Operations Research Forum (2021) 2: 30
https://doi.org/10.1007/s43069-021-00077-w

ORIGINAL RESEARCH

On a Maximum Eigenvalue of Third-Order Pairwise Comparison Matrix in Analytic Hierarchy Process and Convergence of Newton's Method

Shunsuke Shiraishi[1] · **Tsuneshi Obata**[2]

Received: 6 April 2021 / Accepted: 22 June 2021 / Published online: 8 July 2021
© The Author(s) 2021

Abstract
Nowadays, the analytic hierarchy process is an established method of multiple criteria decision making in the field of Operations Research. Pairwise comparison matrix plays a crucial role in the analytic hierarchy process. The principal (maximum magnitude) eigenvalue of the pairwise comparison matrix can be utilized for measuring the consistency of the decision maker's judgment. The simple transformation of the maximum magnitude eigenvalue is known to be Saaty's consistency index. In this short note, we shed light on the characteristic polynomial of a pairwise comparison matrix of third order. We will show that the only real-number root of the characteristic equation is the maximum magnitude eigenvalue of the third-order pairwise comparison matrix. The unique real-number root appears in the area where it is greater than 3, which is equal to the order of the matrix. By applying usual Newton's method to the characteristic polynomial of the third-order pairwise comparison matrix, we see that the sequence generated from the initial value of 3 always converges to the maximum magnitude eigenvalue.

Keywords Decision support systems · Analytic hierarchy process · Pairwise comparison matrix · Newton's method · Convergence

Dedicated to Professor Hidefumi Kawasaki on the occasion of his retirement from Kyushu University.

✉ Tsuneshi Obata
t-obata@oita-u.ac.jp

Shunsuke Shiraishi
shira@eco.u-toyama.ac.jp

1 School of Economics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan

2 Faculty of Science and Technology, Oita University, 700 Dan-noharu, Oita 870-1192, Japan

1 Introduction

Analytic hierarchy process (AHP) is a scaling method for priorities in the hierarchical structure [1, 2]. When people (decision makers) make decisions, they should often treat multiple alternatives. Decision makers' main task is to define the priorities of alternatives.

Pairwise comparison is one of the most tangible ways to prioritize them. Let $C_i, i = 1, \dots, n$ be alternatives to be prioritized. One defines the value a_{ij} in pairs between C_i and C_j. The value represents the priority of C_i as compared to C_j. In the context of AHP, one usually uses its value by Saaty's discrete scale from 1 to 9 and their reciprocals. It is based on the verbal expressions in Table 1 [1].

For instance, suppose that one wishes to determine his/her preference for fruits. In comparison of apples (C_i) and oranges (C_j), if he/she prefers apples to oranges moderately, a_{ij} is assigned to 3 and a_{ji} to 1/3. Logically the value a_{ji} in pairs between C_j and C_i should be $1/a_{ij}$. After all pairwise comparisons, one obtains the following so-called *pairwise comparison matrix*:

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}.$$

In AHP, employing a pairwise comparison matrix enables the treatment of subjective judgment.

In the context of AHP, evaluating the principal (maximum magnitude) eigenvalues of pairwise comparison matrices is a key ingredient. In AHP, solving the following linear system is called by the eigenvector method:

$$Aw = \lambda_{\max} w,$$

where $\lambda_{\max}$ is the maximum magnitude eigenvalue and w is the associated eigenvector. The maximum magnitude eigenvalue defines the first and most popular consistency index of pairwise comparison matrices [2, 3]. The associated eigenvector is utilized as a priority vector (weight vector) [1, 2, 4, 5]. From the viewpoint of mathematical programming, the theoretical foundation of the eigenvector method was established by [6, 7]. In this note, we focus on third-order pairwise comparison matrix. The third-order pairwise comparison matrix has importance in AHP. The reason is that it is a source of inconsistency. There are several research

Table 1 Saaty's discrete scale

Verbal expression	Value of intensity
equal importance	1
moderate importance	3 or 1/3
essential or strong importance	5 or 1/5
demonstrated importance	7 or 1/7
absolute importance	9 or 1/9
intermediate values	2, 4, 6, 8 or their reciprocals

articles devoted to third-order pairwise comparison matrix; see [8–10] and references therein. In this note, we treat inconsistent third-order pairwise comparison matrix A. Inconsistency is characterized by $\det A > 0$.

It is clear that the existence of the real-number root of the characteristic equation because the polynomial to be considered is odd order. In Sect. 2, we will specify the area where the real-number root exists. The root is shown to be the maximum magnitude eigenvalue among all eigenvalues of the pairwise comparison matrix.

The analytic form of the maximum magnitude eigenvalue is known; see [9] and references therein. Nevertheless, in this note, we shed light on Newton's method to obtain the maximum magnitude eigenvalue. We will see mathematical nature of Newton's method for third-order comparison matrix stands out. In Sect. 3, we will construct a sequence by the usual Newton's method. We will take an initial value to be 3, which is equal to the order of the matrix. We rely on the favorable properties of inconsistent third-order pairwise comparison matrix; uniqueness of the real-number root and $\det A > 0$. We will theoretically prove that the sequence generated by Newton's method converges to the maximum magnitude eigenvalue. We also confirm the property of convergence by Newton's method by computational experiment in Sect. 4. Utilization of Newton's method is expected to extend the result to possible application for fourth or more order pairwise comparison matrix.

2 Existence of the Root

We consider the third-order pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{pmatrix},$$

where $a_{ij} > 0$. Its characteristic polynomial has the following form [11]:

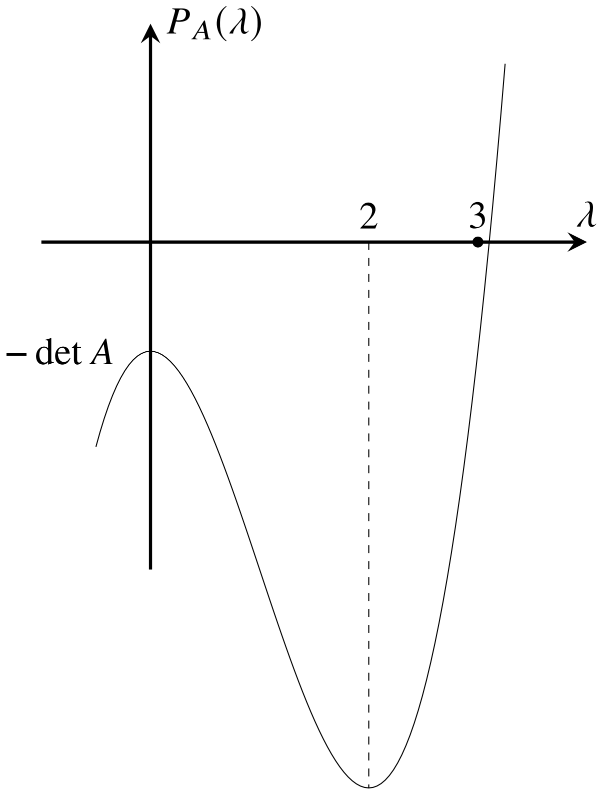
$$P_A(\lambda) = \lambda^3 - 3\lambda^2 - \det A.$$

We can calculate the determinant of the pairwise comparison matrix as follows.

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{vmatrix} \\ &= \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - 2 \geq 0. \end{aligned}$$

The last inequality follows from the relationship between the arithmetic mean and the geometric mean. The equality holds if and only if $\dfrac{a_{12}a_{23}}{a_{13}} = \dfrac{a_{13}}{a_{12}a_{23}}$, which is equivalent to $a_{12}a_{23} = a_{13}$. We say the matrix A is *consistent* if $a_{12}a_{23} = a_{13}$. For the third-order pairwise comparison matrix, as we mentioned above, it is obvious that the consistency of A and $\det A = 0$ is equivalent. If A is consistent, the characteristic

Fig. 1 Graph of $P_A(\lambda)$ in inconsistent case



polynomial is $P_A(\lambda) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$. In this case, the roots of the characteristic equation are $\lambda = 3, 0$ (0 is a multiple root).

Hereafter, we treat the case when A is inconsistent (i.e., $\det A > 0$). Taking the derivatives of $P_A(\lambda)$, we have $P_A'(\lambda) = 3\lambda(\lambda - 2)$. So $P_A(\lambda)$ takes a local maximum at the point $\lambda = 0$ and it takes a local minimum at the point $\lambda = 2$. Moreover, for $\lambda > 2$, $P_A'(\lambda) = 3\lambda(\lambda - 2) > 0$, so $P_A(\lambda)$ is monotone increasing in this area. The vertical axis section is $-\det A < 0$. We indicate the shapes of the graph of $P_A(\lambda)$ in Fig. 1.

An easy calculation shows the following:

$$\begin{aligned} P_A(3) &= 3^3 - 3 \cdot 3^2 - \det A = -\det A < 0, \\ P_A(3 + \det A) &= (3 + \det A)^3 - 3(3 + \det A)^2 - \det A \\ &= 27 + 27 \det A + 9(\det A)^2 + (\det A)^3 \\ &\quad - (27 + 18 \det A + 3(\det A)^2) - \det A \\ &= 8 \det A + 6(\det A)^2 + (\det A)^3 > 0. \end{aligned}$$

We immediately have the following existence theorem of a root for the characteristic equation.

Theorem 1 *For the inconsistent pairwise comparison matrix A, the characteristic equation has a unique real-number root in the interval* $(3, 3 + \det A)$. *This root is the maximum magnitude eigenvalue.*

Proof By the intermediate value theorem (p.206 in [12]), the existence of a real-number root in the interval $(3, 3 + \det A)$ is clear. From the shape of the graph, this root is the unique real-number root. We denote it by λ^*. In general, other two conjugate complex-number roots exist, say $a \pm bi$, $b \neq 0$.

From the general theory of eigenvalues and the trace of the matrix [2], we have $\lambda^* + (a + bi) + (a - bi) = 3$. So we have

$$a = \frac{3 - \lambda^*}{2}.$$

Since $a + bi$ is the root of the characteristic equation, we have

$$(a + bi)^3 - 3(a + bi)^2 - \det A = 0.$$

By expanding the left-hand-side of the formula, we have

$$\begin{cases} 0 = a^3 - 3ab^2 - 3a^2 + 3b^2 - \det A, \\ 0 = 3a^2 b - b^3 - 6ab. \end{cases}$$

Since $b \neq 0$, from the last formula, we have

$$3a^2 - b^2 - 6a = 0.$$

So we finally have

$$b^2 = 3a^2 - 6a.$$

Thus we have

$$\begin{aligned} |\, a \pm bi\,|^2 &= a^2 + b^2 = 4a^2 - 6a \\ &= 4\left(\frac{3 - \lambda^*}{2}\right)^2 - 6\left(\frac{3 - \lambda^*}{2}\right) \\ &= \lambda^{*2} - 3\lambda^* < \lambda^{*2}. \end{aligned}$$

This means λ^* is maximum magnitude.

3 Convergence of Newton's Method

Here we use the usual Newton's method (Chapter 8 in [13]). We give an elementary proof of convergence. We set the initial value to be $\lambda_0 = 3$ and generate a sequence by the following iteration.

$$\lambda_0 = 3,$$
$$\lambda_{n+1} = \lambda_n - \frac{P_A(\lambda_n)}{P_A'(\lambda_n)}, \quad \text{for n} \geq 0.$$

Denote the maximum magnitude eigenvalue guaranteed by Theorem 1 by $\lambda_{\max}$. So $P_A(\lambda_{\max}) = 0$ and $3 < \lambda_{\max} < 3 + \det A$.

Lemma 1 *For all $n \geq 1$, we have $\lambda_n > \lambda_{\max}$.*

Proof We prove it by induction. Set $n = 1$. $\lambda_1 = 3 + \frac{\det A}{9} > 3$ and $P_A(\lambda_1) = \frac{2}{27}(\det A)^2 + (\frac{\det A}{9})^3 > 0$. So we have $\lambda_1 > \lambda_{\max}$.

Assume $\lambda_n > \lambda_{\max}$ is true. Then from this assumption, we can take $\lambda_{\max} \leq \lambda < \lambda_n$. Since $P_A'(\lambda)$ is monotone increasing for $\lambda > 1$ (because $P_A''(\lambda) = 6(\lambda - 1)$) and $\lambda_{\max} > 3$, we have

$$P_A'(\lambda) < P_A'(\lambda_n) \quad \text{for} \lambda \in [\lambda_{\max}, \lambda_n).$$

Thus we have

$$\int_{\lambda_{\max}}^{\lambda_n} P_A'(\lambda)\, d\lambda < \int_{\lambda_{\max}}^{\lambda_n} P_A'(\lambda_n)\, d\lambda.$$

So we conclude

$$P_A(\lambda_n) - P_A(\lambda_{\max}) < P_A'(\lambda_n)(\lambda_n - \lambda_{\max}).$$

Taking account into $P_A(\lambda_{\max}) = 0$, we have

$$\lambda_{\max} < \lambda_n - \frac{P_A(\lambda_n)}{P_A'(\lambda_n)} = \lambda_{n+1}.$$

Lemma 2 *The sequence $\{\lambda_n\}$ is monotone decreasing for $n \geq 1$.*

Proof From Lemma 1, we have $3 < \lambda_{\max} < \lambda_n$. Since $P_A(\lambda)$ is monotone increasing for $\lambda > 3$, we have $P_A(\lambda_n) > P_A(\lambda_{\max}) = 0$. Obviously $P_A'(\lambda_n) > 0$ and $P_A(\lambda_n) > 0$, so we have

$$\lambda_{n+1} = \lambda_n - \frac{P_A(\lambda_n)}{P_A'(\lambda_n)} < \lambda_n.$$

From Lemmas 1 and 2, the sequence is monotone decreasing and bounded below. So it converges, see p.183 in [12].

Theorem 2 *The sequence $\{\lambda_n\}$ converge to $\lambda_{\max}$.*

Proof From Lemmas 1 and 2, the sequence is monotone decreasing and bounded below. So it converges to, say $\hat{\lambda} \geq \lambda_{\max}$. Taking the limit of the iteration

$$\lambda_{n+1} = \lambda_n - \frac{P_A(\lambda_n)}{P'_A(\lambda_n)},$$

we have

$$\hat{\lambda} = \hat{\lambda} - \frac{P_A(\hat{\lambda})}{P'_A(\hat{\lambda})}$$

by the continuity of $P_A(\lambda)$ and $P'_A(\lambda)$. Since $\hat{\lambda} \geq \lambda_{\max} > 3$, the inequality $P'_A(\hat{\lambda}) > 0$ holds. So we have $P_A(\hat{\lambda}) = 0$. Because the root of $P_A(\lambda) = 0$ is unique, $\hat{\lambda}$ is identical to $\lambda_{\max}$.

Now we state the rate of convergence.

Theorem 3 *The sequence* $\{\lambda_n\}$ *converge to* $\lambda_{\max}$ *quadratically.*

Proof For each iteration of λ_n, by the second-order Taylor's theorem (p.254 in [12]), there exists $\xi_n \in (\lambda_{\max}, \lambda_n)$, we have

$$\begin{aligned} 0 &= P_A(\lambda_{\max}) \\ &= P_A(\lambda_n) + P'_A(\lambda_n)(\lambda_{\max} - \lambda_n) + \frac{1}{2} P''_A(\xi_n)(\lambda_{\max} - \lambda_n)^2. \end{aligned}$$

Hence we have

$$\begin{aligned} \lambda_n - \frac{P_A(\lambda_n)}{P'_A(\lambda_n)} - \lambda_{\max} &= \frac{1}{2} \frac{P''_A(\xi_n)}{P'_A(\lambda_n)} (\lambda_n - \lambda_{\max})^2, \\ \lambda_{n+1} - \lambda_{\max} &= \frac{1}{2} \frac{P''_A(\xi_n)}{P'_A(\lambda_n)} (\lambda_n - \lambda_{\max})^2 \\ &\approx \frac{1}{2} \frac{P''_A(\lambda_{\max})}{P'_A(\lambda_{\max})} (\lambda_n - \lambda_{\max})^2. \end{aligned}$$

By Theorem 1, we know $\lambda_{\max} > 3$, so $P'_A(\lambda_{\max}) = 3\lambda_{\max}(\lambda_{\max} - 2) > 0$ and $P''_A(\lambda_{\max}) = 6(\lambda_{\max} - 1) > 0$. We also know $\lambda_{n+1} - \lambda_{\max} > 0$. Hence we have

$$\lambda_{n+1} - \lambda_{\max} \approx \frac{1}{2} \frac{P''_A(\lambda_{\max})}{P'_A(\lambda_{\max})} (\lambda_n - \lambda_{\max})^2,$$

which implies quadratic convergence.

4 Computational Experiment

We have a computational experiment to compare Newton's method with other root-finding methods. The experiment consists of the following steps.

Step 1. Generate an inconsistent pairwise comparison matrix A of order 3, at random.

Step 2. Iterate the following three methods until the differences between $\lambda_{\max}$ and the obtained values become below the threshold and store the numbers of iterations.

– Newton's method with the initial value of $\lambda_0 = 3$.

$$\lambda_{n+1} = \lambda_n - \frac{P_A(\lambda_n)}{P_A'(\lambda_n)}, \qquad n = 0, 1, 2. \dots.$$

– Secant method[1] with initial values of $\lambda_{-1} = 3$ and $\lambda_0 = 3 + \det A$.

$$\lambda_{n+1} = \lambda_n - \frac{P_A(\lambda_n)(\lambda_n - \lambda_{n-1})}{P_A(\lambda_n) - P_A(\lambda_{n-1})}, \qquad n = 0, 1, 2. \dots.$$

– Bisection method[2] with initial interval of $[\lambda_0^{\mathrm{lower}}, \lambda_0^{\mathrm{upper}}] = [3, 3 + \det A]$.

$$\lambda_{n+1} = \frac{\lambda_n^{\mathrm{lower}} + \lambda_n^{\mathrm{upper}}}{2}, \qquad n = 0, 1, 2. \dots,$$

where

$$[\lambda_{n+1}^{\mathrm{lower}}, \lambda_{n+1}^{\mathrm{upper}}] = [\lambda_{n+1}, \lambda_n^{\mathrm{upper}}]$$

if $P_A(\lambda_{n+1})$ and $P_A(\lambda_n^{\mathrm{lower}})$ have the same sign, and

$$[\lambda_{n+1}^{\mathrm{lower}}, \lambda_{n+1}^{\mathrm{upper}}] = [\lambda_n^{\mathrm{lower}}, \lambda_{n+1}],$$

if $P_A(\lambda_{n+1})$ and $P_A(\lambda_n^{\mathrm{upper}})$ have the same sign, for $n = 0, 1, 2. \dots$.
Figure 2 shows the distributions of the numbers of iterations until convergence by each method, after repeating these steps 5,000 times. We set the threshold of convergence to 10^{-12} here.

The summary statistics of the 5,000 numbers of iterations until convergence by each method are shown in Table 2.

For example, for the pairwise comparison matrix of

$$A = \begin{pmatrix} 1 & 5 & 5 \\ 1/5 & 1 & 1/7 \\ 1/5 & 7 & 1 \end{pmatrix},$$

[1] https://encyclopediaofmath.org/wiki/Secant_method
[2] https://encyclopediaofmath.org/wiki/Dichotomy_method

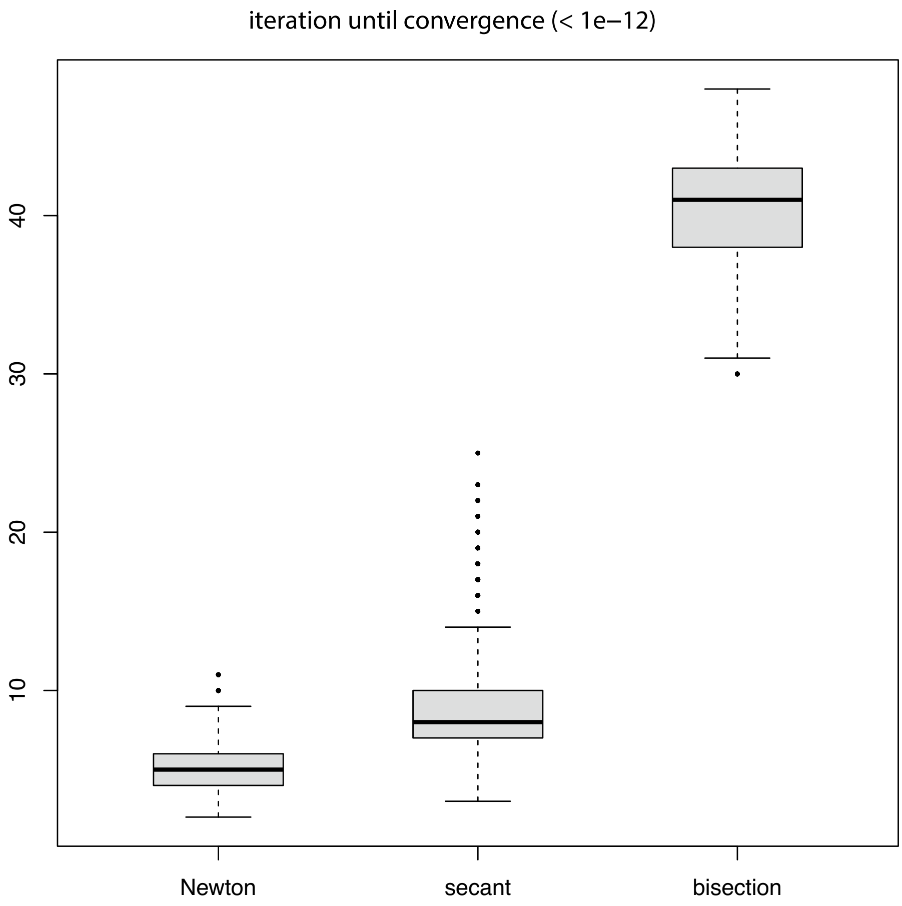


Fig. 2 Distributions of the number of iterations until convergence

obtained values converge in 5, 8, and 41 iterations by Newton's method, the secant method, and the bisection method, respectively. This case can be considered typical because these are the median of each method. The transitions of the values toward convergence for this matrix are shown in Fig. 3.

This experiment confirms the speed of convergence of Newton's method empirically.

Table 2 Summary statistics of the distributions of three methods

	minimum	median	mode	mean	maximum
Newton	2	5	5	5.3926	11
secant	3	8	8	8.9066	25
bisection	30	41	41	40.3576	48

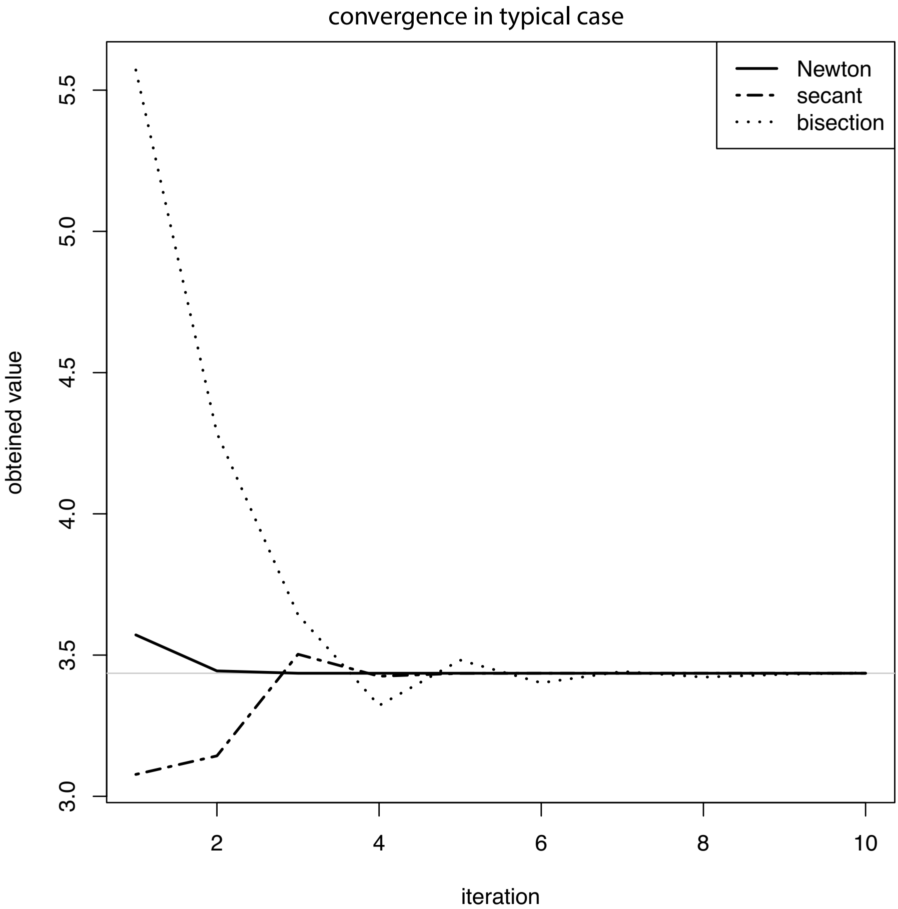


Fig. 3 Transition toward convergence in a typical case

5 Conclusion

In this note, we entirely use the favorable properties from which A is third order. About other properties concerning third-order pairwise comparison matrices, see [9]. The key contribution of this note is findings of the possible area where the root of the characteristic equation and to show the convergence of Newton's method from the initial value $\lambda_0 = 3$. We also confirm the superiority of Newton's method compared to other root-finding methods—the secant method and the bisection method. This is considered as a direct consequence of the quadratic convergence of Newton's method.

For the matrices for fourth and more order dimensions, expanding this note is left to future research.

- For fourth-order pairwise comparison matrix, there usually exist two roots of the characteristic equation. So, if we set the initial value $\lambda_0 = 4$, Newton's method does not generally converge to λ_{max}.
- We need to study the appropriate initial value and the convergence property.

Acknowledgements The authors would like to thank the reviewers and the editor for helpful comments that improved the paper.

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Kułakowski K (2021) Understanding the Analytic Hierarchy Process. CRC Press. https://doi.org/10.1201/9781315392226
2. Saaty TL (1980) The Analytic Hierarchy Process. McGraw-Hill
3. Brunelli M (2018) A survey of inconsistency indices for pairwise comparisons. Int J Gen Syst 47(1):751–771. https://doi.org/10.1080/03081079.2018.1523156
4. Brunelli M (2014) Introduction to the Analytic Hierarchy Process. Springer. https://doi.org/10.1007/978-3-319-12502-2
5. Tone K (1986) The Analytic Hierarchy Process: Decision Making (in Japanese). Japanese Science and Technology Press
6. Sekitani K, Yamaki N (1999) A logical interpretation for the eigenvalue method in AHP. J Oper Res Soc Jpn 42(2):219–232. https://doi.org/10.15807/jorsj.42.219
7. Sekitani K, Yamaki N (2003) A model-based AHP including the case of incomplete information. Asia Pac Manag Rev 8(2):241–258
8. Bozóki S (2003) A method for solving LSM problems of small size in the AHP. CEJOR 11(1):17–33
9. Shiraishi S, Obata T (2021) Some remarks on the maximum eigenvalue of 3rd order pairwise comparison matrices in AHP. Bull Inform Cybern 53(2):1–13. https://doi.org/10.5109/4362571
10. Tummala VMR, Ling H (1998) A note on the computation of the mean random consistency index of the analytic hierarchy process (AHP). Theor Decis 44:221–230. https://doi.org/10.1023/A:1004953014736
11. Shiraishi S, Obata T, Daigo M (1998) Properties of positive reciprocal matrix and their application to AHP. J Oper Res Soc Jpn 41(3):404–414. https://doi.org/10.15807/jorsj.41.404
12. Haier E, Wanner G (1996) Analysis by Its History. Springer. https://doi.org/10.1007/978-0-387-77036-9
13. Ortega JM (1990) Numerical Analysis, A Second Course. SIAM. https://doi.org/10.1137/1.9781611971323