Sustainability over sets and the business cycle

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Abstract

Many rigorous works have examined the behavior of Kaldor business cycle models. In this paper, utilizing the sustainability over sets (SOS) concept, a general theorem is proven establishing necessary and sufficient conditions for the existence and identification of rectangular sustainable sets for a general Kaldor model. These conditions can be readily verified, either in the two-dimensional space of lower and upper bounds on income, or in the one-dimensional space of upper bound on income, thus enabling identification of the region of economic sustainability for the considered Kaldor model. The Theorem's power is illustrated with a case study of a particular Kaldor business cycle model from the literature that can exhibit rich dynamic behavior, and can give rise to economic sustainability regions that encompass multiple stable equilibrium points. Furthermore, a methodology is also presented that identifies non-rectangular sustainable sets, when the imposition of tight lower and upper bounds on one of the economic system variables (e.g., income) prevents the existence of a sustainable rectangular set.

Keywords Sustainability over sets · Business cycle · Positively invariant set

Introduction

Numerous studies over the years have established that the economy is a complex dynamical system, and have focused on the development of mathematical tools that can quantify the dynamic behavior of economic systems. Kaldor in 1940 presented a business cycle model in the Keynesian vein, which accounts for the non-linear relations among the underlying economic variables, and has been analyzed extensively in the literature. Chang and Smyth (1971) set up a simple dynamic formulation of

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the Kaldor model, and demonstrated persistent periodic behavior depending crucially on parameter values, speed of adjustment, initial disturbances, and the relative values of the saving and investment functions. Varian (1979) utilized Catastrophe Theory to analyze both recessions and depressions. Lorenz (1987) identified that a multi-sector Kaldor-type model could possess a strange attractor leading to irregular or chaotic behavior. Both Varian (1979) and Murakami (2014, 2018) applied the Poincare–Hopf and Poincare–Bendixson Theorems to establish the existence of unique equilibrium points and stable limit cycles in Kaldorian business cycle models. Application of these Theorems required them to employ assumptions that helped establish the existence of a positively invariant set in the system’s state-space. A basic assumption for these two Theorems is that there exists a compact region diffeomorphic to a disk in the system’s state-space, such that the system’s vector field points inward on the region’s boundary. This region is thus a positively invariant set. The Varian (1979) work directly assumed that such a region exists, while Murakami (2014, 2018) made other assumptions that were then used to establish the existence of such a region. Once the existence of such a region was at hand, then the two aforementioned Theorems were applied to Kaldorian business cycle models to establish the existence of a unique equilibrium point, and at least one stable limit cycle. Although positively invariant sets are discussed in these references, it is important to emphasize that no necessary and sufficient conditions are presented for a considered region to be a positively invariant set. Discrete time approaches were employed in (Bischi et al. 2001; Dieci et al. 2001; Binter and Vacha 2005; Dobrescu and Opris 2009) and time delay models in (Krawiec and Szydowski 1999; Szydowski and Krawiec 2005; Wu and Wang 2010; Kaddar and Alaoui 2008). Long-term economic behavior that does not approach equilibrium was also demonstrated in (Foster 2006; Farmer 2012). Bashkirtseva et al. (2016) considered a Kaldor-type model with external additive and internal parametric disturbances, while Li et al. (2017), and Wang et al. (2020) examined the impact of financial and investment shocks on Kaldor models. Nevertheless, all these works did not establish necessary and sufficient conditions for a considered region to be a positively invariant set, thus ensuring that economic behavior remains within desirable ranges.

The aim of this work is to propose an approach identifying readily verifiable necessary and sufficient conditions for a considered region to be a positively invariant set, thus ensuring that economic system variables remain within desirable ranges quantified by a predefined set in the system’s state-space. If this predefined, state-space set is deemed to correspond to reality, then the terminology of economic sustainability may be employed. Sustainability was first formally proposed in the report of the World Commission on Environment and Development: our common future. The report suggested that economic and ecological systems are interlocked, and humans must find ways to meet the needs of the present without compromising the ability of future generations to meet their own needs. During the 3 decades that followed, sustainability has been continuously redefined and applied to various areas, such as educational sustainability (Hargreaves 2002), groundwater sustainability (Gleeson et al. 2010), social sustainability (Dempsey et al. 2011), culture–sustainability interactions (Soini and Dessein 2016), and sustainability in the context of process systems engineering (Bakshi and Fiksel 2003).

At the same time, several sustainability assessment tools have been developed. Utilizing Information Theory, a sustainability hypothesis has been put forward that ‘sustainable systems do not lose or gain Fisher information over time (Cabezas and Fath 2002); An ecological system index was then developed, that could be useful in detecting system ‘flips’ associated with a regime change (Fath et al. 2003). Utilizing Fuzzy Logic reasoning, a sensitivity analysis was carried out to identify the most important factors contributing to sustainable development (Andriantatsaholainaina et al. 2004), while a hybrid approach combining Fuzzy Logic, Interval Analysis, and Global Optimization was proposed to exactly quantify a sustainability interval index for general systems (Conner et al. 2012).

In the economic area, when discussing sustainability, the dominant view is the maintenance of a permanent income or an output growth rate, through utilization of non-declining natural and artificial capital stock (Pollitt et al. 2010). Two opposing camps, disagreeing on whether natural capital can be substituted for artificial capital, have developed and are characterized as schools of strong and weak sustainability. However, it is impossible to build a model including natural resources as factors of production, thus making the determination of falsifiability of the two concepts impossible under scientific standards (Neumayer 2003). Even if the limitations imposed by the availability of natural resources are not considered, the maintenance of an output growth rate remains a questionable proposition, in the presence of economic cycles. To address this issue, this paper redefines economic system sustainability, by requiring that all economic system variables remain always within acceptable ranges.

Section 2 briefly outlines the employed conceptual framework of Sustainability Over Sets (SOS); Sect. 3 explains the application of the SOS method to a general Kaldor cycle model, and develops a general theorem that establishes readily verifiable SOS criteria for general Kaldor Cycle models; Sect. 4 details the application of the SOS method to two case studies, involving different instances of the aforementioned general Kaldor cycle; Sect. 5 offers concluding thoughts.

The sustainability over sets (SOS) method

Sustainability over sets (SOS) has been put forward as a rigorous mathematical approach to quantifying sustainability (Manousiouthakis and Jorat 2018). One of the many advantages of the SOS method is the rigorous incorporation of human input in defining a sustainable system, thus enabling the quantification of the sustainability concept. In simple terms, a system is sustainable over a set in its state-space if and only if the considered set is positively invariant for the system, i.e., the system’s trajectories initiated within the set remain within the set for all time. The formal mathematical definition of the SOS concepts for time-invariant systems, whose dynamic behavior is captured by ordinary differential equations (ODEs) is as follows:

$$\dot{x}(t) = f(x(t)) \quad x(0) = x^0, \quad (1)$$

where $x : [0, \infty) \rightarrow \mathbb{R}^n$, $x : t \rightarrow x(t) = [x_1(t) \dots x_n(t)]^T \in \mathbb{R}^n$ is differentiable, $x^0 = [x_1^0 \dots x_n^0]^T \in \mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f : (x(t)) \rightarrow f(x(t)) = [f_1(x_1(t), \dots, x_n(t)) \dots f_n(x_1(t), \dots, x_n(t))]^T$ is infinitely differentiable and locally Lipschitz continuous, and (1) admits the unique solution $x^1(t, x^0) \forall t \in [0, \infty)$ with no finite time escape behavior.

Define:

$$F \triangleq \{x \in \mathbb{R}^n : \Phi_j(x) \leq 0 \forall j = 1, \dots, k\}, \quad S(x) \triangleq \{j = 1, \dots, k : \Phi_j(x) = 0\} \quad \forall x \in F, \quad (2)$$

where $\Phi_j : \mathbb{R}^n \rightarrow \mathbb{R}$; $\Phi_j : x \rightarrow \Phi_j(x)$; $\forall j = 1, \dots, k$ are continuously differentiable. F is a positively invariant set for system (1) if and only if:

$$x^1(t, x^0) \in F \quad \forall t \in [0, \infty) \quad \forall x^0 \in F. \quad (3)$$

Mathematical methods for the identification of invariant sets are discussed in Junge and Kevrekidis (2017). Then, a mathematical sustainability criterion, ensuring that F is a positively invariant set for (1), is: (1) is sustainable over set (SOS) $F \subset \mathbb{R}^n$ iff:

$$\forall x \in F : S(x) \neq \emptyset, \quad [\nabla \Phi_j(x) \cdot f(x)] \leq 0 \quad \forall j \in S(x). \quad (4)$$

Let F^R be a rectangular set defined as:

$$F^R \triangleq \{z = [z_1 \dots z_n] \in \mathbb{R}^n : z_i^l \leq z_i \leq z_i^u, \forall i = 1, \dots, n\}. \quad (5)$$

(1) is SOS F^R iff:

$$\left\{ \left\{ f_j(x^{j,u}) \leq 0 \quad \forall x^{j,u} = \begin{bmatrix} x_1^{j,u} \\ \vdots \\ x_n^{j,u} \end{bmatrix} : \begin{bmatrix} x_j^{j,u} = z_j^u \\ z_i^l \leq x_i^{j,u} \leq z_i^u \quad \forall i = 1, \dots, n; i \neq j \end{bmatrix} \quad \forall j = 1, \dots, n \right\} \right\} \\ \left\{ \left\{ f_j(x^{j,l}) \geq 0 \quad \forall x^{j,l} = \begin{bmatrix} x_1^{j,l} \\ \vdots \\ x_n^{j,l} \end{bmatrix} : \begin{bmatrix} x_j^{j,l} = z_j^l \\ z_i^l \leq x_i^{j,l} \leq z_i^u \quad \forall i = 1, \dots, n; i \neq j \end{bmatrix} \quad \forall j = 1, \dots, n \right\} \right\}. \quad (6)$$

Having outlined the sustainability over sets (SOS) concept and its mathematical quantification, SOS analysis is next carried out for a classical economic system, the Kaldor business cycle.

Sustainability over set (SOS) analysis of an economic system

The Kaldor Model is one of the most interesting theories of business cycles in the Keynesian vein, accounting for non-linear relations among the underlying economic variables used to capture business cycle behavior (Kaldor 1940). At the core of the

Kaldor cycle model is an attempt to quantify the interactions of the important economic forces of investment (I) and savings (S). At a first basic level of analysis, I and S are both considered to be functions of only the level of income (Y). Kaldor divided national income into three stages of high, medium and low, whereby the marginal propensity to investment and savings is different in each stage. As shown in Fig. 1, the resulting investment I and savings S functions of Y are highly non-linear, potentially resulting in three equilibrium points A, B, C.

The shape of the Investment curve I indicates that the marginal propensity to investment in the high- and low-income stages is less than that in the medium-income stage. The reason is that production capacity at low- and high- income levels is underutilized, and when aggregate demand increases, the producers tend to utilize the existing production capacity, without increasing investment. Upon exhaustion of the existing production capacity, further aggregate demand increases can be met by output increases attainable through additional investment.

Similarly, the shape of the Savings curve S indicates that the marginal propensity for savings in the low- and high-income stages is greater than that in the medium-income stage. The reason is that, at low-income levels, once a certain standard of living is attained, further income increases lead to savings increases. Once the income reaches a medium level, an elevation in the standard of living is sought with additional income, at the expense of additional savings. Finally, at the high-income level, further income increases lead again to savings increases, as no further elevation of the standard of living is sought.

As Kaldor pointed out, and can be seen in Fig. 1, the investment (I) and saving (S) functions are such that $S_Y\left(\frac{dS}{dY}\right) > I_Y\left(\frac{dI}{dY}\right)$ at the two equilibrium points A and B, while $S_Y\left(\frac{dS}{dY}\right) < I_Y\left(\frac{dI}{dY}\right)$, at the equilibrium point C. This behavior makes the equi-

Fig.1 Investment (I) and savings (S) as functions of income (Y)

librium points A and B stable, and the equilibrium point C unstable. Indeed, near A , and B , income Y increases (decreases) lead to modest increases (decreases) in investment I and substantial increases (decreases) in savings S , resulting in turn in income decreases (increases). On the other hand, near C , these effects are all reversed leading to instability, i.e., income increases (decreases) leading to further income increases (decreases).

At a more sophisticated level of analysis in Keynesian theory, investment I and savings S are not only influenced by income Y , but also by capital K . Income is a measurement of an entity's (e.g., a country's) yearly income (including wages, interest, rent, and profits), while capital refers to the total amount of capital in the entity at any given time. As already discussed, income Y increases lead to increases in investments I and savings S . The influence of capital K , however, is more circuitous. As K increases, its marginal productivity gradually declines, and the return on investment becomes smaller, so the relationship between investment I and capital K is negative. The impact of capital K on savings S , however, can be more complicated. Kaldor (1940) suggested that the partial derivative of savings to capital stock $\partial S/\partial K$ is positive, arguing that as K rises, additional capacity is used to produce goods, so the price of goods falls, resulting in reduced consumption expenditures, and thus increased savings S . Chang and Smyth (1971) on the other hand suggested that $\partial S/\partial K$ is negative due to a positive wealth effect on consumption.

The above suggest that although the investment and savings functions $I : (Y, K) \rightarrow I(Y, K)$ and $S : (Y, K) \rightarrow S(Y, K)$ may take a variety of forms, the following relations typically hold:

$$\left\{ \frac{\partial I(Y, K)}{\partial Y} > 0, \frac{\partial I(Y, K)}{\partial K} < 0, \frac{\partial S(Y, K)}{\partial K} > 0 \text{ or } \frac{\partial S(Y, K)}{\partial K} < 0, \frac{\partial S(Y, K)}{\partial Y} > 0 \right\} \begin{cases} \forall Y \in (0, \infty) \\ \forall K \in (0, \infty) \end{cases} \quad (7)$$

Kaldor's (1940) business cycle model has been analyzed extensively in the literature. Several investigators established equilibrium point and limit cycle behavior, using restrictive assumptions that enabled application of the Poincaré–Hopf and Poincaré–Bendixson theorems to single sector, Chang and Smyth (1971), Varian (1979), and Murakami (2014), and two-sector, Murakami (2018), Kaldorian business cycle models. These assumptions are only sufficient conditions for establishing the positive invariance of considered sets, and are difficult to verify, as they require repeated global solution of nonconvex optimization problems.

In this work, a general, ODE model of the Kaldor business cycle is considered. It is first placed in context to some of the aforementioned business cycle models, and subsequently, its behavior is analyzed using the aforementioned SOS method. The considered business cycle model is:

$$\begin{cases} \dot{Y} = \alpha \cdot [I(Y, K) - S(Y, K)] \\ \dot{K} = I(Y, K) - \delta \cdot K \end{cases} \quad (8)$$

where $Y\left(\frac{\text{\#currency}}{\text{year}}\right)$ and $K(\text{\#currency})$ denote income and capital, respectively. In this general model, $I : (Y, K) \rightarrow I(Y, K)\left(\frac{\text{\#currency}}{\text{year}}\right)$ and $S : (Y, K) \rightarrow S(Y, K)\left(\frac{\text{\#currency}}{\text{year}}\right)$

describe investments and savings respectively, and are considered to be functions of both income and capital. $\delta \left(\frac{1}{\text{year}} \right)$ is the depreciation rate of capital, satisfying $\delta \in [0, 1]$, and $\alpha (\alpha > 0)$ is the adjustment coefficient denoting the reaction of firm to demand excess, satisfying $\alpha \in (0, \infty)$.

The Varian (1979) business cycle model is identical to the model in Eq. (8), except that S is only a function of Y with form $S(Y) = Y - C(Y)$, where $C : Y \rightarrow C(Y) \left(\frac{\text{\#currency}}{\text{year}} \right)$ describes consumption, and capital depreciation is constant rather than being proportional to the capital itself (i.e., equal to $\delta \cdot K$). Given that in the Varian model, the positive component of the savings function S is equal to Y [rather than being proportional to Y , as $s \cdot Y$ in Eq. (9)], and capital depreciation is a constant, the Varian model's equilibrium point uniqueness results are not applicable to this work.

In this work, theoretical results are first obtained for the following simplified form of the investment and savings functions I and S :

$$\begin{cases} I(Y, K) = f(Y) + g(K) \\ S(Y) = s \cdot Y \end{cases}, \quad (9)$$

where

$$\left\{ \frac{df(Y)}{dY} \triangleq f'(Y) > 0 \forall Y \in (0, \infty), \quad \frac{dg(K)}{dK} \triangleq g'(K) < 0 \forall K \in (0, \infty), \quad s > 0 \right\}. \quad (10)$$

Then, the general business cycle model considered in this work is:

$$\begin{cases} \dot{Y} = \alpha \cdot [f(Y) + g(K) - s \cdot Y] \\ \dot{K} = f(Y) + g(K) - \delta \cdot K \end{cases}. \quad (11)$$

First, an equilibrium analysis is carried out for the above general business cycle model described by (11). Let $(Y_s, K_s) \in \mathbb{R}^2$ be an equilibrium point of (11). It then holds:

$$\{f(Y_s) + g(K_s) = s \cdot Y_s = \delta \cdot K_s\} \Leftrightarrow \begin{cases} K_s = \frac{s}{\delta} \cdot Y_s \\ f(Y_s) + g\left(\frac{s}{\delta} \cdot Y_s\right) - s \cdot Y_s = 0 \end{cases}. \quad (12)$$

Based on (10), and $\delta \in [0, 1]$, it holds:

$$\left\{ \frac{df(Y)}{dY} > 0 \forall Y \in (0, \infty), \quad \frac{d\left[g\left(\frac{s}{\delta} \cdot Y\right) - s \cdot Y\right]}{dY} = \frac{dg\left(\frac{s}{\delta} \cdot Y\right)}{d\left(\frac{s}{\delta} \cdot Y\right)} \frac{s}{\delta} - s < 0 \quad \forall Y \in (0, \infty) \right\}. \quad (13)$$

Equation (13) above suggests that the left-hand side of $f(Y_s) + g\left(\frac{s}{\delta} \cdot Y_s\right) - s \cdot Y_s = 0$ is the sum of a monotonically increasing and a monotonically decreasing function. Therefore, the above equation [and in turn the set of Eq. (12)] can have no solutions, one solution, or multiple solutions, depending on how many times the function values $f(Y_s)$, $s \cdot Y_s - g\left(\frac{s}{\delta} \cdot Y_s\right)$ are equal to each other over the domain $Y_s \in (0, \infty)$. The above presented concepts are illustrated in the case study presented later in this work.

Next, the stability of an equilibrium point $(Y_s, K_s) \in (0, \infty) \times (0, \infty)$ of (11) is analyzed. The Jacobian of the linearized system around the above equilibrium point is:

$$J \triangleq \begin{bmatrix} \frac{\partial[\alpha \cdot [f(Y) + g(K) - s \cdot Y]]}{\partial Y} \Big|_{(Y_s, K_s)} & \frac{\partial[\alpha \cdot [f(Y) + g(K) - s \cdot Y]]}{\partial K} \Big|_{(Y_s, K_s)} \\ \frac{\partial[f(Y) + g(K) - \delta \cdot K]}{\partial Y} \Big|_{(Y_s, K_s)} & \frac{\partial[f(Y) + g(K) - \delta \cdot K]}{\partial K} \Big|_{(Y_s, K_s)} \end{bmatrix}.$$

Given that $\alpha > 0$, the eigenvalues of J are both strictly in the left half plane if and only if:

$$\left\{ \begin{array}{l} \text{tr} J \triangleq \alpha[f'(Y_s) - s] + g'(K_s) - \delta < 0 \\ \det J \triangleq \alpha[s\delta - sg'(K_s) - \delta f'(Y_s)] > 0 \end{array} \right\}, \quad J \triangleq \begin{bmatrix} \alpha[f'(Y_s) - s] & \alpha g'(K_s) \\ f'(Y_s) & g'(K_s) - \delta \end{bmatrix}. \quad (14)$$

The Murakami (2014) business cycle model is even more general than the model considered in Eq. (8), and the further specialized model presented in Eq. (11). Yet, the assumptions employed in arriving at its theoretical results on the existence of a unique equilibrium point within a rectangular subset of the positive orthant in the income-capital (Y, K) space are restrictive and prevent its applicability to (8) and (11). Indeed, condition (2.7) in assumption (2.1) in Murakami (2014) applied to Eq. (11) is effectively equivalent to the requirement that the determinant of the Jacobian $J(Y, K)$ of the linearized system around any arbitrary point (Y, K) in the considered rectangle must satisfy $\det J(Y, K) > 0$. As can be seen in the case study presented below, this condition is violated at one of the equilibrium points identified in Case 2.

Having provided criteria for equilibrium point stability, next, the sustainability of the above general business cycle model (11) is assessed. To this end, consider the following: given that (10) holds, the function f is monotonically increasing over $(0, \infty)$ and the function g is monotonically decreasing over $(0, \infty)$. Thus, the inverse

function f^{-1} exists and is monotonically increasing over $(f(0), f(\infty))$, the range of f , and g^{-1} exists and is monotonically decreasing over $(g(\infty), g(0))$, the range of g .

Let the following functions then be defined:

$$h : K \rightarrow h(K) \triangleq \delta \cdot K - g(K), \quad j : Y \rightarrow j(Y) \triangleq s \cdot Y - f(Y). \quad (15)$$

The function $h : K \rightarrow h(K) \triangleq \delta \cdot K - g(K)$ defined in (15) is also monotonically increasing over $(0, \infty)$, and thus admits a monotonically increasing inverse function h^{-1} over $(h(0), h(\infty))$, the range of h . On the other hand, the function $j : Y \rightarrow j(Y) \triangleq s \cdot Y - f(Y)$ defined in (15) is not necessarily monotonic over $(0, \infty)$, and thus, an inverse function j^{-1} may not exist over the range of j . Consider then a set of N intervals (Y_l^k, Y_u^k) $k = 1, \dots, N$, so that $\bigcup_{k=1}^N (Y_l^k, Y_u^k) = (0, \infty)$, $Y_l^{k+1} = Y_u^k \forall k = 1, \dots, N-1$, $Y_l^1 = 0$, $Y_u^N = \infty$, and the function j is either monotonically increasing or monotonically decreasing over each (Y_l^k, Y_u^k) $k = 1, \dots, N$. In turn, also define:

$$\left\{ \begin{array}{l} j^k : (Y_l^k, Y_u^k) \rightarrow \mathbb{R} \\ j^k : Y \rightarrow j^k(Y) \triangleq j(Y) \end{array} \right\} \forall k = 1, \dots, N \left\{ \begin{array}{l} J^+ \triangleq \left\{ k : k = 1, \dots, n \wedge \frac{dj(Y)}{dY} > 0 \forall Y \in (Y_l^k, Y_u^k) \right\} \\ J^- \triangleq \left\{ k : k = 1, \dots, n \wedge \frac{dj(Y)}{dY} < 0 \forall Y \in (Y_l^k, Y_u^k) \right\} \end{array} \right\}. \quad (16)$$

Then, the following theorem holds:

Theorem Let $0 < Y_l \leq Y_u$, $0 < K_l \leq K_u$.

$$a. \quad (11) \text{ is SOS } S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\} \Leftrightarrow \left\{ \begin{array}{l} \min[g(0), j(Y_u)] \geq g(h^{-1}(f(Y_l))) \geq g(h^{-1}(f(Y_u))) \geq \max[g(\infty), j(Y_l)] \\ \max[f(0), h(0)] \leq f(Y_l) \leq f(Y_u) \leq \min[f(\infty), h(\infty)] \end{array} \right\}. \quad (17)$$

Proof See Appendix A.

$$b. \quad (11) \text{ is SOS } S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\} \Leftrightarrow$$

$$\left. \left\{ \begin{array}{l} g^{-1}(\min[g(0), j(Y_u)]) \leq K_l \leq h^{-1}(f(Y_l)) \\ h^{-1}(f(Y_u)) \leq K_u \leq g^{-1}(\max[g(\infty), j(Y_l)]) \\ \max[Y_l^k, f^{-1}(\max[f(0), h(g^{-1}(\min[g(0), j(Y_u)]))])] \leq Y_l \leq \\ \leq \min[Y_u, Y_u^k, (j^k)^{-1}(\min[g(0), g(h^{-1}(f(Y_u))), j(Y_u^k)])] \\ j(Y_l^k) \leq \min(g(0), g(h^{-1}(f(Y_u)))) \\ h(g^{-1}(\min[g(0), j(Y_u)])) \leq f(\infty) \\ g(\infty) \leq j(Y_u), \quad h(\infty) \geq f(0), \quad h(0) \leq f(\infty) \\ Y_u \leq f^{-1}(\min[f(\infty), h(\infty)]) \end{array} \right\} \right\} k \in J^+ \\
 \left. \left\{ \begin{array}{l} g^{-1}(\min[g(0), j(Y_u)]) \leq K_l \leq h^{-1}(f(Y_l)) \\ h^{-1}(f(Y_u)) \leq K_u \leq g^{-1}(\max[g(\infty), j(Y_l)]) \\ \max \left[\begin{array}{l} Y_l^k, (j^k)^{-1}(\min[g(0), g(h^{-1}(f(Y_u))), j(Y_l^k)]) \\ f^{-1}(\max[f(0), h(g^{-1}(\min[g(0), j(Y_u)]))]) \end{array} \right] \leq \\ \leq Y_l \leq \min \left[\begin{array}{l} Y_u, \\ Y_u^k \end{array} \right] \\ j(Y_u^k) \leq \min(g(0), g(h^{-1}(f(Y_u)))) \\ h(g^{-1}(\min[g(0), j(Y_u)])) \leq f(\infty) \\ g(\infty) \leq j(Y_u), \quad h(\infty) \geq f(0), \quad h(0) \leq f(\infty) \\ Y_u \leq f^{-1}(\min[f(\infty), h(\infty)]) \end{array} \right\} \right\} k \in J^- \\
 \end{array} \right\} \quad (18)$$

Proof See Appendix B.

Having established, in the Theorem's part b, necessary and sufficient conditions for (11) to be sustainable over the set $S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\}$, it can then be readily verified that there always exist, such that $(K_l, K_u) : 0 < K_l \leq K_u$ satisfying (18) if and only if:

$$\left\{ \left\{ \begin{array}{l} \max \left[Y_l^k, f^{-1} \left(\max \left[f(0), h \left(g^{-1} \left(\min \left[g(0), j(Y_u) \right] \right) \right] \right) \right] \leq \right. \\ \leq \min \left[Y_u, Y_u^k, (j^k)^{-1} \left(\min \left[g(0), g \left(h^{-1} \left(f(Y_u) \right) \right), j(Y_u^k) \right] \right) \right] \\ j(Y_l^k) \leq \min \left(g(0), g \left(h^{-1} \left(f(Y_u) \right) \right) \right) \\ h \left(g^{-1} \left(\min \left[g(0), j(Y_u) \right] \right) \right) \leq f(\infty) \\ g(\infty) \leq j(Y_u), \quad h(\infty) \geq f(0), \quad h(0) \leq f(\infty) \\ Y_u \leq f^{-1} \left(\min \left[f(\infty), h(\infty) \right] \right) \end{array} \right\} \quad k \in J^+ \right\} \\
 \left\{ \left\{ \begin{array}{l} \max \left[Y_l^k, (j^k)^{-1} \left(\min \left[g(0), g \left(h^{-1} \left(f(Y_u) \right) \right), j(Y_l^k) \right] \right), \right. \\ \left. f^{-1} \left(\max \left[f(0), h \left(g^{-1} \left(\min \left[g(0), j(Y_u) \right] \right) \right] \right) \right] \right] \leq \min \left[Y_u, Y_u^k \right] \\ j(Y_u^k) \leq \min \left(g(0), g \left(h^{-1} \left(f(Y_u) \right) \right) \right) \\ h \left(g^{-1} \left(\min \left[g(0), j(Y_u) \right] \right) \right) \leq f(\infty) \\ g(\infty) \leq j(Y_u), \quad h(\infty) \geq f(0), \quad h(0) \leq f(\infty) \\ Y_u \leq f^{-1} \left(\min \left[f(\infty), h(\infty) \right] \right) \end{array} \right\} \quad k \in J^- \right\} \quad (19)$$

The importance of the above Theorem is that it provides for the first time readily verifiable necessary and sufficient conditions for the existence of a rectangular sustainable set $S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\}$ for the general Kaldor model (11). The power of the Theorem lies in its ability to express these necessary and sufficient conditions, first in the two-dimensional space (Y_l, Y_u) , inequality (17), and then in the one-dimensional space (Y_u) , inequality (19). As a result, the Theorem enables for the first time the identification of the regions of sustainability for the general Kaldor model (11) in the two-dimensional space (Y_l, Y_u) , and in the one-dimensional space (Y_u) . The Theorem's power is illustrated next with a case study of a particular Kaldor business cycle model.

Case study on SOS analysis of a Kaldor business cycle

In this case study, the following specific Kaldor business cycle model by Lorenz (1987) is considered:

$$\left\{ \begin{array}{l} \dot{Y} = \alpha \cdot [I(Y, K) - S(Y, K)] = \alpha \cdot [f(Y) + g(K) - s \cdot Y] = \alpha \cdot \left[\begin{array}{l} C \cdot 2^{\frac{-1}{(D \cdot Y + e)^2}} + E \cdot Y + \\ + A \cdot \left(\frac{G}{K} \right)^H - s \cdot Y \end{array} \right] \\ \dot{K} = I(Y, K) - \delta \cdot K = f(Y) + g(K) - \delta \cdot K = C \cdot 2^{\frac{-1}{(D \cdot Y + e)^2}} + E \cdot Y + A \cdot \left(\frac{G}{K} \right)^H - \delta \cdot K \end{array} \right\}, \quad (20)$$

where the f, g, h ranges are $f : (0, \infty) \rightarrow (C \cdot 2^{\frac{-1}{\epsilon^2}}, \infty), g : (0, \infty) \rightarrow (0, \infty), h : (0, \infty) \rightarrow (-\infty, \infty)$. To explore the model's sensitivity to changes in parameter values, two cases are considered.

Case 1 $\{A = 5, C = 25, D = 0.015, E = 0.05, G = 320, H = 3, \alpha = 5, \delta = 0.05, \epsilon = 10^{-5}, s = 0.29\}$
Case 2 $\{A = 5, C = 35, D = 0.015, E = 0.05, G = 150, H = 3, \alpha = 5, \delta = 0.05, \epsilon = 10^{-5}, s = 0.29\}$.
 The parameter values used in Case 1 are consistent with those of Lorenz (1987), while only two parameter values are altered from Case 1 to Case 2, namely C is increased from $C = 25$ to $C = 35$, and G is decreased from $G = 320$ to $G = 150$. The impact of these changes is that the function g is increased by a factor of 9.7, while the non-linear component of the function f is increased by a factor of 1.4. Bischi et al. (2001) and Dieci et al. (2001) considered different values for other model parameters, namely α and δ values in the ranges $0 < \alpha < 6$, $0.1 < \delta < 0.2$, and $0 < \delta < 1$, respectively, compared to $\alpha = 5$, $\delta = 0.05$ used in this work. Binter and Vacha (2005) considered both different functional forms and parameter values for the f, g functions. Their form of the g function avoids the Lorenz (1987) model's unrealistically high values of $g(K)$ for small K , yields negative $g(K)$ values for large K , and has a constant component that changes by a factor of 8.3 among their considered cases. The above provide support as to the practical relevance of the two considered cases, since both employ parameter values consistent with those used in the literature. They also provide a glimpse into the sensitivity of the Lorenz (1987) model to changes in its parameter values. As shown below, there are significant changes in business cycle behavior from one case to the other, but, nevertheless, the proposed SOS method will be shown to remain applicable to both cases and to reflect these behavioral changes in the identified set over which the system is sustainable.

Equilibrium points

According to (12), an equilibrium point $(Y_s, K_s) \in \mathbb{R}^2$ of (11), for investment I and savings S functions defined by (20), must satisfy the equations:

$$\left\{ \begin{array}{l} K_s = \frac{s}{\delta} \cdot Y_s \\ C \cdot 2^{\frac{-1}{(D \cdot Y_s + \epsilon)^2}} + E \cdot Y_s + A \cdot \left(\frac{\delta \cdot G}{s \cdot Y_s} \right)^H - s \cdot Y_s = 0 \end{array} \right\}.$$

As can be seen in Fig. 2, the equilibrium points (Y_s, K_s) are:

Case 1 $(Y_s, K_s) = (60.8709, 353.0512)$.

Case 2 $(Y_s, K_s) = (24.7427, 143.5077)$ $(Y_s, K_s) = (55.1948, 320.1298)$ $(Y_s, K_s) = (116.4058, 675.1536)$.

Empirical evidence for the existence of either one or three equilibrium points, as shown in Cases 1 and 2, is also provided in the works of Varian (1979), Bischi et al. (2001), and Binter and Vacha (2005).

Fig. 2 The equilibrium points (Y_s, K_s) for Case1 and Case 2

Local stability analysis

Next, the stability of $(Y_s, K_s) = (60.8709, 353.0512)$ for Case 1, and the stability of $(Y_s, K_s) = (24.7427, 143.5077)$, $(Y_s, K_s) = (55.1948, 320.1298)$, $(Y_s, K_s) = (116.4058, 675.1536)$ for Case 2 are analyzed.

According to (14), the above system's Jacobian is:

$$J = \begin{bmatrix} \alpha[f'(Y_s) - s] & \alpha g'(K_s) \\ f'(Y_s) & g'(K_s) - \delta \end{bmatrix} = \begin{bmatrix} \alpha \left(\frac{-1}{2(DY_s + \epsilon)^2} \frac{2CD \ln 2}{(DY_s + \epsilon)^3} + E - s \right) & -\alpha HAG^H K_s^{(-H-1)} \\ \frac{-1}{2(DY_s + \epsilon)^2} \frac{2CD \ln 2}{(DY_s + \epsilon)^3} + E & -HAG^H K_s^{(-H-1)} - \delta \end{bmatrix},$$

and the system is stable if and only if: $\left\{ \begin{array}{l} \alpha[f'(Y_s) - s] + g'(K_s) - \delta < 0 \\ \alpha[s\delta - sg'(K_s) - \delta f'(Y_s)] > 0 \end{array} \right\} \Leftrightarrow$

$$\left\{ \begin{array}{l} \alpha \left(\frac{-1}{2(DY_s + \epsilon)^2} \frac{2CD \ln 2}{(DY_s + \epsilon)^3} + E - s \right) - HAG^H K_s^{(-H-1)} - \delta < 0 \\ \alpha \left[s\delta + sHAG^H K_s^{(-H-1)} - \delta \left(\frac{-1}{2(DY_s + \epsilon)^2} \frac{2CD \ln 2}{(DY_s + \epsilon)^3} + E \right) \right] > 0 \end{array} \right\}.$$

Substituting the values of the equilibrium points of Case 1 and Case 2, and all system parameters for each case then yield:

Case 1:

$$J = \begin{bmatrix} 0.2868 & -0.1582 \\ 0.3474 & -0.0816 \end{bmatrix}, \quad trJ = 0.2052, \quad detJ = 0.0316, \quad \lambda_1 = 0.1026 - 0.1452i,$$

$\lambda_2 = 0.1026 + 0.1452i$, which indicates that $(Y_s, K_s) = (60.8709, 353.0512)$ is unstable. The above is confirmed in Fig. 3, which also indicates that the system features a stable limit cycle that attracts trajectories initiated both inside and outside the region confined by the limit cycle.

Notes: EP is the equilibrium point EP (60.8709, 353.0512). Trajectories initiated at P1 (40, 310), and P2 (80, 400) are both attracted by the limit cycle.

Fig. 3 State-space of system (11–20) for Case 1

Case 2:

$J = \begin{bmatrix} -0.7354 & -0.5968 \\ 0.1429 & -0.1694 \end{bmatrix}$, $trJ = -0.9048$, $\det J = 0.2099$, $\lambda_1 = -0.4524 - 0.0723i$, $\lambda_2 = -0.4524 + 0.0723i$, which indicates that $(Y_s, K_s) = (24.7427, 143.5077)$ is stable.

$J = \begin{bmatrix} 1.1326 & -0.0241 \\ 0.5165 & -0.0548 \end{bmatrix}$, $trJ = 1.0778$, $\det J = -0.0496$, $\lambda_1 = 1.1221$, $\lambda_2 = -0.0443$, which indicates that $(Y_s, K_s) = (55.1948, 320.1298)$ is unstable.

$J = \begin{bmatrix} -0.6554 & -0.0012 \\ 0.1589 & -0.0502 \end{bmatrix}$, $trJ = -0.7056$, $\det J = 0.0331$, $\lambda_1 = -0.0505$, $\lambda_2 = -0.6551$, which indicates that $(Y_s, K_s) = (116.4058, 675.1536)$ is stable.

The above can be confirmed in Fig. 4, which indicates that for case 2, the considered equilibrium points $EP1$ (24.7427, 143.5077) and $EP3$ (116.4058, 675.1536) are stable, and attract trajectories whose initial conditions may be near them or even afar, in contrast with the equilibrium point $EP2$ (55.1948, 320.1298) which is unstable, and cannot attract trajectories with nearby initial conditions.

Notes $EP1$, $EP2$, and $EP3$ are the equilibrium points (24.7427, 143.5077), (55.1948, 320.1298), and (116.4058, 675.1536). $EP1$ attracts trajectories initiated at $P1$ (30, 200), $P2$ (40, 350), and $P5$ (50, 670). $EP3$ attracts trajectories initiated at $P3$ (70, 370), $P4$ (60, 630), and $P6$ (20, 100).

SOS analysis

To identify whether a rectangular sustainable set exists for the considered Lorenz system (11–20), the SOS criterion developed in the Theorem is applied, with the simplification that the conditions $h(g^{-1}(\min[g(0), j(Y_u)])) \leq f(\infty)$, $h(\infty) \geq f(0)$, $h(0) \leq f(\infty)$, $\max[f(0), h(0)] \leq f(Y_l) \leq f(Y_u) \leq \min[f(\infty), h(\infty)]$ and $Y_u \leq f^{-1}(\min[f(\infty), h(\infty)])$ are not further examined, as they are always true for both Case 1 and Case 2.

Fig. 4 State-space of system (11–20) for Case2

Case 1:

First, the monotonicity regions for the function $j : Y \rightarrow j(Y) \triangleq sY - f(Y) = (s - E)Y - C \cdot 2 \frac{-1}{(DY+\epsilon)^2}$ are determined through identification of all solutions of the equation:

$$\frac{dj(Y)}{dY} \triangleq \left(s - E - \frac{2 \frac{-1}{(DY+\epsilon)^2} 2CD \ln 2}{(DY+\epsilon)^3} \right) = 0.$$

For the above-provided parameter values of Case 1, two solutions are identified: $(Y, j(Y), \frac{dj(Y)}{dY}) = (32.3603, 6.4456, 0)$; $(Y, j(Y), \frac{dj(Y)}{dY}) = (69.9068, 3.4663, 0)$. The resulting set of N intervals (Y_l^k, Y_u^k) $k = 1, \dots, N$ is then $(Y_l^1, Y_u^1) = (0, 32.3603)$, $(Y_l^2, Y_u^2) = (32.3603, 69.9068)$, and $(Y_l^3, Y_u^3) = (69.9068, \infty)$, and the function j is monotonically increasing over (Y_l^1, Y_u^1) and (Y_l^3, Y_u^3) , and monotonically decreasing over (Y_l^2, Y_u^2) , i.e. $J^+ \triangleq \{1, 3\}$; $J^- \triangleq \{2\}$, and $N = 3$.

Next, verification of inequalities (17) in the theorem is carried out, by first plotting the graphs $(Y, j(Y))$ and $(Y, g(h^{-1}(f(Y))))$, as shown in Fig. 5.

The intersection of the graphs $(Y, j(Y))$ $(Y, g(h^{-1}(f(Y))))$ in Fig. 5 occurs at $Y_u = 60.8709$. Then, given that $g(0) = \infty$, and since $Y_l \leq Y_u$, Fig. 5 readily reveals that the inequality $\min[g(0), j(Y_u)] \geq g(h^{-1}(f(Y_l)))$ in (17) is impossible to satisfy for all $Y_u \leq 60.8709$. Repeated selection of fixed values of Y_u (e.g., $Y_u = 130$) then allows the determination of $j(Y_u)$, $g(h^{-1}(f(Y_u)))$ (i.e., $j(130) = 10.3659$, $g(h^{-1}(f(130))) = 0.9091$). Then, given that $g(\infty) = 0$, the inequalities (17) become $10.3659 \geq g(h^{-1}(f(Y_l))) \geq 0.9091 \geq j(Y_l)$. Inspection of Fig. 5 then reveals that there exists no feasible Y_l range; Similarly, selecting $Y_u = 150$, then gives $j(150) = 14.1989$ and $g(h^{-1}(f(150))) = 0.7543$, and the inequalities (17) become $14.1989 \geq g(h^{-1}(f(Y_l))) \geq 0.7543 \geq j(Y_l)$, which yields the feasible Y_l range $0 \leq Y_l \leq 3.1431$; In a similar manner, repeated selection of Y_u values yields corresponding feasible Y_l ranges, which in turn give rise to a region of upper and lower bounds in (Y_u, Y_l) space depicted in Fig. 6, for any element of which the system (11–20) is SOS over the set $S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\}$.

From Fig. 6, it can be seen that for $Y_u \geq 137.8833$, there exists a range of Y_l for which system (11–20) is SOS. Selecting a particular Y_u value, e.g., $Y_u = 137.8833$, gives a Y_l range of $0 \leq Y_l \leq 3.5$. Similarly, when $Y_u = 160$, the Y_l range is $0 \leq Y_l \leq 2.9029$. Selecting $Y_u = 160$, $Y_l = 2$, and based on (18), the K_u and K_l

ranges are identified as: $617.2443 \leq K_u \leq 698.8644$, $216.1029 \leq K_l \leq 239.7571$. Finally, selecting one of the values in the above ranges of K_u, K_l respectively, $K_u = 622$ and $K_l = 220$, gives rise to the rectangular sustainable set $S1 \triangleq \{[Y \ K]^T \in \mathbb{R}^2 : 2 \leq Y \leq 160, 220 \leq K \leq 622\}$. As shown in Fig. 7, the system is sustainable over $S1$, as the system's vector field points inward all along the boundary of $S1$.

The above results can also be obtained using part b of the Theorem, albeit the analysis is carried out through verification of (19) by plotting the graphs $(Y_u, \max[Y_l^k, f^{-1}(\max[f(0), h(g^{-1}(\min[g(0), j(Y_u)]))])])$, $(Y_u, \min[Y_u, Y_u^k, (j^k)^{-1}(\min[g(0), g(h^{-1}(f(Y_u)))]), j(Y_u^k)])$, $(Y_u, \min(g(0), g(h^{-1}(f(Y_u))))$, $(Y_u, j(Y_l^k))$, $(Y_u, g(\infty))$, $(Y_u, j(Y_u))$ $(Y_u, \max[Y_l^k, (j^k)^{-1}(\min[g(0), g(h^{-1}(f(Y_u)))]), j(Y_l^k)])$, $(Y_u, \max[f^{-1}(\max[f(0), h(g^{-1}(\min[g(0), j(Y_u)]))])])$ and $(Y_u, \min[Y_u, Y_u^k])$ and $(Y_u, j(Y_u^k))$ as functions of Y_u , for each $k = 1, 3$ in Fig. 8. The condition $g(\infty) \leq j(Y_u)$ is not further examined, as it is always true for Case 1.

From Fig. 8, it can be seen that only for $k = 1$, there exist Y_u ranges for which inequalities (19) are satisfied. For $k = 1$, the identified feasible Y_u range is $Y_u \geq 137.8833$, which is identical to the one identified using (17). In a similar manner to the one described above, selecting $Y_u = 160$ yields a Y_l range of $0 \leq Y_l \leq 2.9029$, and selecting $Y_u = 160$, $Y_l = 2$ yields the K_u and K_l ranges $617.2443 \leq K_u \leq 698.8644$, $216.1029 \leq K_l \leq 239.7571$.

If one is not satisfied with the size of the identified rectangular sustainable set $S1$, and considers the $Y \in [2, 160]$ range to be too broad, then a tighter range can be considered (e.g., $Y \in [10, 100]$). No rectangular sustainable set exists for this range. Under these circumstances, a non-rectangular sustainable subset $S2$ may be identified, by first imposing the range $Y \in [10, 100]$, and then identifying on the $Y = 10$ and $Y = 100$ lines the points A and C, so that the system trajectories at those points are tangent to these lines. Carrying out backward integration from A, C along these trajectories then identifies the points B, D, respectively, at the trajectory intersections with the $Y = 10$ and $Y = 100$ lines. The resulting set $S2$, with boundary defined by the line segments CB, AD, and the trajectories BA, DC is then a sustainable set (shown in Fig. 9).

Fig. 5 Case 1: $(Y, j(Y))$ and $(Y, g(h^{-1}(f(Y))))$ graphs

Fig. 6 Case 1: region of bounds in (Y_u, Y_l) space for which SOS condition (17) is satisfied

Case 2:

First, the monotonicity regions for the function $j : Y \rightarrow j(Y) \triangleq sY - f(Y) = (s - E)Y - C \cdot 2 \frac{(DY+\epsilon)^2}{(DY+\epsilon)^2}$ are determined through identification of all solutions of the equation:

$$\frac{dj(Y)}{dY} \triangleq \left(s - E - \frac{-1}{2 \frac{(DY+\epsilon)^2}{(DY+\epsilon)^2}} \frac{2CD \ln 2}{(DY+\epsilon)^3} \right) = 0.$$

For the above-provided parameter values of Case 2, two solutions are identified: $(Y, j(Y), \frac{dj(Y)}{dY}) = (29.3891, 6.0645, 0); (Y, j(Y), \frac{dj(Y)}{dY}) = (83.1887, -2.4602, 0)$. The resulting set of N intervals (Y_l^k, Y_u^k) $k = 1, \dots, N$ is then $(Y_l^1, Y_u^1) = (0, 29.3891)$, $(Y_l^2, Y_u^2) = (29.3891, 83.1887)$, and $(Y_l^3, Y_u^3) = (83.1887, \infty)$, and the function j is monotonically increasing over (Y_l^1, Y_u^1) and (Y_l^3, Y_u^3) , and monotonically decreasing over (Y_l^2, Y_u^2) , i.e., $J^+ \triangleq \{1, 3\}; J^- \triangleq \{2\}$, and $N = 3$.

Next, verification of inequalities (17) in the Theorem is carried out, by first plotting the graphs $(Y, j(Y))$ and $(Y, g(h^{-1}(f(Y))))$, as shown in Fig. 10.

The graphs $(Y, j(Y))$ $(Y, g(h^{-1}(f(Y))))$ in Fig. 10 intersect at $Y_u = 24.7426$, $Y_u = 55.1947$, $Y_u = 116.4057$. Given again that $g(0) = \infty$, and $Y_l \leq Y_u$, Fig. 10 suggests that the inequality $\min[g(0), j(Y_u)] \geq g(h^{-1}(f(Y_l)))$ in (17) is impossible to satisfy for all $Y_u \leq 24.7426$. Repeated selection of fixed values of Y_u (e.g., $Y_u = 25$) then allows the determination of $j(Y_u)$, $g(h^{-1}(f(Y_u)))$ (i.e., $j(25) = 5.7468$, $g(h^{-1}(f(25))) = 5.6833$). Then, given that $g(\infty) = 0$, the inequalities (17) become $5.7468 \geq g(h^{-1}(f(Y_l))) \geq 5.6833 \geq j(Y_l)$, which yields a single feasible Y_l range $24.3632 \leq Y_l \leq 24.5653$; However, selecting $Y_u = 170$, gives $j(170) = 9.3389$ and $g(h^{-1}(f(170))) = 0.0330$, and the inequalities (17) become $9.3389 \geq g(h^{-1}(f(Y_l))) \geq 0.0330 \geq j(Y_l)$. There are two feasible Y_l ranges $(0 \leq Y_l \leq 0.1374$ and $57.4117 \leq Y_l \leq 83.1887)$ for which these inequalities are satisfied; In a similar manner, repeated selection of Y_u values yields corresponding feasible Y_l ranges, which in turn give rise to three regions of upper and lower bounds in (Y_u, Y_l) space depicted in Fig. 11, for any element of which the system (11–20) is SOS over the set $S \triangleq \{(Y, K) \in \mathbb{R}^2 : Y \in [Y_l, Y_u], K \in [K_l, K_u]\}$.

From Fig. 11, it can be seen that the Y_u ranges for these three regions are: $24.7426 \leq Y_u \leq 30.6832$ for region 1, $Y_u \geq 116.8387$ for region 2, and

Fig. 7 Trajectories initiated at the four corners of S1, stay always inside S1. Trajectory initiated inside S1 (P5), stays always inside S1

$Y_u \geq 156.8919$ for region 3, respectively. For each Y_u chosen within a region's Y_u range, there exists a Y_l range for which system (11–20) is SOS. For example, for region 1, when $Y_u = 27$, the Y_l range is $21.2649 \leq Y_l \leq 23.0912$. Then, selecting $Y_u = 27$, $Y_l = 22$, and based on (18), the K_u and K_l ranges are identified as: $145.9001 \leq K_u \leq 147.8648$ and $141.4037 \leq K_l \leq 141.7360$. Finally, selecting one of the values in the above ranges of K_u, K_l respectively, $K_u = 146$ and $K_l = 141.5$, gives rise to the rectangular sustainable set $S3 \triangleq \{[Y \ K]^T \in \mathbb{R}^2 : 22 \leq Y \leq 27, 141.5 \leq K \leq 146\}$. As shown in Fig. 12, the system is sustainable over S3, as the system's vector field points inward all along the boundary of S3.

For region 2, when $Y_u = 130$, the Y_l range is $57.3476 \leq Y_l \leq 83.1887$. Selecting $Y_u = 130$ and $Y_l = 60$, and based on (18), the K_u and K_l ranges are identified as: $714.2798 \leq K_u \leq \infty$, $202.4978 \leq K_l \leq 364.4567$. Selecting one of the values in the above ranges of K_u and K_l , respectively, $K_u = 800$ and $K_l = 300$, gives rise to the rectangular sustainable set $S4 \triangleq \{[Y \ K]^T \in \mathbb{R}^2 : 60 \leq Y \leq 130, 300 \leq K \leq 800\}$. As shown in Fig. 13, the system is sustainable over S4, as the system's vector field points inward all along the boundary of S4.

For region 3, when $Y_u = 170$, the Y_l range is $0 \leq Y_l \leq 0.1374$. Selecting $Y_u = 170$, $Y_l = 0.1$, and based on (18), the K_u and K_l ranges are identified as: $799.8807 \leq K_u \leq 889.2233$, $121.8006 \leq K_l \leq 135.5653$. Selecting one of the values in the above ranges of K_u, K_l respectively, $K_u = 820$ and $K_l = 130$, gives rise to the rectangular sustainable set $S5 \triangleq \{[Y \ K]^T \in \mathbb{R}^2 : 0.1 \leq Y \leq 170, 130 \leq K \leq 820\}$. As shown in Fig. 14, the system is sustainable over S5, as the system's vector field points inward all along the boundary of S5. It is important to note that the set S3 includes one stable equilibrium point, and the set S4 includes another stable equilibrium point, while interestingly the set S5 includes both aforementioned stable equilibrium points. The SOS concept is flexible enough to identify that the set S5 is sustainable, even though it contains multiple stable equilibrium points that prevent its classification as a domain of attraction in classic stability analysis. Indeed, as can be seen in Fig. 14, trajectories initiated at some portions of the S5 boundary are attracted to EP1, while others are attracted to EP3, though all of them remain forever within S5.

Fig. 8 Verification of SOS conditions (19) for Case 1

The above results can also be obtained using part b of the Theorem, albeit the analysis is carried out through verification of (19) by plotting the graphs:

$(Y_u, \max[Y_l^k, f^{-1}(\max[f(0), h(g^{-1}(j(Y_u))])])$, $(Y_u, j(Y_u))$, $(Y_u, g(\infty))$, $(Y_u, \min[Y_u, Y_u^k])$, $(Y_u, j(Y_u^k))$, $(Y_u, \min[Y_u, Y_u^k, (j^k)^{-1}(\min[g(h^{-1}(f(Y_u))), j(Y_u^k)])])$, $(Y_u, g(h^{-1}(f(Y_u))))$, $(Y_u, j(Y_l^k))$ and $(Y_u, \max[Y_l^k, (j^k)^{-1}(\min[g(h^{-1}(f(Y_u))), j(Y_l^k)])])$, $f^{-1}(\max[f(0), h(g^{-1}(j(Y_u))])])$ as functions of Y_u , for each $k = 1, 2, 3$ in Fig. 15.

From Fig. 15, it can be seen that for both $k = 1$ and $k = 2$, there exist Y_u ranges for which inequalities (19) are satisfied. For $k = 1$, the identified feasible Y_u ranges are $\{24.7426 \leq Y_u \leq 30.6832\} \cup \{Y_u \geq 156.8919\}$, and for $k = 2$, the identified feasible Y_u range is $\{Y_u \geq 116.8387\}$. These ranges are all identical to the ones identified using (17). In a similar manner to the one described earlier, selecting $Y_u = 27$ yields a Y_l range of $21.2649 \leq Y_l \leq 23.0912$, and selecting $Y_u = 27$, $Y_l = 22$ yields the K_u and K_l ranges $145.9001 \leq K_u \leq 147.8648$, $141.4037 \leq K_l \leq 141.7360$; selecting $Y_u = 130$ yields a Y_l range of $57.3476 \leq Y_l \leq 83.1887$, and selecting $Y_u = 130$, $Y_l = 60$ yields the K_u and K_l ranges $714.2798 \leq K_u \leq \infty$, $202.4978 \leq K_l \leq 364.4567$; However, selecting $Y_u = 170$ yields two Y_l ranges of $0 \leq Y_l \leq 0.1374$ in regions 1 and $57.4117 \leq Y_l \leq 83.1887$ in region 2, and selecting $Y_u = 170$, $Y_l = 0.1$ yields the K_u and K_l ranges $799.8807 \leq K_u \leq 889.2233$, $121.8006 \leq K_l \leq 135.5653$,

Fig. 9 Trajectories initiated at points B, D at boundary of S2, always stay inside S2. Trajectory initiated inside S2 (P10), always stays inside S2

while selecting $Y_u = 170$, $Y_l = 60$ yields the K_u and K_l ranges $799.8807 \leq K_u \leq \infty$, $121.8006 \leq K_l \leq 364.4567$.

Conclusions and discussion

This work discusses the sustainability assessment of economic systems using the novel conceptual framework of sustainability over sets (SOS). SOS analysis determines whether an economy remains within a defined region of its phase space as it evolves over time. If the region of the space coincides with the region representing economic sustainability, then the economy is sustainable over time. Assessing whether an economy is sustainable is often a personal judgment, and thus defining sustainability in a manner that maps well into reality is not easy. This work represents a major stride in making sustainability map well into reality. Indeed, the developed Theorem identifies if and only if criteria expressed in terms of an income range and a subordinately derived capital range, for the existence of a region in the economy's state-space within which the economy's behavior is maintained for ever. If this identified income range contains realistic values, then economic sustainability is well connected with reality. The aforementioned criteria were then applied to two case studies involving particular Kaldor business cycle modeling instances utilizing Investment and Savings functions defined by Lorenz.

The model's dynamic behavior is significantly different for each case. In the first case study, the system features an unstable equilibrium, and a stable limit cycle. In the second case study, the system features two stable equilibria, one unstable equilibrium and no limit cycle. Despite this disparate dynamic behavior, that would tax and challenge traditional analysis methods (e.g., local stability assessment and domain of attraction quantification), SOS analysis is able to identify sustainable rectangular sets in both cases. For the first case study, ranges of rectangular set upper and lower bounds are identified analytically for which sustainability is ascertained. Each member of the resulting family of sustainable rectangular sets contains the stable

Fig. 10 Case 2: $(Y, j(Y))$ and $(Y, g(h^{-1}(f(Y))))$ graphs

limit cycle that dominates the system's long-term behavior. Thus, sustainability is assessed without ever having to identify the limit cycle, an SOS method feature that can be of great benefit in sustainability analysis of higher dimensional systems. For the second case study, ranges of rectangular set upper and lower bounds are again identified analytically for which sustainability is ascertained. These ranges lead to the formation of disconnected regions in the two-dimensional space of upper and lower bounds on income for which the system is deemed sustainable. Some members of the resulting family of sustainable rectangular sets contain only one stable equilibrium point, and others contain another equilibrium point, while others contain both equilibrium points. If one were to pursue a region of attraction analysis for the latter situation where the region contains both equilibrium points, no such region of attraction would be possible to identify, as each such region is by definition associated with a single equilibrium point.

Should the proposed method fail to identify a rectangular sustainable set with desirable characteristics, such as tight lower and upper bounds on the income and savings variables, a methodology is also presented that identifies non-rectangular sustainable sets that enforce exact lower and upper bounds on one of the economic system variables (e.g., income), and then form the non-rectangular portions of the boundary by computing backward system trajectories that are tangential to the aforementioned income exact lower and upper bound boundary lines.

In summary, the original contributions of this work include:

1. A novel theorem is proven establishing readily verifiable necessary and sufficient conditions for the existence of rectangular sustainable sets for a two-dimensional

Kaldor model of the form
$$\begin{cases} \dot{Y} = \alpha \cdot [f(Y) + g(K) - s \cdot Y] \\ \dot{K} = f(Y) + g(K) - \delta \cdot K \end{cases}$$
. The power of the

Theorem lies in its ability to express these necessary and sufficient conditions, first in the two-dimensional space of lower and upper bounds on income (Y_l, Y_u) , inequality (17), and then in the one-dimensional space of upper bound on income (Y_u) , inequality (19).

Fig. 11 Case 2: region of bounds in (Y_u, Y_l) space for which SOS condition (17) is satisfied

Fig. 12 Trajectories initiated at the four corners of S_3 , stay always inside S_3 . Trajectory initiated inside S_3 (P11), stays always inside S_3

Fig. 13 Trajectories initiated at the four corners of S4, stay always inside S4. Trajectory initiated inside S4(P16), stays always inside S4

Fig. 14 Trajectories initiated at the points P19 and P21 are attracted to EP1. Trajectories initiated at the points P17, P18, and P20 are attracted to EP3. All trajectories stay always inside S5

2. The theorem enables identification of the region of economic sustainability for the considered Kaldor model, in the two-dimensional space of lower and upper bounds on income (Y_l, Y_u), and in the one-dimensional space of upper bound on income (Y_u).
3. The theorem's power is illustrated with a case study of a particular Kaldor business cycle model from the literature that can exhibit rich dynamic behavior, and can give rise to economic sustainability regions that encompass multiple stable equilibrium points.
4. A methodology is presented that identifies non-rectangular sustainable sets, when the imposition of tight lower and upper bounds on one of the economic system variables (e.g., income) prevents the existence of a sustainable rectangular set.

Our future research will be focusing on the application of the SOS conceptual framework to higher dimensional economic systems, such as multi-sector business cycle models (Murakami 2018).

Fig. 15 Verification of SOS conditions (19) for Case 2

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1007/s43546-021-00058-5>.

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Data availability The manuscript has no associated data that are not already provided.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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