Sampling Theory, Signal Processing, and Data Analysis (2024) 22:1
https://doi.org/10.1007/s43670-023-00073-7

ORIGINAL ARTICLE

Tiling and weak tiling in $(\mathbb{Z}_p)^d$

Gergely Kiss[1,2] · Dávid Matolcsi[2] · Máté Matolcsi[1,3] · Gábor Somlai[1,2]

Received: 11 December 2022 / Accepted: 6 October 2023

Abstract

We discuss the relation of tiling, weak tiling and spectral sets in finite abelian groups. In particular, in elementary p-groups $(\mathbb{Z}_p)^d$, we introduce an averaging procedure that leads to a natural object of study: a 4-tuple of functions which can be regarded as a common generalization of tiles and spectral sets. We characterize such 4-tuples for $d = 1, 2$, and prove some partial results for $d = 3$.

Keywords Fuglede's conjecture · Translational tiles · Spectral sets · Weak tiling · Abelian p-groups

Mathematics Subject Classification 20K01 · 43A25 · 22B05

Communicated by Azita Mayeli.

✉ Máté Matolcsi
matomate@renyi.hu

Gergely Kiss
kiss.gergely@renyi.hu

Dávid Matolcsi
matolcsidavid@gmail.com

Gábor Somlai
gabor.somlai@ttk.elte.hu

1 Alfréd Rényi Institute of Mathematics, HUN-REN, Reáltanoda u. 13-15, 1053 Budapest, Hungary

2 Eötvös Loránd University, Pázmány Péter sétány 1/C, 1111 Budapest, Hungary

3 Department of Analysis, Institute of Mathematics, Budapest University of Technology and Economics (BME), Müegyetem rkp. 3, 1111 Budapest, Hungary

Published online: 04 December 2023

1 Introduction

The concept of weak tiling was introduced recently in [15], in connection with Fuglede's conjecture for convex bodies in $\mathbb{R}^d$. In this note we will study the relation of tiling, weak tiling and spectral sets in finite abelian groups, with particular attention to elementary p-groups $(\mathbb{Z}_p)^d$. Our motivation is to give a new tool to prove the "spectral $\to$ tile" direction of Fuglede's conjecture in finite abelian groups.

We begin by recalling the necessary notions and fixing our notation.

The cardinality of any finite set A will be denoted by $|A|$. We use the standard notation $A \pm B = \{a \pm b : a \in A,\ b \in B\}$, and $kA = \{ka : a \in A\}$, for any positive integer k. For any n, let $\mathbb{Z}_n$ denote the cyclic group of order n.

Let G be a finite abelian group. A *character* is a homomorphism from G to the complex unit circle $\mathbb{T}$. The dual group, denoted by $\widehat{G}$, is the collection of all characters of G. For a function $f : G \to \mathbb{C}$, the Fourier transform $\widehat{f} : \widehat{G} \to \mathbb{C}$ is defined as

$$\widehat{f}(\gamma) = \sum_{x \in G} f(x)\gamma(x).$$

A set $A \subset G$ is called *spectral* if the function space $L^2(A)$ admits an orthogonal basis consisting of characters restricted to A. Such an orthogonal basis of characters is called a *spectrum* of A. (The spectrum, if it exists, is not necessarily unique.)

We say that a set $A \subset G$ *tiles* G if there exists another set $B \subset G$ such that each $g \in G$ can be written uniquely as $g = a + b$, where $a \in A$, $b \in B$. We usually express this relation as $A \oplus B = G$ or, in the functional notation, $1_A * 1_B = 1_G$, where $*$ denotes convolution. The convolution of any two functions $f, g : G \to \mathbb{C}$ is defined in the standard way as $(f * g)(x) = \sum_{y \in G} f(x - y)g(y)$.

For any function $f : G \to \mathbb{C}$ the function f_- is defined as $f_-(x) = f(-x)$. Some basic properties of the Fourier transform read as follows: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$, $\widehat{f_-}(\gamma) = \overline{\widehat{f}(\gamma)}$, $\widehat{f * f_-} = |\widehat{f}|^2$, and for real-valued even functions $\widehat{\widehat{f}} = |G| f$.

Fuglede's conjecture [7] stated that a set $A \subset G$ is spectral if and only if it tiles G. The conjecture was formulated explicitly in $\mathbb{R}^d$, but Fuglede already mentioned that the notions make sense in any locally compact abelian group G. In fact, the first counterexample by Tao [20] in $\mathbb{R}^5$ was based on a counterexample in the finite group $(\mathbb{Z}_3)^5$. Since then, further counterexamples [5, 12] have been constructed (all based on examples in finite groups) to both directions of the conjecture in $\mathbb{R}^d$, $d \geq 3$. The conjecture (both directions of it) remains open for $\mathbb{R}$ and $\mathbb{R}^2$. Some further connections between the discrete and the continuous settings have been revealed by Dutkay and Lai in [2].

In this paper we will restrict our attention to finite abelian groups, and mostly to the case $G = (\mathbb{Z}_p)^d$ with p being a prime. For finite abelian groups, several results have been discovered in recent years [3, 4, 6, 8–11, 13, 14, 16–19, 21]. For the purposes of this note, we single out the results of [8] and [6, 17]. In [8] the authors prove both directions of Fuglede's conjecture in $(\mathbb{Z}_p)^2$. To the contrary, in [6] and [17] the authors prove (independently of each other) that for odd primes p there exist spectral sets in

$(\mathbb{Z}_p)^d$, $d \geq 4$, which do not tile the group, thus exhibiting a counterexample to the "spectral $\to$ tile" direction of Fuglede's conjecture in these groups.

In connection with Fuglede's conjecture, the notion of *weak tiling* was introduced in [15]. For us, in the case of finite abelian groups, weak tiling can be formulated as follows. A set $A \subset G$ tiles another set $E \subset G$ *weakly*, if there exists a nonnegative function $h : G \to \mathbb{R}$ such that $1_A * h = 1_E$. We will soon see that it makes sense to introduce the further restrictions that $h(0) = 1$, the function h is positive definite (i.e. $\widehat{h} \geq 0$), and restrict our attention to the case $E = G$.

Definition 1.1 We say that a set $A \subset G$ *pd-tiles* G *weakly* if there exists a nonnegative function $h : G \to \mathbb{R}$ such that $h(0) = 1$, $1_A * h = 1_G$, and $\widehat{h} \geq 0$.

In the terminology, "pd" stands for positive definite. Note that, due to positive definiteness, h is necessarily an even function, $h(x) = h(-x)$. The assumption $h(0) = 1$ is essential, otherwise the constant function $h = \frac{1}{|A|}$ would provide a weak pd-tiling for any set $A \subset G$. Note also, as a comparison with the terminology in [15], that if A pd-tiles G weakly then A tiles its complement weakly.

A simple but important observation, also essentially contained in [15], is that if A pd-tiles G weakly with a function h, then

$$A - A \cap \operatorname{supp} h = \{0\}. \tag{1}$$

The reason is that $x = a - a'$ and $h(x) > 0$ would imply $1_A * h(a) \geq h(0)1_A(a) + h(x)1_A(a') = 1 + h(x) > 1$.

Next, we show that a proper tiling always induces a weak pd-tiling of G.

Lemma 1.2 *If $A \oplus B = G$ is a tiling, then A pd-tiles G weakly with $h_1 = \frac{1}{|B|} 1_B * 1_{-B}$.*

Proof $A \oplus B = G$ implies $\widehat{1}_A \cdot \widehat{1}_B = |G|\delta_0$, so that the supports of the functions $\widehat{1}_A$ and $\widehat{1}_B$ are essentially disjoint (only intersect at 0). This implies that $\widehat{1}_A \cdot \frac{1}{|B|}|\widehat{1}_B|^2 = |G|\delta_0$, which, in turn, implies $1_A * \left(\frac{1}{|B|}1_B * 1_{-B}\right) = 1_G$. □

We remark that the notation h_1 above (instead of using simply h) is for later convenience.

The essential observation in [15] is that if a set A is spectral, then it tiles its complement weakly. However, slightly more is true.

Lemma 1.3 *Let G be a finite abelian group. If $A \subset G$ is spectral, then A pd-tiles G weakly.*

Proof Let $S \subset \widehat{G}$ be a spectrum of A. Then $|S| = |A|$ because the space $L^2(A)$ has dimension $|A|$, and S is a basis. Let $h_1 = \frac{1}{|A|^2}|\widehat{1}_S|^2$. Then $h_1 \geq 0$, $h_1(0) = 1$, $\widehat{h}_1 = \frac{|G|}{|A|^2}1_S * 1_{-S} \geq 0$, so h_1 is nonnegative, normalized and positive definite, as required. Also, the weak tiling condition $1_A * h_1 = 1_G$ is most easily seen by taking Fourier transforms. $\widehat{1}_A \cdot \left(\frac{|G|}{|A|^2}1_S * 1_{-S}\right) = |G|\delta(0)$ is true because $\widehat{1}_A(0) = |A|$, $1_S * 1_{-S}(0) = |S| = |A|$, and the support of $1_S * 1_{-S}$ is $S - S$, which is a subset of $\{\widehat{1}_A = 0\} \cup \{0\}$ by the orthogonality of S in $L^2(A)$. □

By this lemma, we can establish the "spectral $\to$ tile" direction of Fuglede's conjecture in a finite abelian group G, if we prove that any set that pd-tiles G weakly, actually tiles G properly. This motivates the following definition.

Definition 1.4 Assume that a finite abelian group G has the property that whenever a set A pd-tiles G weakly then A tiles G properly. Then we call the group G *pd-flat*.

In the remainder of this note we will focus our attention on elementary p-groups $G = (\mathbb{Z}_p)^d$. As mentioned above, for odd primes and $d \geq 4$, there exist spectral sets in G which do not tile G. Therefore, $(\mathbb{Z}_p)^d$ is not pd-flat for $d \geq 4$.

In Sect. 2 we introduce an averaging technique which leads to a natural generalization of spectral sets and tiles in $(\mathbb{Z}_p)^d$. Finally, in Sect. 3 we show that $(\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, and we give some partial results and conjectures for $d = 3$.

2 Averaging

It is natural to go one step further with the generalization of tiling.

Definition 2.1 Let $f, h : G \to \mathbb{R}$ be nonnegative functions such that $f(0) = h(0) = 1$, $\widehat{f} \geq 0$, $\widehat{h} \geq 0$. We say that the pair (f, h) is a *functional pd-tiling* of G if $f * h = 1_G$.

It turns out that this notion is very flexible, and for $G = (\mathbb{Z}_p)^d$ it gives rise to a natural averaging procedure. In connection with this, we introduce a special class of functions.

Definition 2.2 We say that a function $f : (\mathbb{Z}_p)^d \to \mathbb{C}$ is *ray-type*, if for any $\mathbf{x} \in (\mathbb{Z}_p)^d$, and any $k = 1, \ldots, p-1$ we have $f(k\mathbf{x}) = f(\mathbf{x})$.

That is, a ray-type function is constant on any punctured line through the origin (but may have a different value at the origin).

We also need some information on the zeroes of the Fourier transform of the indicator function 1_A of a set $A \subset (\mathbb{Z}_p)^d$. We make the identification $G = \{0, 1, \ldots, p-1\}^d$, and the same for the dual group $\widehat{G} = \{0, 1, \ldots, p-1\}^d$. It is sometimes useful to think of the elements of G as *column* vectors of length d, and elements of $\widehat{G}$ as *row* vectors of length d. Also, in notation, we will use boldface letters for elements of G and $\widehat{G}$ to indicate that they are vectors. With these identifications in mind, the action of a character $\mathbf{t} \in \widehat{G}$ on an element $\mathbf{x} \in G$ is given by $e^{2i\pi\langle \mathbf{t}, \mathbf{x}\rangle/p}$.

For a function $f : G \to \mathbb{C}$, and $\mathbf{t} \in \widehat{G}$, we have $\widehat{f}(\mathbf{t}) = \sum_{\mathbf{x}\in G} f(\mathbf{x}) e^{2i\pi\langle \mathbf{t}, \mathbf{x}\rangle/p}$. In particular, for $A \subset G$, the Fourier transform of 1_A takes the form $\widehat{1}_A(\mathbf{t}) = \sum_{\mathbf{a}\in A} e^{2i\pi\langle \mathbf{t}, \mathbf{a}\rangle/p}$.

The important point here is that for any $\mathbf{t} \neq 0$ we either have $\widehat{1}_A(k\mathbf{t}) = 0$ for all $k = 1, 2, \ldots, p-1$, or $\widehat{1}_A(k\mathbf{t}) \neq 0$ for all $k = 1, 2, \ldots, p-1$. This is well-known, and the reason is that $\widehat{1}_A(\mathbf{t}) = 0$ if and only if the sum $\sum_{\mathbf{a}\in A} e^{2i\pi\langle \mathbf{t}, \mathbf{a}\rangle/p}$ contains the same number of terms for each pth root of unity, in which case $\sum_{\mathbf{a}\in A} e^{2i\pi\langle k\mathbf{t}, \mathbf{a}\rangle/p}$ also contains the same number of terms of each pth root of unity.

Therefore, the zeros of the Fourier transform $\widehat{1}_A$ consist of punctured lines through the origin (for $\mathbf{t} \neq 0$ a punctured line $\dot{L}$ is given by $\dot{L} = \{\mathbf{t}, 2\mathbf{t}, \ldots, (p-1)\mathbf{t}\}$; note that $0 \notin \dot{L}$, and we use the notation $L = \dot{L} \cup \{0\}$). The same is true for the zeroes of the Fourier transform of $f = \frac{1}{|A|}1_A * 1_{-A}$, because $\widehat{f} = \frac{1}{|A|}|\widehat{1}_A|^2$.

We can now perform the first half of the averaging.

Lemma 2.3 *Let $G = (\mathbb{Z}_p)^d$, and $1_A * h_1 = 1_G$ be a weak pd-tiling of G by a set A. Then, for any $k = 1, \dots, p-1$, $1_{kA} * h_1 = 1_G$ is also a weak pd-tiling. Furthermore, with the notation $f_k = \frac{1}{|A|}1_{kA} * 1_{-kA}$, we have that $f_k * h_1 = 1_G$ is a functional pd-tiling. Finally, for $f = \frac{1}{p-1}\sum_{k=1}^{p-1} f_k$ we have that f is a ray-type function,* supp $f \cap$ supp $h_1 = \{0\}$*, and $f * h_1$ is also a functional pd-tiling.*

Proof Note that $\widehat{1}_{kA}(\mathbf{t}) = \widehat{1}_A(k\mathbf{t})$, so the zeroes of these functions are the same punctured lines through the origin.

The assumption $1_A * h_1 = 1_G$ is equivalent to $\widehat{1}_A\widehat{h}_1 = |G|\delta_0$. Due to the fact that the zeroes of $\widehat{1}_A$ and $\widehat{1}_{kA}$ coincide, and $|A| = |kA|$, we have $\widehat{1}_{kA}\widehat{h}_1 = |G|\delta_0$, and therefore $1_{kA} * h_1 = 1_G$ is a weak pd-tiling.

The function $f_k = \frac{1}{|A|}1_{kA} * 1_{-kA}$ is nonnegative, $f_k(0) = 1$, $\widehat{f_k} = \frac{1}{|A|}|\widehat{1}_{kA}|^2 \geq 0$, and the zeroes of $\widehat{f_k}$ and $\widehat{1}_{kA}$ coincide. Also, $\widehat{f_k}(0) = |A| = \widehat{1}_{kA}(0)$, therefore $\widehat{f_k}\widehat{h}_1 = |G|\delta_0$, which implies that $f_k * h_1$ is a functional pd-tiling. Also, supp $f_k = kA - kA$, and hence supp $f_k \cap$ supp $h_1 = \{0\}$ by (1).

Finally, for the average $f = \frac{1}{p-1}\sum_{k=1}^{p-1} f_k$, we clearly have $f \geq, 0$, $\widehat{f} \geq 0$, $f(0) = 1$ and $f * h_1 = |G|\delta_0$ is a functional pd-tiling since these properties hold for each f_i. Also, by averaging, it is clear that f is ray-type, and the property supp $f \cap$ supp $h_1 = \{0\}$ is inherited. □

We can perform a second step of averaging for the function h_1.

Lemma 2.4 *Let $G = (\mathbb{Z}_p)^d$, and $1_A * h_1 = 1_G$ be a weak pd-tiling of G by a set A, and define the ray-type function f as in Lemma 2.3. For any $k = 1, \dots, p-1$, let $h_k(\mathbf{x}) = h_1(k\mathbf{x})$, and let $h = \frac{1}{p-1}\sum_{k=1}^{p-1} h_k$. Then, for each k, $f * h_k = 1_G$ is a functional pd-tiling, the average h is a ray-type function, and $f * h$ is also a functional pd-tiling such that* supp $f \cap$ supp $h = \{0\}$.

Proof Recall from Lemma 2.3 that f is a ray-type function. We claim that $\widehat{f}$ is also ray-type. To see this, note that f can be written in the form $f = c\delta_0 + \sum_i c_i 1_{L_i}$ for some lines L_i (for convenience, we use proper lines in this decomposition, not punctured lines; the difference is absorbed in the constant c at the origin). The Fourier transform of 1_{L_i} is $p1_{H_i}$ with the hyperplane H_i being orthogonal to L_i, and therefore the $\widehat{f} = c1_{\widehat{G}} + p\sum_i 1_{H_i}$, which is clearly ray-type.

By Lemma 2.3 we have that $f * h_1 = 1_G$, which implies $\widehat{f}\widehat{h}_1 = |G|\delta_0$. This means that supp $\widehat{f} \cap$ supp $\widehat{h}_1 = \{0\}$. As $\widehat{f}$ is ray-type, its support is a union of lines R_i, and hence $\widehat{h}_1$ must be 0 on the punctured lines $\dot{R}_i$. But $\widehat{h}_k(\mathbf{t}) = \widehat{h}_1(\mathbf{t}/k)$, so $\widehat{h}_k$ is also 0 on $\dot{R}_i$, and hence $\widehat{f}\widehat{h}_k = |G|\delta_0$ holds (for the value at zero we have $\widehat{h}_1(0) = \widehat{h}_k(0) = |G|/|A|$). In turn, this implies that $f * h_k = 1_G$ is indeed a functional pd-tiling.

Also, supp $f \cap$ supp $h_1 = \{0\}$ by Lemma 2.3, and f is ray-type, so the support of h_1 is contained in rays where $f = 0$. This clearly implies that the support of h_k is also contained in these rays, and therefore supp $f \cap$ supp $h_k = \{0\}$

After averaging, it is clear that h is ray-type, and the functional pd-tiling property $f * h$ is preserved, as well as the condition $\operatorname{supp} f \cap \operatorname{supp} h = \{0\}$. □

We see from Lemmas 1.2 and 1.3 that A pd-tiles G weakly in both cases when A tiles G, or A is spectral in G. After applying Lemmas 2.3 and 2.4 we arrive at the following corollary.

Corollary 2.5 *Let $G = (\mathbb{Z}_p)^d$, and assume A pd-tiles G weakly with $1_A * h_1 = 1_G$ (in particular, this is the case if A tiles G, or A is spectral in G). For $k = 1, \ldots, p-1$, let $f_k = \frac{1}{|A|} 1_{kA} * 1_{-kA}$, $h_k(\mathbf{x}) = h_1(k\mathbf{x})$, and $f = \frac{1}{p-1} \sum_{k=1}^{p-1} f_k$, $h = \frac{1}{p-1} \sum_{k=1}^{p-1} h_k$. Then*

$$|A| = \sum_{x \in G} f(x), \tag{2}$$

and the 4-tuple of functions $(f, h, \widehat{f}, \widehat{h})$ satisfy the following properties:

1. $f, h, \widehat{f}, \widehat{h}$ are all ray-type functions,
2. $f \geq 0$, $h \geq 0$,
3. $f(0) = 1$, $h(0) = 1$,
4. $f * h = 1_G$,
5. $\operatorname{supp} f \cap \operatorname{supp} h = \{0\}$,
6. $\widehat{f} \geq 0$, $\widehat{h} \geq 0$,
7. $\widehat{f}(0) = |A|$, $\widehat{h}(0) = |G|/|A|$,
8. $\widehat{f} * \widehat{h} = |G| 1_{\widehat{G}}$,
9. $\operatorname{supp} \widehat{f} \cap \operatorname{supp} \widehat{h} = \{0\}$.

Proof Only 8 needs further explanation, and it follows by taking Fourier transform of the equation, noting that $\widehat{\widehat{f}} = |G| f$ (and the same for h), and applying 5 and 3. □

This corollary motivates the following definition.

Definition 2.6 A 4-tuple of functions $(f, h, \widehat{f}, \widehat{h})$ is called a *complementary 4-tuple* if they satisfy conditions (i)-(ix) of Corollary 2.5. (We do not necessarily require that f be constructed from a set A.)

The notion of complementary 4-tuples is very appealing, because it puts the functions f, h and their Fourier transform $\widehat{f}, \widehat{h}$ on equal footing. When studying spectral sets and tiles in $(\mathbb{Z}_p)^d$ it is natural to try to characterize all complementary 4-tuples. We will do this for $d = 1, 2$ in the next section, and present some partial results for $d = 3$.

We also remark that an analogous averaging procedure can be carried out in cyclic groups $\mathbb{Z}_n$, leading to a similar notion of complementary 4-tuples in those groups. Studying these 4-tuples could lead to some insights concerning the Coven-Meyerowitz conjecture. This is subject to future research.

3 Complementary 4-tuples in $(\mathbb{Z}_p)^d$

In this section we analyze complementary 4-tuples in $(\mathbb{Z}_p)^d$, and prove that $(\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, and also give some partial results for $d = 3$.

For convenience, we recall here what is known about Fuglede's conjecture in these groups. Both directions of the conjecture are true for $d = 1, 2$, with $d = 1$ being fairly trivial, and $d = 2$ being treated in [8]. A simpler proof of the case $d = 2$ can be found in [10]. For $d = 3$ it is known that all tiles are spectral [1], but it is not known whether all spectral sets are tiles. For $d \geq 4$ (and p being odd) there are examples of spectral sets which do not tile the group [6, 17] but the tile-spectral direction of the conjecture is open.

We now turn to the main results of this section. We emphasize here that a group G being pd-flat is *formally* stronger than the "spectral $\rightarrow$ tile" direction of Fuglede's conjecture in G.

Proposition 3.1 *$G = (\mathbb{Z}_p)^d$ is pd-flat for $d = 1, 2$, that is, every set A which pd-tiles G weakly, tiles G properly. Also, G is not pd-flat for $d \geq 4$.*

Proof Assume A pd-tiles G weakly. Then there exists a complementary 4-tuple $(f, h, \widehat{f}, \widehat{h})$ with the properties listed in Corollary 2.5, and f is constructed from 1_A as in Corollary 2.5.

For $d = 1$ the support of any ray-type function is either $\{0\}$ or G. If $\operatorname{supp} f = \{0\}$ then $|A| = 1$, and A tiles G trivially. If $\operatorname{supp} f = G$ then necessarily $\operatorname{supp} h = \{0\}$, and $h(0) = 1$ implies $h = \delta_0$, and hence f must be the constant function 1. Therefore, $|A| = p$ by (2), and hence $A = G$.

For $d = 2$ we must perform a case-by-case analysis of the function f. Let $c = \inf_{x \in G} f(x) \geq 0$, and let L_i be the lines in the support of f. Then f can be decomposed uniquely as $f = c1_G + \sum_i d_i 1_{L_i} + m\delta_0$ with some $d_i \geq 0$ and $m \in \mathbb{R}$ (note that L_i denotes the full line here, not the punctured line).

If $c > 0$ then $\operatorname{supp} f = G$, and hence $\operatorname{supp} h = \{0\}$, $h = \delta_0$, so $f = 1_G$. By (2) we obtain that $|A| = p^2$ and $A = G$.

If $c = 0$ and $m > 0$ then $\widehat{f} > 0$ everywhere on $\widehat{G}$, so $\operatorname{supp} \widehat{h} = \{0\}$. Thus h is constant, and $h(0) = 1$ implies that h is constant 1. This implies that $f = \delta_0$, and $|A| = 1$.

If $c = 0$ and $m = 0$, then the total mass of f is exactly p times the mass at 0, because this is true for all 1_{L_i}. The mass at zero is $f(0) = 1$, and hence we have $|A| = p$, by equation (2). Also, $c = 0$ implies that f must be zero on some punctured line $\dot{R}$, so A can have at most one element in each coset of the line R since $A - A \subset \operatorname{supp}(f)$. But $|A| = p$, so A must have exactly one element in each coset of R, and therefore $A \oplus R = G$ is a tiling.

Finally, we claim that $c = 0$ and $m < 0$ is not possible. Indeed, $c = 0$ means that f must be zero on some punctured line $\dot{R}$, and therefore $\widehat{f} = m < 0$ on the orthogonal punctured line $\dot{R}^{\perp}$, contradicting the nonnegativity of $\widehat{f}$.

For $d \geq 4$ we know that there exist spectral sets in $(\mathbb{Z}_p)^d$ which do not tile the group [6, 17]. Therefore, $(\mathbb{Z}_p)^d$ cannot be pd-flat. □

For $d = 3$, we conjecture that $(\mathbb{Z}_p)^3$ is pd-flat, but unfortunately we cannot give a complete characterization of complementary 4-tuples, we can only prove some partial results in this direction. Therefore, the "spectral $\to$ tile" direction of Fuglede's conjecture remains open in this case. We have the following partial result.

Proposition 3.2 *Let $G = (\mathbb{Z}_p)^3$, and assume $A \subset G$ pd-tiles G weakly with $1_A * h_1 = 1_G$. Let $(f, h, \widehat{f}, \widehat{h})$ be the complementary 4-tuple constructed in Corollary 2.5. Then either A tiles G properly, or the functions $f, h, \widehat{f}, \widehat{h}$ all have the following "dispersive" property: for any 2-dimensional subspaces $S \subset G$, $S' \subset \widehat{G}$ the intersections* $\operatorname{supp} f \cap S$, $\operatorname{supp} h \cap S$, $\operatorname{supp} \widehat{f} \cap S'$ *and* $\operatorname{supp} \widehat{h} \cap S'$ *are all non-trivial (i.e. not equal to* $\{0\}$ *or the whole plane).*

Proof Exploiting the fact that all appearing functions are ray-type, we can represent each function in the following normalized form. Let us label the planes and lines through the origin as $S_1, S_2, \ldots, S_{p^2+p+1}$ and $L_1, L_2, \ldots, L_{p^2+p+1}$. Then f can be represented uniquely as

$$f = w 1_G + \sum_i (c_i 1_{S_i} + d_i 1_{L_i}) + m\delta_0, \tag{3}$$

where the coefficients $w, c_i, d_i \geq 0$, $m \in \mathbb{R}$ are defined by the following greedy algorithm: w is the maximal value such that $f - w1_G \geq 0$, c_1 is the maximal value such that $f - w1_G - c_1 1_{S_1} \geq 0$, c_2 is the maximal value such that $f - w1_G - c_1 1_{S_1} - c_2 1_{S_2} \geq 0$, etc, and subsequently d_1 is the maximal value such that $f - w1_G - (\sum_{i=1}^{p^2+p+1} c_i 1_{S_i}) - d_1 1_{L_1} \geq 0$, etc. We can represent $f, h, \widehat{f}, \widehat{h}$ in this manner uniquely, after fixing the order of planes and lines in G and $\widehat{G}$.

We will show that if any of $f, h, \widehat{f}, \widehat{h}$ does not have the dispersive property (stated in the proposition), then A tiles G. This is done by a case-by-case analysis, where we will use the properties of complementary 4-tuples stated in Corollary 2.5.

Case I. Let us first consider the trivial case when the support of some of the appearing functions is the whole underlying group.

(a) If $\operatorname{supp} f = G$, then $\operatorname{supp} h = \{0\}$ by 5, and hence $h = \delta_0$ by 3, and $f = 1_G$ by 4, and $|A| = p^3$ by (2). Therefore, $A = G$, and A tiles the group trivially.

(b) Similarly, if $\operatorname{supp} h = G$ then $\operatorname{supp} f = \{0\}$, and $|A| = 1$, and A tiles G trivially.

(c) If $\operatorname{supp} \widehat{f} = \widehat{G}$, then $\operatorname{supp} \widehat{h} = \{0\}$ by 9, and hence h is a constant function, $h = 1_G$ by 3, and therefore $f = \delta_0$ by 4, and again $|A| = 1$.

(d) Similarly, if $\operatorname{supp} \widehat{h} = \widehat{G}$, then $\widehat{f} = \{0\}$, and hence f is constant 1, and $|A| = p^3$ by (2).

Having dealt with Case I, we can assume for the rest of the argument that $w = 0$ in equation (3). Moreover, this also implies that $m \leq 0$ in (3) because $m > 0$ would imply $\operatorname{supp} \widehat{f} = \widehat{G}$. Altogether, we can conclude that the representation of f takes the form

$$f = \sum_i (c_i 1_{S_i} + d_i 1_{L_i}) + m\delta_0, \tag{4}$$

where $c_i, d_i \geq 0$, and $m \leq 0$. The same is true for the representations of h, $\widehat{f}$, $\widehat{h}$.

Case II. Next, assume that the support of one of the appearing functions intersects a plane only at 0.

(a) Let supp $f \cap S = \{0\}$. Let $\dot{L} \subset \widehat{G}$ denote the punctured line orthogonal to S. Then $\widehat{f} = m \leq 0$ on $\dot{L}$, and the nonnegativity of $\widehat{f}$ implies $m = 0$. Also, the "empty" plane S intersects every other plane S_i, so every S_i has at least one "empty" line $S \cap S_i$, and hence the weight c_i must be 0 in equation (4) for every i. Hence, f can be represented in the form $f = \sum_i d_i 1_{L_i}$, and therefore the total mass of f is p times the mass at 0 (as this is the case for every L_i), and hence $|A| = p$ is implied by (2) and $f(0) = 1$. Furthermore, the fact that S is empty means that $A - A \cap S = \{0\}$, so there is exactly one point of A on each coset of S. Therefore, $A \oplus S = G$ is a tiling.

(b) If supp $h \cap S = \{0\}$, then the same argument as in (a) shows that h has a decomposition $h = \sum_i \tilde{d}_i L_i$, and therefore $\frac{\sum_{\mathbf{x} \in G} h(\mathbf{x})}{h(0)} = p$. Considering $h(0) = 1$, this implies $\sum_{\mathbf{x} \in G} h(x) = p$, and hence $|A| = p^2$ by 4. Also, supp $f \neq G$, so there exists a line L such that $A - A \cap L = \{0\}$, and hence $A \oplus L = G$ is a tiling.

(c) If supp $\widehat{f} \cap S' = \{0\}$, then the same argument as in (a) shows that $\widehat{f}$ has a decomposition $\widehat{f} = \sum_i d_i' L_i'$, and therefore $\frac{\sum_{\mathbf{t} \in \widehat{G}} \widehat{f}(t)}{\widehat{f}(0)} = p$. Considering that $\widehat{f}(0) = \sum_{\mathbf{x} \in G} f(x)$ and $\sum_{\mathbf{t} \in \widehat{G}} \widehat{f}(t) = p^2 f(0)$, we obtain $\sum_{\mathbf{x} \in G} f(\mathbf{x}) = p^2$, that is $|A| = p^2$ by (2). We can finish the argument as in (b): as supp $f \neq G$, there exists a line L such that $A - A \cap L = \{0\}$, and hence $A \oplus L = G$ is a tiling.

(d) If supp $\widehat{h} \cap S' = \{0\}$, then the same argument as in (a) shows that $\widehat{h}$ has a decomposition $\widehat{h} = \sum_i \tilde{d}_i' L_i'$, and therefore $\frac{\sum_{\mathbf{t} \in \widehat{G}} \widehat{h}(t)}{\widehat{h}(0)} = p$. This implies that $\sum_{\mathbf{x} \in G} h(\mathbf{x}) = p^2$, and $\sum_{\mathbf{x} \in G} f(\mathbf{x}) = p$. This implies that in the decomposition (4) all coefficients $c_i = 0$ and $m = 0$, otherwise the total mass of f would be greater than p times the mass at 0 (recall that $c_i \geq$ and $m \leq 0$). Therefore, $f = \sum_i d_i 1_{L_i}$. Also, as supp $\widehat{f} \neq \widehat{G}$, we have a line $L' \subset \widehat{G}$ such that $\widehat{f} = 0$ on $\dot{L}'$, which implies that $f = 0$ on the punctured plane $S = L'^{\perp}$. Finally, as $|A| = p$ and $A - A \cap S = \{0\}$ we have that $A \oplus S = G$ is a tiling.

Case III. Finally, assume that the support of one of the appearing functions contains a whole plane. In this case, we can invoke 5 and 9 to conclude that the support of some other function intersects the same plane at 0 only, and we are done by Case II above. □

It is worth summarizing here what this result means for spectral sets. For a spectral set $A \subset (\mathbb{Z}_p)^3$, perform the avearaging procedure leading to the complementary 4-tuple in Corollary 2.5. If any of the functions $f, h, \widehat{f}, \widehat{h}$ does not have the dispersive property described in Proposition 3.2 then A necessarily tiles $(\mathbb{Z}_p)^3$. If all appearing functions have the dispersive property, they must have a representation

$$\sum_i d_i 1_{L_i} + m\delta_0, \tag{5}$$

where the lines L_i are situated such that they do not cover any plane fully, and do not leave any plane empty. Furthermore, $d_i > 0$, $m < 0$ (the latter is true because $m \geq 0$ would imply the Fourier transform being positive on the planes $L_i^{\perp}$).

It would be tempting to conjecture that this situation cannot occur, that is, complementary 4-tuples $(f, h, \widehat{f}, \widehat{h})$ with all functions having the dispersive property do not exist. However, unfortunately, the following example shows that this is not the case.

Example 3.3 Let S_1, S_2, S_3 be three different planes through the origin in $(\mathbb{Z}_p)^3$, and let $L_1 = S_2 \cap S_3$, $L_2 = S_1 \cap S_3$, $L_3 = S_1 \cap S_2$ be the lines of intersection. For the sake of easier calculations we will use a representation for f and h different from (5). (It is not difficult to re-write the representations below according to equation (5), and we invite the reader to do so.)

Let $f_0 = 2 \cdot 1_G - 2(1_{S_1} + 1_{S_2} + 1_{S_3}) + p(1_{L_1} + 1_{L_2} + 1_{L_3})$, and $h_0 = (1_{S_1} + 1_{S_2} + 1_{S_3}) - 2(1_{L_1} + 1_{L_2} + 1_{L_3}) + 2p\delta_0$.

Then, $\widehat{f_0} = p^2(1_{L_1^\perp} + 1_{L_2^\perp} + 1_{L_3^\perp}) - 2p^2(1_{S_1^\perp} + 1_{S_2^\perp} + 1_{S_3^\perp}) + 2p^3\delta_0$, and $\widehat{h}_0 = 2p1_{\widehat{G}} - 2p(1_{L_1^\perp} + 1_{L_2^\perp} + 1_{L_3^\perp}) + p^2(1_{S_1^\perp} + 1_{S_2^\perp} + 1_{S_3^\perp})$.

Based on these formulae, it is easy (but somewhat tedious) to check that the normalized functions $f = \frac{1}{3p-4} f_0$, $h = \frac{1}{2p-3} h_0$, and their Fourier transforms $\widehat{f}, \widehat{h}$ form a complementary 4-tuplet $(f, h, \widehat{f}, \widehat{h})$, such that none of the appearing functions have the dispersive property of Proposition 3.2.

It may be easier to visualize this example if we identify the punctured lines $\dot{L}_i$ of $(\mathbb{Z}_p)^3$ with points of the projective plane $PG(2, \mathbb{Z}_p)$. In this picture, S_i become lines on the projective plane, and the functions f and h can be described in terms of a triangle in the projective plane. For example, the function h is positive on the lines of a triangle (with the exception of the vertices, where h is 0), and zero everywhere outside. We leave the details to the reader.

Notice, however, that in this example neither f nor h can come from the averaging of an indicator function of a set A as in Lemma 2.3. The reason is that by Eq. (2) we would have $|A| = \frac{p^2(2p-3)}{3p-4}$ or $|A| = \frac{p(3p-4)}{2p-3}$, neither of which is an integer for $p \neq 2, 3$, and for $p = 2, 3$ it is easy to check (with a finite case-by-case analysis) that f and h cannot be a result of averaging an indicator function of any set A.

Example 3.3 can be generalized: one can take k points on the projective plane, with $k - 1$ lying on a line, and one point being outside. Then, a complementary 4-tuple $(f, h, \widehat{f}, \widehat{h})$ can be constructed, such that the function h is positive on all the lines connecting these points (with exception of the points themselves, where h is 0), and zero everywhere outside. The function f is then supported on the complement supp h. The exact formulae are somewhat cumbersome, so we choose to omit them.

Given that such examples exist, we cannot yet decide whether the "spectral → tile" direction of Fuglede's conjecture holds in these groups.

As a final remark we mention here Rédei's conjecture which concerns the structure of tilings in $(\mathbb{Z}_p)^3$. It states that in any normalized tiling $A \oplus B = (\mathbb{Z}_p)^3$ (normalized meaning that $0 \in A, B$ is assumed) we must have that either A or B is contained in a proper subgroup.

A complete characterization of complementary 4-tuples in $(\mathbb{Z}_p)^3$ could, in principle, lead to the solution of both Fuglede's and Rédei's conjecture in these groups.

Author Contributions Each author has contributed in 25% to the manuscript.

Funding Open access funding provided by Budapest University of Technology and Economics. G.K. was supported by NKFIH Grants K124749 and K142993, by Bolyai János Research Fellowship of the Hungarian Academy of Sciences and by ÚNKP-22-5 New National Excellence Program of the Ministry for Culture and Innovation. M.M. is the corresponding author. He was supported by NKFIH Grants K129335 and K132097. G.S. was supported by NKFIH Grants 138596 and 132625, and G.S is a holder of János Bolyai Research Fellowship. The work of G.S. on the project leading to this application has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 741420).

Declarations

Conflict of interest To comply with the journal's policies, we declare that the authors have no competing interest.

Ethical approval Not applicable.

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