equiv J_0/\hbar$ the Rabi frequency and $\phi_0(t) = -\dot{\zeta}_0(t)/\Omega_{\mathrm{R}}$ the zero-mode of the relative phase. The function $\tilde{F}(\zeta_0, \phi_0)$ is given by

$$\tilde{F}(\zeta_0, \phi_0) = -\Omega_{\mathrm{R}}\Omega^2\zeta_0\phi_0. \tag{37}$$

The linearized equation (35) involves a harmonic potential and gives no quantum correction that stems from the higher-order derivatives of the external potential. The stationary solution is given by

$$\mathcal{P}_{\mathrm{stat}}(\zeta_0, \phi_0) = Z^{-1}\exp\left[-\frac{\hbar\Omega}{2k_{\mathrm{B}}T_{\mathrm{eff}}}\bar{N}\left(\frac{\Omega}{\Omega_{\mathrm{R}}}\zeta_0^2 + \frac{\Omega_{\mathrm{R}}}{\Omega}\phi_0^2\right)\right]. \tag{38}$$

We show the stationary solution in Fig. 2(b) for $\Omega/\Omega_{\mathrm{R}} = 0.5$ and $k_{\mathrm{B}}T/(\hbar\Omega) = 0.1$. Figure 2(b) shows that the relatively localized ϕ_0 around the origin and the delocalized ζ_0 are realized as a stationary configuration. As one decreases the ratio between the interaction energy and the tunneling energy $\Omega/\Omega_{\mathrm{R}} = \sqrt{2g\bar{\rho}/J_0}$ less than one, the population imbalance ζ_0 is delocalized and the relative phase ϕ_0 is highly localized in the stationary configuration given by Eq. (38).

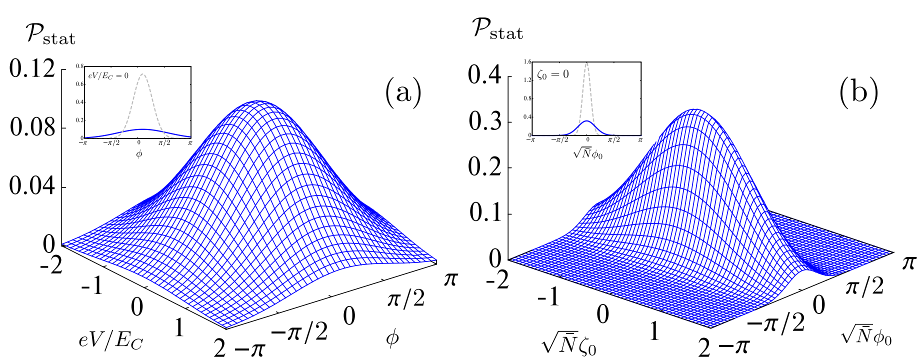

Fig. 2 Stationary solutions of the semiclassical Fokker-Planck equations of a RCSJ junction (**a**) and BJJ (**b**). The left panel (**a**) illustrates the result of Eq. (34) for $E_J/E_C = 1$, $k_{\mathrm{B}}T/E_C = 0.1$, and $\Phi_0 I_{\mathrm{ext}}/(2\pi E_J) = 0.3$. The right panel (**b**) shows the result of Eq. (38) for $\Omega/\Omega_{\mathrm{R}} = 0.5$ and $k_{\mathrm{B}}T/(\hbar\Omega) = 0.1$. The gray dashed curves in the insets stand for the probability in the classical limit $T_{\mathrm{eff}} \to T$

7 Conclusions

In the first part of this paper, we have derived the Schwinger-Keldysh action of a particle under the effect of a deterministic external potential and a stochastic Ohmic bath, which contains both thermal and quantum fluctuations. Contrary to previous papers [10–12], our derivation has been performed starting from the Langevin equation of the system. In the second part of the paper, we have then adopted the Schwinger-Keldysh action to include the velocity of the particle by using a Hubbard-Stratonovich transformation and to derive the fully analytical semiclassical and quantum Fokker-Planck equations for the time-dependent probability of the particle in the quantum-thermal Ohmic bath. The semiclassical Fokker-Planck equation involves the effective temperature associated with the frequency of harmonic potential. The obtained results can be applied to various contexts. In Section 6, we wrote down the quantum Fokker-Planck equations for the Josephson mode in an atomic Josephson junction and for the superconducting phase in a superconducting Josephson circuit. We showed the stationary solution of the semiclassical Fokker-Planck equation for each of the Josephson systems. These Josephson systems have been experimentally realized and attracted marked attention. We expect that our work would also contribute to the understanding of such a noisy Josephson junction. For instance, the escape rate of the superconducting phase from a local potential minimum is related to the temperature that appeared in the Fokker-Planck equation [10]. Experimental measurements imply that the escape temperature deviates from the absolute temperature at a low-temperature regime. The deviation of the escape temperature is explained by macroscopic quantum tunneling [47–49]. In addition to the macroscopic quantum tunneling, within the underdamped limit, our obtained effective temperature originating from the quantum fluctuations would give a considerable contribution to this escape temperature in a superconducting Josephson circuit, which would be useful to verify our result. To obtain a quantum Fokker-Planck equation without higher-order derivatives, it could be useful to apply the approach of effective action [50]. The effective action includes the quantum fluctuations, and it would enable us to derive a quantum Fokker-Planck equation.

Appendix A: Derivation of Fokker-Planck equation

In order to derive the semiclassical Fokker-Planck equation for $\mathcal{P}(q,v,t)$ from Eqs. (20) and (21), let us consider an infinitesimal time interval $\varepsilon = t' - t$. The probability $\mathcal{P}(q',v',t+\varepsilon)$ satisfies, to $\mathcal{O}(\varepsilon)$ [34, 35],

$$
\begin{aligned}
\mathcal{P}(q',v',t+\varepsilon) &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv K(q',v',t+\varepsilon|q,v,t)\mathcal{P}(q,v,t) \\
&= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv \int_{q(t)=q}^{q(t')=q'} D[q(\bar{t})] \int_{v(t)=v}^{v(t')=v'} D[v(\bar{t})] \int D[\tilde{q}(\bar{t}),\lambda(\bar{t})] \\
&\times \exp\left[\int_t^{t+\varepsilon} dt_1 i\Big[-m\tilde{q}(t_1)\dot{v}(t_1) - m\tilde{q}(t_1)\frac{\partial F(q,v)}{\partial v} + \frac{i}{2}\int_t^{t+\varepsilon} dt_2 \tilde{q}(t_1)C(t_1-t_2)\tilde{q}(t_2) + \lambda(t_1)[\dot{q}(t_1) - v(t_1)]\Big]\right].
\end{aligned} \tag{39}
$$

Taking into account Eq. (10), Eq. (39) gives

$$
\begin{aligned}
\mathcal{P}(q',v',t+\varepsilon) &= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv \int D[\tilde{q}(t),\lambda(t)] \\
&\times \exp\left[-i\varepsilon\tilde{q}m\left(\frac{v'-v}{\varepsilon} + \frac{\partial F}{\partial v}\right) - \varepsilon\gamma k_{\mathrm{B}} T_{\mathrm{eff}}\tilde{q}^2 + \mathcal{O}(\varepsilon^2) + i\varepsilon\lambda\left(\frac{q'-q}{\varepsilon} - v\right)\right]\mathcal{P}(q,v,t) \\
&= \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dv \int D[\tilde{q}(t)]\delta\big(q'-q-\varepsilon v\big)e^{-im\tilde{q}(v'-v)} \\
&\times \left[1 - i\varepsilon\tilde{q}\left(m\frac{\partial F}{\partial v} - i\gamma k_{\mathrm{B}} T_{\mathrm{eff}}\tilde{q}\right) + \mathcal{O}(\varepsilon^2)\right]\mathcal{P}(q,v,t),
\end{aligned} \tag{40}
$$

In the above calculation, we used $\int D[\lambda(t)]e^{i\lambda(t)[q'(t)-q(t)-\varepsilon v(t)]/\hbar} = \delta\big[q'(t) - q(t) - \varepsilon v(t)\big]$. Hence, performing the integrals in Eq. (40), one obtains

$$
\begin{aligned}
\mathcal{P}(q',v',t+\varepsilon) &= \left[1+\varepsilon\partial_{v'}\left(\frac{\partial F}{\partial v'}+\frac{\gamma}{m^2}k_{\mathrm{B}}T_{\mathrm{eff}}\partial_{v'}\right)+\mathcal{O}(\varepsilon^2)\right]\mathcal{P}(q'-v'\varepsilon,v',t) \\
&= \left[1+\varepsilon\partial_{v'}\left(\frac{\partial F}{\partial v'}+\frac{\gamma}{m^2}k_{\mathrm{B}}T_{\mathrm{eff}}\partial_{v'}\right)+\mathcal{O}(\varepsilon^2)\right](1-\varepsilon v'\partial_{q'})\mathcal{P}(q',v',t) \\
&= \left[1+\varepsilon\left[-v'\partial_{q'}+\partial_{v'}\left(\frac{\partial F}{\partial v'}+\frac{\gamma}{m^2}k_{\mathrm{B}}T_{\mathrm{eff}}\partial_{v'}\right)\right]+\mathcal{O}(\varepsilon^2)\right]\mathcal{P}(q',v',t)\,.
\end{aligned}
\tag{41}
$$

For $\varepsilon \to +0$ one finally finds Eq. (22), which is our semiclassical Fokker-Planck equation.

Acknowledgements
The authors acknowledge Pietro Faccioli, Amos Maritan, Fabio Sattin, and Sandro Azaele for useful suggestions.

Authors' contributions
KF and LS equally contributed to all aspects of the manuscript. Both authors read and approved the final manuscript.

Funding
KF is supported by a PhD fellowship of the Fondazione Cassa di Risparmio di Padova e Rovigo.

Availability of data and materials
The data generated during the current study are available from the contributing author upon reasonable request.

Declarations

Ethics approval and consent to participants
The authors declare they have upheld the integrity of the scientific record.

Consent for publication
The authors give their consent for publication of this article.

Competing interests
The authors declare no competing interests.

Received: 17 April 2023 Accepted: 9 August 2023
Published online: 19 September 2023

References

1. A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. Ann. Phys **17**, 549 (1905). https://doi.org/10.1002/andp.19053220806
2. A.M. Smoluchowski, Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen. Ann. Phys. **21**, 756 (1906). https://doi.org/10.1002/andp.19063261405
3. P. Langevin, Sur la théorie du mouvement brownien. C. R. Acad. Sci. Paris. **146**, 530 (1908). https://doi.org/10.1119/1.18725
4. J.B. Johnson, Thermal agitation of electricity in conductors. Phys. Rev. **32**, 97 (1928). https://doi.org/10.1103/physrev.32.97
5. H. Nyquist, Thermal agitation of electric charge in conductors. Phys. Rev. **32**, 110 (1928). https://doi.org/10.1103/physrev.32.110
6. H.B. Callen, T.A. Welton, Irreversibility and generalized noise. Phys. Rev. **83**, 34 (1951). https://doi.org/10.1103/physrev.83.34
7. R.H. Koch, D.J. Van Harkingen, J. Clarke, Quantum-noise theory for the resistively shunted Josephson junction. Phys. Rev. Lett. **45**, 2132 (1980). https://doi.org/10.1103/PhysRevLett.45.2132
8. G.W. Ford, M. Kac, On the quantum Langevin equation. J. Stat. Phys. **46**, 803 (1987). https://doi.org/10.1007/BF01011142
9. H. Metiu, G. Schön, Description of quantum noise by a Langevin equation. Phys. Rev. Lett. **53**, 13 (1985). https://doi.org/10.1103/PhysRevLett.53.13
10. A. Kamenev, *Field Theory in Non-Equilibrium Systems* (Cambridge University Press, Cambridge, 2011). https://doi.org/10.1017/CBO9781139003667
11. J. Schwinger, Brownian motion of a quantum oscillator. J. Math. Phys. **2**, 407 (1961). https://doi.org/10.1063/1.1703727
12. L.V. Keldysh, Diagram technique for nonequilibrium processes. Sov. Phys. JETP **20**, 1018 (1965)
13. P.C. Martin, E. Siggia, H. Rose, Statistical dynamics of classical systems. Phys. Rev. A **8**, 423 (1973). https://doi.org/10.1103/PhysRevA.8.423
14. A.D. Fokker, Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld. Ann. Phys. **43**, 810 (1914). https://doi.org/10.1002/andp.19143480507
15. M. Planck, Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie. Sitzungsberichte Preuss. Akad. Wiss. Berlin **24**, 324 (1917)
16. L.D. Chang, D. Waxman, Quantum Fokker-Planck equation. J. Phys. C Solid State Phys. **18**, 5873 (1985). https://doi.org/10.1088/0022-3719/18/31/019
17. A. Arnold, J.L. Lopez, P.A. Markowich, J. Soler, An analysis of quantum Fokker-Planck models: a Wigner function approach. Rev. Mat. Iberoam. **20**, 771 (2004)
18. S. Jang, Generalized quantum Fokker-Planck equation for photoinduced nonequilibrium processes with positive definiteness condition. J. Chem. Phys. **144**, 214102 (2016). https://doi.org/10.1063/1.4952477
19. H. Collins, R. Holman, T. Vardanyan, The quantum Fokker-Planck equation of stochastic inflation. JHEP **11**, 065 (2017). https://doi.org/10.1007/JHEP11(2017)065
20. P.J. Colmenares, Fokker-Planck equation of the reduced Wigner function associated to an Ohmic quantum Langevin dynamics. Phys. Rev. E **97**, 052126 (2018). https://doi.org/10.1103/PhysRevE.97.052126
21. C. De Dominicis, Techniques de renormalisation de la theorie des champs et dynamique des phenomenes critiques. J. Phys. Colloq. **37**, C1-247 (1976). https://doi.org/10.1051/jphyscol:1976138
22. H.K. Janssen, On a Lagrangean for classical field dynamics and renormalization group calculations of dynamical critical properties. Z. Phys. B **23**, 377 (1976). https://doi.org/10.1007/BF01316547
23. C. De Dominicis, L. Peliti, Field-theory renormalization and critical dynamics above T_c: Helium, antiferromagnets, and liquid-gas systems. Phys. Rev. B **18**, 353 (1978). https://doi.org/10.1103/PhysRevB.18.353
24. C. Aron, G. Biroli, L. F. Cugliandolo, Symmetries of generating functionals of Langevin processes with colored multiplicative noise. J. Stat. Mech. P11018 (2010). https://doi.org/10.1088/1742-5468/2010/11/p11018
25. U.C. Täuber, *Critical dynamics: a field theory approach to equilibrium and non-equilibrium scaling behavior* (Cambridge University Press, Cambridge, 2014). https://doi.org/10.1017/CBO9781139046213
26. R. Araújo, S. Wald, M. Henkel, Axiomatic construction of quantum Langevin equations. J. Stat. Mech. 053101 (2019). https://doi.org/10.1088/1742-5468/ab11dc
27. L. Onsager, S. Machlup, Fluctuations and irreversible processes. Phys. Rev. **91**, 1505 (1953). https://doi.org/10.1103/PhysRev.91.1505
28. R. Olender, R. Elber, Yet another look at the steepest descent path. J. Mol. Struct. (Theochem) **398-399**, 63 (1997). https://doi.org/10.1016/S0166-1280(97)00038-9
29. B. Huard, Interactions between electrons, mesoscopic Josephson effect and asymmetric current fluctuations. PhD thesis, Université Pierre et Marie Curie - Paris VI (2006)
30. B. Huard, H. Pothier, N.O. Birge, D. Esteve, X. Waintal, J. Ankerhold, Josephson junctions as detectors for non-Gaussian noise. Ann. Phys. Wiley **16**(10–11), 736–750 (2007). https://doi.org/10.1002/andp.200751910-1107
31. K. Furutani, L. Salasnich, Quantum and thermal fluctuations in the dynamics of a resistively and capacitively shunted Josephson junction. Phys. Rev. B **104**, 014519 (2021). https://doi.org/10.1103/PhysRevB.104.014519
32. U. Eckern, G. Schön, V. Ambegaokar, Quantum dynamics of a superconducting tunnel junction. Phys. Rev. B **30**, 6419 (1984). https://doi.org/10.1103/PhysRevB.30.6419
33. E.G. Dalla Torre, E. Demler, T. Giamarchi, E. Altman, Quantum critical states and phase transitions in the presence of non-equilibrium noise. Nat. Phys. **6**, 806 (2010). https://doi.org/10.1038/nphys1754
34. A.K. Das, S. Panda, J.R.L. Santos, A path integral approach to the Langevin equation. Int. J. Mod. Phys. A **30**, 1550028 (2015). https://doi.org/10.1142/S0217751X15500281
35. H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, Singapore, 2004). https://doi.org/10.1142/7305

36. N. Hashitsume, A statistical theory of linear dissipative systems. Progr. Theoret. Phys. **8**, 461 (1952). https://doi.org/10.1143/ptp/8.4.461
37. R. Kubo, The fluctuation-dissipation theorem. Rep. Prog. Phys. **29**, 255 (1966). https://doi.org/10.1088/0034-4885/29/1/306
38. H. Grabert, U. Weiss, P. Talkner, Quantum theory of the damped harmonic oscillator. Z. Phys. B Condens. Matter **55**, 87–94 (1984). https://doi.org/10.1007/BF01307505
39. R.F. Pawula, Approximation of the linear Boltzmann equation by the Fokker-Planck equation. Phys. Rev. **162**, 186 (1967). https://doi.org/10.1103/PhysRev.162.186
40. F.T. Brandt, J. Frenkel, J.C. Taylor, Noise in resistively shunted Josephson junctions. Phys. Rev. B **82**, 014515 (2010). https://doi.org/10.1103/PhysRevB.82.014515
41. A. Schmid, Diffusion and localization in a dissipative quantum system. Phys. Rev. Lett. **51**, 1506 (1983). https://doi.org/10.1103/PhysRevLett.51.1506
42. S.A. Bulgadaev, Phase diagram of a dissipative quantum system. JETP Lett. **39**, 315 (1984)
43. M.H. Devoret, J.M. Martinis, J. Clarke, Measurements of macroscopic quantum tunneling out of the zero-voltage state of a current-biased josephson junction. Phys. Rev. Lett. **55**, 1908 (1985). https://doi.org/10.1103/PhysRevLett.55.1908
44. J. Polo, V. Ahufinger, F.W.J. Hekking, A. Minguzzi, Damping of Josephson oscillations in strongly correlated one-dimensional atomic gases. Phys. Rev. Lett. **121**, 090404 (2018). https://doi.org/10.1103/PhysRevLett.121.090404
45. F. Binanti, K. Furutani, L. Salasnich, Dissipation and fluctuations in elongated bosonic Josephson junctions. Phys. Rev. A **103**, 063309 (2021). https://doi.org/10.1103/PhysRevA.103.063309
46. M. Pigneur, T. Berrada, M. Bonneau, T. Schumm, E. Demler, J. Schmiedmayer, relaxation to a phase-locked equilibrium state in a one-dimensional bosonic Josephson junction. Phys. Rev. Lett. **120**, 173601 (2018). https://doi.org/10.1103/PhysRevLett.120.173601
47. J.M. Martinis, M.H. Devoret, J. Clarke, Energy-level quantization in the zero-voltage state of a current-biased Josephson junction. Phys. Rev. Lett. **55**, 1543 (1985). https://doi.org/10.1103/PhysRevLett.55.1543
48. J. Clarke, A.N. Cleland, M.H. Devoret, D. Esteve, J.M. Martinis, Quantum mechanics of a macroscopic variable: the phase difference of a Josephson junction. Science **239**, 992 (1988). https://doi.org/10.1126/science.239.4843.992
49. J.A. Blackburn, M. Cirillo, N.G. Jensen, A survey of classical and quantum interpretations of experiments on Josephson junctions at very low temperatures. Phys. Rep. **611**, 1–33 (2016). https://doi.org/10.1016/j.physrep.2015.10.010
50. J. Stapmanns, T. Kühn, D. Dahmen, T. Luu, C. Honerkamp, M. Helias, Self-consistent formulations for stochastic nonlinear neuronal dynamics. Phys. Rev. E **101**, 042124 (2020). https://doi.org/10.1103/PhysRevE.101.042124