Sampling Theory, Signal Processing, and Data Analysis (2022) 20:12
https://doi.org/10.1007/s43670-022-00030-w

ORIGINAL PAPER

Some applications of modular convergence in vector lattice setting

Antonio Boccuto[1] · Anna Rita Sambucini[1]

Received: 16 May 2022 / Accepted: 29 June 2022 / Published online: 21 July 2022

Abstract
The main purpose of this paper is to apply the theory of vector lattices and the related abstract modular convergence to the context of Mellin-type kernels and (non)linear vector lattice-valued operators, following the construction of an integral given in earlier papers.

Keywords Integration · Modular convergence · Vector lattice · Orlicz space · Urysohn-type integral operator

Mathematics Subject Classification Primary 41A35 · 28B05 · Secondary 46A19

1 Introduction

In this paper we continue the study started in [3, 10, 11, 16–19] were we have extended to the vector lattice setting the problem of approximating a function f by means of Urysohn-type integral operators in the setting of modular convergence. These operators are particularly useful in order to approximate a continuous or analog signal by means of discrete samples, and therefore they are widely applied for instance in reconstructing images, see for example [1, 2, 5, 6, 29–32]. Professor Paul Leo Butzer is certainly

This article is part of the topical collection "Sampling theory, approximation and their applications. An appreciation to Paul Butzer's scientific work" edited by Carlo Bardaro, Rudolf Stens and Gianluca Vinti.

To Professor Paul Leo Butzer, the Scientist and the Man, with deep esteem and admiration for His notable contributions in Mathematics.

✉ Anna Rita Sambucini
anna.sambucini@unipg.it

Antonio Boccuto
antonio.boccuto@unipg.it

[1] Department of Mathematics and Computer Sciences, University of Perugia, 1, Via Vanvitelli, Perugia 06123, Italy

one of the Masters and pioneers in Approximation Theory and Signal Analysis and his works have formed many generations of researchers. He has worked continuously on these topics, as evidenced by his scientific production [4–9, 20–22, 24–28]. We met him in 1990, in Capri during the conference "Quarto Convegno di Analisi Reale e Teoria della Misura", in which he presented the results of [23] in the main conference *"The sampling theorem and its unique role in various branches of mathematics"*, which has influenced our research in this area and related topics and applications. In [19] we have considered vector lattice-valued functions defined on a metric space, vector lattice-valued measures and we have constructed a vector lattice-valued integral, involving a triple of vector lattices, linked by a suitable "product" structure. The results obtained are extensions of those on the integrals investigated in [14], where only finite measures are considered, and in [16], where the vector lattices considered there have suitable properties. In this paper we continue the investigation started in [19], and give examples and applications on moment and Mellin-type kernels, as well as linear and nonlinear vector lattice-valued operators.

In Sect. 2 we recall some properties of vector lattices and introduce an axiomatic definition of limit and limit superior together with the definition of the integral. In Sect. 3 we recall the main properties of vector lattice-valued modulars, while in Sect. 4 we give our structural assumptions on operators and apply our results to the case of Mellin-type kernels, and operators which can be linear or nonlinear.

2 Axiomatic convergence and integration in vector lattices

Let $(\mathbf{X}, \leq_X)$ be a vector lattice, and let $\vee$, $\wedge$ indicate the lattice suprema and infima in $\mathbf{X}$. The space $\mathbf{X}$ is *Dedekind complete* if and only if each $\emptyset \neq A \subset \mathbf{X}$ which is order bounded from above, has a lattice supremum in $\mathbf{X}$, called $\bigvee A$. In this paper $\mathbf{X}$ will be a Dedekind complete vector lattice, and $\mathbf{X}^+$ will be its positive cone. For any $x \in \mathbf{X}$, put $|x| = x \vee (-x)$. An extra element $+\infty$ is added to $\mathbf{X}$, which extends in a natural way the operations and the structure of order. The symbol $\overline{\mathbf{X}}$ denotes the set $\mathbf{X} \cup \{+\infty\}$, and we suppose $0 \cdot (+\infty) = 0$, by convention. Let $\mathcal{T} := \{(x_n)_n \subset \mathbf{X}\}$ and $\mathcal{T}^+ = \{(x_n)_n \in \mathcal{T} \colon x_n \geq_X 0, \ \forall\, n \in \mathbb{N}\}$. An (o)*-sequence* $(\sigma_l)_{l \in \mathbb{N}}$ in $\mathbf{X}^+$ is a decreasing sequence such that $\wedge_l \sigma_l = 0$. For what unexplained we refer, for instance, to [14, 19, 33–37]. Now we recall the axioms given in [19] in order to present the abstract convergence in the vector lattice context (see, e.g., [14, Definition 2.1] and [3]).

Axioms 2.1 ([19, Axioms 2.1]) Given a linear subspace $\mathcal{S}$ of $\mathcal{T}$, a convergence is a pair $(\mathcal{S}, \ell)$, with $\ell : \mathcal{S} \to \mathbf{X}$, which satisfies the conditions of linearity and monotonicity and the following properties for every $(x_n)_n, (y_n)_n, (z_n)_n \in \mathcal{S}$:

2.1.a) If $(x_n)_n$ has the property that $x_n = l$ definitely, then $(x_n)_n \in \mathcal{S}$ and $\ell((x_n)_n) = l$; if the set $\{n \in \mathbb{N} : x_n \neq y_n\}$ is finite and $(x_n)_n \in \mathcal{S}$, then $(y_n)_n \in \mathcal{S}$ and $\ell((y_n)_n) = \ell((x_n)_n)$.
2.1.b) If $(x_n)_n \in \mathcal{S}$, then $(|x_n|)_n \in \mathcal{S}$ and $\ell((|x_n|)_n) = |\ell((x_n)_n)|$.
2.1.c) If $(x_n)_n, (z_n)_n \in \mathcal{S}$, $\ell((x_n)_n) = \ell((z_n)_n)$, and $x_n \leq_X y_n \leq_X z_n$ definitely, then $(y_n)_n \in \mathcal{S}$ and $\ell((x_n)_n) = \ell((y_n)_n) = \ell((z_n)_n)$).

2.1.d) If $u \in \mathbf{X}^+$, then $\left(\frac{1}{n}u\right)_n \in \mathcal{S}$ and $\ell\left(\left(\frac{1}{n}u\right)_n\right) = 0$.

An axiomatic approach for a "limit superior"-type vector lattice-valued operator satisfying Axioms 2.1 is the following:

Axioms 2.2 (see [19, Axioms 2.2]) Let $\mathcal{T}, \mathcal{S}$ be as in Axioms 2.1, $(x_n)_n, (y_n)_n \in \mathcal{T}^+$, and define a function $\overline{\ell} : \mathcal{T}^+ \to \overline{\mathbf{X}}^+$ satisfying the conditions of subadditivity and monotonicity, and the next properties:

2.2.a) If $x_n = y_n$ definitely, then $\overline{\ell}((x_n)_n) = \overline{\ell}((y_n)_n)$.
2.2.b) If $(x_n)_n \in \mathcal{S}$, then $\overline{\ell}((x_n)_n) = \ell((x_n)_n)$.
2.2.c) If $(x_n)_n$ has the property that $\overline{\ell}((x_n)_n) = 0$, then $(x_n)_n \in \mathcal{S}$ and $\ell((x_n)_n) = 0$.

In order to introduce an abstract integral for vector lattice-valued functions with respect to (possibly infinite) vector lattice-valued measures, extending the integrals given in [14–16], we assume some relations between the order and the addition and product operations.

Assumption 2.3 ([19, Axioms 3.1]) Let $\ell, \ell_{\mathbb{R}}$ be two convergences, satisfying Axioms 2.1. We say that $(\mathbf{X}, \mathbb{R}, \mathbf{X})$ is a *product triple* iff there is a "product operation" $\cdot : \mathbf{X} \times \mathbb{R} \to \mathbf{X}$, fulfilling the distributivity laws with respect to the sum, compatible with respect to the order and to the operations of supremum and infimum, and satisfying the following two additional conditions:

2.3.1) if $(x_n)_n \subset \mathbf{X}$ with $\ell((x_n)_n) = 0$ and $y \in \mathbb{R}$, then $\ell((x_n \cdot y)_n) = 0$;
2.3.2) if $x \in \mathbf{X}$ and $(y_n)_n \subset \mathbb{R}$ with $\ell_{\mathbb{R}}((y_n)_n) = 0$, then $\ell((x \cdot y_n)_n) = 0$.

Let $G = (G, d)$ be a metric space, $\mathcal{P}(G)$ be the family of all subsets of G, $\mathcal{A} \subset \mathcal{P}(G)$ be an algebra, $\mu : \mathcal{A} \to \overline{\mathbb{R}}^+$ be a finitely additive measure; μ is said to be *σ-finite* if and only if there is an (increasing) sequence $(B_n)_n$ from $\mathcal{A}$, such that $\mu(B_n) \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} B_n = G$.

Now we introduce an integral for $\mathbf{X}$-valued functions with respect to a positive, finitely additive and σ-finite extended real-valued measure μ, associated with convergences $\ell, \ell_{\mathbb{R}}$, fulfilling Axioms 2.1. The integral will be an element of $\mathbf{X}$. Similar constructions were given also in [14–16], in some particular cases.

We denote by $\mathscr{S}$ the set of all simple functions defined on G, which assume a finite number of values and with support contained in a set of finite measure μ. For such functions, the integral is defined in a natural way (see, e.g., [14, 19]).

Now, to introduce the integral for more general functions, we use the definitions of uniform convergence and convergence in measure and in L^1.

Definitions 2.4 (see [19, Definition 3.2]) Let $A \in \mathcal{A}$ and $(f_n)_n \subset \mathbf{X}^G$. We say that:

- $(f_n)_n$ is *uniformly convergent* to $f \in \mathbf{X}^G$ on A if and only if

$$\ell\left(\left(\bigvee_{g \in A} |f_n(g) - f(g)|\right)_n\right) = 0;$$

- $(f_n)_n$ *converges in μ-measure* to $f \in \mathbf{X}^G$ on A if and only if there exists $(A_n)_n \subset \mathcal{A}$, such that $\ell_{\mathbb{R}}((\mu(A \cap A_n))_n) = 0$ and

$$\ell\Big(\Big(\bigvee_{g \in A \setminus A_n} |f_n(g) - f(g)|\Big)_n\Big) = 0;$$

- $(f_n)_n \subset \mathscr{S}$ *converges in L^1* to $f \in \mathscr{S}$ if and only if

$$\ell\left(\left(\int_G |f_n(g) - f(g)|\, d\mu(g)\right)_n\right) = 0.$$

Convergence in measure is a consequence of the uniform convergence and, if $\mathbf{X} = \mathbb{R}$, the convergences here defined are equivalent to the classical ones (see, e.g., [14, Remark 3.3]).

Definition 2.5 (see [19, Definition 3.5]) A sequence $(f_n)_n \subset \mathscr{S}$ is *μ-equiabsolutely continuous* if and only if

2.5.1) $\ell\left(\left(\int_{A_n} |f_n(g)|\, d\mu(g)\right)_n\right) = 0$ whenever $\ell_{\mathbb{R}}((\mu(A_n))_n) = 0$;

2.5.2)] there exists an increasing sequence $(B_m)_m \subset \mathcal{A}$ with $\mu(B_m) \in \mathbb{R} < +\infty$ for each $m \in \mathbb{N}$, and

$$\ell\Big(\Big(\overline{\ell}\Big(\Big(\int_{G \setminus B_m} |f_n(g)|\, d\mu(g)\Big)_n\Big)\Big)_m\Big) = 0.$$

We say that a sequence $(f_n)_n$ in $\mathscr{S}$ is *defining for* $f \in \mathbf{X}^G$ if and only if it converges in μ-measure to f on each set $A \in \mathcal{A}$ of finite measure μ, and their integrals are μ-equiabsolutely continuous.

Definition 2.6 A positive function $f \in \mathbf{X}^G$ is *integrable* on G if and only if there are a defining sequence $(f_n)_n$ for f and a function $l : \mathcal{A} \to \mathbf{X}$, such that

$$\ell\Big(\Big(\bigvee_{A \in \mathcal{A}} \Big|\int_A f_n(g)\, d\mu(g) - l(A)\Big|\Big)_n\Big) = 0,$$

and we put

$$\int_A f(g)\, d\mu(g) := l(A) \quad \text{for every } A \in \mathcal{A}. \tag{1}$$

For not necessarily positive functions the integral, as usual, is given using $f^+(g) = f(g) \vee 0$, $f^-(g) = (-f(g)) \vee 0$, for every $g \in G$, namely:

Definition 2.7 A function $f : G \to \mathbf{X}$ is *integrable* on G if and only if the functions $f^{\pm}$ are integrable on G, and in this case we define

$$\int_A f(g)\, d\mu(g) = \int_A f^+(g)\, d\mu(g) - \int_A f^-(g)\, d\mu(g), \quad A \in \mathcal{A}. \tag{2}$$

For the properties of this integral and the convergence results, we refer to [19, Section 3]; in particular, when $\mathbb{X} = \mathbb{R}$ and $\ell_{\mathbb{R}}$ is the usual limit, this integral coincides with the Lebesgue one. We denote by $\mathscr{L}(\mathbf{X}, \mu)$ the space of all functions $f : G \to \mathbf{X}$ which are integrable in the sense of formulas (1) and (2).

Now we recall uniform continuity in the vector lattice context (see, e.g., [16, 19]).

- We say that $f : G \to \mathbf{X}$ is *uniformly continuous on* G if and only if $\exists\ u \in \mathbf{X}^+ :$ $\forall\ \varepsilon \in \mathbb{R}^+,\ \exists\ \delta \in \mathbb{R}^+$ such that

$$|f(g_1) - f(g_2)| \leq_X \varepsilon\, u \quad \forall\ g_1, g_2 \in G\ :\ d(g_1, g_2) \leq \delta.$$

- A function $\psi : \mathbf{X} \to \mathbf{X}$ is *uniformly continuous on* $\mathbf{X}$ if and only if $\forall\ u \in \mathbf{X}^+$ and $\varepsilon \in \mathbb{R}^+$, $\exists\ w \in \mathbf{X}^+$ and $\exists\ \delta \in \mathbb{R}^+$ such that

$$|\psi(x_1) - \psi(x_2)| \leq_X\ \varepsilon\, w, \quad \text{whenever } |x_1 - x_2| \leq_X \delta\, u.$$

If $G = \mathbf{X} = \mathbb{R}$ with the usual topology, the two definitions of uniform continuity coincide with the classical one (see [16, 19]).

3 Modulars

Let T be a vector subspace of $\mathbf{X}^G$ such that, if $f \in T$ and $A \in \mathcal{A}$, then $|f| \in T$ and $f \cdot \chi_A \in T$, where the symbol χ_A denotes the *characteristic function* associated with the set A, that is the function which associates the value 1 to every element of A and the value 0 to every element of $G \backslash A$.

A functional $\rho : T \to \overline{\mathbf{X}}^+$ is a *modular* on T if and only if the following properties hold:

(m_0) $\rho(0) = 0$;
(m_1) $\rho(-f) = \rho(f)$ for each $f \in T$;
(m_2) $\rho(c_1 f + c_2 h) \leq_X \rho(f) + \rho(h)$ for every $f, h \in T$ and for all $c_1, c_2 \in \mathbb{R}_0^+$ such that $c_1 + c_2 = 1$.

A modular ρ is *monotone* if and only if $\rho(f) \leq_X \rho(h)$ for every $f, h \in T$ such that $|f| \leq_X |h|$. If $f \in T$, then $|f| \in T$ and $\rho(f) = \rho(|f|)$ (see, e.g., [12, 16]); while ρ is *convex* if and only if $\rho(c_1 f + c_2 h) \leq_X c_1\, \rho(f) + c_2\, \rho(h)$ for all $f, h \in T$, $c_1, c_2 \in \mathbb{R}_0^+$ with $c_1 + c_2 = 1$. For a related literature on modulars see, e.g., [12, 16, 17, 33] and the references therein. Proceeding now analogously as in [16, Proposition 3.1], it is possible to see that, if $\varphi : \mathbf{X} \to \mathbf{X}$ is increasing on $\mathbf{X}^+$ and $\varphi(0) = 0$, then it makes sense to define the set

$$\mathcal{L}^{\varphi} = \left\{ f \in \mathbf{X}^G :\ \int_G \varphi(|f(g)|)\, d\mu(g) \text{ exists in } \mathbf{X} \right\},$$

and then the $\mathbf{X}$-valued operator ρ^φ defined by

$$\rho^\varphi(f) = \int_G \varphi(|f(g)|)\, d\mu(g), \quad f \in \mathcal{L}^\varphi,$$

is a monotone modular and, if φ is convex, then ρ^φ is convex on the set of the positive functions of $\mathcal{L}^\varphi$. The set

$$L^\varphi(G) = \Big\{ f \in \mathbf{X}^G : \bigwedge_{a \in \mathbb{R}^+} \rho^\varphi(a\, f) = 0 \Big\}$$

is the *Orlicz space* generated by φ. Note that $L^\varphi(G)$ is a vector space and, if $a \in \mathbb{R}^+$ and $a\, f \in L^\varphi(G)$, then $b\, f \in L^\varphi(G)$ whenever $b \in]0, a[$. The subspace $E^\varphi(G)$ of $L^\varphi(G)$, defined by setting

$$E^\varphi(G) := \{ f \in L^\varphi(G) : \rho^\varphi(a\, f) \in \mathbf{X} \text{ for each } a \in \mathbb{R}^+ \},$$

is the *space of the finite elements of* $L^\varphi(G)$.

We say that a sequence $(f_n)_n$ from $L^\varphi(G)$ is *modularly convergent* to $f \in L^\varphi(G)$ if and only if there exists $a \in \mathbb{R}^+$ with

$$\ell((\rho^\varphi(a(f_n - f)))_n) = 0. \tag{3}$$

4 Structural assumptions on operators and examples

As examples and applications of the results given in the previous sections, we consider (non)linear Mellin-type operators with values in vector lattices. In this setting, we take $G = (\mathbb{R}^+, d_{\ln})$, where $d_{\ln}(t_1, t_2) = |\ln t_1 - \ln t_2|$, $t_1, t_2 \in \mathbb{R}^+$, and for each measurable set $S \subset \mathbb{R}^+$ we put $\mu(S) = \displaystyle\int_S \frac{dt}{t}$. Let $\mathcal{M}$ be the set of all sequences of functions $\widetilde{L}_n$ defined on $\mathbb{R}^+$, non-negative and μ-integrable, and such that for any $n \in \mathbb{N}$ and every $s \in \mathbb{R}^+$ (resp., $t \in \mathbb{R}^+$) the map $t \mapsto \widetilde{L}_n\Big(\dfrac{t}{s}\Big)$ (resp., $s \mapsto \widetilde{L}_n\Big(\dfrac{t}{s}\Big)$) is (μ-integrable and) bounded. We now introduce some structural assumptions.

Assumptions 4.1 4.1.a) Let Ψ be the class of all functions $\psi : \mathbf{X}^+ \to \mathbf{X}^+$ with the property that

4.1.a.1) ψ is uniformly continuous and increasing on $\mathbf{X}^+$, $\psi(0) = 0$ and $\psi(v) \in \mathbf{X}^+ \setminus \{0\}$ for all $v \in \mathbf{X}^+ \setminus \{0\}$.

Let $\Xi = (\psi_n)_n \subset \Psi$ be a sequence of functions, satisfying the following conditions:

4.1.a.2) $(\psi_n)_n$ is *equicontinuous at* 0, namely for each $u \in \mathbf{X}^+ \setminus \{0\}$ and $\varepsilon \in \mathbb{R}^+$ there exist $w \in \mathbf{X}^+ \setminus \{0\}$ and $\delta \in \mathbb{R}^+$ such that $\psi_n(x) \leq_X \varepsilon\, w$ whenever $x \leq_X \delta\, u$ and for all $n \in \mathbb{N}$;

4.1.a.3) for each $v \in \mathbf{X}^+$ the sequence $(\psi_n(v))_n$ is *order equibounded*, that is there is $A_v \in \mathbf{X}^+ \backslash \{0\}$ such that $\psi_n(v) \leq_X A_v$ for all $n \in \mathbb{N}$.

4.1.b) Let $\Xi = (\psi_n)_n \subset \Psi$ be as in 4.1.a), and denote by $\widetilde{\mathcal{K}}_\Xi$ the set of all sequences of functions $\widetilde{K}_n : \mathbb{R}^+ \times \mathbf{X} \to \mathbf{X}$, $n \in \mathbb{N}$, such that

4.1.b.1) $\widetilde{K}_n(\cdot, u) \in \mathscr{L}(\mathbf{X}, \mu)$ for each $u \in \mathbf{X}$ and $n \in \mathbb{N}$, and $\widetilde{K}_n(t, 0) = 0$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$;

4.1.b.2) there are sequences $(\widetilde{L}_n)_n \subset \mathcal{M}$ and $(\psi_n)_n \subset \Psi$ with

$$|\widetilde{K}_n(t, u) - \widetilde{K}_n(t, v)| \leq_X \widetilde{L}_n(t)\, \psi_n(|u - v|)$$

for all $n \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $u, v \in \mathbf{X}$.

Let $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n \in \widetilde{\mathcal{K}}_\Xi$, and consider a sequence $\widetilde{\mathbf{T}} = (\widetilde{T}_n)_n$ of nonlinear Mellin-type operators defined by

$$(\widetilde{T}_n f)(s) = \int_0^{+\infty} \widetilde{K}_n\Big(\frac{t}{s}, f(t)\Big)\frac{dt}{t}, \tag{4}$$

$n \in \mathbb{N}$, $s \in \mathbb{R}^+$, $f \in \mathrm{Dom}\, \widetilde{\mathbf{T}} = \bigcap_{n=1}^{\infty} \mathrm{Dom}\, \widetilde{T}_n$, where $\mathrm{Dom}\, \widetilde{T}_n$ is the set of the functions f for which the integral in (4) makes sense. The concept of singularity in the context of Mellin operators can be introduced in the following form, given in [19, Definition 6.7].

Definition 4.2 We say that $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n$ is *U-singular* if and only if there exist an infinite set $H \subset \mathbb{N}$ and a positive real number $D^{(1)}$, fulfilling the following conditions:

4.2.1) for each $n \in H$, $\displaystyle\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} \leq D^{(1)}$;

4.2.2) for every $(a_n)_n \subset \mathbb{R}$, $\overline{\ell}((a_n)_{n\in\mathbb{N}}) = \overline{\ell}((a_n)_{n\in H})$;

4.2.3) $\displaystyle\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} > 0$ for each $n \in \mathbb{N}$;

4.2.4) for any $\delta \in \mathbb{R}^+$, $\delta > 1$, one has

$$\ell_{\mathbb{R}}\Big(\Big(\int_{\mathbb{R}^+ \backslash [1/\delta, \delta]} \widetilde{L}_n(t)\, \frac{dt}{t}\Big)_n\Big) = 0;$$

4.2.5) there exist $z \in \mathbf{X}^+ \setminus \{0\}$ and an (o)-sequence $(\varepsilon_n)_n$ in $\mathbb{R}^+$ such that

$$\bigvee_{u \in \mathbf{X}\backslash\{0\}} \Big| \int_0^{+\infty} \widetilde{K}_n(t, u)\, \frac{dt}{t} - u \Big| \leq_X \varepsilon_n\, z \quad \text{for any } n \in H.$$

In general, in the examples and applications, it is not very easy to verify 4.2.5) directly. So, we give a sufficient condition to have 4.2.5), which is easier to handle and verify in the practice, in the vector lattice setting.

Proposition 4.3 *Suppose that* $\widetilde{K}_n(t,u) = \widetilde{L}_n(t)\, \Upsilon_n(u)$, *where* $\Upsilon_n : \mathbf{X} \to \mathbf{X}$,

4.3.1) $\displaystyle\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} = 1$ *for all* $n \in H$, *and*

4.3.2) *there are* $v \in \mathbf{X}^+ \setminus \{0\}$ *and an* (o)*-sequence* $(\sigma_n)_n$ *in* $\mathbb{R}^+$ *such that*

$$\bigvee_{u \in \mathbf{X}\setminus\{0\}} |\Upsilon_n(u) - u| \leq_X \sigma_n\, v \quad \textit{for any } n \in H.$$

Then, 4.2.5) *holds.*

Proof Let v and $(\sigma_n)_n$ be related to 4.3.2). One has

$$\begin{aligned}
\bigvee_{u \in \mathbf{X}\setminus\{0\}} \left| \int_0^{+\infty} \widetilde{K}_n(t,u)\, \frac{dt}{t} - u \right| &= \bigvee_{u \in \mathbf{X}\setminus\{0\}} \left| \int_0^{+\infty} \widetilde{L}_n(t)\, \Upsilon_n(u)\, \frac{dt}{t} - u \right| \\
&= \bigvee_{u \in \mathbf{X}\setminus\{0\}} \left| \left(\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} \right) \Upsilon_n(u) - \left(\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} \right) u \right| \\
&= \bigvee_{u \in \mathbf{X}\setminus\{0\}} |\Upsilon_n(u) - u| \leq_X \sigma_n\, v,
\end{aligned}$$

getting 4.2.5). □

For the convenience of the reader we recall also the following result, which is a particular case of [19, Theorems 6.5 and 6.9], that will be used in the sequel. For a more detailed description we refer to [19, Section 6] (https://arxiv.org/pdf/2112.12085.pdf).

Theorem 4.4 *Under Assumptions* 4.1, *assume that* $\widetilde{\mathbb{K}}$ *is* U*-singular. Then, for every* $f \in \mathcal{C}_c(\mathbb{R}^+)$, *the sequence* $(\widetilde{T}_n f)_n$ *is uniformly convergent to* f *on* $\mathbb{R}^+$, *and modularly convergent to* f *with respect to the modular* ρ^φ, *where the constant* a *in* (3) *can be chosen independently of* f.

Now, we are ready to give some examples of functions $\widetilde{L}_n$ and vector lattice-valued maps Υ_n, such that the functions

$$\widetilde{K}_n(t,u) = \widetilde{L}_n(t)\, \Upsilon_n(u), \quad n \in \mathbb{N},\ t \in \mathbb{R}^+,\ u \in \mathbf{X}$$

satisfy all conditions on U-singularity and Assumptions 4.2, and such that

$$\int_a^b \widetilde{L}_n\Big(\frac{t}{\cdot}\Big) \frac{dt}{t} \in L^q(\mathbb{R}^+) \tag{5}$$

for every $[a,b] \subset \mathbb{R}^+$, $q \geq 1$ and n large enough, depending on q (see also [18, Corollary 5.4] for kernels, when $\Upsilon_n(u) = u$ and $\mathbf{X} = \mathbb{R}$). Moreover, from U-singularity and using [19, Remark 6.8.b], we get that for any compact subset $C \subset \mathbb{R}^+$ there is a set $B \subset \mathbb{R}^+$ of finite μ-measure such that

$$\bar{\ell}_{\mathbb{R}}\left(\left(\sup_{t \in C} \int_{\mathbb{R}^+ \setminus B} \widetilde{L}_n\Big(\frac{s}{t}\Big) \frac{ds}{s} \right)_n \right) = 0. \tag{6}$$

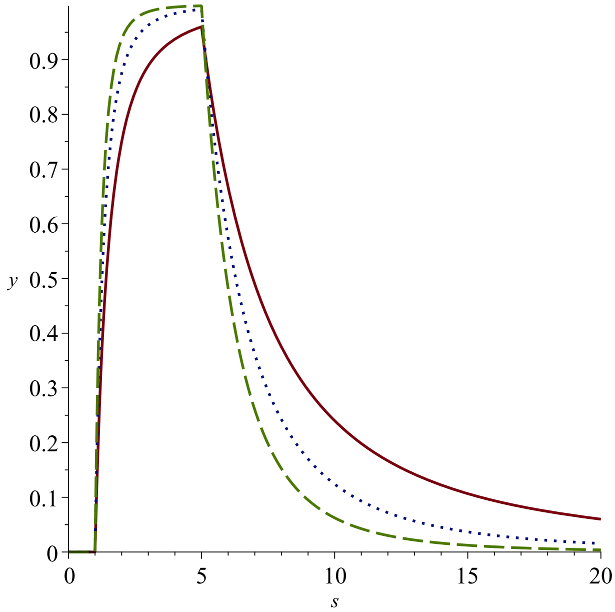

Fig. 1 Moment kernel: $a = 1$, $b = 5$, $n = 2$ (line), $n = 3$ (point), $n = 4$ (dash)

So by Theorem 4.4 we obtain that the sequence $(\widetilde{T}_n f)_n$ converges uniformly to f, and we find a positive real number a such that $\ell((\rho^{\varphi}(a(\widetilde{T}_n f - f))_n)) = 0$ for each $f \in \mathcal{C}_c(\mathbb{R}^+)$.

Example 4.5 *Moment kernel.* This kernel is defined by setting $\widetilde{L}_n(t) = n\, t^n \chi_{]0,1[}(t)$, $n \in \mathbb{N}, t \in \mathbb{R}^+$. In [18, Section 5] it is shown that $\int_0^{+\infty} \widetilde{L}_n(t)\, \frac{dt}{t} = 1$ (getting 4.3.1)), and

$$\int_{\mathbb{R}^+ \setminus [1/\delta, \delta]} \widetilde{L}_n(t)\, \frac{dt}{t} = \frac{1}{\delta^n} \leq \frac{1}{n}$$

for each $n \in \mathbb{N}$ and $\delta \in]1, +\infty[$. From this and Axioms 2.1.c), 2.1.d) we obtain 4.2.4). Moreover $0 \leq \widetilde{L}_n(t) \leq n$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. From this, since $\widetilde{L}_n \in L^1(\mathbb{R}^+, \mu)$ for every $n \in \mathbb{N}$, it follows that $(\widetilde{L}_n)_n \subset \mathcal{M}$. Relation (5) follows from the fact that, for each $[a, b] \subset \mathbb{R}^+$, for every $n \in \mathbb{N}$ and $s \in \mathbb{R}^+$ it is

$$\int_a^b \widetilde{L}_n\Big(\frac{t}{s}\Big)\frac{dt}{t} = \begin{cases} \dfrac{b^n - a^n}{s^n} & \text{if } s \geq b, \\ \dfrac{s^n - a^n}{s^n} & \text{if } a \leq s < b, \\ 0 & \text{if } 0 < s < a \end{cases}$$

(see also [18, formula (5.20)]).

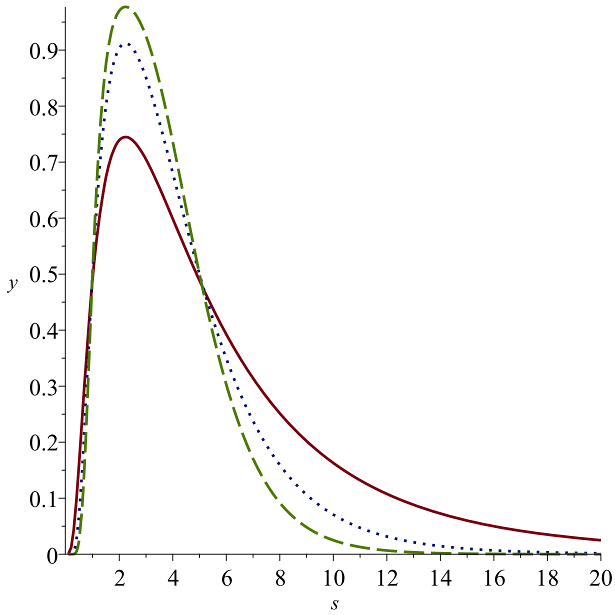

Fig. 2 Mellin–Gauss–Weierstrass kernel: $a = 1$, $b = 5$, $n = 2$ (line), $n = 3$ (point), $n = 4$ (dash)

Example 4.6 *Mellin–Gauss–Weierstrass kernel.* Let

$$\widetilde{L}_n(t) = \frac{n}{2\sqrt{\pi}} e^{-\frac{n^2}{4}\ln^2 t}, \quad n \in \mathbb{N}, \ t \in \mathbb{R}^+.$$

In [18, formula (5.17)] it is proved that $\displaystyle\int_0^{+\infty} \widetilde{L}_n(t)\,\frac{dt}{t} = 1$ for any $n \in \mathbb{N}$, and that for every $\delta > 1$ there exists a positive integer $n_0 = n_0(\delta)$ such that

$$\int_{\mathbb{R}^+ \setminus [1/\delta, \delta]} \widetilde{L}_n(t)\,\frac{dt}{t} \le \frac{2}{\sqrt{\pi}} e^{-\frac{n \ln \delta}{2}} \le \frac{1}{n}$$

whenever $n \ge n_0$. From this and Axioms 2.1c), 2.1d) we get 4.2.4).

As before, $0 \le \widetilde{L}_n(t) \le n$ for any $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. From this and the μ-integrability of the $\widetilde{L}_n$, we obtain that $(\widetilde{L}_n)_n \subset \mathcal{M}$.

We now prove that formula (5) holds for the Mellin–Gauss–Weierstrass kernel.

Proposition 4.7 *For every* $[a, b] \subset \mathbb{R}^+$ *and* $q \ge 1$ *one has*

$$s \mapsto \frac{n}{2\sqrt{\pi}} \int_a^b e^{-\frac{n^2}{4}\ln^2\left(\frac{t}{s}\right)} \frac{dt}{t} \in L^q(\mathbb{R}^+), \quad s \in \mathbb{R}^+$$

for $n \in \mathbb{N}$ *large enough, depending on* q.

Proof Let $[a, b] \subset \mathbb{R}^+$. For every $s \in \mathbb{R}^+$ it is

$$\int_a^b \widetilde{L}_n\Big(\frac{t}{s}\Big)\frac{dt}{t} = \frac{1}{\sqrt{\pi}}\int_{\frac{n}{2}\ln\left(\frac{a}{s}\right)}^{\frac{n}{2}\ln\left(\frac{b}{s}\right)} e^{-w^2}\,dw.$$

Now we prove that

$$s \mapsto \int_a^b \widetilde{L}_n\Big(\frac{t}{s}\Big)\frac{dt}{t} \in L^q([b\,e^2, +\infty[) \tag{7}$$

for each $q \geq 1$ and n large enough, depending on q. If $s \geq b\,e^2$, then $\ln\Big(\frac{b}{s}\Big) \leq -2$, and hence

$$\frac{n}{2}\ln\Big(\frac{a}{s}\Big) < \frac{n}{2}\ln\Big(\frac{b}{s}\Big) \leq -1.$$

Therefore

$$\frac{1}{\sqrt{\pi}}\int_{\frac{n}{2}\ln\left(\frac{a}{s}\right)}^{\frac{n}{2}\ln\left(\frac{b}{s}\right)} e^{-w^2}\,dw \leq \frac{1}{\sqrt{\pi}}\int_{\frac{n}{2}\ln\left(\frac{a}{s}\right)}^{\frac{n}{2}\ln\left(\frac{b}{s}\right)} e^{w}\,dw = \frac{1}{\sqrt{\pi}}\Big(\Big(\frac{b}{s}\Big)^{n/2} - \Big(\frac{a}{s}\Big)^{n/2}\Big),$$

getting (7). Now we claim that

$$s \mapsto \int_a^b \widetilde{L}_n\Big(\frac{t}{s}\Big)\frac{dt}{t} \in L^q(]0, a\,e^{-2}]) \tag{8}$$

for all $q \geq 1$ and for every n. If $0 < s \leq a\,e^{-2}$, then $\ln\Big(\frac{a}{s}\Big) \geq 2$, and thus

$$\frac{n}{2}\ln\Big(\frac{b}{s}\Big) > \frac{n}{2}\ln\Big(\frac{a}{s}\Big) \geq 1.$$

Hence,

$$\frac{1}{\sqrt{\pi}}\int_{\frac{n}{2}\ln\left(\frac{a}{s}\right)}^{\frac{n}{2}\ln\left(\frac{b}{s}\right)} e^{-w^2}\,dw \leq \frac{1}{\sqrt{\pi}}\int_{\frac{n}{2}\ln\left(\frac{a}{s}\right)}^{\frac{n}{2}\ln\left(\frac{b}{s}\right)} e^{-w}\,dw = \frac{1}{\sqrt{\pi}}\Big(\Big(\frac{s}{a}\Big)^{n/2} - \Big(\frac{s}{b}\Big)^{n/2}\Big),$$

obtaining (8). Furthermore, we have

$$s \mapsto \int_a^b \widetilde{L}_n\Big(\frac{t}{s}\Big)\frac{dt}{t} \in L^q([a\,e^{-2}, b\,e^2]) \tag{9}$$

for all $q \geq 1$ and for every $n \in \mathbb{N}$, since the function in (9) is continuous on $([a\,e^{-2}, b\,e^2])$. Thus, the assertion follows from (7), (8) and (9). □

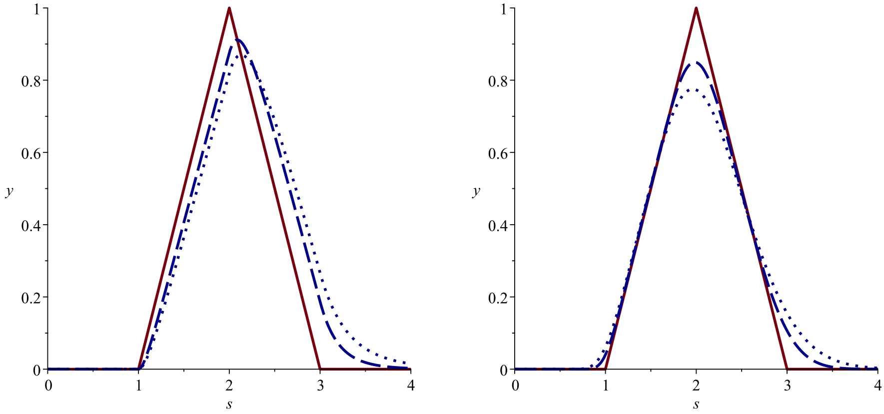

Fig. 3 Moment (Mellin–Gauss–Weierstrass) kernels applied to f: f (line), $\widetilde{T}_{10} f$ (point), $\widetilde{T}_{15} f$ (dash)

The graphs show how the moment and the Mellin–Gauss–Weierstrass kernels approximate the function $f \in \mathcal{C}_c(\mathbb{R}^+)$, $f(t) = (t-1)\chi_{[1,2]} + (3-t)\chi_{[2,3]}$, for $n = 10, 15$.

Moreover we give the following

Example 4.8 *Mellin–Poisson–Cauchy kernel.* Fix $p \in \mathbb{N}$, $p \geq 2$ and put, for every $n \in \mathbb{N}$, $t \in \mathbb{R}^+$,

$$\widetilde{L}_n(t) = C_p \, \frac{n}{(1 + n^2 \ln^2 t)^p}, \quad \text{where} \quad C_p = \frac{2^{p-1}(p-1)!}{\pi (2p-3)!!}.$$

In [18, formulas (5.10)–(5.14)] it is shown that

$$\int_0^{+\infty} \widetilde{L}_n(t) \, \frac{dt}{t} = 1 \quad \text{for all } n \in \mathbb{N}$$

and that, for any $\delta > 1$ and $n \in \mathbb{N}$, it is

$$\int_{\mathbb{R}^+ \setminus [1/\delta, \delta]} \widetilde{L}_n(t) \, \frac{dt}{t} \leq C_p \, (\pi - 2 \arctan(n \, \ln \delta)). \tag{10}$$

Thanks to de l'Hôpital's rule, it is possible to show that

$$\lim_{n \to +\infty} \frac{\pi - 2 \arctan(n \, \ln \delta)}{1/n} = \frac{2}{\ln \delta}. \tag{11}$$

From (10) and (11) we find a positive real number $K_{\delta,p}$ and an integer $n_0 = n_0(\delta, p)$ with

$$\int_{\mathbb{R}^+ \setminus [1/\delta, \delta]} \widetilde{L}_n(t) \, \frac{dt}{t} \leq K_{\delta,p} \, \frac{1}{n} \tag{12}$$

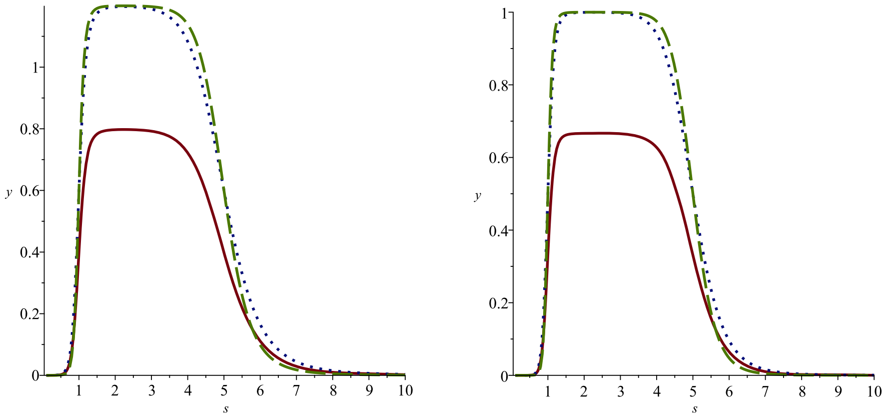

Fig. 4 Mellin–Poisson–Cauchy kernel: $a = 1$, $b = 5$, $n = 2$ (line), $n = 3$ (point), $n = 4$ (dash), $p = 3, 4$

whenever $n \geq n_0$. From (12) and Axioms 2.1.c), 2.1.d) we deduce 4.2.4).

Moreover $0 \leq \widetilde{L}_n(t) \leq n$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. From this and the μ-integrability of $\widetilde{L}_n$ we deduce that $(\widetilde{L}_n)_n \subset \mathcal{M}$. Furthermore, we get

$$s \mapsto \int_a^b \widetilde{L}_n\left(\frac{t}{s}\right)\frac{dt}{t} = \frac{n2^{p-1}(p-1)!}{\pi(2p-3)!!}\int_a^b \frac{1}{(1+n^2(\ln s - \ln t)^2)^p}\frac{dt}{t} \in L^q(\mathbb{R}^+), \quad s \in \mathbb{R}^+$$

for any $[a, b] \subset \mathbb{R}^+$, $q \geq 1$ and n large enough, depending on q (see [18, formula (5.21)]), getting relation (5).

Example 4.9 *Linear operators*. The simplest example of functions Υ_n satisfying 4.3.2) is given by

$$\Upsilon_n(u) = u, \qquad n \in \mathbb{N}, \quad u \in \mathbf{X}.$$

This case is widely studied in the literature when $\mathbf{X} = \mathbb{R}$ (see, e.g., [12] and the references therein). From 4.3.1), 4.3.2) and Proposition 4.3 we obtain 4.2.5). Therefore, $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n$ is U-singular, with $H = \mathbb{N}$ and $D^{(1)} = 1$.

Now observe that, for every $n \in \mathbb{N}$, the function

$$\widetilde{K}_n(t, u) = \widetilde{L}_n(t)\ u, \quad t \in \mathbb{R}^+, \ u \in \mathbf{X}, \tag{13}$$

is μ-integrable, since $\widetilde{L}_n$ is. Moreover, we get that $\widetilde{K}_n(t, 0) = 0$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. So, condition 4.1.b.1) holds. Furthermore, it is straightforward to see that 4.1.b.2) is satisfied by taking $\psi_n(u) = u$ for all $n \in \mathbb{N}$ and $u \in \mathbf{X}^+$, since $(\widetilde{L}_n)_n \subset \mathcal{M}$. Thus, Assumptions 4.1 are fulfilled.

Example 4.10 *Nonlinear operators*. Let $0 \leq p \leq \infty$, $\mathbf{X} = L^p([0, 1], \Sigma, \nu)$, where Σ is the σ-algebra of all measurable subsets of $[0, 1]$ and ν is the Lebesgue measure on [0, 1]. Note that such spaces satisfy Axioms 2.1 and 2.2, and are widely studied in Stochastic Integration, Stochastic Processes and Brownian Motions, and it is possible to see that our integration theory includes also some types of stochastic integration (see also [16, 19]). For every $u \in \mathbf{X}$, $n \in \mathbb{N}$ and $t \in [0, 1]$, set

$$\Upsilon_n(u(t)) = \frac{n\, u(t)\, |u(t)|}{n\, |u(t)| + 1}.$$

This case is an extension of some examples studied in the literature when $\mathbf{X} = \mathbb{R}$ (see [10, Sections 5 and 6]). First we note that, for every $n \in \mathbb{N}$, Υ_n is well-defined: indeed, for each $u \in \mathbf{X}$ and $n \in \mathbb{N}$, we get

$$|\Upsilon_n(u(t))| = \frac{n\ |u(t)|^2}{n\, |u(t)| + 1} \leq |u(t)|$$

when $u(t) \neq 0$, and $\Upsilon_n(u(t)) = 0$ whenever $u(t) = 0$. From this it follows that

$$|\Upsilon_n(u)| \leq |u| \tag{14}$$

for any $u \in \mathbf{X}$ and $n \in \mathbb{N}$, and hence we deduce that $\Upsilon_n(u) \in \mathbf{X}$ whenever $u \in \mathbf{X}$.

Now, observe that for each $u \in \mathbf{X}$, $t \in [0, 1]$ and $n \in \mathbb{N}$ it is

$$|\Upsilon_n(u(t)) - u(t)| = \left| \frac{n\, u(t)\, |u(t)|}{n\, |u(t)| + 1} - u(t) \right| = \frac{|u(t)|}{n\, |u(t)| + 1} \leq \frac{1}{n} = \frac{1}{n} \cdot \mathbf{1},$$

where $\mathbf{1}$ is the function which associates the real number 1 to every $t \in [0, 1]$. Since $\mathbf{1} \in \mathbf{X}$, then the Υ_n's satisfy condition 4.3.2) with $v = \mathbf{1}$ and $\sigma_n = \dfrac{1}{n}$, $n \in \mathbb{N}$. From this, 4.3.1) and Proposition 4.3 we obtain 4.2.5). Hence, $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n$ is U-singular, with $H = \mathbb{N}$ and $D^{(1)} = 1$.

Moreover, from (14) and the integrability of the functions $\widetilde{K}_n$ in (13), arguing analogously as in [13, Theorem 3.21], it follows that, for each $n \in \mathbb{N}$, the function

$$\widetilde{K}_n(t, u) = \widetilde{L}_n(t)\, \Upsilon_n(u), \quad t \in \mathbb{R}^+, \ u \in \mathbf{X},$$

is integrable with respect to μ. Furthermore, since $\Upsilon_n(0) = 0$, we obtain that $\widetilde{K}_n(t, 0) = 0$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$. Thus, condition 4.1.b.1) is satisfied.

Now, analogously as in [10, Section 5, Example II], it is possible to see that for each $u, v \in \mathbf{X}$, $t \in [0, 1]$ and $n \in \mathbb{N}$ one has

$$|\Upsilon_n(u(t)) - \Upsilon_n(v(t))| = \left| \frac{n\, u(t)\, |u(t)|}{n\, |u(t)| + 1} - \frac{n\, v(t)\, |v(t)|}{n\, |v(t)| + 1} \right| \leq |u(t) - v(t)|,$$

and hence

$$|\Upsilon_n(u) - \Upsilon_n(v)| \leq |u - v|.$$

So, condition 4.1.b.2) holds, with $\psi_n(u) = u$ for every $n \in \mathbb{N}$ and $u \in \mathbf{X}^+$, taking into account that $(\widetilde{L}_n)_n \subset \mathcal{M}$. Hence, Assumptions 4.1 are satisfied.

Then we deduce that

Corollary 4.11 *Let $q \in \mathbb{N}$, $\varphi(u) = |u|^q$ and assume that $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n$ is defined by*

$$\widetilde{K}_n(t, u) = \widetilde{L}_n(t)\, \Upsilon_n(u), \quad n \in \mathbb{N},\ t \in \mathbb{R}^+,\ u \in \mathbf{X},$$

where $\widetilde{L}_n(t)$ is as in Examples 4.5–4.10. Then, for any $f \in \mathcal{C}_c(\mathbb{R}^+)$, the sequence $(\widetilde{T}_n f)_n$ is uniformly convergent to f on $\mathbb{R}^+$ and modularly convergent to f with respect to ρ^φ, where the number a obtained by the last convergence (in formula (3)) can be taken independently of f.

Proof It is a consequence of U-singularity, the argument in (6) and [19, Theorems 6.5 and 6.9]. □

We would like to highlight that our theory can be used also in the following contexts.

Remarks 4.12 a) In 4.2.2), we consider an infinite subset H of natural numbers. This set can be different from $\mathbb{N}$, and thus our theory includes filter convergence, which is in general strictly weaker than the classical one.
We remember, for the reader's simplicity, that a *filter* on $\mathbb{N}$ is a family $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ such that $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, and for any $A \in \mathcal{F}$ and $B \subset \mathbb{N}$ with $A \subset B$, one has $B \in \mathcal{F}$. We say that a filter on $\mathbb{N}$ is *free* iff it contains $\mathcal{F}_{\text{cofin}}$, which is the filter of all cofinite subsets of $\mathbb{N}$. If $\mathcal{F}$ is any fixed free filter on $\mathbb{N}$ different from $\mathcal{F}_{\text{cofin}}$, then it is advisable to take in 4.2.2) an infinite set $H \in \mathcal{F}$, such that $\mathbb{N} \setminus H$ is infinite (see also [19]). In [18] some other examples are presented, where the involved "Mellin-type" kernels satisfy U-singularity conditions with respect to $\mathcal{F}$-convergence, and not with respect to the usual convergence.

b) Notice that our theory can include also the multidimensional Mellin convolution operators, by considering $G = (\mathbb{R}^+)^N$ endowed with the scalar product, where the distance d_N on $G \times G$ is given by

$$d_N(s, t) = \max\{d_{\ln}(s_1, t_1), d_{\ln}(s_2, t_2), \ldots, d_{\ln}(s_N, t_N)\},$$

and $\mu(A) = \mu_N(A) = \displaystyle\int_A \frac{(dt)^N}{\prod_{j=1}^N t_j}$, $\quad A \in \mathcal{M}^N$, where $\mathcal{M}^N$ is the family of all measurable subsets of $(\mathbb{R}^+)^N$ (see, e.g., [2, 7]). Moreover, our theory can include also some types of non-convolution operators (see, for instance, [11, Example 3]).

c) As in [12, Example 3.3], it is possible to give some examples of nonlinear singular operators, in which the associated sequence $(K_n)_n$ satisfies Lipschitz-type conditions with respect to more general functions ψ_n, $n \in \mathbb{N}$, as in 4.1.1).

5 Conclusion

In this article, by continuing the investigation started in earlier papers, we have recalled an axiomatic theory of convergence, an abstract integral (with respect to possibly infinite finitely additive measures) and modulars in vector lattice setting. We have given some examples of Mellin-type kernels and corresponding operators (moment, Mellin-Gauss-Weierstrass, Mellin-Poisson-Cauchy) and we have illustrated the behavior of the corresponding convergences by means of figures.

Acknowledgements The authors are also members of the University of Perugia and of the GNAMPA—INDAM (Italy). This research has been accomplished within the UMI Group TAA "Approximation Theory and Applications". The authors gratefully thank to the Referees for the constructive comments and recommendations which definitely help to improve the readability and quality of the paper.

Author Contributions All authors have contributed equally to this work for writing, review and editing. All authors have read and agreed to the published version of the manuscript.

Funding Open access funding provided by Università degli Studi di Perugia within the CRUI-CARE Agreement. This research was supported by Ricerca di Base (2018) dell'Università degli Studi di Perugia "Metodi di Teoria dell'Approssimazione, Analisi Reale, Analisi Nonlineare e loro Applicazioni"; Ricerca di Base (2019) dell'Università degli Studi di Perugia-"Integrazione, Approssimazione, Analisi Nonlineare e loro Applicazioni" and "Metodi di approssimazione, misure, Analisi Funzionale, Statistica e applicazioni alla ricostruzione di immagini e documenti"; "Metodi e processi innovativi per lo sviluppo di una banca di immagini mediche per fini diagnostici" funded by the Fondazione Cassa di Risparmio di Perugia (FCRP), (2018); "Metodiche di Imaging non invasivo mediante angiografia OCT sequenziale per lo studio delle Retinopatie degenerative dell'Anziano (M.I.R.A.)", funded by FCRP, (2019).

Avaibility of matherials and data The authors confirm that the data supporting the results of this study are available within the article [and/or] its supplementary material in the bibliography.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Angeloni, L., Costarelli, D., Seracini, M., Vinti, G., Zampogni, L.: Variation diminishing-type properties for multivariate sampling Kantorovich operators. Boll. Unione Mat. Ital. **13**(4), 595–605 (2020)
2. Angeloni, L., Vinti, G.: Rate of approximation for nonlinear integral operators with application to signal processing. Differ. Integral Equ. **18**(8), 855–890 (2005)
3. Bardaro, C., Boccuto, A., Dimitriou, X., Mantellini, I.: Abstract Korovkin-type theorems in modular spaces and applications. Cent. Eur. J. Math. **11**(10), 1774–1784 (2013)

4. Bardaro, C., Butzer, P.L., Mantellini, I.: The exponential sampling theorem of signal analysis and the reproducing kernel formula in the Mellin transform setting. Sampl. Theory Signal Image Process. **13**(1), 35–66 (2014)
5. Bardaro, C., Butzer, P.L., Mantellini, I.: The Mellin-Parseval formula and its interconnections with the exponential sampling theorem of optical physics. Integral Transf. Spec. Funct. **27**(1), 17–29 (2016)
6. Bardaro, C., Butzer, P.L., Mantellini, I., Schmeisser, G.: Mellin analysis and its basic associated metric—applications to sampling theory. Anal. Math. **42**(4), 297–321 (2016)
7. Bardaro, C., Butzer, P. L., Stens, R. L., Vinti, G.: Convergence in variation and rates of approximation for Bernstein-type polynomials and singular convolution integrals. Analysis, (Munich) **23**(4), 299–340 (2003)
8. Bardaro, C., Butzer, P.L., Stens, R.L., Vinti, G.: Kantorovich-type generalized sampling series in the setting of Orlicz spaces. Sampl. Theory Signal Image Process. **6**(1), 29–52 (2007)
9. Bardaro, C., Butzer, P.L., Stens, R.L., Vinti, G.: Prediction by samples from the past with error estimates covering discontinuous signals. IEEE Trans. Inf. Theory **56**(1), 614–633 (2010)
10. Bardaro, C., Mantellini, I.: Pointwise convergence theorems for nonlinear Mellin convolution operations. Int. J. Pure Appl. Math. **27**(4), 431–447 (2006)
11. Bardaro, C., Mantellini, I.: Uniform modular integrability and convergence properties for a class of Urysohn integral operators in function spaces. Math. Slovaca **56**(4), 465–482 (2006)
12. Bardaro, C., Musielak, J., Vinti, G.: Nonlinear Integral Operators and Applications. de Gruyter, Berlin (2003)
13. Boccuto, A.: Integration in Riesz Spaces with respect to (D)-convergence. Tatra Mt. Math. Publ. **5**, 107–124 (1995)
14. Boccuto, A., Candeloro, D.: Integral and Ideals in Riesz Spaces. Inf. Sci. **179**, 2891–2902 (2009)
15. Boccuto, A., Candeloro, D.: Defining limits by means of integrals. Integral Equ. Oper. Theory Ser. Oper. Theory Adv. Appl. **201**, 79–87 (2009)
16. Boccuto, A., Candeloro, D., Sambucini, A.R.: Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes. J. Math. Anal. Appl. **419**, 818–838 (2014). https://doi.org/10.1016/j.jmaa.2014.05.014
17. Boccuto, A., Candeloro, D., Sambucini, A.R.: L_p-spaces in vector lattices and applications. Math. Slovaca **67**(6), 1409–1426 (2017)
18. Boccuto, A., Dimitriou, X.: Modular convergence theorems for integral operators in the context of filter exhaustiveness and applications. Mediterranean J. Math. **10**(2), 823–842 (2013)
19. Boccuto, A., Sambucini, A. R.: Abstract integration with respect to measures and applications to modular convergence in vector lattice setting. arXiv:2112.12085 (2021)
20. Butzer, P.L.: A survey of work on approximation at Aachen, 1968–1972, Approximation theory (Proc. Internat. Sympos., Univ. Texas, Austin, Tex.,: 31–100, p. 1973. Academic Press, New York (1973)
21. Butzer, P.L., Engels, W., Ries, S., Stens, R.L.: The Shannon sampling series and the reconstruction of signals in terms of linear, quadratic and cubic splines. SIAM J. Appl. Math. **46**(2), 299–323 (1986)
22. Butzer, P.L., Ferreira, P.J.S.G., Schmeisser, G., Stens, R.L.: The summation formulae of Euler-Maclaurin, Abel-Plana, Poisson, and their interconnections with the approximate sampling formula of signal analysis. Results Math. **59**(3–4), 359–400 (2011)
23. Butzer, P.L., Fisher, A., Stens, R.L., Generalized sampling approximation of multivariate signals; general theory, Atti del Quarto Convegno di Analisi Reale e Teoria della Misura, Capri,: Atti Sem. Mat. Fis. Univ. Modena **40**(1992), 1–21 (1990)
24. Butzer, P.L., Hinsen, G.: Reconstruction of bounded signals from pseudo-periodic, irregularly spaced samples. Signal Process. **17**(1), 1–17 (1989)
25. Butzer, P.L., Lei, L.: Approximation of signals using measured sampled values and error analysis. Commun. Appl. Anal. **4**(2), 245–255 (2000)
26. Butzer, P.L., Schmeisser, G., Stens, R.L.: Sobolev spaces of fractional order, Lipschitz spaces, readapted modulation spaces and their interrelations; applications. J. Approx. Theory **212**, 1–40 (2016)
27. Butzer, P.L., Stark, E.L.: On a trigonometric convolution operator with kernel having two zeros of simple multiplicity. Acta Math. Acad. Sci. Hungar. **20**, 451–461 (1969)
28. Butzer, P.L., Stens, R.L.: Reconstruction of signals in $L^p(\mathbb{R})$-space by generalized sampling series based on linear combinations of B-splines. Integral Transf. Spec. Funct. **19**(1–2), 35–58 (2008)
29. Cantarini, M., Coroianu, L., Costarelli, D., Gal, S. G., Vinti, G.: Inverse result of approximation for the max-product neural network operators of the kantorovich type and their saturation order. Mathematics, **10**, 63 (2022). https://doi.org/10.3390/math10010063

30. Coroianu, L., Costarelli, D., Gal, S.G., Vinti, G.: Approximation by max-product sampling Kantorovich operators with generalized kernels. Anal. Appl. **19**(2), 219–244 (2021)
31. Costarelli, D., Sambucini, A. R.: Approximation results in Orlicz spaces for sequences of Kantorovich max-product neural network operators. Results Math. **73**(1) (2018). https://doi.org/10.1007/s00025-018-0799-4
32. Costarelli, D., Sambucini, A.R., Vinti, G.: Convergence in Orlicz spaces by means of the multivariate max-product neural network operators of the Kantorovich type. Neural Comput. Appl. **31**(9), 5069–5078 (2019). https://doi.org/10.1007/s00521-018-03998-6
33. Kozlowski, W.M.: Modular Function Spaces. Pure Applied Mathematics, Marcel Dekker, New York (1988)
34. Luxemburg, W.A.J., Zaanen, A.C.: Riesz Spaces I. North-Holland Publ. Co., Amsterdam (1971)
35. Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin (1991)
36. Schaefer, H.H.: Banach Lattices and Positive Operators. Springer, Berlin (1974)
37. Vulikh, B.Z.: Introduction to the Theory of Partially Ordered Spaces. Wolters-Noordhoff Sci. Publ, Groningen (1967)