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ORIGINAL RESEARCH

Worst-Case Analysis of a New Heuristic for the Travelling Salesman Problem

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Abstract

An $O(n^3)$ heuristic algorithm is described for solving D-city travelling salesman problems (TSP) whose cost matrix satisfies the triangularity condition. The algorithm involves as substeps the computation of a shortest spanning tree of the graph G defining the TSP and the finding of a minimum cost perfect matching of a certain induced subgraph of G. A worst-case analysis of this heuristic shows that the ratio of the answer obtained to the optimum TSP solution is strictly less than 3/2. This represents a 50% reduction over the value 2 which was the previously best known such ratio for the performance of other polynomial growth algorithms for the TSP.

1 Introduction

Heuristic algorithms with polynomial rates of growth in the number of variables can be used to provide approximate solutions to combinatorial problems. The question then arises as to what is the worst possible ratio of the value of the answer obtained by the heuristic to the value of the optimum solution. We will denote this worst-case ratio by R_w.

Values of R_w for the graph-coloring problem have been investigated by Garey and Johnson [4] who showed that finding a polynomial growth graph-coloring algorithm with $R_w < 2$ is just as hard as finding a polynomial algorithm for optimal coloring. For the loading (packing) problem [3, 5], Johnson et al. described an algorithm with $R_w \leq 11/9$. Rosenkrantz, Stearns, and Lewis [7] investigated a variety of heuristics

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for the travelling salesman problem. For the best of the algorithms investigated in [7], $R_w \to 2$, as n, the number of cities in the travelling salesman problem (TSP), tends to be ∞.

In this paper, we describe a heuristic algorithm with $O(n^3)$ growth rate and for which $R_w < 3/2$ for all n. This represents an improvement of 50% over the previously best known value of R_w for the TSP.

2 The Main Result

Consider the n-city TSP defined on the complete graph $G=(X, A)$ where X is the set of vertices and A is the set of links. Let the link cost matrix be $[c_{ij}]$ which satisfies the triangle inequality.

Let $T^* = \left(X, A_{T^*}\right)$ be the shortest spanning tree (SST) of the graph G, and let $C(T^*)$ be the cost of T^*. Let:

$$X^{\mathrm{O}}(T^*) = \left\{x_i | d_i(T^*)\mathrm{odd}\right\}$$

where $d_i(T^*)$ is the degree of vertex $x_i^* \in X$ with respect to the T^*. The cardinality $\left|X^{\mathrm{O}}(T^*)\right|$ of the set $X^{\mathrm{O}}(T^*)$ is always even [1].

Consider now the subgraph $<X^{\mathrm{O}}(T^*)>$ induced by the set $X^{\mathrm{O}}(T^*)$ of vertices. Since $\left|X^{\mathrm{O}}(T^*)\right|$ is even, a perfect matching in $<X^{\mathrm{O}}(T^*)>$ always exists. A matching is called "perfect" [1] if it contains exactly $1/2\left|X^{\mathrm{O}}(T^*)\right|$ links. Let $M_{\mathrm{O}}^* = \left(X^{\mathrm{O}}(T^*), A_{M_{\mathrm{O}}^*}\right)$ be the minimum-cost perfect matching of $<X^{\mathrm{O}}(T^*)>$ and $C\left(M_{\mathrm{O}}^*\right)$ be its cost.

We can now state the following theorem:

Theorem 1 *A Hamiltonian circuit Φ_{H} of G can be found with cost $C\left(\Phi_{\mathrm{H}}\right) \leq C(T^*) + C\left(M_{\mathrm{O}}^*\right) < \frac{3}{2}C(\Phi^*)$ where $C(\Phi^*)$ is the optimal value of the TSP tour Φ^*.*

In the proof of Theorem 1, we will make use of the following Lemma.

Lemma 1 *For an n-city TSP with n even, we have $C(M^*) \leq \frac{1}{2}C(\Phi^*)$, where M^* is the minimum-cost perfect matching of the graph G defining the TSP and Φ^* is the optimal TSP tour.*

Proof Consider $\Phi^* = \left(x_{i_1}, x_{i_2}, \ldots, x_{i_n}\right)$. Starting from vertex x_{i_1} and travelling round the circuit Φ^*, allocate the links traversed in an alternating manner to two sets M_1 and M_2. Starting with M_1, for example:

$$M_1 = \left\{\left(x_{i_1}, x_{i_2}\right), \left(x_{i_3}, x_{i_4}\right), \ldots, \left(x_{i_{n-1}}, x_{i_n}\right)\right\}$$

and

$$M_2 = \left\{\left(x_{i_2}, x_{i_3}\right), \left(x_{i_4}, x_{i_5}\right), \ldots, \left(x_{i_n}, x_{i_1}\right)\right\}$$

M_1 and M_2 are matchings of G and:

$$C(M_1) + C(M_2) = C(\Phi^*)$$

Since M_1 and M_2 are defined arbitrarily, we can assume $C(M_1) \leq C(M_2)$ without loss of generality, and so we have:

$$C(M^*) \leq C(M_1) \leq \frac{1}{2}C(\Phi^*)$$

Hence the Lemma.

Proof of Theorem 1 It is well known [2] that for a graph G.

$$C(T^*) \leq C\left(\Phi_p^*\right) < C(\Phi^*) \tag{1}$$

where Φ_p^* is the shortest Hamiltonian path of G. (The last inequality becoming $\leq$ if zero-cost links are allowed.)

The graph $G^e = \left(X, A_{T^*} \cup A_{M_O^*}\right)$—which is a ε partial graph of G—is Eulerian, i.e., has all vertices of even degree, since M_O^* is a matching of all odd degree vertices of T*. Hence, G^e contains an Eulerian circuit $\Phi^e = \left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$. Since Φ^e traverses all the links of G^e it also visits all the vertices $x_i^* \in X$ at least once. Let $C(\Phi^e)$ be the cost of Φ^e, i.e.,

$$C(\Phi^e) = C(T^*) + C\left(M_O^*\right) \tag{2}$$

If Φ_O^* is the TSP solution to the problem defined by the induced subgraph $< X^O(T^*) >$, then we have from Lemma $1, C\left(M_O^*\right) \leq \frac{1}{2}C\left(\Phi_O^*\right)$ and since $C\left(\Phi_O^*\right) \leq C(\Phi^*)$ we immediately obtain

$$C\left(M_O^*\right) \leq \frac{1}{2}C(\Phi^*) \tag{3}$$

From expressions (1), (2), and (3), it follows that:

$$C(\Phi^e) < \frac{3}{2}C(\Phi^*) \tag{4}$$

Consider the traversal of Φ^e starting from x_{i_1} up to the point when a vertex x_{i_r} is reached which has been visited previously — i.e., $x_{i_r} = \left[x_{i_1}, \ldots, x_{i_{r-1}}\right]$. Let x_{i_s} be the first vertex following x_{i_r} in the sequence of Φ^e which has not been previously visited and consider the circuit $\Phi_1 = \left(x_{i_1}, \ldots, x_{i_{r-1}}, x_{i_n} \ldots, x_{i_k}\right)$ derived from Φ^e by replacing the path $P_{rs} = \left(x_{i_{r-1}}, x_{i_r} \ldots, x_{i_{s-1}}, x_{i_s}\right)$ with the single link $\left(x_{i_{r-1}}, x_{i_s}\right)$. Because of the triangularity condition, we have:

$$c_{i_{r-1} i_s} \leq \sum_{(x_i, x_j) \in P_{rs}} c_{ij}$$

where P_{rs} is also used as an unordered set of the links on the path P_{rs}. Hence, we have $C(\Phi^e) \geq C(\Phi_1)$.

In the same way, starting with a traversal of Φ_1 a circuit Φ_2 can be produced with a path of Φ_1 replaced by a direct link and $C(\Phi_1) \geq C(\Phi_2)$. Eventually, a Hamiltonian circuit Φ_H of G will result with the following:

$$C(\Phi_H) \leq \ldots \leq C(\Phi_1) \leq C(\Phi^e) < \frac{3}{2}C(\Phi^*)$$

Hence the Theorem.

The algorithm implied by Theorem 1 consists of two parts: the calculation of an SST and finding a minimum-cost perfect matching. Several good $0(n^2)$ algorithms exist for finding the SST of a graph [1]. The best known algorithm for calculating minimum matchings is one developed by Lawler [6] and has growth rate $0(n^3)$. The overall growth rate of the proposed algorithm is — therefore —$0(n^3)$. (Note that the last step of converting Φ^e to a Hamiltonian circuit Φ_H can be done in linear time.)

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