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The finite Jung constant in Banach spaces

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Abstract
We study in this paper the finite Jung constant, its interplay with Kottman's constant and its meaning regarding the geometry of Banach spaces.

Keywords Jung constants · Kottman's constant · Complex interpolation · Lindenstrauss spaces

Mathematics Subject Classification 46B04 · 46B20 · 46B26 · 46M18

1 Introduction: the Jung constants

Given a bounded subset $A \subset X$ the diameter of A is defined as $\delta(A) = \sup\{\|a - b\| : a, b \in A\}$, while the radius of A in X is defined by $r_X(A) = \inf_{b \in X} \sup_{a \in A} \|a - b\|$. If the infimum is attached at a point b then this point is called a center for A; if only $\sup_{a \in A} \|a - b\| \leq r_X(A) + \varepsilon$ then b will be called an ε-center. The Jung constant [16] of A is defined as

$$J(A) = \frac{2r_X(A)}{\delta(A)}$$

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while the Jung constant of X is the supremum $J(X) = \sup J(A)$ taken over all closed bounded sets A with $\delta(A) > 0$. A combination of results by Davis [11], Franchetti in [14] and Lindenstrauss [19] show that a Banach space is 1-injective if and only if $J(X) = 1$. Recall that a Banach space X is λ-injective if for every Banach space F and every subspace E of F every operator $t : E \to X$ has an extension $T : F \to X$ with $\|T\| \leq \lambda\|t\|$.

Two important variations of this notion [2, 4] are λ-separable injectivity, when the property above holds when F is separable; and local λ-injectivity, when the preceding property holds when F is finite dimensional. We can consider the corresponding variation of Jung's constant for separability and obtain the *separable* Jung constant $J_s(\cdot)$, introduced in [7] as

$$J_s(X) = \sup J(A)$$

where the supremum is taken over all separable closed bounded sets A with $\delta(A) > 0$. In this paper we will consider the finite Jung constant introduced by Amir [1] (see also [5]) and defined as

$$J_f(X) = \sup J(A)$$

where the supremum is taken over all finite sets A with $\delta(A) > 0$.

The first type of characterization we are interested in this paper was obtained by Davis [11]: a Banach space X is 1-injective if and only if $J(X) = 1$. We obtained in [7] the corresponding characterization for J_s: a Banach space X is separably 1-injective if and only if $J_s(X) = 1$. Our first set of results in this paper provided in Sect. 2 deal with the characterization of the spaces X for which $J_f(X) = 1$. It was (implicitly) proved by Bayod and Masa [3] that $J_f(X) = 1$ if and only if X is a Lindenstrauss space. This fact was reproved in [25], while in [13] it was observed that a careful reading of [19] yields the same characterization. Moreover, they show [13, Theorem 2.7] that $J_f(X) = 1$ if and only if every four-point set of diameter 1 has radius 1/2 and a center.

2 Banach spaces with finite Jung constant 1

In this Section, we will prove the Lindenstrauss–Bayod–Masa characterization of Lindenstrauss spaces through a new equality $J_f(X) = J_s(X_{\mathcal{U}})$ for some ultrapower of X. This characterization will have a few interesting consequences.

Semenov and Franchetti [26, Lemma 2.4] show that if Y, X are Banach spaces such that for each $\varepsilon > 0$ the space X contains $(1+\varepsilon)$-isomorphic $(1+\varepsilon)$-complemented copy of Y then $J(Y) \leq J(X)$. In particular, if Y is a 1^+-complemented subspace of X then $J(Y) \leq J(X)$. We generalize this:

Lemma 2.1 *Let $Y \subset X$ and Z be Banach spaces.*

(1) If Y is λ^+-complemented in X then $J(Y) \leq \lambda J(X)$, $J_s(Y) \leq \lambda J_s(X)$ and $J_f(Y) \leq \lambda J_f(X)$.

(2) If Y is locally λ^+-complemented in X then $J_f(Y) \le \lambda J_f(X)$
(3) $J(X \oplus_\infty Z) = \max\{J(X), J(Z)\}$; $J_s(X \oplus_\infty Z) = \max\{J_s(X), J_s(Z)\}$; $J_f(X \oplus_\infty Z) = \max\{J_f(X), J_f(Z)\}$

Proof (1) To avoid further confusion, given $Y \subset X$ and $A \subset Y$ let $r_Y(A)$ (resp. $r_X(A)$) denote the radius of A in Y (resp. in X). Thus, we need to show that $r_Y(A) \le \lambda r_X(A)$ for every $A \subset Y$ with $0 < \delta(A) \le 1$. Let P be a projection on X onto Y with $\|P\| \le \lambda$. If $x \in X$ and $\|a - x\| \le r$ for each $a \in A$, taking $y = Px \in Y$ we have $\|a - y\| = \|P(a - x)\| \le \lambda \|a - x\| \le r$ and the result follows.

(2) Assume now that $A \subset Y$ is finite and Y is locally λ^+-complemented in X. If $x \in X$ and $\|a - x\| \le r$ for each $a \in A$, pick $P : A + [x] \longrightarrow Y$ a projection with $\|P\| \le \lambda + \varepsilon$ so that $\|a - Py\| = \|P(a - x)\| \le (\lambda + \varepsilon)\|a - x\| \le (\lambda + \varepsilon)r$ and the result follows as well.

(3) We make the proof for J, but the proofs for J_s and J_f are analogous. By (1), $J(X \oplus_\infty Y) \ge \max\{J(X), J(Y)\}$. Conversely, let $A \subset X \oplus_\infty Y$ with $0 < \delta(A) \le 1$. Let π_X the canonical projection onto X. The sets $B = \pi_X(A) \subset X$ and $C = (I - \pi_X)(A) \subset Y$ satisfy $\delta(B), \delta(C) \le 1$. If $\delta(B) = 0$ then $\delta(C) = \delta(A)$ and $r_Y(C) = r_{X \oplus_\infty Y}(A)$, and analogously when $\delta(C) = 0$. Assume then that $\delta(B) > 0$ and $\delta(C) > 0$. If we fix $\varepsilon > 0$ and pick $x_0 \in X$ and $y_0 \in Y$ such that $\|x_0 - b\| < r_X(B) + \varepsilon$ for each $b \in B$ and $\|y_0 - c\| < r_Y(C) + \varepsilon$ for each $c \in C$ then $\|(x_0, y_0) - (b, c)\| = \max\{\|x_0 - b\|, \|y_0 - c\|\} < \max\{r_X(B), r_Y(C)\} + \varepsilon$ for each $(b, c) \in A$. On the other hand, $\delta(A) = \max\{\delta_X(B), \delta_Y(C)\}$. Therefore

$$J(X \oplus_\infty Y) = \sup \frac{2r(A)}{\delta(A)} \le \sup \frac{2\max\{r_X(B), r_Y(C)\}}{\delta(A)} \le \max\{J(X), J(Y)\}.$$

□

It is clear that this result says nothing for $\lambda \ge 2$. One could suspect that the parameter λ plays no role in either (1) or (2). Let us show it is not so. We discuss (2) first, and recall from [4, Chapter 9] the existence of an exact sequence $0 \to C[0, 1] \to \Omega \to c_0 \to 0$ in which Ω cannot be renormed to be a Lindenstrauss space. This sequence can be placed in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[0,1] & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/C[0,1] & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & C[0,1] & \xrightarrow{J} & \Omega & \longrightarrow & c_0 & \longrightarrow & 0 \end{array}$$

which therefore (see again [4]) yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C[0,1] & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/C[0,1] & \longrightarrow & 0 \\ & & {\scriptstyle J}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega & \longrightarrow & P & \longrightarrow & \ell_\infty/C[0,1] & \longrightarrow & 0 \end{array}$$

in which the lower sequence locally 1^+-splits and the space P is isomorphic to $\ell_\infty \oplus_\infty c_0$. Thus, after renorming, we have a locally λ^+-split sequence

$$0 \longrightarrow \Omega \longrightarrow \ell_\infty \oplus_\infty c_0 \longrightarrow \ell_\infty/C[0,1] \longrightarrow 0$$

in which $J_f(\ell_\infty \oplus_\infty) = 1$ but $J_f(\Omega) > 1$. The bidual sequence

$$0 \longrightarrow \Omega^{**} \longrightarrow \ell_\infty^{**} \oplus_\infty \ell_\infty \longrightarrow (\ell_\infty/C[0,1])^{**} \longrightarrow 0$$

provides a counterexample for (1): it splits by [4], namely, Ω^{**} is complemented in $\ell_\infty^{**} \oplus_\infty \ell_\infty$; it cannot be 1-complemented because otherwise, by the Principle of Local Reflexivity, Ω would be locally 1^+-complemented in $\ell_\infty \oplus_\infty c_0$, which is not the case. Hence Ω^{**} is not 1-injective, and therefore $J(\Omega^{**}) > 1$, while $J(\ell_\infty^{**} \oplus_\infty \ell_\infty) = 1$. The same works, under the Continuum Hypothesis, regarding J_s: if Ω were 1-separably injective then it would be universally 1-separably injective (see [2]), hence 1-complemented in $\ell_\infty \oplus_\infty c_0$, that we know it is not.

As a consequence of the results in [7, 9], $J(X)$ (resp. $J_s(X)$) and $J(X^{**})$ (resp. $J_s(X^{**})$) can be different since $J(c_0) = 2 = J_s(c_0)$ while $J(\ell_\infty) = J_s(\ell_\infty) = 1$. In general, given a countably incomplete ultrafilter $\mathcal{U}$ on $\mathbb{N}$, one has $J_s(C[0,1]) = 2$ but $J_s(C[0,1]_\mathcal{U}) = 1$ since according to [2] the ultrapower of a Lindenstrauss space is 1-separably injective. On the other hand, $J(\ell_\infty) = 1$ but $J((\ell_\infty)_\mathcal{U}) > 1$ since, again according to [2] no infinite-dimensional ultrapower is injective. The finite Jung constant behaves, however, differently:

Proposition 2.2 $J_f(X) = J_f(X^{**})$.

Proof The inequality $J_f(X) \le J_f(X^{**})$ follows from (2) in the previous Lemma. Next observe that for a certain ultrafilter $\mathcal{U}$ the space X^{**} is 1-complemented in $X_\mathcal{U}$, and therefore $J_f(X^{**}) \le J_f(X_\mathcal{U})$ by part (1) of the Lemma above. It remains to show that $J_f(X_\mathcal{U}) \le J_f(X)$. Let $\varepsilon > 0$ be fixed and pick a finite set $A = \{a^1, \dots, a^m\} \subset X_\mathcal{U}$ with $\delta(A) = \|a^u - a^v\| = 1$ such that $J_f(X_\mathcal{U}) \le 2r(A) + \varepsilon$ and let a be an ε-approximate center for A; namely $\|a^j - a\| \le r(A) + \varepsilon$ for $1 \le j \le m$. Assume that $a^j = [a_1^j, \dots, a_n^j, \dots]$ and $a = [a_1, \dots, a_n, \dots]$. Since the sets $U_j = \{n : \|a_n^j - a_n\| \le r(A) + 2\varepsilon\}$ belong to $\mathcal{U}$ for $j = 1, ..., m$ as well as $U = \{n : \|a_n^u - a_n^v\| \ge 1 - \varepsilon\}$ so does $\bigcap_j U_j \cap U$. Thus, picking n in this set, $B = \{a_n^1, \dots, a_n^m\} \subset X$ has diameter at least $1 - \varepsilon$ and the point a_n is a 2ε-approximate center for B. All this yields

$$J_f(X) \ge \frac{J_f(X_\mathcal{U}) + 2\varepsilon}{1 - \varepsilon}.$$

□

It is then immediate from the previous argument that also $J_f(X) = J_f(X_\mathcal{U})$ for every ultrafilter $\mathcal{U}$. When $\mathcal{U}$ is countably incomplete, the argument can be improved to

Theorem 2.3 *If $\mathcal{U}$ is a countably incomplete ultrafilter then $J_f(X) = J_s(X_\mathcal{U})$.*

Proof We just need to prove the inequality $J_s(X_{\mathcal{U}}) \leq J_f(X)$. Let $\varepsilon > 0$ be fixed and pick a countable set $A = \{a^m : m \in \mathbb{N}\} \subset X_{\mathcal{U}}$ with $\delta(A) = 1$ such that $J_s(X_{\mathcal{U}}) \leq 2r(A) + \varepsilon$. There is no loss of generality in assuming that $\|a^i - a^j\| = 1$ for all $i, j \in \mathbb{N}$ just to simplify future choices. Let a be an ε-approximate center for A; namely $\|a^m - a\| \leq r(A) + \varepsilon$ for all m. Set as before $a^m = [a_1^m, \dots, a_n^m, \dots]$ and $a = [a_1, \dots, a_n, \dots]$. The sets $U_m = \{n : \|a_n^m - a_n\| \leq r(A) + 2\varepsilon\}$ belong to $\mathcal{U}$ for all m as well as $U_{u,v} = \{n : \|a_n^u - a_n^v\| \geq 1 - \varepsilon\}$ for all $u, v \in \mathbb{N}$. Now proceed orderly: pick $k \in \bigcap_{m=1}^k U_m \cap \bigcap_{1 \leq u,v \leq k} U_{u,v} \in \mathcal{U}$ and form the set $B_k = \{a_k^1, \dots, a_k^k\} \subset X$, who has diameter at least $1 - \varepsilon$ and the point a_k is a 2ε-approximate center for B_k. The only problem that could appear is if some $b_k \in X$ yields a "better" center for B_k, namely $\|b - b_k\| \leq \alpha < J_s(X_{\mathcal{U}})$ for some α and all $b \in B_k$. But if this happens for an infinite set $M \subset \mathbb{N}$ then the element $b1_M$ having b_k at the corresponding place of 1_M is a "better" center for A in $X_{\mathcal{U}}$, namely $\|a - b1_M\| \leq \alpha$, which is a contradiction as $\varepsilon \to 0$. Therefore

$$J_f(X) \geq \frac{J_s(X_{\mathcal{U}}) + 2\varepsilon}{1 - \varepsilon}.$$

□

We draw now some consequences. The first of them is a new proof for the Lindenstrauss–Bayod–Masa characterization of Lindenstraus spaces:

Proposition 2.4 *A Banach space X is a Lindenstrauss space if and only if $J_f(X) = 1$*

Proof If X is a Lindenstrauss space, $X_{\mathcal{U}}$ is 1-separably injective [2] and therefore $J_s(X_{\mathcal{U}}) = 1$ according to [7], which proves the necessity. On the other hand, if $J_f(X) = 1$ then also $J_s(X_{\mathcal{U}}) = 1$ and thus $X_{\mathcal{U}}$ is 1-separably injective [7]. It must therefore be Lindenstrauss space [2], as well as X by the principle of local reflexivity. □

Amir [1, p.5] shows that $J_f(X^*) = J(X^*)$ for every dual space. Moreover:

Corollary 2.5 *If X is 1-complemented in X^{**} then $J_f(X) = J_s(X)$.*

Proof $J_s(X) \leq J_s(X^{**}) \leq J_s(X_{\mathcal{U}}) = J_f(X)$. □

3 The interplay between the finite Jung and Kottman constants

In [21, Theorem 6] it is shown that $2 \leq J_s(X)K(X)$. If X is an infinite-dimensional Banach space with unit ball $B(X)$, the finite Kottman constant of X is defined as

$$K_f(X) = \sup\{r > 0 : \forall n \in \mathbb{N} \quad \exists\, A : |A| = n \text{ and } \inf_{i \neq j} \|x_i - x_j\| \geq r\}.$$

Since $J_f(X) \leq J_s(X)$ but $K(X) \leq K_f(X)$ it is worth checking the finite analog. A combinatorial argumentation could be: if X is not reflexive then $K_f(X) = 2$ [18]; while if X is reflexive then [1] $J_f(X) = J(X)$ and therefore $K_f(X)J_f(X) = K_f(X)J(X) \geq K(X)J(X) \geq 2$. Let us present a straight proof.

Lemma 3.1 $2 \leq J_f(X)K_f(X)$.

Proof Given $\varepsilon > 0$, pick a set $A = \{x_1, \ldots, x_N\}$ such that $K_f(X) + \varepsilon \geq \|x_i - x_j\| \geq K_f(X) - \varepsilon$ (use [7, Lemma 5]). Since $2r_A/\delta(A) \leq J_f(X)$ one has $r_X(A) \leq \frac{1}{2}J_f(X)\delta(A) \leq \frac{1}{2}J_f(X)(K_f(X) + \varepsilon)$. Pick p such that $\|x_i - p\| \leq \frac{1}{2}(J_f(X) + \varepsilon)(K_f(X)+\varepsilon)$ and therefore the ball centered at p with radius $\frac{1}{2}(J_f(X)+\varepsilon)(K_f(X)+\varepsilon)$ contains a finite set $(K_f(X) - \varepsilon)$-separated, and therefore the unit ball contains a finite set $\frac{K_f(X)-\varepsilon}{\frac{1}{2}(J_f(X)+\varepsilon)(K_f(X)+\varepsilon)}$-separated; hence

$$\frac{K_f(X) - \varepsilon}{\frac{1}{2}(J_f(X) + \varepsilon)(K_f(X) + \varepsilon)} \leq K_f(X)$$

and, therefore,

$$K_f(X) \leq \frac{1}{2}J_f(X)K_f(X)^2 \Longrightarrow 2 \leq J_f(X)K_f(X).$$

. □

One deduces from here and Proposition 2.4 that $K_f(X) = 2$ for every Lindenstrauss space. It had been however shown in [6, Proposition 3.4] that $K(X) = 2$ for every $\mathcal{L}_\infty$-space. We continue our study recalling the following result from Pichugov [23, Assertion]. We present it in its original formulation even if some terms appear unexplained in our context. The consequence we seek, namely, Pichugov's inequality (3.1) is however clear:

Lemma 3.2 *Let a closed convex set M in X^n have Chebyshev radius r. Then the point y is its Chebyshev center if and only if there is a natural number $N \leq n + l$, such that*

(a) there are points x_i in M $(i = 1 \ldots N)$ such that $\|y_i - y\| = r$.
(b) there are functionals f_i in $(X^n)^*$ $(i = 1 \ldots N)$ such that $\|f_i\| = 1$ and $\langle f_i, x_i - y\rangle = \|x_i - y\|$.
(c) there are numbers a_i $(i = 1 \ldots N)$, $\sum_i^N a_i = 1$, $a_i \geq 0$ such that $\sum_i^N a_i f_i = 0$.

From there one deduces the following version of Pichugov's inequality (see [15]):

$$J(A) \leq \sup \sum_{i,j=1}^{n} \alpha_i \alpha_j \|f_i - f_j\|_{X^*} \tag{3.1}$$

for $A \subset X$ a finite set of cardinality n, and the supremum is taken over finite families $f_1, \ldots, f_n$ of elements of X^* and scalars $\alpha_1, \ldots, \alpha_n$ with $\|f_i\|_{X^*} \leq 1$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$ and $(\sum \alpha_i f_i)|_A = 0$.

Proof Pichugov's inequality [15, (5)] yields $2r_X(A) = \sum_{i,j=1}^{n} \alpha_i \alpha_j \langle x_i - x_j, f_i - f_j\rangle$ from where

$$2r_X(A) \leq \sup \sum_{i,j=1}^{n} \alpha_i\alpha_j \|x_i - x_j\|\|f_i - f_j\| \leq \delta(A) \sup \sum_{i,j=1}^{n} \alpha_i\alpha_j \|f_i - f_j\|$$

□

One gets

Proposition 3.3 *If X is infinite-dimensional, $J_f(X) \leq K_f(X^*)$.*

Proof We begin recalling from [6] that $K_f(X) = K(X_{\mathcal{U}})$ for every countably incomplete ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Set for each $u \in \mathbb{N}$ a finite set $A_u \subset X$ such that $J(X) \leq J(A_u) + \varepsilon u^{-1}$ and $J(A_u) \leq J(A_v)$ when $u \leq v$. Pick for each u elements $f_1^u, \ldots, f_n^u$ as in Pichugov's inequality and form the elements $F_n = [f_n^1, f_n^2, f_n^3, \ldots,] \in X^*_{\mathcal{U}}$ where we understand that $f_n^k = 0$ for $n > n(k)$. Since $\|F_i - F_j\|_{X^*_{\mathcal{U}}} = \lim_{\mathcal{U}} \|f_i^u - f_j^u\|$ means that for every $\varepsilon > 0$, $\{u : |\|f_i^u - f_j^u\|_{X^*} - \|F_i - F_j\|_{X^*_{\mathcal{U}}}| \leq \varepsilon\} \in \mathcal{U}$, we get that $\|f_i^u - f_j^u\|_{X^*} \leq \|F_i - F_j\|_{X^*_{\mathcal{U}}} + \varepsilon$ for all u in a set of $\mathcal{U}$. Thus

$$\begin{aligned} J(X) &= \sup J(A_u) \\ &= \lim_{\mathcal{U}} J(A_u) \\ &\leq \lim_{\mathcal{U}} \sum_{i,j} \alpha_i^u \alpha_j^u \|f_i^u - f_j^u\|_{X^*} \\ &\leq \lim_{\mathcal{U}} \sum_{i,j} \alpha_i^u \alpha_j^u \left(\|F_i - F_j\|_{X^*_{\mathcal{U}}} + \varepsilon\right) \end{aligned}$$

and since no infinite subset $M \subset \mathbb{N}$ exists such that $\|F_m - F_n\| > K(X^*_{\mathcal{U}})$ for $m, n \in M$ we get

$$\begin{aligned} &\leq \lim_{\mathcal{U}} \sum_{i,j} \alpha_i^u \alpha_j^u \left(K(X^*_{\mathcal{U}}) + \varepsilon\right) \\ &\leq K(X^*_{\mathcal{U}}) + \varepsilon. \end{aligned}$$

□

The converse is obviously false since $J_f(c_0) = 1$ and $K_f(\ell_1) = 2$. The inequality above belongs to the world of finite constants since

- it is not true that $J_f(X) \leq K(X^*)$ as the example of ℓ_p-spaces, $1 < p < 2$, shows.
- Consequently it is not true that $J(X) \leq K(X^*)$ either.

In [9] we showed that $K(X)$ and $K(X^{**})$ are not necessarily equal. However

Proposition 3.4 $K_f(X) = K_f(X^{**})$.

Proof We need the following version of the Principle of Local Reflexivity (see [20]): for each finite-dimensional subspace $E \subset X^{**}$ and each $\varepsilon > 0$ there is a $(1+\varepsilon)$-isometry $T : E \to X$ such that $T|_X = id_{E \cap X}$. Pick now a finite set $\{x_1^{**}, \ldots, x_N^{**}\}$

such that $\|x_n^{**} - x_m^{**}\| \geq K_f(X^{**}) - \varepsilon$; hence

$$\|Tx_n^{**} - Tx_m^{**}\| \geq (1-\varepsilon)\|x_n^{**} - x_m^{**}\| \geq (1-\varepsilon)(K_f(X^{**}) - \varepsilon),$$

for $\|Tx_n^{**}\| \leq (1+\varepsilon)$, which is enough to conclude. □

A combination of the inequality above with Proposition 3.3 provides a remarkable symmetry:

Corollary 3.5 *If X is infinite-dimensional, $J_f(X)J_f(X^*) \leq K_f(X)K_f(X^*)$.*

A combination of Proposition 3.3 with the estimates in [7] yields

(1) $K_f(Y)J_f(X) \leq 2e_1^f(Y, X)$.
(2) $K_f(Y)J_s(X) \leq 2e_1^s(Y, X^{**})$.

Here $e_1^f(Y, X)$ (resp. $e_1^s(Y, X)$) is the infimum of all $\lambda > 0$ such that for finite (resp. separable) subset M of Y and every $y \in Y$, every Lipschitz map $f : M \to Z$ admits a Lipschitz extension $F : M \cup \{y\} \to Z$ with $Lip(F) \leq \lambda Lip(f)$.

Corollary 3.6 $J_f(Y)J_f(X) \leq 2e_1^f(Y^*, X)$. *In particular,* $J_f(X) \leq \sqrt{2e_1^f(X^{**}, X)}$.

4 Jung constants and interpolation

In [10] we studied the behavior of Kottman's constants regarding complex interpolation obtaining the continuity of $K(\cdot)$ with respect to the interpolation parameter and the interpolation inequality: if (X_0, X_1) is an interpolation pair and $X_\theta = (X_0, X_1)_\theta$ is the complex interpolation space obtained at θ then $K(X_\theta) \leq K(X_0)^{1-\theta}K(X_1)^\theta$. The behavior of the Jung constants regarding interpolation is necessarily quite different since an inequality $J(X_\theta) \leq J(X_0)^{1-\theta}J(X_1)^\theta$ does not hold since $J(L_\infty) = 1$, $J(L_p) = 2^{1-1/p}$ for $2 \leq p < \infty$ and $L_3 = (L_2, L_\infty)_\theta$ for $\theta = 1/3$. Moreover, the characterizations of spaces X with $J_f(X)$, $J_s(X)$ or $J(X)$ equal to 1 makes an interpolation inequality such as $J(X_\theta) \leq J(X_0)^{1-\theta}J(X_1)^\theta$ impossible since one can obtain reflexive spaces as interpolation between injective spaces. The following explicit example was provided to us by Manuel González.

Example 4.1 Pick the space $\ell_\infty(1/n) = \{x : \sup_n \frac{1}{n}x_n < \infty\}$ endowed with the sup norm and consider the pair $(\ell_\infty, \ell_\infty(1/n))$. The canonical inclusion $\ell_\infty \longrightarrow \ell_\infty(1(n))$ is compact, hence there are reflexive interpolation spaces, whose Jung constants must be greater than 1. However $J(\ell_\infty) = J(\ell_\infty(1/n)) = 1$.

Observe that this shows that $J(\cdot)$ does not satisfy an interpolation inequality for either the real or complex methods. However, if we denote the complex interpolation space as $X_\theta = (X_0, X_1)_\theta$, we have

Proposition 4.2 *The Jung and finite Jung constants are continuous with respect to the interpolation parameter on* $(0, 1)$*, but not on* $[0, 1]$.

Proof The lack of continuity at the extremes has already been shown. In fact, [26, p. 870] already observed that $J(X_\theta)$ can be discontinuous at the boundary points. To show the continuity at the interior, let us recall the definition of the Kadets metric. Let M, N be closed subspaces of a Banach space Z, and let B_M denote the unit ball of M. The *gap* $g(M, N)$ *between* M *and* N is defined by

$$g(M, N) = \max\left\{\sup_{x\in B_M} \operatorname{dist}(x, B_N), \sup_{y\in B_N} \operatorname{dist}(y, B_M)\right\},$$

The *Kadets metric* $d_K(X, Y)$ between two Banach spaces X and Y is the infimum of the gap $g(i(X), j(Y))$ taken over all the isometric embeddings of i, j of X, Y into a common Banach space. It turns out that $J(\cdot)$, $J_s(\cdot)$ and $J_f(\cdot)$ are continuous with respect to the Banach–Mazur metric: if $T : X \longrightarrow Y$ is an isomorphism with $\max \|T\|\|T^{-1}\| \le \alpha$ then $|J(X)-J(Y)| \le (\alpha^2-1)\min\{J(X), J(Y)\}$. We show now the continuity of the Jung constants with respect to the Kadets metric. We will need a few general facts that will be useful. Given a bounded set $A \subset X$ with approximate center a, a translation $x \to x-a$ allows us to work with the set $A' = A-a$ contained in the ball of radius $r_M(A)$ and center 0. We can change now A' by $A'' = r_X(A)^{-1}A'$ and still $J(A'') = J(A)$. In other words, with regard to the calculus of the Jung constants of Z there is no loss of generality in assuming that A has radius (or diameter) 1 and is contained in the unit ball has has approximate center 0. What is not true, as simple examples show, is that a subset of the ball must have its center inside the ball. Let us show now:

Claim. J_f, J_s and J are continuous with respect to the gap.

Let us make the proof for J_f. Let $M, N \subset X$ two closed subspaces of a Banach space X. Fix M and $\varepsilon > 0$. Let us call $g : B_M \to B_N$ (resp. $g' : B_N \to B_M$) a function such that $\|x - g(x)\| \le g(M, N) + \varepsilon$ (resp. $\|x - g'(x)\| \le g(M, N) + \varepsilon$). There is no loss of generality in assuming that g, g' are homogeneous since $\|\frac{x}{2} - \frac{1}{2}g(x)\| \le g(M, N) + \varepsilon$. Pick a finite set $A = \{a_1, \dots, a_n\}$ in B_M with ε-center 0 and such that $J_f(M) < J(A) + \varepsilon$. Form the set $B = \{g(a_1), \dots, g(a_n)\}$ and let b be an ε-center for it. It is easy to check that $\|b\| \le 2 + 2\varepsilon$ (see the comment after the proof).

$$\begin{aligned} r(A) &\le \sup \|a_i - g'\left(\frac{b}{2}\right)\| \\ &= \sup \left\|\frac{a_i}{2} + \frac{a_i}{2} - g\left(\frac{a_i}{2}\right) + g\left(\frac{a_i}{2}\right) - \frac{b}{2} + \frac{b}{2} - g'\left(\frac{b}{2}\right)\right\| \\ &\le \sup \left(\left\|\frac{a_i}{2}\right\| + \left\|\frac{a_i}{2} - g\left(\frac{a_i}{2}\right)\right\| + \left\|g\left(\frac{a_i}{2}\right) - \frac{b}{2}\right\| + \left\|\frac{b}{2} - g'\left(\frac{b}{2}\right)\right\|\right) \\ &\le \frac{1}{2}r(A) + \sup\left\|g\left(\frac{a_i}{2}\right) - \frac{b}{2}\right\| + 2g(M, N) + \varepsilon \\ &\le \frac{1}{2}r(A) + \frac{1}{2}r(B) + 2g(M, N) + 2\varepsilon \end{aligned}$$

which yields

$$r(A) \leq r(B) + 4g(M, N) + 4\varepsilon.$$

On the other hand

$$\begin{aligned} \delta(A) &= \sup \|a_i - a_j\| \\ &= \|a_i - g(a_i) + g(a_i) - g(a_j) + g(a_j) - a_j\| \\ &\geq \delta(B) - 2g(M, N) - 2\varepsilon. \end{aligned}$$

Consequently,

$$\frac{2r_N(A)}{\delta(A)} \leq \frac{2r(B) + 8g(M, N) + 8\varepsilon}{\delta(B) - 2g(M, N) - 2\varepsilon}$$

and thus

$$J_f(M) \leq J_f(N) + F$$

for some positive and continuous function F such that $f(0) = 0$. Doing the same replacing M by N we get the other inequality, and thus

$$\lim_{g(M,N)\to 0} \left|J_f(M) - J_f(N)\right| = 0$$

which is the continuity (not the uniform continuity, as it is the case of the Kottman's constants) of J_f with respect to the gap.

The continuity with respect to the Kadets metric is now immediate taking into account that if i, j are isometric embeddings, $J(X) = J(iX)$ and $J(Y) = J(jY)$. The continuity with respect to the interpolation parameter follows as we proved in [10] for the Kottman's constants: Kalton and Ostrovskii [17] proved that the Kadets metric is continuous with respect to the interpolation parameter; precisely,

$$d_K(X_\theta, X_\eta) \leq 2 \left| \frac{\sin(\pi(\theta - \eta)/2)}{\sin(\pi(\theta + \eta)/2)} \right|.$$

Thus, the Jung constants are continuous with respect to the interpolation parameter. □

The following is an example of a set in the unit ball of c_0 with a center having norm 2: pick $A = \{\sum_i^n e_i : n \in \mathbb{N}\}$ and set $2e_1$. The continuity with respect to the Kadets metric in combination with the fact that given an exact sequence $0 \to Y \to Z \to X \to 0$ one has $d_K(Z, Y \oplus_\infty X) = 0$ (see [10]) yields that Z can be renormed to have (finite, separable) Jung constant $\max\{J(Y), J(X)\}$. However, the example $0 \to C[0,1] \to \Omega \to c_0 \to 0$ presented earlier shows that, however, no renorming of Z such that $J(Z) = \max\{J(Y), J(X)\}$ can, in general, be achieved.

5 Open questions

Most of our open questions wheel around the validity of the interpolation inequality

$$J_f(X_\theta) \leq J_f(X_0)^{1-\theta} J_f(X_1)^\theta$$

which is false. Could it be true on a restricted context? Say, when also X_0, X_1 are superreflexive interpolation spaces, or infinite-dimensional spaces with a common unconditional basis?

- Does the inequality $2 \leq J_f(X)J_f(X^*)$ hold for infinite-dimensional spaces? The inequality fails for finite-dimensional spaces since $J(\ell_1^n) = 2n/n+1$ (see [12, 15]) and $J(\ell_\infty^n) = 1$. Observe that when X, X^* have a common unconditional basis then $(X, X^*)_{1/2} = \ell_2$ and thus the interpolation inequality would yield $\sqrt{2} \leq J_f(X)^{1/2} J_f(X^*)^{1/2}$, which is the inequality above.
- Does the interpolation formula hold for pairs (E_0, E_1) of rearrangement invariant Banach lattices with $E_0, E_1 \neq L_\infty$? R.i. Banach lattices can be seen as a generalized form of Banach spaces with unconditional or symmetric basis.
- A Banach lattice E is said to be a θ-Hilbert space $(0 < \theta < 1)$ if $E = (F, L_2)_\theta$ for some r.i. space F. Each θ-Hilbert space is a r.i. space (see [24]). Is it true that $J(E) \leq J(L_2)^\theta J(F)^{1-\theta} = J(F)^{1-\theta} 2^{\theta/2}$ for E a θ-Hilbert space? This would generalize the inequality $J(E) \leq 2 \cdot 2^{\theta/2}$ in [26, Theorem 2.2.].
- Can a reflexive space X be renormed in such a way that $J(X) = J(X^*)$? Recall that there are reflexive spaces for which $J(X) \neq J(X^*)$, say $J\left(\ell_2(\ell_1^n)\right) = 2$ and $J\left(\ell_2(\ell_\infty^n)\right) = \sqrt{2}$. This example is from Amir [1, 2.15.b], although he erroneously writes that this space has Jung constant 1.

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