On the energy losses due to tracks vibrations in rubber track crawler vehicles

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Abstract

In spite of an increasing number of rubber-tracked vehicles, there are no engineering models for predicting and optimizing the energy consumption of vehicles of this type. To formulate those models, the models of the phenomena resulting in the internal losses of rubber-track systems need to be developed. This article presents a model describing the losses caused by the transverse vibrations of rubber tracks. The predictions made using the model are discussed against the background of the preliminary experimental tests on a sample rubber track for heavy off-road vehicles. The model predictions and the experimental tests suggest that the losses caused by the 1st mode vibration of rubber tracks are marginal in relation to the total internal resistance of rubber-track systems. However, according to the model predictions, a significant increase in the rubber-tracked undercarriage internal resistance is expected as a result of the high-amplitude track vibrations corresponding to the higher-order modes. To make the model applicable in practice, a method for determining the essential parameters of the model, including the bending stiffness and the decrement of oscillation damping, is demonstrated. The accuracy of the method is confirmed by the computations, where the sag and the frequency of the 1st mode free vibration of a sample track are predicted with an error of 10% and 1.8%, respectively. The parameter values obtained by this method are suitable for modeling a wide variety of off-road vehicles. The method can be applied to many other types of reinforced rubber belts, e.g., conveyor belts.

Keywords Tracked undercarriage · Rubber track · Track vibrations · Energy efficiency · Internal motion resistance

1 Introduction

While designing a tracked crawler vehicle, the actual power demand of the vehicle must be predicted with appropriate accuracy to distinguish a power unit that will be suitable for a given vehicle. Designers also pay a lot of attention to maximizing the energy efficiency of the new generation tracked undercarriages. Those problems might be effectively solved if models of the external and internal motion resistance of tracked undercarriages are known. The models

describing the external resistance are broadly available in the literature, as seen in the following examples.

The interaction between tracks and a soft ground results in a relatively high rolling resistance arising from soil compaction and the bulldozing effect. To roughly estimate this resistance for a sample vehicle, the weight of the vehicle should be multiplied by an equivalent coefficient of rolling resistance. The experiments by Cleare [7] reveal that the rolling resistance coefficient should be estimated at 0.10 on dry sandy and loamy grounds, whereas 0.17 should be assumed for muddy terrain, where the vehicle sinks more deeply into the ground. On the other hand, according to the handbook by Pieczonka [28], the values of 0.10–0.15 and 0.15–0.20 are suggested for loamy and sandy grounds, respectively. Bekker [2] presents a method, where the rolling resistance of tracks on soft grounds is calculated on the basis of a theoretical formula describing the energy losses attributed to soil compaction as a function of the mechanical parameters of the ground as well as the unit pressure and soil sinkage under the tracks. In this method, the track footprint

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is assumed to be a rigid plate and the pressure distribution under the track is approximated by a simple geometric shape, for example, a rectangle or trapezoid. Wong [41] and Pieczonka [28] use the Bekker's model to formulate the equations illustrating how to minimize the rolling resistance of tracks on soft grounds. Wong [41] shows that the resistance arising from soil compaction can be minimized by maximizing the area of the track footprint. According to Pieczonka's findings [28], the more uniform the pressure under the track, the lower the rolling resistance is. Apart from the reference to the Bekker's model, Wong [41] presents the most sophisticated method for calculating the rolling resistance of tracks on soft grounds. In this method, the horizontal resultant reaction force in the interface between the track and the ground is interpreted as the rolling resistance. To calculate the resultant reaction force, the track is represented by a flexible belt, whereas the pressure distribution under the track is determined on the basis of the mechanical parameters of the ground, track tension, dimensions of the track footprint, and the number, spacing, and diameter of the road wheels. Therefore, those parameters can be adjusted to provide the lowest rolling resistance of the track possible.

Cleare [7] presents a simple model, where the turning resistance of tracked vehicles is calculated as a product of the vehicle weight, apparent coefficient of friction between the track and the ground, and a correction factor defined by the vehicle characteristics. Furthermore, the experiments presented by Cleare [7] show that turning resistance decreases with the increasing turning circle radius. On the other hand, Pieczonka [28] presents an analytical model, where the turning resistance is calculated as a sum of the two components: the sliding friction in the contact patch between the track and the ground surface, and the lateral forces acting on the track edges from the walls of the ruts due to the internal friction of soil. According to the model equations, the contact pressure under the track should preferably be concentrated at the midpoint of the track footprint to minimize the turning resistance. Furthermore, the shorter the track footprint, the lower the turning resistance is. The same conclusions are drawn by Dudziński and Gładysiewicz [9], where the model computations of the turning resistance are carried out for a sample tracked vehicle with different patterns of pressure distribution under the track.

The grade resistance is defined only by the vehicle weight and the maximum rated ground inclination. Consequently, not a lot can be done with the structure of a tracked undercarriage to reduce this resistance. The same can be claimed regarding the aerodynamic drag, which is affected by the vehicle speed, the area of the vehicle transverse cross-section, and the drag coefficient. Furthermore, Wong [41] claims that for the vehicles operating at speeds below 48 km/h, the aerodynamic drag is negligibly small. Namely, the power loss arising from the aerodynamic drag

at the speed of 48 km/h can be estimated at about 11 kW for the vast majority of the state-of-the-art track layers. The power loss of this order of magnitude can be represented by an equivalent coefficient of rolling resistance of about 0.0017–0.0083, which is 5 to 15 times smaller than the respective coefficient representing the internal motion resistance of tracked vehicles (see Table 1). Very similar results and conclusions are provided by Cleare [7] and Ogorkiewicz [25].

In the vast majority of papers and handbooks, the internal motion resistance of tracked vehicles is expressed with the coefficient of internal resistance defined by the following equation:

$$C_R = \frac{F_{R,int}}{Q} \quad (1)$$

$F_{R,int}$ (N) is the driving force that needs to be applied to the driving wheel of a tracked vehicle to overcome the internal motion resistance of the vehicle, Q (N) is the overall weight of the tracked vehicle.

In practice, the coefficient of internal resistance is determined by experiments, where a test vehicle is moved along a flat paved test track. The internal resistance $F_{R,int}$ is usually determined by one of the two following methods. In the first one, called the towing test, the drawbar needed to tow the test vehicle with another one is measured. The second approach adopts the torque meters fitted to the drive shafts of the test vehicle. The experiments involving the torque meters might be carried out at zero or non-zero drawbar pull delivered by the test vehicle.

The values of the coefficient of internal resistance for sample state-of-the-art vehicles are summarized in Table 1. On the other hand, Fig. 1 shows the experimental characteristics, where the coefficient of internal resistance for multiple sample vehicles is displayed as a function of the vehicle speed. Those data lead to a conclusion that the energy efficiency of a tracked undercarriage is strictly related to the undercarriage layout. It depends, for example, on the number and arrangement of the road and idler wheels. Furthermore, track systems with a front-drive sprocket usually manifest higher energy consumption than their rear-drive-sprocket counterparts. The energy consumption of track systems is also affected by the design of the undercarriage components, including the design features of the track and drive wheels defined by the power transmission method (i.e., the positive, negative, or friction-drive) as well as the structure and material of the belts of the rubber tracks or the pin bushings of the link tracks. Finally, the internal resistance of track systems increases with the vehicle speed and driving force transmitted by the tracks, and, according to the experiments by Cleare [7], the higher the initial track tension, the higher the internal losses of the undercarriage are.

Table 1 Values of the coefficient of internal resistance for sample state-of-the-art tracked vehicles

Vehicle type	C_R (-)	Test method	References
10.5 tonne military high-speed vehicle, rubber-bushed metal-link tracks, front drive sprocket	0.040	Towing test	[7]
	0.040	Driving torque measurement, zero drawbar pull	
	0.050	Driving torque measurement, non-zero drawbar pull	
17.2 tonne high-speed military vehicle, rubber-bushed metal-link tracks, front drive sprocket, no rear idler	0.054	Towing test	[7]
	0.055	Driving torque measurement, zero drawbar pull	
	0.065	Driving torque measurement, non-zero drawbar pull	
37.4 tonne high-speed military vehicle, rubber-bushed metal-link tracks, rear drive sprocket	0.037	Towing test	[7]
	0.037	Driving torque measurement, zero drawbar pull	
	0.037	Driving torque measurement, non-zero drawbar pull	
40.0 tonne T-62 battle tank	0.025	Towing test	[20]
39.6 tonne Leopard I battle tank	0.024–0.046 ^(v)	Towing test	[20]
49.4 tonne Leopard II battle tank	0.032–0.038 ^(v)	Towing test	[20]
47.2 tonne M 48 A2 battle tank	0.040	Towing test	[20]
55.0 tonne Chieftain battle tank	0.046	Towing test	[20]
Earthmoving machine with conventional metal-link tracks	0.03–0.06	General case	[8]
720 kg articulated body vehicle with 4 delta rubber-tracked systems	0.14–0.28	Driving torque measurement, zero drawbar pull	[8]
12.5 tonne 157 kW rubber-tracked Track Marshall TM200 tractor	0.042	Driving torque measurement, zero drawbar pull	[11]
6.7 tonne 75 kW half-track prototype FORD 7610 tractor with rubber tracks	0.048	Driving torque measurement, zero drawbar pull	[11]
2.0 tonne prototype track system with Bridgestone rubber track	0.08–0.10 ^(t)	Towing test	[11]
1.7 tonne prototype track system with Colmant-Cuvelier rubber belt	0.15–0.21 ^(t,v)	Towing test	[26]

^(t)—depending on the track tension, ^(v)—depending on the vehicle speed

Fig. 1 Empirical relationships between the coefficient of internal resistance and vehicle speed for different tracked vehicles: 1—agricultural tractor with metal link tracks [2, 41], 2—curve for the general case [2, 41], 3—metal link track with rubber bushings [2, 41], 4—half-track with rubber tracks [2, 41], 5—all-steel link track [2, 41], 6—lubricated all-steel track, overlapping road wheels [2, 41],

7—T23 battle tank [25], 8—M113 Hotrod armored personal carrier [23], 9—36.9 tonne tank with rear drive wheel [7], 10—Leopard II battle tank [20, 25], 11—M41 battle tank [25], 12—LVTP-7 amphibious carrier [25], 13—Cromwell battle tank [25], 14—T236 self-propelled gun [25], 15—3/4-track carrier with needle bearing pin joints [25]

In practice, the internal resistance of tracked undercarriages is affected by so many factors that the coefficient of internal resistance cannot be used to optimize the energy consumption of the new generation undercarriages. To do that, more sophisticated models are needed. Namely, each phenomenon contributing to the internal losses of a track system, including the resistance arising from bending of the tracks, rolling resistance of the road wheels, bearings friction, etc., should be described by a separate model. Each elementary model should preferably be an explicit function of the aforementioned vehicle design features and operating parameters. The elementary models should be joined together to create a more advanced model. Such advanced models are presented by Pieczonka [28], Merhof and Hackbarth [20], Kolkov et al. [15], and Rehorn [30].

The model developed by Pieczonka [28] is applicable mainly to the low-speed heavy vehicles. Pieczonka's model equations suggest that the resistance arising from bending of the tracks, friction in the idler and the sprocket wheel bearings, and friction in the interface between the guide flanges of those wheels and the guide lugs of the tracks might be minimized by reducing the initial track tension. Bending resistance of the tracks can be also reduced by fitting the vehicle with the idler and sprocket wheels with the largest diameter possible.

The model by Kolkov et al. [15] refers to the high-speed military vehicles with metal-link tracks. It also suggests that bending resistance of the tracks can be minimized by maximizing the diameter of the undercarriage wheels. Furthermore, according to that model, bending resistance of the tracks is proportional to the square of the vehicle velocity. This explains to some extent why the internal motion resistance of tracked vehicles increases with the vehicle speed (see Fig. 1).

Merhof and Hackbarth [20] developed another model of the internal resistance for the high-speed metal-link-tracked vehicles. The conclusions drawn by analyzing this model are similar to the findings about the models by Pieczonka and Kolkov et al. Namely, the energy consumption of a tracked undercarriage can be minimized by minimizing the track tension and maximizing the diameter of the idler and sprocket wheels, because they result in a decrease in the bending resistance of the tracks. The model by Merhof and Hackbarth confirms also that bending resistance of the tracks increases with the vehicle speed, which is attributed to the increase in the dynamic tension of the track.

Rehorn [30] presents a comprehensive model of the internal losses for an unconventional track layer with the tracks made from conveyor belts. Some of the elementary models introduced by Rehorn, including the model describing bending resistance of the tracks, have been obtained by fitting empirical formulas to experimental data. Thus, they are not entirely representative for the conventional rubber-tracked

vehicles. Nevertheless, the experiments carried out by Rehorn [30] show that bending resistance of the rubber reinforced belt structures is likely to increase with an increasing belt tension and speed. Furthermore, according to those experiments, the resistance arising from bearings friction in the road-wheel assemblies increases with the vehicle speed, which is yet another explanation for the trends observed in Fig. 1.

Apart from the above-mentioned comprehensive models, the studies focusing on the particular phenomena resulting in the internal losses in the undercarriages of track layers are known. Watanabe et al. [39] present the experimental study on the internal resistance for a sample light-weight undercarriage with conventional rubber tracks, where the resistance to motion of each wheel of the test undercarriage has been measured separately. According to this study, the motion resistance of every wheel increases with an increasing track tension and speed, which agrees with the results provided by Rehorn [30].

Chołodowski and Dudziński [3] carried out quasi-static experimental tests on rubber-band tracks for heavy agricultural vehicles. According to that study, bending resistance of rubber tracks decreases with an increase in the diameter of the idler and drive wheels embodied in the track systems. On the other hand, the influence of the track tension on the bending resistance has been found to be negligibly small. The last mentioned result has been confirmed by the high-speed tests on a conventional rubber-track system for heavy agricultural vehicles discussed by Chołodowski et al. [4], where it has also been shown that bending resistance of rubber tracks increases with an increase in the track speed. Furthermore, the experiments presented by Chołodowski et al. [4] revealed that the increase in the initial tension and speed of the track leads to an increase in the friction of the idler and drive wheel bearings. Therefore, the increase in the overall undercarriage motion resistance resulting from an increase in the track tension is mainly attributed to the increase in the idler and sprocket wheel's bearings friction.

Since multiple similarities between the rubber-track systems and the belt conveyors exist, the motion resistance models formulated for the belt conveyors can be adapted for rubber-tracked undercarriages. Gładysiewicz [13] introduces the models describing rolling resistance and bearings friction of the idlers, bearings friction in the drive and tail pulley assemblies, and the bending resistance of conveyor belts. According to the model equations given by Gładysiewicz [13], the bearings friction of the drive and tail pulley is proportional to the pulley load, which is mainly defined by the belt tension. On the other hand, the bearings friction in the idler assemblies is described by an empirical formula, where the friction increases with the belt speed. A similar formula is suggested by Lachmann [17] and Vierling [36]. In [13], bending resistance of conveyor belts is also expressed

by an empirical formula, and the parameter values applicable to a wide range of commercially available belts are provided. Therefore, sample computations can be done using those parameters to prove that bending resistance of conveyor belts decreases with increasing diameter of drive and tail pulleys. The same conclusion has been made by Vierling [36] on the basis of an extensive experimental research. Finally, the analytical derivations presented by Gładysiewicz [13] show that the coefficient of the rolling resistance resulting from the indentation of the idlers into the rubbery surface of conveyor belts depends on the vertical load of the idlers. This is confirmed by the experimental results provided by Wheeler and Menzenberger [40]. Lachmann [17] has formulated a similar model that also accounts for the fact that rolling resistance of belt conveyor idlers increases with an increase in the belt speed. An empirical model that accounts for this phenomenon is presented also by Schwarz [32].

Summing up the above review, a comprehensive model for predicting and optimizing the internal resistance of conventional rubber-tracked off-road vehicles still needs to be developed. Unfortunately, it cannot be formulated using only the data provided by the state-of-the-art literature. Therefore, the authors of this article have been conducting theoretical and experimental research in this field for many years. This paper focuses on the research on the resistance caused by the transverse vibrations of rubber-band tracks. This topic has been undertaken for the following reasons.

According to Fig. 1, the internal resistance of some tracked vehicles manifests its maxima at the vehicle speeds that are substantially lower than the maximum vehicle speed. Ogorkiewicz [25] claims that this phenomenon arises from the resonant vibrations of tracked running gear but does not provide more details on this issue. Merhof and Hackbarth [20] give the same general explanation. This hypothesis has been first confirmed by the experiments on the vehicles with metal-link tracks carried out by Cleare [7]. Those experiments reveal that while the road wheels of a tracked vehicle are passing over the joints connecting the consecutive links of the track, vibrations are induced in the suspension system of each road wheel. If the excitation frequency, defined by the vehicle speed and the track pitch, coincides with the resonant frequency of the suspension system, an increase in the vehicle motion resistance is observed.

The tracks are yet another component of track systems that are subjected to oscillations during vehicle operation. The experiments by Takano et al. [35] prove that transverse vibrations are induced in rubber-band tracks as a result of non-uniform mass distribution along the tracks or, to a minor extent, due to the eccentricity of drive sprockets. Scholar and Perkins [31] have formulated an analytical model for predicting the amplitude and frequency of longitudinal and transverse vibrations of flexible rubber-bushed link tracks induced by road bumps and the so-called polygon effect.

According to Wang et al. [38], transverse track vibrations adversely affect the energy consumption of tracked vehicles. Although no explicit evidence for this hypothesis is presented by Wang et al. [38], it can be explained as follows. The shape of a track subjected to transverse vibrations periodically changes, resulting in the losses attributed to the sliding friction in dry and lubricated pin joints or the mechanical hysteresis of rubber belts or rubber-bushed pin joints. In spite of an extensive literature review, the authors have not found any detailed theoretical or experimental research on the losses caused by the vibrations of rubber-band tracks for off-road vehicles; however, this problem is worth considering. According to Fig. 1, the energy consumption of the high-speed vehicles with flexible rubber-bushed link tracks, e.g., the Leopard II battle tank, is significantly affected by the undercarriage resonant vibrations. Furthermore, an increasing number of high-speed military vehicles are designed to adopt rubber-band tracks offered by a number of specialized manufacturers, including Soucy Defense (Canada), Diehl Defence (Germany), and DST Defence Service Tracks GmbH (Germany). Multiple modern rubber-tracked commercial vehicles can be classified as high-speed vehicles as well. According to the technical specifications provided by Claas GmbH, the maximum speed of the most up-to-date rubber-tracked Lexion 8000–7000 series combine harvesters [6] and Axion 900 series tractors [5] is rated at 40 km/h. Furthermore, Takano et al. [35] presented a custom commercial rubber-tracked vehicle called Robot Systems RD3, which might deliver maximum speed of about 100 km/h.

The main objective of this research is to evaluate the severity of the losses resulting from the transverse vibrations of conventional rubber-band tracks for off-road vehicles. A simple engineering model for predicting those losses is formulated in Sect. 2 for the purpose of this research. Although no model pertaining to exactly this issue was found in the literature, the authors came across a number of models that are potentially suitable for predicting the deflection shape, amplitude, and frequency of track vibration. The models of this type are often developed for the purpose of eliminating noise emission, reducing fatigue wear, or improving ride quality of tracked vehicles. The authors found those models particularly suited to the approach used to formulate the model presented in Sect. 2. The authors used the following literature review as a reference in developing their own model.

Basically, two alternative approaches are adopted in modeling tracks. In the first one, called the multibody approach, the track is represented by a chain composed of rigid bodies connected to each other with discrete joints. Regarding the characteristics of rubber tracks, a multibody model with the shortest links and the largest number of joints possible is required for the best accuracy. Furthermore, the definition of the joints should account for at least the elastic properties

of the track. Wallin et al. [37] discuss multiple joint definition types. According to the tests carried out by Wallin et al. [37], the multibody track models, where the joints are defined by the algebraic equations describing the kinematic constraints governing the relative motion of the consecutive track links require the least computation time. This arises from the fact that the number of degrees of freedom of the model decreases when the joint constraint equations are introduced. Unfortunately, this joint definition type does not account for the elastic properties of the track. According to Wallin et al. [37], stiffness of the track can be implemented into the multibody track model if the joints are defined by the so-called compliant discrete joint elements. Unfortunately, the joint elements with this definition type do not affect the number of degrees of freedom of the model, which increases the computation time demand. High computational complexity of the multibody models of rubber tracks has been confirmed by Mężyk et al. [22], where a multibody track model is implemented into a simulation model of a rubber-tracked vehicle with an active track tensioning system. Dynamic behavior of that vehicle has been tested for a number of control algorithms implemented in the tensioning system. In [22], the rubber track was first modeled as a chain composed of short-pitched rigid links connected to each other by flexible pivots. Unfortunately, this model required so much computation time that the flexible bushings were finally given up in favor of simple revolute joints. It should be pointed out that it is time consuming to build a universal multibody model that would be suitable for representing a wide variety of tracks, which also discourages the authors from adopting the multibody approach in the research presented herein.

The concern regarding high computational complexity of the multibody models of rubber tracks is also raised by Scholar and Perkins [31] and Wang et al. [38]. They do not discuss this issue in much detail; however, they introduce an alternative approach in tracks modeling, called the continuum approach. In this method, tracks are represented by the continuous structures with non-zero elasticity modulus and density, e.g., strings, beams, etc. Scholar and Perkins [31] present a model, where the track is represented by a string made from perfectly elastic material with a constant elasticity modulus. The model passed the verification tests, where the natural frequencies of the first two vibration modes for a representative flexible rubber-bushed link track were predicted with an excellent accuracy, whereas the frequencies for the third and fourth modes were underestimated by 8% and 18%, respectively. Scholar and Perkins [31] attribute those discrepancies to the fact that the model does not allow for the track bending stiffness. After the verification was completed, forced vibration response of a track subjected to two different excitation sources was calculated as a superposition of several low-order modes of the track. The results of

those computations indicate that the dynamic loads acting on tracks in real conditions result in vibration responses whose bandwidths remain mostly below the natural frequency of the third mode. Thus, the limitations revealed for the model of Scholar and Perkins are unlikely to badly influence the results of the model computations for real tracks.

Although Wang et al. [38] adopt a beam model to describe dynamic behavior of flexible tracks, the assumptions made by Wang et al. are quite similar to those of Scholar and Perkins [31]. In Wang et al. [38], bending and axial stiffness of a track are assumed to be constant values, whereas damping properties of the track are neglected. In spite of those simplifications, the model by Wang et al. predicted the natural frequencies of the first, the second, the third, and the fourth mode of a sample flexible rubber-bushed link track with an error of 2%, 3%, 11%, and 6%, respectively.

A model beam with a constant bending stiffness is also used by Takano et al. [35] to describe dynamic behavior of a rubber-band track. In contrast to the models discussed by Scholar and Perkins [31] and Wang et al. [38], the model by Takano et al. allows for damping of the track, which is represented by the modal damping ratio ζ , i.e., the standard measure defining the ratio of the actual and the critical damping for a given mechanical system. Nevertheless, a damping ratio as small as $\zeta = 0.03$ has been determined by experiments on the rubber track investigated by Takano et al. [35]. Using the above-mentioned model, Takano et al. designed a control algorithm for an active track tensioning system that adjusts the track tension to eliminate transverse track vibrations. The effectiveness of that system has been proved by the experiments, where a 30% decrease in the vibration amplitude has been revealed for a sample track.

The beam models are actually adopted to model transverse vibrations of a variety of technical objects, including serpentine belts, conveyor belts, band saws, paper industry machines, etc. This issue is broadly discussed in the state-of-the-art literature. For example, Marynowski and Kapitaniak [18] implement the Kelvin-Voight and the Břrgers material models into the differential equation of motion of a beam subjected to transverse vibrations. On the other hand, in [19] they present the analogous considerations involving the Zener material model. The above-mentioned material models capture damping properties of materials, which corresponds well with the viscoelastic behavior of rubber. However, according to Marynowski and Kapitaniak [18, 19], the deflection shape of a beam with a viscoelastic material model can be derived from a nonlinear differential equation of motion whose approximate solution can be found by solving a set of linear differential equations. The resultant sets of linear differential equations obtained by Marynowski and Kapitaniak [18, 19] have been solved by numerical integration, and their solutions have not been compared with the results of laboratory tests. Therefore, it is rather

hard to find a method for estimating the parameters of the models discussed by Marynowski and Kapitaniak [18, 19] that could be based on relatively simple experimental tests involving a rubber track with a composite structure. The difference between the accuracy of those models and the models formulated by Scholar and Perkins [31], Wang et al. [38], and Takano et al. [35] is hard to estimate, whereas the accuracy of the above-mentioned, more simple, models is in fact acceptable for the purpose of engineering computations.

To make the model introduced in Sect. 2 applicable in practice, an effective method for determining the essential model parameters, including the bending stiffness and the ratio of successive amplitudes of damped vibration of rubber tracks, is demonstrated in Sect. 3. The method is illustrated with experiments on a sample track for heavy agricultural machines. In Sect. 4.1, the model formulated in Sect. 2 and the parameter values determined in Sect. 3 are involved in predicting the losses resulting from the transverse vibrations of the sample track as a function of the mode number and amplitude of the track vibration as well as the linear speed of the track. In Sect. 4.2, the results of the model computations are discussed against the background of the results of the very preliminary experimental tests, where an attempt was made to estimate the losses resulting from the track vibrations in more explicit way.

2 Theoretical model of the losses due to rubber track vibrations

If a single strand of a rubber track strung between two wheels of a track system is subjected to transverse vibrations, some amount of energy is lost due to viscoelastic properties of rubber and the internal friction of reinforcing layers of the track. Since rubber tracks and belts are continuous mechanical systems, in practice, vibration motion of a single track strand might be decomposed into multiple harmonic components or, in other words, normal modes. To estimate the overall resistance caused by the vibrations of the track strand, the losses attributed to each harmonic component should be calculated and a sum of all those values should be computed afterwards. As it was mentioned earlier, a similar approach was adopted in [31], where the forced vibration response of a flexible track to dynamic excitation was predicted by the superposition of several low-order modes of the track. The superposition of component waves was also successfully used by Kong and Parker [16] to develop an efficient approximate method for determining the resultant deflection shape of beams subjected to transverse vibration.

In the considerations presented herein, it is assumed that the losses resulting from the n th mode track vibration of constant amplitude are represented by the general formulas given by Eqs. 2 and 3. The derivations presented below

focus on formulating the equations representing the energy lost as a result of a single cycle of the n th mode track vibration and the frequency of that vibration:

$$M_{tv}^{(n)} = P_{tv}^{(n)} / \omega_{dw} \quad (2)$$

$$P_{tv}^{(n)} = \Delta U_{tv}^{(n)} \cdot f_{tv}^{(n)} \quad (3)$$

$M_{tv}^{(n)}$ (Nm) is the torque applied to a driving wheel of a rubber track system to overcome the losses caused by the n th mode track vibration of given amplitude, $P_{tv}^{(n)}$ (W) is the power loss resulting from the track vibrations defined above, $\Delta U_{tv}^{(n)}$ (J) is the energy dissipated in one cycle of the vibrations defined above, $f_{tv}^{(n)}$ (Hz) is the n th mode track vibration frequency, ω_{dw} (rad/s) is the angular velocity of the driving wheel.

For the purpose of the considerations presented herein a rubber track strand is represented by a beam depicted in Fig. 2. The bending stiffness and the unit mass of the beam are denoted by EI and $m_{tr,unt}$, respectively. The model allows also for the track tension (F_{ta}), because it affects the track resonant frequency, which will be discussed later.

It should be emphasized that the derivations presented below are aimed at formulating a model that would be suitable for the purpose of engineering computations conducted, for example, to determine the performance of power units and drivetrain components of the new generation rubber-tracked vehicles. Consequently, the main idea behind those derivations is to describe the discussed phenomenon by the most simple model possible. The model should preferably include the smallest possible number of parameters whose values can be determined using relatively simple experimental methods. The advantage of this approach will clearly manifest itself in the circumstances, where the losses resulting from the transverse vibrations of a large number of sample tracks need to be estimated in a short period of time. To fulfill the above-mentioned requirement, several simplifications were made while formulating the model discussed herein. For example, bending stiffness (EI) of the beam representing a rubber track strand is assumed to be a constant value, although in fact the elasticity modulus of rubber depends on the

Fig. 2 Model beam representing a rubber track strand subjected to transverse vibrations

deformation which is claimed by multiple handbooks, e.g., [12] and [21]. The research by Scholar and Perkins [31] and Wang et al. [38] indicate that, in spite of this assumption, dynamic properties, including the resonant frequencies, of flexible rubber-bushed metal-link tracks can be predicted with the accuracy that is acceptable for engineering purposes. The same has been also proved by Takano et al. [35] for the rubber-band tracks. Therefore, the authors find this assumption acceptable.

Bearing in mind the nature of the mechanisms underlying the energy dissipation in rubber tracks, i.e., the mechanical hysteresis, it was assumed that the energy lost as a result of a single cycle of the n th mode vibration of a rubber track is given by Eq. 4. Namely, it is calculated as a certain fraction of the strain energy $U^{(n)}$ of the model beam in the position of maximum deflection (see Fig. 2). The loss factor ψ_{tv} represents the structural damping of the track:

$$\Delta U_{tv}^{(n)} = \psi_{tv} \cdot U^{(n)} \quad (4)$$

The above-mentioned assumption might be justified by the fact that in the position of maximum deflection, the velocity of the beam elements subjected to transverse vibrations is zero. Therefore, the strain energy calculated for that position represents the total mechanical energy of the system.

Equations 5 and 6 give the strain energy and the deflection of the track in the position of maximum deflection, respectively. These equations are derived on the basis of the theory applicable to the perfectly elastic beams. Equation 5 is actually one of the fundamental equations of the theory of the strength of materials and can be found in [24]. On the other hand, Eq. 5 has been derived by Arczewski et al. [1]. Substituting Eqs. 5 and 6 into Eq. 4 and solving the obtained integral yields Eq. 7:

$$U^{(n)} = \frac{1}{2} \int_0^{l_0} EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \quad (5)$$

$$y(x) = A_n \sin \left(\frac{\pi n}{l_0} x \right) \quad (6)$$

$$\Delta U_{tv}^{(n)} = \frac{1}{4} EI A_n^2 \frac{(\pi n)^4}{l_0^3} \psi_{tv} \quad (7)$$

l_0 (m) is the length of a track subjected to transverse vibrations, $y(x)$ (m) is the deflection of a track subjected to transverse vibrations in the position of maximum deflection, A_n (m) is the amplitude of the n th mode track vibration.

The authors decided that the loss factor ψ_{tv} will be calculated on the basis of the ratio of successive amplitudes of the 1st mode damped free vibration of the track given by Eq. 8:

$$\delta = \frac{A_m}{A_{m+1}} \quad (8)$$

The above-mentioned δ factor was found to be the most suitable for representing damping properties of rubber tracks from the point of view of the derivations presented below. Nevertheless, it should be noted that Eq. 9 might be used to convert the δ factor into the logarithmic decrement Λ , which is commonly used in the literature as a standard measure of damping in mechanical systems:

$$\Lambda = \ln \frac{A_m}{A_{m+1}} = \ln \delta \quad (9)$$

The energy lost as a result of one cycle of the 1st mode damped free vibration of the rubber track strand can be described by Eq. 7; however, it might be also represented by Eq. 10. Furthermore, substituting Eq. 8 into Eq. 10 yields Eq. 11:

$$\Delta U_{tv(fr)}^{(1)} = \frac{1}{4} EI \frac{\pi^4}{l_0^3} (A_m^2 - A_{m+1}^2) \quad (10)$$

$$\Delta U_{tv(fr)}^{(1)} = \frac{1}{4} EI A_m^2 \frac{\pi^4}{l_0^3} \left(1 - \frac{1}{\delta^2} \right) \quad (11)$$

The comparison of the formulas given by Eqs. 7 and 11 leads to a conclusion that the loss factor ψ_{tv} can be calculated according to the following equation:

$$\psi_{tv} = \left(1 - \frac{1}{\delta^2} \right) \quad (12)$$

Thus, Eq. 13 can predict driving torque representing the internal losses of a rubber track system due to the n th mode vibration of a track section strung between two consecutive wheels of the system:

$$M_{tv}^{(n)} = \frac{1}{4} EI A_n^2 \frac{(\pi n)^4}{l_0^3} \left(1 - \frac{1}{\delta^2} \right) \cdot f_{tv}^{(n)} / \omega_{dw} \quad (13)$$

In Eq. 13, the amount of energy lost due to damping properties of the track is defined only by Eq. 12. The formula representing the frequency of the track vibration in mode n has to be substituted into Eq. 13 only for the purpose of calculating the power loss resulting from a single cycle of the track vibration. Namely, to calculate the power loss, the energy loss attributed to that phenomenon has to be divided by the vibration period which is in reciprocal relationship to the frequency.

According to the derivations presented in [1], Eq. 14 represents the n th mode vibration frequency for a perfectly elastic beam:

$$f_{tv}^{(n)} = \frac{1}{2\pi} \left(\frac{\pi n}{l_0} \right)^2 \sqrt{\frac{EI}{m_{tr,unt}}} \sqrt{1 + \frac{F_{ta} l_0^2}{\pi^2 n^2 EI}} \quad (14)$$

For the purpose of the study presented herein, an assumption is made that Eq. 14 is also suitable for representing the frequency of the n th mode vibration of real rubber tracks, which in fact exhibit some amount of damping. This assumption can be justified as follows. Rubber tracks for off-road vehicles are intentionally made from low-damping rubbery materials to avoid overheating arising from repetitive bending cycles induced on the tracks while passing over the undercarriage wheels. Consequently, they exhibit a relatively small amount of damping. As it was mentioned earlier, in the experimental study by Takano et al. [35], a modal damping ratio for a sample commercially available rubber-band track was estimated at no more than $\zeta = 0.03$. According to the handbook by Osiński [27], if damping of a given mechanical system is small, it does not lead to a significant change in the frequency response of the system. This has been confirmed by Scholar and Perkins [31], and Wang et al. [38] for sample flexible link tracks with rubber-bushed pin joints. The research by Scholar and Perkins [31], and Wang et al. [38] indicate that the theoretical models that do not account for damping properties of the tracks can predict the frequency response of the state-of-the-art flexible tracks with the accuracy that is acceptable for the purpose of engineering computations, which has been discussed in more details in Sect. 1. Furthermore, the independent considerations undertaken by the authors to estimate the influence of

structural damping on the frequency response of the track investigated in this article are presented in Appendix 1. It has been shown that the frequency of the 1st mode vibration of the track investigated in this article is smaller than the frequency of the 1st mode vibration of a similar idealized track with no damping by no more than 0.035%.

3 Determination of model parameters

Computations according to the algorithm derived in Sect. 2 can be carried out if the length and the unit mass of a track strand subjected to transverse vibrations as well as the bending stiffness and the ratio of successive amplitudes of the 1st mode damped free vibration of the track are known. The consecutive sections of the article present the materials and methods developed to determine those parameters.

3.1 Determination of the track bending stiffness

Bending stiffness of the track investigated in the scope of this article was identified by a test stand presented in Fig. 3. The stand is a rubber-track system including a rubber-band track wrapped around an idler and a driving wheel. The idler wheel is coupled with a hydraulic cylinder for adjusting track tension. Since the road wheels, visible in Fig. 3, could badly affect the results of the experiments, they had been dismantled before the experiments started. While propelling the track, zero or finite amount of traction force can be applied to the lower track section

Fig. 3 Test stand involved in determination of bending stiffness and the ratio of successive amplitudes of vibrations as well as the resistance arising from vibrations of the Goodyear TM 8276 rubber belt for heavy duty agricultural machinery; the stand funded by German companies IAMT (Weischlitz), IBAF (Bochum) and Intertractor (Gevelsberg); *CB* fiber string, *DW* driving wheel, *LS* laser position sensor, *IW*

idler wheel, *TD* tensioning device, *BR* brake, *HPU* hydraulic power unit, *WG* weights, F_s track tension sensor, M_n driving torque sensor, ω_{dw} drive wheel angular velocity sensor, y_L , y_C track deflection measured with a laser position sensor and a caliper; *DAQ*, *PC* HBM QuantumX MX840 data acquisition system and a computer

by means of a break pressed against the outer surface of the track. More detailed description of the stand might be found in [10].

Two types of experiments were carried out to determine bending stiffness of the track. Firstly, the sag, i.e., the deflection of the upper track strand, resulting from the weight of the track was investigated. The sag was measured at the midpoint between the axles of the idler and the driving wheel with the SENSOPART FT 50 RLS 220-SIL8 laser position sensor and a standard caliper measuring tool featuring a depth rod, i.e., a tip for measuring the depth of holes, grooves, etc. The laser position sensor was mounted to the frame of the test stand with the laser beam pointing at the inner track surface at the right angle to that surface (see *LS*, Fig. 3). Since the relationship between the sag of the track and the signal delivered by the laser position sensor was unknown at the beginning of the experiments, additional measurements involving the caliper measuring tool were simultaneously conducted. A thin fiber string was strung on the top surface of the upper track strand for the purpose of those measurements. The tension of the string was maintained by the weights attached to each end of the string hanging on the opposite sides of the test stand, as depicted in Fig. 3. To determine the amount of the sag, the distance between the string and the outer surface of the upper track strand was carefully measured with the depth rod of the caliper.

Secondly, the frequency of the 1st mode damped free vibration of the upper track strand was investigated. To induce the vibration motion of the track, the track was manually displaced from the static equilibrium position and released so that it could return to that position again. The vibration frequency was calculated using Eq. 15, where $T_{iv,exp}$ denotes the track vibration period:

$$f_{iv,exp} = 1/T_{iv,exp} \quad (15)$$

The track vibration period was determined on the basis of the vibration traces recorded by the laser position sensor that was previously used to determine the sag of the track. A sample vibration trace obtained as a result of the experiments is shown in Fig. 4. The vibration period $T_{iv,exp}$ for a given experimental dataset was calculated as the average of the vibration periods determined for at least 30 arbitrary vibration cycles recorded during the experiment.

All the tests were carried out at zero track linear velocity and multiple different values of track tension. The tension of the track was measured indirectly, using a load cell embedded in the bolt connecting the idler wheel assembly and the rod of the hydraulic cylinder for adjusting the track tension.

After the experiments, bending stiffness of the track was estimated at the value minimizing the function given by Eq. 16:

$$\text{err}(\text{EI}) = \frac{\text{err}_y(\text{EI})}{\min [\text{err}_y(\text{EI})]} + \frac{\text{err}_f(\text{EI})}{\min [\text{err}_f(\text{EI})]} \quad (16)$$

$$\text{err}_y(\text{EI}) = \sum_{i=1}^N [y_{\text{est},i}(\text{EI}) - y_{\text{exp},i}]^2 \quad (17)$$

$$\text{err}_f(\text{EI}) = \sum_{i=1}^N [f_{\text{tv,est},i}(\text{EI}) - f_{\text{tv,exp},i}]^2 \quad (18)$$

$N(-)$ is the number of experimental datasets involved in estimating bending stiffness of the investigated track; each dataset includes the values representing the tension, the sag, and the frequency of the 1st mode damped free vibration of the track, $y_{\text{exp},i}$ (m) is the track deflection measured in the i th experiment at the actual track tension $F_{\text{ta},i}$, $y_{\text{est},i}$ (m) is the theoretical track deflection calculated with Eq. 19 assuming the track tension $F_{\text{ta},i}$ and some specific bending stiffness of the track EI, $f_{\text{tv,exp},i}$ (m) is the frequency of the 1st mode damped free vibration of the upper track strand measured in the i th experiment at the actual track tension $F_{\text{ta},i}$, $f_{\text{tv,est},i}$ (m) is the theoretical frequency of the 1st mode damped free vibration of the upper track strand calculated with Eq. 14 assuming the track tension $F_{\text{ta},i}$ and some specific bending stiffness of the track EI.

The theoretical frequency of the 1st mode damped free vibration of the upper track strand $f_{\text{tv,est},i}$ was calculated according to Eq. 14, substituting $n=1$. On the other hand, the theoretical deflection of the track $y_{\text{est},i}$ is predicted with Eq. 19:

$$y\left(\frac{l_0}{2}\right) = \frac{2\text{EI}q}{F_{\text{ta}}^2} \cdot \frac{\exp\left(l_0/2 \cdot \sqrt{F_{\text{ta}}/\text{EI}}\right)}{\exp\left(l_0 \cdot \sqrt{F_{\text{ta}}/\text{EI}}\right) + 1} + \frac{ql_0^2}{8F_{\text{ta}}} - \frac{\text{EI}q}{F_{\text{ta}}^2} \quad (19)$$

Equation 19 was derived assuming that the upper track strand could have been represented by a model beam presented in Fig. 5. According to this assumption, deflection of the track was described by the differential equation given by Eq. 20, whose solution is represented by Eqs. 22 and 23. The boundary conditions assumed for the purpose of determining the D_1 and D_2 parameters are given by Eq. 21.

$$\frac{d^2y}{dx^2} - \frac{F_{\text{ta}}}{\text{EI}}y = -\frac{ql_0}{2\text{EI}}x + \frac{q}{2\text{EI}}x^2 \quad (20)$$

$$y(0) = 0, y(l_0) = 0 \quad (21)$$

$$y(x) = D_1 \exp\left(x\sqrt{F_{\text{ta}}/\text{EI}}\right) + D_2 \exp\left(-x\sqrt{F_{\text{ta}}/\text{EI}}\right) - \frac{q}{2F_{\text{ta}}}x^2 + \frac{ql_0}{2F_{\text{ta}}}x - \frac{\text{EI}q}{F_{\text{ta}}^2} \quad (22)$$

Fig. 4 Vibration traces used to determine the frequency of the 1st mode damped free vibration and damping properties of the track investigated in the article—sample data recorded at the track tension of $F_s = 39$ kN

Fig. 5 Model beam involved in estimation of bending stiffness of a rubber track

$$\begin{cases} D_1 = \frac{EIq}{F_{ta}^2} \cdot \frac{1}{\exp(l_0 \sqrt{F_{ta}/EI}) + 1} \\ D_2 = \frac{EIq}{F_{ta}^2} \cdot \frac{\exp(l_0 \sqrt{F_{ta}/EI})}{\exp(l_0 \sqrt{F_{ta}/EI}) + 1} \end{cases} \quad (23)$$

3.2 Determination of the track damping properties

The δ factor defined by Eq. 8, i.e., the ratio of successive amplitudes of the 1st mode damped free vibration of the track investigated in this paper, was also determined using the test stand depicted in Fig. 3. The δ factor was calculated on the basis of the same vibration traces that were used to determine the frequency of the track free vibration (see Fig. 4b). The vibration traces were analyzed to find the influence of the vibration amplitude and track tension on the damping properties of the track (see Figs. 4b and 7). Since the method for processing those data is driven by the results of the experiments, more detailed description of this method is presented in Sect. 3.4.

It should be noted that in practice the δ factor was calculated according to Eq. 24, i.e., on the basis of the vibration amplitude before and after three complete cycles of track vibration:

$$\delta = \sqrt[3]{\frac{A_m}{A_{m+3}}} \quad (24)$$

Equation 24 was obtained as a result of the following derivations. First, Eq. 8 was extended to the representation given by Eq. 25. After that Eq. 25 was split into three separate equations: Eqs. 26, 27, and 28:

$$\delta = \frac{A_m}{A_{m+1}} = \frac{A_{m+1}}{A_{m+2}} = \frac{A_{m+2}}{A_{m+3}} \quad (25)$$

$$A_m = \delta A_{m+1} \quad (26)$$

$$A_{m+1} = \delta A_{m+2} \quad (27)$$

$$A_{m+2} = \delta A_{m+3} \quad (28)$$

To find the relationship between the vibration amplitude before and after three complete cycles of the track vibration, Eq. 28 was substituted into Eq. 27. The resultant formula was substituted into Eq. 26, which yielded Eq. 29. Finally, Eq. 29 was transformed into Eq. 24:

$$A_m = \delta^3 A_{m+3} \quad (29)$$

3.3 Determination of the track length and unit mass

To calculate the unit mass of the investigated track, the overall mass of the track was divided by its total length. Weight of the track was determined by means of the Dini Argeo MCWNT1 B1W2 load cell. Furthermore, an assumption has been made that the length of the upper track strand l_0 is equal to the distance between the axles of the idler and the driving wheel of the track system depicted in Fig. 3.

3.4 Results of the experiments

The materials and methods discussed in Sects. 3.1, 3.2, 3.3 were involved in estimating the model parameters for the 7.57 m long, 0.51 m wide Goodyear TM 8276 rubber track. The track does not have any stiffening metal molds embedded inside the rubbery envelope. It is actually a rubber belt reinforced with metal cords. The overall weight of the track is estimated at approximately 300 kg. Since the outer surface of the track features tread blocks, the thickness of the track changes periodically from 35 to 64 mm. The tracks of this type are broadly implemented in heavy-duty agricultural machines, e.g., 10 to 15 tonne tractors.

During the experiments, the actual track tension F_{ta} was set at several different values, where the minimum and the maximum tensions were 11.2 kN and 34.6 kN, respectively. Since mechanical properties of the track could vary depending on the investigated track section, the experiments at every considered track tension were repeated on at least five different track sections. To identify only the properties of the upper track strand, the lower one was held still by the break installed in the test stand (see BR, Fig. 3).

In Fig. 6a, the deflection of the investigated track is plotted against the actual track tension. On the other hand, Fig. 6b presents the frequency of the 1st mode damped free vibration of the track as a function of the track tension. The data obtained by the experiments described in Sect. 3.1 are compared with the results of model computations. The deflection of the upper track at the midpoint of the track span, represented by the curve shown in Fig. 6a, was calculated using Eq. 19, whereas the model computations of the frequency of damped free vibration of the track presented in Fig. 6b were carried out according to Eq. 14. The following model parameters were involved in those computations: $EI = 1758 \text{ Nm}^2$, $m_{tr,unt} = 39.71 \text{ kg/m}$, and $l_0 = 2.65 \text{ m}$.

According to further analyses, mean estimation error of the track deflection did not exceed 10% of the values determined experimentally. Referring to the theoretical

Fig. 6 Comparison of the experimental results and model computations carried out for a 2.65 m long section of the Goodyear TM 8276 rubber track; **a** deflection at the midpoint of the track section as a function of the actual track tension F_{ta} (see Fig. 5), **b** frequency of the 1st mode damped free vibration of the track section as a function of the actual track tension; the model computations carried out with the model parameters estimated at: $EI = 1758 \text{ Nm}^2$, $m_{tr,unt} = 39.71 \text{ kg/m}$, $l_0 = 2.65 \text{ m}$

computations of the frequency of the 1st mode track vibration, the analogous error was 1.8%. It should be noted that estimation of bending stiffness of the investigated track as well as the estimation error calculations were performed on the basis of two separate subsets of experimental data. Hence, the accuracy of track bending stiffness estimation might be claimed to be satisfactory.

The results of the experimental identification of the ratio of successive amplitudes of the 1st mode damped free vibration of the track investigated herein are presented in Fig. 7. As it was mentioned in Sect. 3.2, every vibration trace obtained as a result of the experiments described in Sect. 3.1 was used to identify the influence of the vibration amplitude and track tension on the damping properties of the investigated track. At first, the vibration traces were split into several datasets based on the amount of the tension applied to the track during the experiment. Each dataset was analyzed separately from the remaining ones. The δ factor was calculated for every recorded vibration cycle according to Eq. 24 and the method demonstrated in Fig. 4b. Afterwards, the δ factor was displayed as a function of the vibration amplitude. Figure 7a presents sample relationships between the δ factor and the amplitude of the track vibration obtained by processing the data recorded during several tests performed at the track tension of about 11.2 kN–14.9 kN. The results obtained for the remaining load cases are similar to the ones presented in Fig. 7a from a qualitative point of view. Therefore, it has been concluded that the δ factor is not significantly affected by the vibration amplitude.

After that the influence of the track tension on the δ factor was analyzed. According to Eq. 13, governing the model derived in Sect. 2, the low-amplitude vibrations of tracks are unlikely to result in significant energy losses. For that reason, this analysis was based on the values of the δ factor calculated with the data recorded at the track vibration amplitude exceeding an arbitrary threshold value, as depicted in Fig. 7a. The maximum ($\delta_{\max}$), the minimum

($\delta_{\min}$) and the average (δ_{mean}) value of the δ factor was found for each vibration trace, and Fig. 7b was obtained by plotting those values against the actual track tension. The curve presented in Fig. 7b was fitted to the $\delta_{\max}$ characteristics. The relatively small deviation of the maximum and the minimum values from the average ones suggests that the δ factor is not significantly affected by the track vibration amplitude. On the other hand, according to Fig. 7b, the δ factor decreases with an increase in the track tension, and it does not exceed $\delta = 1.176$.

4 Discussion

4.1 Model computations of losses due to rubber track vibrations

This section discusses the results of model computations of the losses caused by transverse vibrations of the upper strand of the Goodyear TM 8276 rubber track installed in the track system depicted in Fig. 3. The losses were calculated as a function of the mode number and the amplitude of track vibration as well as the tension and the linear velocity of the track. They are represented by the γ factor defined by the following equation:

$$\gamma(v_{tr}, F_{ta}) = \frac{M_{tv}^{(n)}(\omega_{dw}, F_{ta})}{M_{bnd}(\omega_{dw}, F_{ta})} \cdot 100\% \quad (30)$$

The $M_{tv}^{(n)}$ moment denotes the driving torque that needs to be applied to the drive wheel of the track system presented in Fig. 3 to overcome the losses caused by the n th mode track vibration of given amplitude. On the other hand, the $M_{tv}^{(n)}$ moment was predicted using the model given by Eqs. 13 and 14. The computations were carried out involving the parameters given in Sect. 3.4, including the characteristics presented in Fig. 7b.

Fig. 7 Experimental relationships between the δ coefficient (see Eq. 24) and: **a** the amplitude of the Goodyear TM 8276 rubber track transverse vibration in mode 1 at the track tension of 11.2 kN–14.9 kN (see Fig. 2), **b** the actual tension of the Goodyear TM 8276 rubber track

The M_{bnd} moment, introduced as a reference quantity in Eq. 30, represents the losses caused by bending of the investigated rubber track, while it passes over the wheels of the track system shown in Fig. 3. This quantity was chosen as the reference one, because the losses attributed to transverse vibrations and bending of rubber tracks are caused by mechanical hysteresis. The M_{bnd} moment was described by an empirical model fitted to the results of the experiments, where the track was running over the drive and the idler wheel of the track system depicted in Fig. 3 with no contact with the brake embodied in the stand. The tests were carried out in various conditions defined by multiple combinations of linear velocity and tension of the track. The torque applied to the drive wheel (M_n), the force delivered by the tensioning device (F_s), and the angular speed of the drive wheel (ω_{dw}) were measured in every trial using the sensors installed in the test stand (see Fig. 3).

It might be noted that bending resistance of the investigated track was not the only cause of the driving torque (M_n) determined during those tests. The driving torque resulted also from the resistance attributed to: friction in the bearings of the drive and the idler wheel, sliding friction in the interface between the guide flanges of the wheels and the guide lugs of the track, limited efficiency of transmitting power from the drive wheel to the track, and transverse vibrations of the track. Nevertheless, only the losses arising from bearings friction were regarded as severe disruption to the experiments. To eliminate them, experimental tests analogous to those that were carried out to determine bending resistance of the investigated track were conducted. The only difference was that the polyester belts with negligible bending resistance were installed in the test stand instead of the track. Afterwards, an empirical formula representing bearings friction was formulated and fitted to the measurement data. More elaborate considerations on the mathematical formulas describing the losses attributed to the bending resistance of the investigated track as well as the friction in the bearings of the drive and the idler wheel are not crucial for clarity of the considerations presented herein. They are described in more details in [4].

The losses caused by limited efficiency of transmitting power from the drive wheel to the track were neglected, because power transmission was provided by the friction in the interface between the inner track surface and the outer surface of the drive wheel. In the friction-drive track systems those losses are smaller than in the positive-drive ones, which is proved by the research experience of the Silsoe Research Institute summarized by Okello et al. in [26]. On the other hand, according to the literature review provided by Rehorn in [30], for the positive-drive rubber-track systems they can be estimated at a 1% of the force transmitted from the drive wheel to the track. Furthermore, in the friction-drive track systems, the efficiency of transmitting

power from the drive wheel to the track is governed mainly by the slip in the interface between the wheel and the track which, according to the experimental research on the belt drives presented by Hohmann [14], is likely to occur only in the conditions, where the difference in the track tension at the point of running onto and running away from the drive wheel is high relative to the initial track tension. During the experiments this difference was rather small.

To minimize the resistance arising from sliding friction in the interface between the track guide lugs and the guide flanges of the drive and the idler wheel, the position of the idler wheel relative to the drive wheel was adjusted so that the track had no tendency to move either to the left- or to the right-hand side of the wheels while running. Furthermore, guide lugs of the track were covered with talc powder to reduce friction.

To assure that the model describing the resistance arising from the bending of the investigated track was not badly affected by the energy losses resulting from the track vibrations, it was fitted only to the data recorded, while the amplitude of the track vibration was smaller than some a priori defined threshold value. The amplitude was measured using the laser position sensor depicted in Fig. 3.

Eventually, the experiments showed that the M_{bnd} depends on the linear velocity and the tension of the investigated track; however, the influence of the track tension was negligibly small. An example relationship between the M_{bnd} moment and the track linear speed at the actual track tension of about 14 kN–17 kN is presented in Fig. 8. The experimental data are represented by the points highlighted in black, whereas the values calculated according to the empirical model are illustrated with the dashed curve. The values calculated according to the model represented by the curve were substituted into Eq. 30. The results of the computations according to Eq. 30 are shown in Figs. 9, 10 and 11.

The results of the model computations correspond well with Eq. 13 from a qualitative point of view. The losses caused by transverse vibrations of the Goodyear TM 8276 rubber track predicted by the model are proportional to the vibration amplitude raised to the power of 2 and inversely proportional to the track linear velocity. From a quantitative point of view, according to the model predictions, the 1st mode track vibration should not lead to a noticeable increase in the internal motion resistance of a vehicle equipped with tracks similar to the investigated ones, regardless of the vibration amplitude. On the other hand, harmonic components corresponding to the higher-order mode vibrations are expected to result in a significant increase in the undercarriage internal resistance, especially at low vehicle speed. This situation might be explained by the fact that the energy lost in one complete cycle of track vibration (see Eq. 7) as well as the vibration frequency (see Eq. 14) are described by the functions, where the

Fig. 8 Resistance arising from bending of the Goodyear TM 8276 rubber track while passing over the drive and idler wheels of the track system depicted in Fig. 3 as a function of the track linear speed, determined at the track tension of about 14 kN–17 kN

mode number n is raised to the power that is greater than 1. Consequently, if the track is subjected to high-order mode vibrations, a relatively large amount of energy is lost in a short period of time. Hence, the power loss attributed to track vibrations is high. Considering higher-order mode track vibrations of certain amplitude, it might be concluded that an increase in track tension results in a decrease in the losses due to track vibrations predicted by the model, which might be explained by the fact that the ψ_{tv} coefficient representing damping of the track decreases with increasing track tension (see Eq. 12 and Fig. 7b).

Fig. 9 Results of theoretical computations of the losses caused by the 1st mode vibration of the upper section of the Goodyear TM 8276 rubber belt installed in the track system depicted in Fig. 3 as a function of vibration amplitude at: **a** track linear speed of 10 km/h, **b** track tension of 15 kN; the losses expressed with the γ factor defined by Eq. 30

4.2 Preliminary tests on the losses due to rubber track vibrations

This section presents a discussion on the results of the preliminary experimental tests on the Goodyear TM 8276 rubber belt installed in the track system depicted in Fig. 3, where the M_{bnd} moment defined in the previous section was determined in the conditions, where the track was subjected to vibrations of noticeably high amplitude. In those conditions, the M_{bnd} moment represents the resistance arising not only from bending of the track but also from its vibrations.

During those preliminary tests it was possible to induce only the 1st mode vibration on the track. The results of the tests are represented by the points highlighted in light grey in Fig. 8, where they are compared with their counterparts obtained as a result of the tests at low-amplitude track vibrations. On the other hand, Fig. 12 brings the information on the root mean square amplitude of the vibration induced on the track, where the abscissa represents the velocity of the track while running over the wheels. The following observations might be made on the basis of Figs. 8 and 12.

- High-amplitude vibrations of the investigated track result in an increase in the M_{bnd} moment.
- However, the above-mentioned increase is smaller than the deviation of the results of the experimental tests on the bending resistance of the track.
- High-amplitude track vibrations are recorded in very narrow speed ranges.

As a result of those observations, the following approach was adopted during the preliminary tests on the losses arising from the vibrations of the investigated track.

- Every experiment was carried out at gradually changing track speed so that the M_{bnd} moment could have been determined at the speed either exactly equal or slightly

Fig. 10 Results of theoretical computations of the losses caused by the 2nd mode vibration of the upper section of the Goodyear TM 8276 rubber belt installed in the track system depicted in Fig. 3 as a function of vibration amplitude at: **a** track linear speed of 10 km/h, **b** track tension of 15 kN

Fig. 11 Results of theoretical computations of the losses caused by the 3rd mode vibrations of the upper section of the Goodyear TM 8276 rubber belt installed in the track system depicted in Fig. 3 as a function of vibration amplitude at: **a** track linear speed of 10 km/h, **b** track tension of 15 kN

Fig. 12 Root mean square amplitude of the 1st mode vibration of the upper strand of the Goodyear TM 8276 rubber track while running over the wheels of the track system depicted in Fig. 3 as a function of the track linear speed; the characteristics recorded during preliminary tests carried out at the track tension of 14 kN–17 kN

different from the one that corresponded to the resonant vibrations of the track.

- Every dataset obtained as a result of a single experiment was analyzed independently from the remaining data. A chart similar to the one presented in Fig. 13 was prepared for every dataset and the resistance resulting from track vibrations of certain amplitude was calculated on the basis of the distinguishing points of the chart (Lo and Hi , Fig. 13).

Table 2 summarizes the results of the analyses for several datasets.

According to Fig. 12, root mean square amplitude of the 1st mode vibration induced on the investigated track during the preliminary experiments presented herein did not exceed 25 mm. The resistance arising from this phenomenon was estimated at no more than 8.1% of the bending resistance exhibited by the track while passing over the wheels of the

Fig. 13 Example relationship between root mean square amplitude of the 1st mode vibration of the upper strand of the Goodyear TM 8276 rubber track while running over the wheels of the track system depicted in Fig. 3 and the M_{bnd} moment representing the resistance due to bending and vibrations of the track; the data recorded at the track linear speed of 7.72 m/s and the track tension of 14 kN–17 kN (see Table 2, row 8)

test stand shown in Fig. 3. According to the model computations performed in the previous section, in the conditions considered in the experimental study discussed herein those losses should not exceed 4%. Although the model underestimates the results of the preliminary experimental tests, the results of the model computations and the experimental tests are similar in magnitude, which is quite promising from the point of view of more detailed verification that will be carried out by the authors in the future.

5 Summary and conclusions

The article might be summarized with the following conclusions.

- The authors derive a theoretical model for predicting the internal motion resistance of rubber-tracked vehicles due to transverse vibrations of the tracks. The model is

intended to be used mainly for engineering purposes, for example, to determine the demanded performance of power units and drivetrain components of the new generation rubber-tracked vehicles. Thanks to the simplifying assumptions that have been made while formulating the model, it includes a small number of parameters that can be determined with relatively simple experimental methods. In spite of those simplifications, the model is supposed to provide the accuracy that is suitable for potential engineering applications.

- An experimental method for determining bending stiffness as well as the coefficient representing structural damping of rubber tracks was discussed and demonstrated in the article. Both of those parameters are essential for predicting the losses arising from rubber-track vibrations using the model formulated herein. They are crucial for estimating the resistance to motion in rubber-tracked vehicles arising from other phenomena as well. For example, they are required to calculate the soil sinkage under the tracks with the model developed by Stefanow [34]. Since the method proposed herein exhibits a relatively small estimation error, it might be involved in similar laboratory tests on other types of reinforced rubber belts implemented in a number of machines, for example, in the tests of conveyor belts. The experiments presented in Sect. 3 were carried out on the Goodyear TM 8276 rubber track, which is a representative sample of tracks broadly implemented in heavy duty off-road vehicles, e.g., agricultural or earthmoving mobile machines. Consequently, the values of the parameters determined herein might be useful in model computations pertaining to a wide variety of vehicles.
- The resistance arising from vibrations of the rubber track investigated in the article was computed using the aforementioned model and parameters. To some extent, it was also determined in preliminary experimental tests. The results of the model computations and the preliminary tests lead to a conclusion that the losses caused by the 1st mode vibration of the investigated track are negli-

Table 2 Increase in internal resistance of the rubber track system depicted in Fig. 3 resulting from the 1st mode track vibration of noticeably high amplitude

l.p	Track linear speed (m/s)	RMS track vibrations amplitude		Increase in track vibrations amplitude (mm)	M_{bnd}		δM_{bnd} (%)
		Lo (mm)	Hi (mm)		Lo (mm)	Hi (mm)	
1	2.23	3.7	5.5	1.8	121.8	125.8	3.3
2	3.46	2.5	15.8	13.3	110.6	112.0	1.3
3	7.85	3.5	14.8	11.3	139.4	143.2	2.7
4	7.71	2.7	15.4	12.7	131.2	137.8	5.0
5	7.72	3.6	20.8	17.2	124.7	127.0	1.8
6	7.72	4.0	21.3	17.3	125.1	129.3	3.4
7	7.72	2.8	23.8	21.0	127.9	134.5	5.2
8	7.72	3.9	16.8	12.9	131.4	141.9	8.0

gibly small in relation to the bending resistance manifested by the track while passing over drive and idler wheels, regardless of the amplitude of vibration and the linear speed of the track. On the other hand, according to the model computations, those losses are supposed to be exceptionally high if the track were subjected to higher-order mode high-amplitude vibrations at low linear speed of the track. In the future, the authors will conduct experiments to confirm this conclusion. These studies will preferably involve the field tests on a real rubber-tracked vehicle so that it will be possible to verify whether high-order mode high-amplitude vibrations of rubber tracks might occur in real operating conditions.

Appendix 1: On the influence of structural damping on the frequency response of rubber tracks

According to Skrodzka [33], vibrations of a multiple-degree-of-freedom (multi-DOF) mechanical system can be represented by a superposition of a number of independent single-DOF oscillators, where the mass, the stiffness, and the damping properties of every single-DOF oscillator is selected such that each oscillator represents one harmonic component of the multi-DOF system. This approach is the basis of the experimental modal analysis and can be successfully applied to the linear mechanical systems and those non-linear ones that can be linearized. Furthermore, according to the handbook by Pretlove [29], the above theory is applicable also to the continuous mechanical systems, e.g., beams, plates, etc. In the previous sections of the article, an assumption has been made that rubber-band tracks, including the sample track investigated in this paper, can be represented by a linear continuous mechanical system, i.e., a model beam with constant bending stiffness EI (see Fig. 2). Consequently, the authors adopted the above-mentioned approach to discuss in quantitative terms the influence of damping properties revealed for the track investigated in this article on the frequency response of that track.

Figure 4 presents a sample vibration trace representing the 1st mode damped free vibration of the rubber-band track investigated in this paper. According to Fig. 7, very similar traces were obtained as a result of all the experimental tests discussed herein. Assuming that the 1st mode vibration of the track investigated in this article might be characterized as vibration of an equivalent single-DOF oscillator (see Fig. 14) with properly tuned mass (m), stiffness (k), and damping (b), the damped free vibration of that track in mode 1 can be described by Eq. 31. In other words, the trace presented in Fig. 4 is governed by a function that fulfills the differential equation given below:

Fig. 14 Equivalent single DOF harmonic oscillator representing the 1st mode damped free vibration of a rubber track

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \quad (31)$$

Equation 31 can be used to estimate how the frequency of the 1st mode free vibration of the track investigated in the article is affected by its damping properties. In particular, the relationship might be found between the free vibration frequencies of the real track and the idealized one that is similar to the real track in terms of mass and stiffness but exhibits no damping. To do so, let Eq. 31 be first transformed into Eq. 32 by applying the standard substitutions given by Eqs. 33 and 34. The ω_0 denotes the angular frequency of the free vibration of the system with no damping:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad (32)$$

$$\beta = \frac{b}{2m} \quad (33)$$

$$\omega_0^2 = \frac{k}{m} \quad (34)$$

Figure 4 shows that structural damping of the track investigated herein is so small that the track oscillates with gradually decreasing amplitude while returning to the static equilibrium position. From the point of view of mathematical derivations, the solution of Eq. 32 is represented by a periodic function if the roots of the characteristic equation of Eq. 32 are given by Eq. 35 and Eq. 36, i.e., they belong to the complex numbers:

$$r_1 = -\beta - i\sqrt{\omega_0^2 - \beta^2} \quad (35)$$

$$r_2 = -\beta + i\sqrt{\omega_0^2 - \beta^2} \quad (36)$$

The exact solution of Eq. 32 is given by Eq. 37 in those circumstances. Therefore, the angular frequency of free vibration (ω) of a damped single-DOF oscillator that is equivalent to the investigated rubber track is given by Eq. 38:

$$x(t) = e^{-\beta t} \left\{ A \sin \left[\left(\sqrt{\omega_0^2 - \beta^2} \right) t \right] + B \cos \left[\left(\sqrt{\omega_0^2 - \beta^2} \right) t \right] \right\} \quad (37)$$

$$\omega = \sqrt{\omega_0^2 - \beta^2} \quad (38)$$

Considering a single-DOF oscillator with a certain mass and stiffness, a formula describing the ratio of the free vibration frequencies of the oscillator with zero and non-zero damping will be derived on the basis of Eq. 38. Damping properties of the oscillator will be expressed with the logarithmic decrement Λ defined by Eq. 9 for the purpose of those derivations.

According to the transformations given by Eq. 39, it might be shown that the relation between the logarithmic decrement (Λ), the vibration period of the oscillator with non-zero damping (T), and the β factor defined by Eq. 33 is governed by Eq. 40. Equation 39 was obtained by merging Eqs. 9 and 37:

$$\Lambda = \ln \frac{A_m}{A_{m+1}} = \ln \frac{x(t)}{x(t+T)} = \ln \frac{\exp(-\beta t)}{\exp(-\beta(t+T))} = \ln \frac{1}{\exp(-\beta T)} \quad (39)$$

$$\Lambda = \beta T \quad (40)$$

Equation 40 can be transformed into Eq. 42 if the standard formula given by Eq. 41 is substituted into Eq. 40:

$$T = \frac{2\pi}{\omega} \quad (41)$$

$$\beta = \frac{\Lambda\omega}{2\pi} \quad (42)$$

Furthermore, substituting Eq. 42 into Eq. 38 yields Eq. 43. Finally, Eq. 46 might be obtained by the transformations given below:

$$\omega = \sqrt{\omega_0^2 - \frac{\Lambda^2\omega^2}{4\pi^2}} \quad (43)$$

$$\omega^2 = \omega_0^2 - \frac{\Lambda^2\omega^2}{4\pi^2} \quad (44)$$

$$\omega^2 \cdot \left(1 + \frac{\Lambda^2}{4\pi^2}\right) = \omega_0^2 \quad (45)$$

$$\frac{f_0}{f} \equiv \frac{\omega_0}{\omega} = \sqrt{1 + \frac{\Lambda^2}{4\pi^2}} \quad (46)$$

Summing up the above derivations, Eq. 46 represents the ratio of free vibration frequencies of two single-DOF oscillators with identical mass and stiffness, where f_0 and f denote the free vibration frequency of the oscillator with zero and non-zero damping, respectively. Equation 46 draws a conclusion that the smaller the damping of a rubber track, the higher the frequency of the 1st mode damped free vibration

Fig. 15 Values of logarithmic decrement describing damping properties of the rubber track investigated in the article as a function of the track tension; the values calculated with the data presented in Fig. 7b and Eq. 9

Fig. 16 Ratio of the free vibration frequencies of the equivalent single-DOF oscillator representing the rubber track investigated in the article assuming zero and non-zero damping; the values calculated with the data presented in Fig. 15 and Eq. 46

of the track. Consequently, considering the free vibration frequencies for a group of tracks with given stiffness and mass properties, the idealized track with no damping will manifest the highest possible free vibration frequency. The difference between the free vibration frequencies of such idealized track and the real rubber-band track investigated in the scope of this article will be discussed in quantitative terms on the basis of Figs. 15 and 16.

Figure 15 presents the experimental relationship between the logarithmic decrement describing structural damping of the rubber track investigated in this article and the actual tension of that track. The values presented in Fig. 15 were calculated by applying Eq. 9 to the data depicted in Fig. 7b. It might be noted that the logarithmic decrement describing damped free vibration of the track investigated herein does not exceed 0.17. On the other hand, Fig. 16 illustrates the results of the computations involving the formula given by Eq. 46 and the data presented in Fig. 15. They suggest that the influence of structural damping on the frequency of the 1st mode free vibration of

the track investigated in this article is in fact negligibly small. Namely, if the investigated track manifested no damping, the frequency of the 1st mode free vibration of that track would exceed the one that has been actually revealed by experiments by no more than 0.035%. The analysis presented above can be generalized to the higher-order mode vibrations. It should be emphasized that the model describing the resistance arising from the transverse vibrations of rubber tracks formulated in this article has been developed mainly for engineering applications, e.g., for the purpose of determining the demanded performance of power units and drive train components of rubber-tracked vehicles. The error of the above-mentioned magnitude is not significant for this application. Furthermore, according to the analysis presented in this section, Eq. 14 is likely to slightly overestimate the actual frequency of the n th mode vibration for real rubber tracks. Consequently, Eq. 13 will overestimate the resistance arising from the transverse vibrations of rubber tracks in a very similar way. The error of this kind can be seen as a safety factor. Therefore, it might be concluded that although Eq. 14 represents in fact the n th mode vibration frequency for perfectly elastic beams that do not exactly correspond with the nature of real rubber tracks, it provides the accuracy that is acceptable from the point of view of the potential practical applications of the model formulated in this article as well.

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Declarations

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