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Measure of non-compactness and limiting interpolation with slowly varying functions

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Abstract
We give estimates for the measure of non-compactness of an operator interpolated by the limiting methods involving slowly varying functions. As applications we establish estimates for the measure of non-compactness of operators acting between Lorentz–Karamata spaces.

Keywords Limiting interpolation · Slowly varying functions · Measure of non-compactness · Lorentz–Karamata spaces · Generalized Lorentz–Zygmund spaces

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1 Introduction

The real interpolation method $(A_0, A_1)_{\theta,q}$ has found important applications in Operator Theory, Approximation Theory, Function Spaces and Harmonic Analysis. See, for

To Professor William B. Johnson on the occasion of his 80th birthday.

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example, the monographs by Butzer and Berens [8], Bergh and Löfström [4], Triebel [43, 44], König [32] and Bennett and Sharpley [3]. The real method is very flexible, admitting several equivalent definitions, what is very useful in applications.

The real method applied to the couple of Lebesgue spaces (L_1, L_∞) yields Lorentz spaces $L_{p,q}$. It is possible to obtain more general spaces if we modify the definition of the real method. So, logarithmic perturbations of the real method produce Lorentz–Zygmund spaces $L_{p,q}(\log L)_a$ (see [20, 27, 28]) and perturbations involving slowly varying functions $(L_1, L_\infty)_{\theta,q;b}$ give Lorentz–Karamata spaces $L_{p,q;b}$(see [29]).

We are interested here in the limit cases when $\theta = 0, 1$ of the perturbations with slowly varying functions $(A_0, A_1)_{\theta,q;b}$. These spaces are very close to A_0 when $\theta = 0$ and to A_1 when $\theta = 1$. They have received attention from a number of authors either to study limiting embeddings between function spaces or to establish limiting properties of operators (see, for example, [26, 29, 36]).

Among the classical problems for any interpolation method, a prominent one is to describe the behavior of properties that operators may have. First of all boundedness but then other useful properties of operators. For example, techniques used by Davis, Figiel, Johnson and Pelczyński [23] in the proof of their famous factorization theorem for weakly compact operators motivated the investigation on the behavior of weak compactness under interpolation (see, for example, [1, 18, 31, 34, 35]).

The behavior under interpolation of compactness have been also deeply studied (see [9, 17, 22] and the references given there). Quantitative estimates in terms of the measure of non-compactness have been also established. Concerning the real method, the first result in this direction is due to Edmunds and Teixeira [42]. They assume an approximation condition for the couple in the target. The case of general Banach couples has been studied by Cobos, Fernández-Martínez and Martínez [15]. Results for the real method with a function parameter and $0 < \theta < 1$ are due to Cordeiro [21], Szwedek [41] and Cobos, Fernández-Cabrera and Martínez [11]. Besides, the case of limiting methods involving logarithms have been considered by Cobos, Fernández-Cabrera and Martínez [12, 14] and Besoy and Cobos [5].

Our aim here is to establish estimates for the measure of non-compactness of operators interpolated by the limiting perturbations of the real method involving slowly varying functions. As applications we derive estimates for the measure of non-compactness of operators acting between certain Lorentz–Karamata spaces. In particular, one of our results can be considered as a quantitative extension of a compactness result of Edmunds and Opic [26] for operators acting between Lorentz–Zygmund spaces.

We work with quasi-Banach couples (A_0, A_1). Our techniques are based on the vector-valued sequence spaces that come up with the definition of $(A_0, A_1)_{0,q;b}$ and with its description as a J-space. These ideas originated in the papers on compactness by Cobos and Peetre [19] and Cobos, Kühn and Schonbek [17]. In the context of the measure of non-compactness, they were developed by Cobos, Fernández-Martínez and Martínez [15], Cobos, Fernández-Cabrera and Martínez [14] and Besoy and Cobos [5] among other authors.

2 Limiting real interpolation spaces

Let $(A, \|\cdot\|_A)$ be a quasi-Banach space and let $c_A \geq 1$ be its constant in the quasi-triangle inequality. Let $0 < p \leq 1$ such that $c_A = 2^{1/p-1}$. According to the Aoki-Rolewicz theorem (see [33, Section 15.10]) there is another quasi-norm $|||\cdot|||$ on A which is equivalent to $\|\cdot\|_A$ and such that $|||\cdot|||^p$ satisfies the triangle inequality. Then $(A, |||\cdot|||)$ is called a *p-Banach space*. Note that if $0 < r < p$, then $(A, |||\cdot|||)$ is also an r-Banach space and that any p-Banach space satisfies the quasi-triangle inequality with constant $2^{1/p-1}$.

If B is another quasi-Banach space, we write $A = B$ if $A \hookrightarrow B$ and $B \hookrightarrow A$, where $\hookrightarrow$ means continuous embedding.

For $0 < q \leq \infty$, let ℓ_q be the space of q-summable sequences with $\mathbb{Z}$ as index set. If $(w_m)_{m\in Z}$ is a sequence of positive numbers, we denote by $\ell_q(w_m)$ the space of all scalars sequences (ξ_m) such that $(w_m\xi_m) \in \ell_q$.

Let (W_m) be a sequence of quasi-Banach spaces with the same constant in the quasi-triangle inequality. We put

$$\ell_q(w_m \mathrm{W_m}) = \{\mathrm{w} = (\mathrm{w_m}) : \mathrm{w_m} \in \mathrm{W_m} \text{ and } \\ \|\mathrm{w}\|_{\ell_q(w_m W_m)} = \|(w_m\|\mathrm{w_m}\|_{\mathrm{W_m}})\|_{\ell_q} < \infty\}.$$

A quasi-Banach space $(\Gamma, \|\cdot\|_\Gamma)$ of real valued sequences with $\Gamma \hookrightarrow \ell_q + \ell_q(2^{-m})$ is said to be a *quasi-Banach sequence lattice* if Γ contains all the sequences with only finitely many non-zero coordinates and whenever $(\eta_m) \in \Gamma$ and $|\xi_m| \leq |\eta_m|$ for each $m \in \mathbb{Z}$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \leq \|(\eta_m)\|_\Gamma$.

We define $\Gamma(\mathrm{W_m})$ as the collection of all sequences $\mathrm{w} = (\mathrm{w_m})$ such that $\mathrm{w_m} \in \mathrm{W_m}$ and $\|\mathrm{w}\|_{\Gamma(\mathrm{W_m})} = \|(\|\mathrm{w_m}\|_{\mathrm{W_m}})\|_\Gamma < \infty$.

Subsequently, if b and v are non-negative functions on $(0, \infty)$, we say that b and v are *equivalent* (and write $b(t) \approx v(t)$) if there are positive constants c, C such that $cb(t) \leq v(t) \leq \mathrm{C}b(t)$ for any $t > 0$.

A positive, finite and Lebesgue-measurable function b on $(0, \infty)$ is said to be *slowly varying* ($b \in SV(0, \infty)$) if, for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is equivalent to a positive non-decreasing measurable function and $t^{-\varepsilon}b(t)$ is equivalent to a positive non-increasing measurable function. Important examples of slowly varying functions are powers of iterated logarithms and broken logarithmic functions $v(t) = \ell^{\mathbb{A}}(t)$ where $\ell(t) = (1 + |\log(t)|)$, $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $\ell^{\mathbb{A}}(t) = \ell^{\alpha_0}(t)$ if $0 < t \leq 1$ and $\ell^{\mathbb{A}}(t) = \ell^{\alpha_\infty}(t)$ if $1 < t < \infty$.

We refer to [29] for properties of slowly varying functions. We only recall here that if $\varepsilon > 0$, then there are positive constant c_ε, C_ε such that

$$c_\varepsilon \min\{s^{-\varepsilon}, s^\varepsilon\}b(t) \leq b(st) \leq \mathrm{C}_\varepsilon \max\{s^\varepsilon, s^{-\varepsilon}\}b(t) \text{ for every } s, t > 0, \tag{2.1}$$

(see [29, Proposition 2.2]). Put

$$\overline{b}(s) = \sup_{t>0} \frac{b(st)}{b(t)}.$$

The function $\overline{b}$ satisfies that $\overline{b}(st) \leq \overline{b}(s)\overline{b}(t)$. Moreover, using (2.1) with $\varepsilon = 1/2$, we have

$$s\bar{b}(s) \leq \mathrm{C}_{1/2}s^{1/2} \to 0 \text{ as } s \to 0. \tag{2.2}$$

Another consequence of (2.1), this time with $\varepsilon = 1$, is that

$$c_1/2 \leq \bar{b}(s) \leq 2\mathrm{C}_1 \text{ for any } 1/2 \leq s \leq 1. \tag{2.3}$$

Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, $v(t) = \ell^{\mathbb{A}}(t)$ and $\mathbb{B} = (\alpha_0^+ + (-\alpha_\infty)^+, \alpha_\infty^+ + (-\alpha_0)^+)$ with $\alpha^+ = \max\{0, \alpha\}$. It follows from [14, Lemma 2.1] and [5, (2.6)] that $\bar{v}(s) \leq \ell^{\mathbb{B}}(s)$, $s \in (0, \infty)$.

For $0 < q \leq \infty$ and $b \in SV(0, \infty)$, the quasi-Banach sequence space $\ell_q(b(2^m))$ will be of special interest for us.

If $k \in \mathbb{Z}$, the *shift operator* τ_k is defined by $\tau_k\xi = (\xi_{m+k})_{m\in\mathbb{Z}}$ for $\xi = (\xi_m)$. We have

$$\|\tau_k\xi\|_{\ell_q(b(2^m))} = \|(b(2^m)|\xi_{m+k}|)\|_{\ell_q} \leq \overline{b}(2^{-k})\|(b(2^{m+k})|\xi_{m+k}|)\|_{\ell_q}.$$

Hence $\tau_k : \ell_q(b(2^m)) \to \ell_q(b(2^m))$ is bounded with

$$\|\tau_k\|_{\ell_q(b(2^m)),\ell_q(b(2^m))} \leq \bar{b}(2^{-k}). \tag{2.4}$$

We say that $\overline{A} = (A_0, A_1)$ is a (p*-Banach*) *quasi-Banach couple* if A_0 and A_1 are (p-Banach) quasi-Banach spaces which are continuously embedded in the same Hausdorff topologic vector space.

For $t > 0$ and $a \in A_0 + A_1$, the Peetre's K-functional is given by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}.$$

If $a \in A_0 \cap A_1$, the J-functional of Peetre is

$$J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}.$$

Note that $K(1, \cdot)$ and $J(1, \cdot)$ are the quasi-norms of A_0+A_1 and $A_0\cap A_1$, respectively.

If $(A_j, \|\cdot\|_{A_j})$ is a p-Banach space for $j = 0, 1$, then $J(t, \cdot)$ is also a p-norm, as well as

$$\begin{aligned} K_p(t, a) &= K_p(t, a; A_0, A_1) \\ &= \inf\{(\|a_0\|_{A_0}^p + t^p\|a_1\|_{A_1}^p)^{1/p} : a = a_0 + a_1, a_j \in A_j\}. \end{aligned}$$

This last functional is equivalent to the K-functional. In fact

$$K(t, a) \leq K_p(t, a) \leq 2^{1/p-1}K(t, a), \quad a \in A_0 + A_1. \tag{2.5}$$

Note that if $\xi = (\xi_m) \in \ell_p + \ell_p(2^{-m})$ then

$$K_p(2^r, \xi; \ell_p, \ell_p(2^{-m})) = \left(\sum_{m=-\infty}^{\infty} [\min\{1, 2^{r-m}\}|\xi_m|]^p \right)^{1/p}.$$

This expression will be useful later.

A quasi-Banach space A is said to be an *intermediate space* with respect to the couple $\overline{A}$ if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. We write A° for the closure of $A_0 \cap A_1$ in A. The fundamental lemma (see [4, Lemma 3.3.2] and [37, Lemma 2.4]) yields that

$$a \in (A_0 + A_1)^\circ \text{ if and only if } \min\left\{1, \frac{1}{t}\right\} K(t, a) \to 0 \\ \text{as } t \to 0 \text{ and as } t \to \infty. \tag{2.6}$$

For $0 \le \theta \le 1,\ 0 < q \le \infty$ and $b \in SV(0, \infty)$, the space $\bar{A}_{\theta,q;b} = (A_0, A_1)_{\theta,q;b}$ consists of all those $a \in A_0 + A_1$ that have a finite quasi-norm

$$\|a\|_{\bar{A}_{\theta,q;b}} = \|a\|_{(A_0,A_1)_{\theta,q;b}} = \left(\sum_{m=-\infty}^{\infty} [2^{-\theta m} b(2^m) K(2^m, a)]^q \right)^{1/q}$$

(the sum should be replaced by the supremum when $q = \infty$). See [29, 37]. If $b \equiv 1$ and $0 < \theta < 1$, we recover the classical real interpolation space $(A_0, A_1)_{\theta,q;b}$ (see [3, 4, 8, 43]). If $0 < \theta < 1$ then $(A_0, A_1)_{\theta,q;b}$ is a special case of the real method with function parameter (see [30, 40]). If $\theta = 0, 1,\ \alpha_0, \alpha_\infty \in \mathbb{R}$ and

$$b(t) = \begin{cases} (1 + |\log t|)^{\alpha_0} & \text{if } 0 < t \le 1, \\ (1 + |\log t|)^{\alpha_\infty} & \text{if } 1 < t < \infty, \end{cases}$$

then we recover the logarithmic interpolation spaces $\bar{A}_{\theta,q;(\alpha_0,\alpha_\infty)}$ (see [10, 16, 20, 27, 28]).

We are mainly interested here in the limiting spaces $\overline{A}_{0,q;b}$ and $\overline{A}_{1,q;b}$. Since $K(t, a; A_0, A_1) = tK(t^{-1}, a; A_1, A_0)$, they are related by the equality

$$(A_0, A_1)_{0,q;b} = (A_1, A_0)_{1,q;v} \text{ where } v(t) = b(1/t). \tag{2.7}$$

Note that v is also slowly varying on $(0, \infty)$. Due to equality (2.7), in what follows we focus on the case $\theta = 0$.

As it is shown in [28], $(A_0, A_1)_{0,q;b}$ is an intermediate space with respect to $\overline{A}$ if and only if

$$\left(\int_1^\infty b(t)^q \,\mathrm{d}t/t\right)^{1/q} < \infty. \tag{2.8}$$

Let $\overline{B} = (B_0, B_1)$ be another quasi-Banach couple. We write $T \in \mathcal{L}(\overline{A}, \overline{B})$ to mean that T is a bounded linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that the restrictions $T : A_j \to B_j$ are bounded for $j = 0, 1$. Then the restriction

$$T : (A_0, A_1)_{0,q;b} \to (B_0, B_1)_{0,q;b}$$

is also bounded. Indeed, if M_j is bigger than or equal to the norm of $T : A_j \to B_j$, $j = 0, 1$, then

$$K(t, Ta; B_0, B_1) \le M_0 K\left(\frac{tM_1}{M_0}, a; A_0, A_1\right).$$

Therefore, if $M_1 \le M_0$, we obtain that $\|T\|_{\bar{A}_{0,q;b},\bar{B}_{0,q;b}} \le M_0$. If $M_0 < M_1$ then we can find $r \in \mathbb{N} \cup \{0\}$ such that $2^r \le M_1/M_0 < 2^{r+1}$. Hence

$$\begin{aligned}\|Ta\|_{\bar{B}_{0,b;q}} &\le \left(\sum_{m=-\infty}^{\infty} [b(2^m) M_0 K(2^{(m+r+1)}, a)]^q\right)^{1/q}\\ &\le M_0\, \bar{b}(2^{-r-1}) \|a\|_{\bar{A}_{0,b;q}}\\ &\le c M_0\, \bar{b}\left(\frac{M_0}{M_1}\right) \|a\|_{\bar{A}_{0,b;q}}\end{aligned}$$

where we have used (2.3) in the last inequality. Therefore

$$\|T\|_{\bar{A}_{0,q;b},\bar{B}_{0,q;b}} \le \begin{cases} M_0 & \text{if } M_1 \le M_0,\\ cM_0 \bar{b}\left(\frac{M_0}{M_1}\right) & \text{if } M_0 < M_1,\end{cases} \tag{2.9}$$

where $c > 0$ is a constant depending only on b.

If $(T_n) \subseteq \mathcal{L}(A_0 + A_1, B_0 + B_1)$ with

$$\sup\{\|T_n\|_{A_1,B_1} : n \in \mathbb{N}\} < \infty \text{ and } \lim_{n\to\infty} \|T_n\|_{A_0,B_0} = 0,$$

then it follows from (2.9) and (2.1) that

$$\lim_{n\to\infty} \|T_n\|_{\bar{A}_{0,q;b},\bar{B}_{0,q;b}} = 0. \tag{2.10}$$

Next we show a sufficient condition on b for the inclusion $(A_0, A_1)_{0,q;b} \subseteq (A_0 + A_1)^\circ$. Let $0 < q \le \infty$ and take any $a \in (A_0, A_1)_{0,q;b}$. Then

$$\left(\sum_{n=-\infty}^{\infty} \left[\frac{b(2^n)}{\min\{1, 2^{-n}\}} \min\{1, 2^{-n}\} K(2^n, a)\right]^q\right)^{1/q} = \|a\|_{\bar{A}_{0,q;b}} < \infty. \tag{2.11}$$

Since $tb(t)$ is equivalent to a non-decreasing function, we have $\left(\sum_{n=0}^{\infty}[2^n b(2^n)]^q\right)^{1/q} = \infty$. Hence, from (2.11) it follows that

$$\lim_{t\to\infty}\frac{1}{t}K(t,a) = \lim_{n\to\infty}\frac{1}{2^n}K(2^n,a) = 0.$$

On the other hand, if we assume

$$\begin{cases} \left(\int_0^1 b(t)^q \,\mathrm{d}t/t\right)^{1/q} = \infty & \text{if } 0 < q < \infty, \\ \lim_{t\to 0} b(t) = \infty & \text{if } \ q = \infty, \end{cases} \tag{2.12}$$

then we also have that $\lim_{t\to 0} K(t,a) = 0$. Having in mind (2.6), it turns out that if (2.12) is satisfied then $(A_0, A_1)_{0,q;b} \subseteq (A_0 + A_1)^\circ$.

The *Gagliardo completion* $A_j^\sim$ of A_j consists of all those $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{A_j^\sim} = \sup\{t^{-j}K(t,a) : t > 0\} \quad \text{(see [4,3]).}$$

We have that $A_j \hookrightarrow A_j^\sim$ for $j = 0, 1$. The quasi-Banach couple $\overline{A}$ is called *mutually closed* if $A_j = A_j^\sim$ for $j = 0, 1$.

If Γ is a quasi-Banach sequence lattice and $\overline{A} = (A_0, A_1)$ is a p-Banach couple, then the J-space $\overline{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is formed by all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $(u_m) \subseteq A_0 \cap A_1$ and $(J(2^m, u_m)) \in \Gamma$. We endow $\overline{A}_{\Gamma;J}$ with the quasi-norm

$$\|a\|_{\overline{A}_{\Gamma;J}} = \|a\|_{(A_0,A_1)_{\Gamma;J}} = \inf\left\{\|(J(2^m, u_m))\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m\right\}$$

(see [37]).

Next, we give a description of $(A_0, A_1)_{0,q;b}$ by means of the J-functional.

Theorem 2.1 *Let* $\overline{A} = (A_0, A_1)$ *be a mutually closed p-Banach couple* $(0 < p \le 1)$. *Let* $0 < q \le \infty$ *and let* $b \in SV(0,\infty)$ *satisfying (2.8) and (2.12). Put* $\Lambda = (\ell_p, \ell_p(2^{-m}))_{0,q:b}$. *Then we have with equivalent quasi-norms*

$$(A_0, A_1)_{0,q:b} = (A_0, A_1)_{\Lambda;J}.$$

Proof Let $a \in (A_0, A_1)_{0,q:b}$. By the assumption on b, we know that $(A_0, A_1)_{0,q:b} \subseteq (A_0 + A_1)^\circ$. Hence, according to [37, Theorem 3.2], there exists $(u_m) \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ (in $A_0 + A_1$) and

$$\left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p\right)^{1/p} \leq cK(2^k, a), \quad k \in \mathbb{Z},$$

where c only depends on p. Whence

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\Lambda;J}} &\leq \|(J(2^m, u_m))\|_{(\ell_p,\ell_p(2^{-m}))_{0,q;b}} \\
&\leq \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) K_p(2^k, ((J(2^m, u_m)); \ell_p, \ell_p(2^{-m}))\right]^q\right)^{1/q} \\
&\leq \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p\right)^{1/p}\right]^q\right)^{1/q} \\
&\leq c \left(\sum_{k=-\infty}^{\infty} [b(2^k) K(2^k, a)]^q\right)^{1/q} \\
&= c\|a\|_{(A_0,A_1)_{0,q;b}} .
\end{aligned}$$

Conversely, take any $a \in (A_0, A_1)_{\Lambda;J}$. We can find a J-representation $a = \sum_{m=-\infty}^{\infty} u_m$ with $\|(J(2^m, u_m))\|_\Lambda \leq 2\|a\|_{(A_0,A_1)_{\Lambda;J}}$. Since

$$K_p(2^k, a) \leq \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p\right)^{1/p}, \quad k \in \mathbb{Z},$$

we obtain that

$$\begin{aligned}
&\|a\|_{(A_0,A_1)_{0,q;b}} \\
&\leq \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p J(2^m, u_m)^p\right)^{1/p}\right]^q\right)^{1/q} \\
&\leq 2^{1/p-1} \|(J(2^m, u_m))\|_{(\ell_p,\ell_p(2^{-m}))_{0,q;b}} \\
&\leq 2^{1/p} \|a\|_{(A_0,A_1)_{\Lambda;J}} .
\end{aligned}$$

□

In Theorem 2.1, the sequence space that defines $(A_0, A_1)_{0,q;b}$ as a J-space is not explicitly described, it appears as the interpolation space $\Lambda = (\ell_p, \ell_p(2^{-m}))_{0,q:b}$ instead, what is enough for our aims here. Assuming extra conditions on the couple (A_0, A_1) and on the function b, there are several papers in the literature where the sequence space Λ is explicitly described. More precisely, in the case of logarithmic interpolation spaces, explicit descriptions as J-spaces have been obtained by Cobos and Kühn [16] for the case of ordered Banach couples, by Cobos and Segurado [20] and Besoy, Cobos and Fernández–Cabrera [7] for general Banach couples and by

Besoy and Cobos [6] for quasi-Banach couples. If (A_0, A_1) is a Banach couple and $1 \le q \le \infty$, an explicit description of $(A_0, A_1)_{0,q;b}$ as a J-space has been recently established by Grover and Opic [38].

The following estimate for the norm of the shift operator τ_k on Λ will be useful later.

Lemma 2.2 *Let $0 < p \le 1$, $0 < q \le \infty$ and let $b \in SV(0, \infty)$ satisfying (2.8). Put $\Lambda = (\ell_p, \ell_p(2^{-m}))_{0,q;b}$. Then, for any $k \in \mathbb{Z}$, we have*

$$\|\tau_k\|_{\Lambda,\Lambda} \le 2^{1/p-1}\overline{b}(2^{-k}).$$

Proof Given any $\xi = (\xi_m) \in \Lambda$, we have

$$\begin{aligned}
\|\tau_k \xi\|_\Lambda &\le \left(\sum_{n=-\infty}^{\infty} \left[b(2^n) K_p(2^n, \tau_k\xi; \ell_p, \ell_p(2^{-m}))\right]^q \right)^{1/q} \\
&= \left(\sum_{n=-\infty}^{\infty} \left[b(2^n) \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{n-m})^p |\xi_{m+k}|^p \right)^{1/p} \right]^q \right)^{1/q} \\
&\le \overline{b}(2^{-k}) \left(\sum_{n=-\infty}^{\infty} \left[b(2^{n+k}) \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{n+k-m})^p |\xi_m|^p \right)^{1/p} \right]^q \right)^{1/q} \\
&\le 2^{1/p-1}\overline{b}(2^{-k}) \left(\sum_{n=-\infty}^{\infty} [b(2^n) K(2^n, \xi)]^q \right)^{1/q} \\
&= 2^{1/p-1}\overline{b}(2^{-k}) \|\xi\|_\Lambda
\end{aligned}$$

where we have used (2.5) in the penultimate inequality. □

3 Measure of non-compactness

Let A, B be quasi-Banach spaces and $T \in \mathcal{L}(A, B)$. The (ball) *measure of non-compactness* $\beta(T) = \beta(T : A \to B)$ is defined to be the infimum of the set of numbers $\sigma > 0$ for which there is a finite subset $\{z_1, \dots, z_n\} \subseteq B$ such that

$$T(U_A) \subseteq \bigcup_{j=1}^{n} \{z_j + \sigma U_B\}.$$

Here U_A, U_B are the closed unit balls of A and B, respectively. See [24] for details on the measure of non-compactness. Note that $\beta(T) \le \|T\|_{A,B}$ and that $\beta(T) = 0$ if and only if T is compact. That is, $\beta(T) = 0$ means that T transforms each bounded set of A into a set whose closure is compact in B.

If T_1 is another operator belonging to $\mathcal{L}(A, B)$, then it is not difficult to check that

$$\beta(T + T_1 : A \to B) \le c_B\big(\beta(T : A \to B) + \beta(T_1 : A \to B)\big).$$

If we assume that E, X are other quasi-Banach spaces and that $S \in \mathcal{L}(B, E)$ and $R \in \mathcal{L}(X, A)$, then we have

$$\beta(STR : X \to E) \le \|R\|_{X,A}\beta(T : A \to B)\|S\|_{B,E}.$$

Furthermore, if $\|Sb\|_E = \|b\|_B$ for all $b \in B$, then

$$\beta(T : A \to B) \le 2c_E\beta(ST : A \to E).$$

If for any $a \in A$ with $\|a\|_A < 1$, there is $x \in X$ with $\|x\|_X < 1$ such that $Rx = a$, then

$$\beta(T : A \to B) \le \beta(TR : X \to B).$$

We will use freely these properties in our later computations.

Next we establish the main result of the paper. It shows an estimate for the measure of non-compactness of an operator interpolated using parameters $0, q, b$.

Theorem 3.1 *Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\overline{A}, \overline{B})$. Let $0 < q \le \infty$ and $b \in SV(0, \infty)$ satisfying (2.8) and (2.12). Then we have*

(i) $\beta(T : \overline{A}_{0,q;b} \to \overline{B}_{0,q;b}) = 0$ *if* $\beta(T : A_0 \to B_0) = 0$,

(ii)

$$\begin{aligned}\beta(T :\overline{A}_{0,q;b} \to \overline{B}_{0,q;b}) \le C\beta(T : A_0 \to B_0) \quad &\textit{if} \\ 0 \le \beta(T : A_1 \to B_1) < \beta(T : A_0 \to B_0),&\end{aligned}$$

(iii)

$$\begin{aligned}&\beta(T :\overline{A}_{0,q;b} \to \overline{B}_{0,q;b})\\ &\quad\le C \max\left\{\beta(T : A_0 \to B_0), \beta(T : A_0 \to B_0)\overline{b}\left(\frac{\beta(T : A_0 \to B_0)}{\beta(T : A_1 \to B_1)}\right)\right\}\end{aligned}$$

if $0 < \beta(T : A_0 \to B_0) \le \beta(T : A_1 \to B_1)$.

Here C is a constant independent of T.

Proof *Step* 1. Consider the mutually closed quasi-Banach couples $\overline{A^{\sim}} = (A_0^{\sim}, A_1^{\sim})$, $\overline{B^{\sim}} = (B_0^{\sim}, B_1^{\sim})$. The arguments of [3, Theorem V.1.5] may be modified to give that

$$K(t, a; A_0^{\sim}, A_1^{\sim}) \leq K(t, a; A_0, A_1) \leq \max\{c_{A_0}, c_{A_1}\} K(t, a; A_0^{\sim}, A_1^{\sim}).$$

Therefore,

$$(A_0, A_1)_{0,q;b} = (A_0^{\sim}, A_1^{\sim})_{0,q;b} \quad \text{and} \quad (B_0, B_1)_{0,q;b} = (B_0^{\sim}, B_1^{\sim})_{0,q;b}\,.$$

Besides, $T \in \mathcal{L}(\overline{A^{\sim}}, \overline{B^{\sim}})$ and, according to [5, Lemma 3.1],, we have

$$\beta(T : A_j^{\sim} \to B_j^{\sim}) \leq \max\{c_{B_0}, c_{B_1}\} \beta(T : A_j \to B_j), \quad j = 0, 1.$$

Consequently, without loss of generality we may assume in the following that the couples $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ are mutually closed. We may also assume that the spaces A_0, A_1, B_0, B_1 are p-Banach for some $0 < p \leq 1$. Therefore, we can use Theorem 2.1.

Step 2. In this step we will introduce vector-valued sequence spaces and projections which will allow to split the operator T.

Let $\Lambda = (\ell_p, \ell_p(2^{-m}))_{0,q;b}$. By Theorem 2.1, we know that $(A_0, A_1)_{0,q;b} = (A_0, A_1)_{\Lambda;J}$. Consider the vector-valued sequence space $\Lambda(G_m)$ where $G_m = (A_0 \cap A_1, J(2^m, \cdot))$, $m \in \mathbb{Z}$. Let $\pi : \Lambda(G_m) \to (A_0, A_1)_{\Lambda;J}$ be the linear operator defined by $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$). Then π is surjective and induces the quasi-norm of $(A_0, A_1)_{\Lambda;J}$. Note also that $\pi \in \mathcal{L}(\ell_p(2^{-mj} G_m), A_j)$, $j = 0, 1$, and its norm is less than or equal to 1.

Put $\overline{\ell_p(G)} = (\ell_p(G_m), \ell_p(2^{-m} G_m))$. The following projections will be useful. For $n \in \mathbb{N}$ and $u = (u_m) \in \ell_p(G_m) + \ell_p(2^{-m} G_m)$ let

$$\begin{aligned}
P_n u &= (\ldots, 0, 0, u_{-n}, u_{-n+1}, \ldots, u_{n-1}, u_n, 0, 0, \ldots),\\
P_n^+ u &= (\ldots, 0, 0, u_{n+1}, u_{n+2}, u_{n+3}, \ldots),\\
P_n^- u &= (\ldots, u_{-n-3}, u_{-n-2}, u_{-n-1}, 0, 0, \ldots).
\end{aligned}$$

Then the identity operator I on $\ell_p(G_m) + \ell_p(2^{-m} G_m)$ can be decomposed as $I = P_n + P_n^+ + P_n^-$, $n \in \mathbb{N}$. These projections are bounded from $\ell_p(2^{-mj} G_m)$ into $\ell_p(2^{-mj} G_m)$ with norm less than or equal to 1 for $j = 0, 1$, and the same happens on $\Lambda(G_m)$. Furthermore,

$$\|P_n^+\|_{\ell_p(G_m), \ell_p(2^{-m} G_m)} = 2^{-(n+1)} = \|P_n^-\|_{\ell_p(2^{-m} G_m), \ell_p(G_m)}\,. \tag{3.1}$$

Write $F_m = (B_0 + B_1, K(2^m, \cdot))$, $m \in \mathbb{Z}$. Then the linear operator $\iota b = (\ldots., b, b, b, \ldots.)$ is a metric injection from $(B_0, B_1)_{0,q;b}$ into $\ell_q(b(2^m) F_m)$. Consider the couple $\overline{\ell_\infty(F)} = (\ell_\infty(F_m), \ell_\infty(2^{-m} F_m))$. Note that $\iota : B_j \to \ell_\infty(2^{-mj} F_m)$ is bounded with norm less than or equal to 1. On $\overline{\ell_\infty(F)}$ we can consider the corresponding sequences of projections that we denote by (Q_n), (Q_n^+), (Q_n^-). They enjoy analogous properties as (P_n), (P_n^+) and (P_n^-). In particular, we have

$$\|Q_n^+\|_{\ell_\infty(F_m), \ell_\infty(2^{-m} F_m)} = 2^{-(n+1)} = \|Q_n^-\|_{\ell_\infty(2^{-m} F_m), \ell_\infty(F_m)}\,. \tag{3.2}$$

The following diagram illustrates the situation

$$\frac{\begin{array}{c}\ell_p(G_m)\xrightarrow{\pi} A_0 \xrightarrow{T} B_0 \xrightarrow{\iota} \ell_\infty(F_m)\\ \ell_p(2^{-m}G_m) \xrightarrow{\pi} A_1 \xrightarrow{T} B_1 \xrightarrow{\iota} \ell_\infty(2^{-m}F_m)\end{array}}{\Lambda(G_m) \xrightarrow{\pi} \overline{A}_{0,q;b} \xrightarrow{T} \overline{B}_{0,q;b} \xrightarrow{\iota} \ell_q(b(2^m)F_m)}.$$

In this diagram, the first three spaces of the last line are obtained by interpolation of the couple above and the fourth space contains the corresponding interpolation space. That is to say, we have

$$(\ell_p(G_m), \ell_p(2^{-m}G_m))_{0,q;b} = \Lambda(G_m) \text{ and}$$

$$(\ell_\infty(F_m), \ell_\infty(2^{-m}(F_m))_{0,q;b} \hookrightarrow \ell_q(b(2^m)F_m). \tag{3.3}$$

To establish the first formula we proceed as in the case when $b(t) = (1 + |\log t|)^{\mathbb{A}}$ (see [5, Lemma 3.2]). Take any $u = (u_m) \in (\ell_p(G_m), \ell_p(2^{-m}G_m))_{0,q;b}$. For any $k \in \mathbb{Z}$ and $0 < \varepsilon < 1$, there are $u^j = (u_{j,m}) \in \ell_p(2^{-mj}G_m)$ such that $u = u^0 + u^1$ and

$$\|u^0\|_{\ell_p(G_m)} + 2^k\|u^1\|_{\ell_p(2^{-m}G_m)} \le (1+\varepsilon)K(2^k, u; \ell_p(G_m), \ell_p(2^{-m}G_m)).$$

Then

$$\begin{aligned}\left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p \|u_m\|_{G_m}^p\right)^{1/p} &\le (\|u^0\|_{\ell_p(G_m)}^p + 2^{kp}\|u^1\|_{\ell_p(2^{-m}G_m)}^p)^{1/p}\\ &\le 2^{1/p-1}(1+\varepsilon)K(2^k, u; \ell_p(G_m), \ell_p(2^{-m}G_m))\end{aligned}$$

and thus

$$\begin{aligned}\|u\|_{\Lambda(G_m)} &= \left(\sum_{k=-\infty}^{\infty}\left[b(2^k)K(2^k, (\|u_m\|_{G_m}); \ell_p, \ell_p(2^{-m}))\right]^q\right)^{1/q}\\ &\le \left(\sum_{k=-\infty}^{\infty}\left[b(2^k)\left(\sum_{m=-\infty}^{\infty}(\min(1, 2^{k-m})\|u_m\|_{G_m})^p\right)^{1/p}\right]^q\right)^{1/q}\\ &\le 2^{1/p-1}(1+\varepsilon)\left(\sum_{k=-\infty}^{\infty}\left[b(2^k)K(2^k, u; \ell_p(G_m), \ell_p(2^{-m}G_m))\right]^q\right)^{1/q}\\ &\le 2^{1/p}\|u\|_{(\ell_p(G_m), \ell_p(2^{-m}G_m))_{0,q;b}}.\end{aligned}$$

Reciprocally, if $u = (u_m) \in \Lambda(G_m)$, given any $k \in \mathbb{Z}$ we can decompose $u = u^0 + u^1$ with

$$u_{0,m} = \begin{cases} u_m & \text{if } m \le k, \\ 0 & \text{if } m > k, \end{cases}, \quad u_{1,m} = \begin{cases} 0 & \text{if } m \le k, \\ u_m & \text{if } m > k. \end{cases}$$

Then $u^0 \in \ell_p(G_m)$, $u^1 \in \ell_p(2^{-m}G_m))$ and we have

$$\begin{aligned} & K(2^k, u; \ell_p(G_m), \ell_p(2^{-m}G_m)) \\ & \le \left(\sum_{m=-\infty}^{k} \|u_m\|_{G_m}^p \right)^{1/p} + 2^k \left(\sum_{m=k+1}^{\infty} (2^{-m}\|u_m\|_{G_m})^p \right)^{1/p} \\ & \le 2 \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p \|u_m\|_{G_m}^p \right)^{1/p}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|u\|_{(\ell_p(G_m), \ell_p(2^{-m}G_m))_{0,q;b}} \\ & = \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) K(2^k, u; \ell_p(G_m), \ell_p(2^{-m}G_m)) \right]^q \right)^{1/q} \\ & \le 2 \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) \left(\sum_{m=-\infty}^{\infty} \min(1, 2^{k-m})^p \|u_m\|_{G_m}^p \right)^{1/p} \right]^q \right)^{1/q} \\ & = 2 \left(\sum_{k=-\infty}^{\infty} \left[b(2^k) K_p \left(2^k, (\|u_m\|_{G_m}); \ell_p, \ell_p(2^{-m}) \right) \right]^q \right)^{1/q} \\ & \le 2^{1/p} \|u\|_{\Lambda(G_m)}. \end{aligned}$$

To establish the second embedding in (3.3), take any $x = (x_m) \in (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{0,q;b}$. Give any decomposition $x = x^0 + x^1$ with $x^0 = (x_{0,m}) \in \ell_\infty(F_m)$ and $x^1 = (x_{1,m}) \in \ell_\infty(2^{-m}F_m)$, and any $k \in \mathbb{Z}$, we obtain

$$\begin{aligned} \|x_k\|_{F_k} & \le c(\|x_k^0\|_{F_k} + \|x_k^1\|_{F_k}) \\ & \le c(\|x^0\|_{\ell_\infty(F_m)} + 2^k \|x^1\|_{\ell_\infty(2^{-m}F_m)}). \end{aligned}$$

It follows that

$$\|x_k\|_{F_k} \le cK(2^k, x; \ell_\infty(F_m), \ell_\infty(2^{-m}F_m)), \quad k \in \mathbb{Z}.$$

Therefore, $\|x\|_{\ell_q(b(2^m)F_m)} \le c\|x\|_{(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{0,q;b}}$ as we wanted.

Put $\hat{T} = \iota T \pi$. Since

$$\begin{aligned} \beta(T : \bar{A}_{0,q;b} \to \bar{B}_{0,q;b}) & \le c_1 \beta(\iota T : \bar{A}_{0,q;b} \to \ell_q(b(2^m)F_m)) \\ & \le c_2 \beta(\hat{T} : \Lambda(G_m) \to \ell_q(b(2^m)F_m)), \end{aligned}$$

it suffices to estimate the measure of non-compactness of $\hat{T}$ acting between the vector-valued sequence spaces. With this aim, for $n \in \mathbb{N}$ we decompose $\hat{T}$ as

$$\begin{aligned}\hat{T} &= \hat{T}P_n + \hat{T}(P_n^+ + P_n^-)\\ &= \hat{T}P_n + Q_n\hat{T}(P_n^+ + P_n^-) + Q_n^-\hat{T}P_n^+ + Q_n^+\hat{T}P_n^- + Q_n^-\hat{T}P_n^- + Q_n^+\hat{T}P_n^+\end{aligned}$$

and we proceed to estimate the measure of non-compactness of each of these six operators acting from $\Lambda(G_m)$ into $\ell_q(b(2^m)F_m)$.

Step 3. We start with $Q_n^-\hat{T}P_n^+$. We are going to show that $\beta(Q_n^-\hat{T}P_n^+ : \Lambda(G_m) \to \ell_q(b(2^m)F_m))$ tends to 0 as $n \to \infty$.

Using the factorization

$$\ell_p(G_m) \xrightarrow{P_n^+} \ell_p\left(2^{-m}G_m\right) \xrightarrow{\hat{T}} \ell_\infty(2^{-m}F_m) \xrightarrow{Q_n^-} \ell_\infty(F_m)$$

and (3.1) and (3.2), we get

$$\|Q_n^-\hat{T}P_n^- : \ell_p(G_m) \to \ell_\infty(F_m)\| \le 2^{-(n+1)}\|T : A_1 \to B_1\|2^{-(n+1)} \to 0 \text{ as } n \to \infty.$$

In addition, the factorization

$$\ell_p(2^{-m}G_m) \xrightarrow{P_n^+} \ell_p(2^{-m}G_m) \xrightarrow{\hat{T}} \ell_\infty\left(2^{-m}F_m\right) \xrightarrow{Q_n^-} \ell_\infty\left(2^{-m}F_m\right)$$

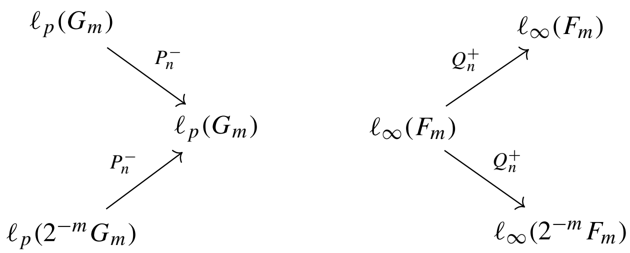
yields that

$$\|Q_n^-\hat{T}P_n^+ : \ell_p(2^{-m}G_m) \to \ell_\infty(2^{-m}F_m)\| \le \|T : A_1 \to B_1\| \text{ for any } n \in \mathbb{N}.$$

Therefore, by formulae (3.3) and (2.10), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\beta(Q_n^-\hat{T}P_n^+ : \Lambda(G_m) \longrightarrow \ell_q(b(2^m)F_m))\\ &\qquad \le c_1 \lim_{n\to\infty}\|Q_n^-\hat{T}P_n^+\|_{\overline{\ell_p(G)}_{0,q;b},\overline{\ell_\infty(F)}_{0,q;b}} = 0\,.\end{aligned}$$

Step 4. Consider $Q_n^+\hat{T}P_n^-$. Using the factorizations



and having in mind estimates (3.1), (3.2) and formulae (3.3) we get that $\|P_n^- : \Lambda(G_m) \to \ell_p(G_m)\| \le c_2$ and $\|Q_n^+ : \ell_\infty(F_m) \to \ell_q(b(2^m)F_m)\| \le c_3$. Hence, with the help of the diagram

$$
\begin{array}{ccc}
\Lambda(G_m) & \xrightarrow{Q_n^+ \hat{T} P_n^-} & \ell_q(b(2^m)F_m) \\
\downarrow P_n^- & & Q_n^+ \uparrow \\
\ell_p(G_m) & \xrightarrow{\hat{T}} & \ell_\infty(F_m)
\end{array}
$$

we derive

$$
\begin{aligned}
\beta(Q_n^+ \hat{T} P_n^- : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) &\le c_2 c_3 \beta(\hat{T} : \ell_p(G_m) \to \ell_\infty(F_m)) \\
&\le c_2 c_3 \beta(T : A_0 \to B_0).
\end{aligned}
$$

Step 5. Now we proceed with $Q_n^- \hat{T} P_n^-$. Take any $\sigma_j > \beta(T : A_j \to B_j)$, $j = 0, 1$. First we are going to compare $\| Q_n^- \hat{T} P_n^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)}$ with σ_1. We have

$$
\| Q_n^- \hat{T} P_n^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)} \le \| \hat{T} P_n^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)}
$$

and

$$
\| \hat{T} P_1^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)} \ge \| \hat{T} P_2^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)} \ge \ldots \ge 0.
$$

Therefore, the sequence $(\| \hat{T} P_n^- \|_{\ell_p(2^{-m}G_m), \ell_\infty(2^{-m}F_m)})$ is convergent, say, to $\tau \ge 0$. Let $(v_n) \subseteq U_{\ell_p(2^{-m}G_m)}$ such that $\lim_{n\to\infty} \| \hat{T} P_n^- v_n \|_{\ell_\infty(2^{-m}F_m)} = \tau$. To relate τ and σ_1, let $\{z_1, \ldots, z_r\} \subseteq B_1$ such that

$$
T\pi(U_{\ell_p(2^{-m}G_m)}) \subseteq \cup_{k=1}^r \{z_k + \sigma_1 U_{B_1}\}.
$$

We can find a subsequence $(v_{n'})$ of (v_n) and some $1 \le k \le r$ such that $\| T\pi P_{n'}^- v_{n'} - z_k \|_{B_1} \le \sigma_1$ for all n'. Then, for any $s \in \mathbb{Z}$, we have

$$
\begin{aligned}
K(2^s, z_k) &\le \| T\pi P_{n'}^- v_{n'} \|_{B_0} + 2^s \| z_k - T\pi P_{n'}^- v_{n'} \|_{B_1} \\
&\le \| P_{n'}^- v_{n'} \|_{\ell_p(G_m)} \| T \|_{A_0, B_0} + 2^s \sigma_1 \\
&\le 2^{-n'} \| T \|_{A_0, B_0} + 2^s \sigma_1 \to 2^s \sigma_1 \text{ as } n' \to \infty.
\end{aligned}
$$

It follows that

$$
\| \iota z_k \|_{\ell_\infty(2^{-m}F_m)} = \sup_{s \in \mathbb{Z}} \{2^{-s} K(2^s, z_k)\} \le \sigma_1.
$$

Hence,

$$
\begin{aligned}
\tau &= \lim_{n' \to \infty} \| \hat{T} P_{n'}^- v_{n'} \|_{\ell_\infty(2^{-m}F_m)} \\
&\le \max\{c_{B_0}, c_{B_1}\} \sup_{n'} \{ \| \hat{T} P_{n'}^- v_{n'} - \iota z_k \|_{\ell_{\infty(2^{-m}F_m)}} + \| \iota z_k \|_{\ell_\infty(2^{-m}F_m)} \}
\end{aligned}
$$

$$\leq \max\{c_{B_0}, c_{B_1}\} \sup_{n'}\{\|T\pi P_{n'}^- v_{n'} - z_k\|_{B_1} + \sigma_1\}$$
$$\leq 2\max\{c_{B_0}, c_{B_1}\}\sigma_1.$$

Since the sequence $(\|\hat{T} P_n^-\|_{\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}F_m)})$ is decreasing, we conclude that there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$\begin{aligned}\|Q_n^- \hat{T} P_n^-\|_{\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} &\leq \|\hat{T} P_n^-\|_{\ell_p(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \\ &\leq 3\max\{c_{B_0}, c_{B_1}\}\sigma_1.\end{aligned}$$

Next we compare $\|Q_n^- \hat{T} P_n^-\|_{\ell_p(G_m),\ell_\infty(F_m)}$ with σ_0. Since sequences having a finite number of coordinates different from 0 are dense in $\ell_p(G_m)$, we can find $\{d_1, \dots, d_s\} \subseteq U_{\ell_p(G_m)}$ such that each d_k has a finite number of coordinates different from 0 and with

$$\hat{T}(U_{\ell_p(G_m)}) \subseteq \bigcup_{k=1}^{s}\{\hat{T} d_k + c_4\sigma_0 U_{\ell_\infty(F_m)}\}$$

where $c_4 = 3\max\{c_{B_0}, c_{B_1}\}^2$. We can also find $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ we have

$$\|Q_n^- \hat{T} d_k\|_{\ell_\infty(F_m)} \leq 2^{-(n+1)}\|\hat{T} d_k\|_{\ell_\infty(2^{-m}F_m)} \leq \sigma_0 \text{ for any } 1 \leq k \leq s.$$

Take any $n \geq N_2$ and any $u \in U_{\ell_p(G_m)}$. Then $P_n^- u \in U_{\ell_p(G_m)}$ and so there is $1 \leq k \leq s$ such that $\|\hat{T} P_n^- u - \hat{T} d_k\|_{\ell_\infty(F_m)} \leq c_4\sigma_0$. Therefore, $\|Q_n^- \hat{T} P_n^- u\|_{\ell_\infty(F_m)} \leq \|Q_n^- \hat{T} P_n^- u - Q_n^- \hat{T} d_k\|_{\ell_\infty(F_m)} + \|Q_n^- \hat{T} d_k\|_{\ell_\infty(F_m)} \leq 2c_4\sigma_0$.

Finally, using (3.3) and (2.9), we derive that there is $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\begin{aligned}\beta(Q_n^- \hat{T} P_n^- : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) &\leq c_5\|Q_n^- \hat{T} P_n^-\|_{\overline{\ell_p(G)}_{0,q;b},\overline{\ell_\infty(F)}_{0,q;b}} \\ &\leq \begin{cases} c_6\sigma_0 & \text{if } \sigma_1 \leq \sigma_0, \\ c_6\sigma_0\overline{b}\left(\frac{\sigma_0}{\sigma_1}\right) & \text{if } \sigma_0 < \sigma_1. \end{cases}\end{aligned}$$

With similar arguments one can show that there is a constant $c_7 > 0$ such that

$$\beta(Q_n^+ \hat{T} P_n^+ : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) \leq \begin{cases} c_7\sigma_0 & \text{if } \sigma_1 \leq \sigma_0, \\ c_7\sigma_0\overline{b}\left(\frac{\sigma_0}{\sigma_1}\right) & \text{if } \sigma_0 < \sigma_1. \end{cases}$$

Step 6. Given any quasi-Banach sequence lattice Γ, we can define a quasi-norm $\|\cdot\|_{\widetilde{\Gamma}}$ in $\mathbb{R}^{2n+1}$ by $\|x\|_{\widetilde{\Gamma}} = \|\tilde{x}\|_\Gamma$, where $x = (x_k)_{-n\leq k\leq n} \in \mathbb{R}^{2n+1}$, $\tilde{x} = \sum_{k=-n}^{n} x_k e_k$, $e_k = (\delta_m^k)_{m\in\mathbb{Z}}$ and δ_m^k is the Kronecker delta. Compactness of the unit ball $U = U_{(\mathbb{R}^{2n+1},\|\cdot\|_{\widetilde{\Gamma}})}$ in $(\mathbb{R}^{2n+1}, \|\cdot\|_{\widetilde{\Gamma}})$ will be useful to estimate the measure of non-compactness of the remaining operators.

Let $\sigma_j > \beta(T : A_j \to B_j)$, $j = 0, 1$. We can find finite sets $\Sigma_j = \{h_1^j, \ldots, h_{L_j}^j\} \subseteq B_j$ such that

$$T(U_{A_j}) \subseteq \bigcup_{l=1}^{L_j} \{h_l^j + \sigma_j U_{B_j}\}, \quad j = 0, 1. \tag{3.4}$$

Let $N \in \mathbb{N}$ such that $2^{N-1} \leq \sigma_1/\sigma_0 < 2^N$ if $\sigma_0 \leq \sigma_1$ and let $N = 0$ if $\sigma_1 < \sigma_0$.

As for $\hat{T} P_n$, consider the quasi-norm $\|\cdot\|_{\widetilde{\Lambda}}$ on $\mathbb{R}^{2n+1}$ and let $\eta = \left\| \sum_{k=-n}^{n} \frac{e_k}{\|e_k\|_\Lambda} \right\|_\Lambda^{-1}$. By compactness of $U = U_{(\mathbb{R}^{2n+1}, \|\cdot\|_{\widetilde{\Lambda}})}$, we can find a finite set $\Upsilon = \{\lambda^1, \ldots, \lambda^s\} \subseteq U$ such that

$$U \subseteq \bigcup_{d=1}^{s} \{\lambda^d + \eta U\}.$$

We associate to each $\lambda^d = \left(\lambda_k^d\right)_{-n \leq k \leq n}$ the numbers

$$\varphi_k^j = \varphi_{k,\lambda^d}^j = \left(\frac{\eta}{\|e_k\|_\Lambda} + \left|\lambda_k^d\right| \right) 2^{-kj}, \quad j = 0, 1.$$

Next, for $-n \leq k \leq n$, $\lambda^d \in \Upsilon$, $h_l^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$ in (3.4), pick any g_k in the intersection $(\varphi_k^0 h_l^0 + \varphi_k^0 \sigma_0 U_{B_0}) \cap (\varphi_k^1 h_y^1 + \varphi_k^1 \sigma_1 U_{B_1})$ provided it is non-empty and let $g_k = 0$ otherwise. Consider the finite set Φ formed by all sums $\sum_{k=-n}^{n} g_k$. We look at $\overline{B}_{0,q;b}$ as a J-space. We have

$$\beta(\hat{T} P_n : \Lambda(G_m) \to \ell_q(b\left(2^m\right) F_m)) \leq c_1 \beta(T \pi P_n : \Lambda(G_m) \to \overline{B}_{\Lambda;J}).$$

We are going to estimate the last term with the help of Φ.

For any $u = (u_m) \in U_{\Lambda(G_m)}$, we can find $\lambda^d \in \Upsilon$ such that

$$|J(2^k, u_k) - \lambda_k^d| \|e_k\|_\Lambda \leq \|(J(2^k, u_k) - \lambda_k^d)\|_{\widetilde{\Lambda}} \leq \eta, \quad -n \leq k \leq n.$$

It follows that $|J(2^k, u_k)| \leq \frac{\eta}{\|e_k\|_\Lambda} + |\lambda_k^d| = 2^{kj} \varphi_k^j$. This yields that $\|u_k\|_{A_j} \leq \varphi_k^j$, $-n \leq k \leq n$, $j = 0, 1$. By (3.4), there are $h_l^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$ such that

$$\|T u_k - \varphi_k^0 h_l^0\|_{B_0} \leq \varphi_k^0 \sigma_0$$

and

$$\|T u_k - \varphi_k^1 h_y^1\|_{B_1} \leq \varphi_k^1 \sigma_1.$$

Hence, the intersection $(\varphi_k^0 h_l^0 + \varphi_k^0 \sigma_0 U_{B_0}) \cap (\varphi_k^1 h_y^1 + \varphi_k^1 \sigma_1 U_{B_1})$ is not empty and for the g_k corresponding to that intersection we have

$$\begin{aligned}J(2^{k-N}, Tu_k - g_k) \leq \max\{&\|Tu_k - \varphi_k^0 h_l^0\|_{B_0}^p + \|\varphi_k^0 h_l^0 - g_k\|_{B_0}^p,\\ &2^{(k-N)p}(\|Tu_k - \varphi_k^1 h_y^1\|_{B_1}^p + \|\varphi_k^1 h_y^1 - g_k\|_{B_1}^p)\}^{1/p}\\ &\leq 2^{1/p} \max\{\sigma_0, 2^{-N}\sigma_1\}\varphi_k^0.\end{aligned}$$

Then, $g = \sum_{k=-n}^{n} g_k$ belongs to Φ and

$$\begin{aligned}&\|T\pi P_n u - g\|_{\overline{B}_{\Lambda;J}} = \left\| \sum_{k=-n}^{n} (Tu_k - g_k) \right\|_{\overline{B}_{\Lambda;J}}\\ &\leq \|\tau_N(\dots 0, 0, J(2^{-n-N}, Tu_{-n} - g_{-n}), \dots, J(2^{n-N}, Tu_n - g_n), 0, 0, \dots)\|_\Lambda\\ &\leq 2^{1/p}\|\tau_N\|_{\Lambda,\Lambda} \max\{\sigma_0, 2^{-N}\sigma_1\}\|(\dots, 0, 0, \varphi_{-n}^0, \dots, \varphi_n^0, 0, 0, \dots)\|_\Lambda\\ &\leq c_2\overline{b}(2^{-N}) \max\{\sigma_0, 2^{-N}\sigma_1\}\end{aligned}$$

where we have used Lemma 2.2 and definition of φ_k^0 in the last inequality. Whence, according to the choice of N and (2.3), we obtain that

$$\begin{aligned}\beta(\hat{T} P_n : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) &\leq c_1\beta(T\pi P_n : \Lambda(G_m) \to \bar{B}_{\Lambda;J})\\ &\leq c_3\overline{b}(2^{-N}) \max\{\sigma_0, 2^{-N}\sigma_1\}\\ &\leq \begin{cases} c_4\sigma_0 & \text{if } \sigma_1 \leq \sigma_0,\\ c_4\sigma_0\overline{b}\left(\frac{\sigma_0}{\sigma_1}\right) & \text{if } \sigma_0 < \sigma_1.\end{cases}\end{aligned}$$

Next we consider $Q_n\hat{T}(P_n^+ + P_n^-)$. This time we work with $\overline{A}_{0,q;b}$ and $\overline{B}_{0,q;b}$ realized as K-spaces. We put $\Delta = \ell_q(b(2^m))$. We have

$$\beta(Q_n\hat{T}(P_n^+ + P_n^-) : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) \leq c_5\beta(Q_n\iota T : \overline{A}_{0,q;b} \to \Delta(F_m)).$$

Let now $\eta = \left\| \sum_{k=-n}^{n} \frac{e_k}{\|e_k\|_\Delta} \right\|_\Delta^{-1}$ and consider on $\mathbb{R}^{2n+1}$ the quasi-norm $\|\cdot\|_{\widetilde{\Delta}}$. There is a finite set $\Psi = \{\mu^1, \dots, \mu^s\} \subseteq U = U_{(\mathbb{R}^{2n+1}, \|\cdot\|_{\widetilde{\Delta}})}$ such that

$$U \subseteq \bigcup_{d=1}^{s} \left\{\mu^d + \eta U\right\}.$$

Starting from $\mu^d = (\mu_k^d)_{-n\leq k\leq n}$ we define the numbers

$$\psi_k^j = \psi_{k,\mu^d}^j = \overline{b}(2^{-N})\left(\frac{\eta}{\|e_k\|_\Delta} + |\mu_k^d|\right)2^{-(k+N)j}, \quad j = 0, 1,$$

where N was defined in the following line to (3.4). Let Ω be the finite subset of $\Delta(F_m)$ formed by all vectors $z^{d,l,y} = (z_m^{d,l,y})_{m\in\mathbb{Z}}$ where

$$z_m^{d,l,y} = \begin{cases} 0 & \text{if } m \notin [-n,n] \\ \psi_m^0 h_l^0 + \psi_m^1 h_y^1 & \text{if } -n \le m \le n, \end{cases}$$

where $h_\ell^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$ are the vectors of (3.4). We refer to $z^{d,l,y}$ as the element of Ω associated to μ^d, h_l^0 and h_y^1.

Given any $a \in U_{\overline{A}_{0,q;b}}$, using the shift operator τ_N and (2.4), we have

$$\|(K(2^{m+N},a))\|_\Delta \le \|\tau_N\|_\Delta \|a\|_{\overline{A}_{0,q;b}} \le \overline{b}(2^{-N}).$$

Therefore, there is $\mu^d \in \Psi$ such that

$$\|(K(2^{m+N},a) - \overline{b}(2^{-N})\mu_m^d)_{-n\le m\le n}\|_{\widetilde{\Delta}} < \eta\overline{b}(2^{-N}).$$

Hence

$$|K(2^{m+N},a) - \overline{b}(2^{-N})\mu_m^d| \|e_m\|_\Delta < \eta\overline{b}(2^{-N}), \quad -n \le m \le n,$$

and so $K(2^{m+N},a) < \psi_m^0$ for $-n \le m \le n$. It follows that we can decompose $a = a_{0,m} + a_{1,m}$ with $a_{j,m}$ belonging to A_j and such that $\|a_{0,m}\|_{A_0} + 2^{m+N}\|a_{1,m}\|_{A_1} < \psi_m^0$. Therefore, there are $h_l^0 \in \Sigma_0$ and $h_y^1 \in \Sigma_1$ such that

$$\|Ta_{0,m} - \psi_m^0 h_l^0\|_{B_0} \le \psi_m^0 \sigma_0$$

and

$$\|Ta_{1,m} - \psi_m^1 h_y^1\|_{B_1} \le \psi_m^1 \sigma_1, \quad -n \le m \le n.$$

If we take $z = z^{d,l,y}$ the element of Ω associated to μ^d, h_l^0, and h_y^1, then we have

$$\begin{aligned} \|Q_n\iota Ta - z\|_{\Delta(F_m)} &= \|(K(2^m, Ta - z_m^{d,l,y}))_{-n\le m\le n}\|_{\widetilde{\Delta}} \\ &\le \left\|(\|Ta_{0,m} - \psi_m^0 h_l^0\|_{B_0} + 2^m\|Ta_{1,m} - \psi_m^1 h_y^1\|_{B_1})_{-n\le m\le n}\right\|_{\widetilde{\Delta}} \\ &\le \|(\psi_m^0\sigma_0 + 2^m\psi_m^1\sigma_1)_{-n\le m\le n}\|_{\widetilde{\Delta}} \\ &\le c_6\overline{b}(2^{-N})(\sigma_0 + 2^{-N}\sigma_1). \end{aligned}$$

Consequently,

$$\begin{aligned} \beta(Q_n\hat{T}(P_n^+ + P_n^-) : \Lambda(G_m) \to \ell_q(b(2^m)F_m)) &\le c_5\beta(Q_n\iota T : \overline{A}_{0,q;b} \to \Delta(F_m)) \\ &\le c_7\overline{b}(2^{-N})(\sigma_0 + 2^{-N}\sigma_1) \end{aligned}$$

$$\leq \begin{cases} c_8\sigma_0 & \text{if } \sigma_1 \leq \sigma_0, \\ c_8\sigma_0 \overline{b}\left(\frac{\sigma_0}{\sigma_1}\right) & \text{if } \sigma_0 < \sigma_1, \end{cases},$$

where we have used the value of N and (2.3) in the last inequality.
Step 7. Collecting the estimates of the Steps 3 to 6, we conclude that there is a constant $C > 0$ independent of T such that if we split the operator as in the Step 2 and we take a suitable n, then for $\sigma_j > \beta(T : A_j \to B_j)$, we have

$$\beta(\hat{T} : \Lambda(G_m) \to \ell_q(2^m F_m)) \leq \begin{cases} C\sigma_0 & \text{if } \sigma_1 \leq \sigma_0, \\ C \max\left\{\sigma_0, \sigma_0 \overline{b}\left(\frac{\sigma_0}{\sigma_1}\right)\right\} & \text{if } \sigma_0 < \sigma_1. \end{cases}$$

Then, if $\beta(T : A_0 \to B_0) = 0$, letting $\sigma_0 \to 0$ and using (2.2) we obtain case (i) of the statement. If $0 \leq \beta(T : A_1 \to B_1) < \beta(T : A_0 \to B_0)$, letting $\sigma_0 \to \beta(T : A_0 \to B_0)$ we get the case (ii). Finally, if $0 < \beta(T : A_0 \to B_0) \leq \beta(T : A_1 \to B_1)$, taking $\sigma_j = (1+\varepsilon)\beta(T : A_j \to B_j)$ and letting ε goes to 0 we derive the case (iii). This finishes the proof. □

Remark 3.2 On the contrary to the case of the real method (see [17, 22]), if $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_1 \to B_1$ is compact, then $T : \bar{A}_{0,q;b} \to \bar{B}_{0,q;b}$ might not be compact. A counterexample can be found in [13, Remark 2.4].

For limiting methods with $\theta = 1$ we have the following direct consequence of (2.7) and Theorem 3.1.

Theorem 3.3 *Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be quasi-Banach couples and let $T \in \mathcal{L}(\overline{A}, \overline{B})$. Let $0 < q \leq \infty$ and $v \in SV(0, \infty)$ satisfying*

$$\left(\int_0^1 v(t)^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty, \quad \textit{and also that}$$

$$\left(\int_1^\infty v(t)^q \frac{\mathrm{d}t}{t}\right)^{1/q} = \infty \textit{ if } q < \infty \textit{ and } \lim_{t\to\infty} v(t) = \infty \textit{ if } q = \infty.$$

Then we have

(i) $\beta(T : \overline{A}_{1,q;v} \to \overline{B}_{1,q;v}) = 0$ *if* $\beta(T : A_1 \to B_1) = 0$,
(ii) $\beta(T : \overline{A}_{1,q;v} \to \overline{B}_{1,q;v}) \leq C\beta(T : A_1 \to B_1)$ *if* $0 \leq \beta(T : A_0 \to B_0) < \beta(T : A_1 \to B_1)$,
(iii)

$$\beta(T : \overline{A}_{1,q;v} \to \overline{B}_{1,q;v}) \leq C \max\Big\{\beta(T : A_1 \to B_1),$$

$$\beta(T : A_1 \to B_1)\overline{v}\left(\frac{\beta(T : A_1 \to B_1)}{\beta(T : A_0 \to B_0)}\right)\Bigg\}$$

if $0 < \beta(T : A_1 \to B_1) \le \beta(T : A_0 \to B_0)$.

Here C *is a constant independent of* T.

4 Applications

Let (R, μ) be a non-atomic σ-finite measure space. For $0 < p, q \le \infty$ and $b \in SV(0, \infty)$, the *Lorentz–Karamata space* $L_{p,q;b}(R)$ is formed by all (equivalent classes of) measurable functions f on R which have a finite quasi-norm

$$\|f\|_{L_{p,q;b}(R)} = \left(\int_0^\infty [t^{1/p} b(t) f^*(t)]^q \frac{\mathrm{d}t}{t}\right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here f^* stands for the non-increasing rearrangement of f defined by

$$f^*(t) = \inf\{s > 0 : \mu\{x \in R : |f(x)| > s\} \le t\}.$$

We refer to [25] and [29] for properties of Lorentz–Karamata spaces. Note that if $b(t) = (1 + |\log t|)^a$ we get the *Lorentz–Zygmund spaces* $L_{p,q}(\log L)_a$ (see [2, 3]). If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and

$$b(t) = \ell^{\mathbb{A}}(t) = \begin{cases} (1 + |\log t|)^{\alpha_0} & \text{for } 0 < t \le 1, \\ (1 + |\log t|)^{\alpha_\infty} & \text{for } 1 < t < \infty, \end{cases}$$

then we obtain the *generalized Lorentz–Zygmund spaces* $L_{p,q}(\log L)_{\mathbb{A}}(R)$ (see [39]). If $b \equiv 1$ then we obtain the *Lorentz spaces* $L_{p,q}(R)$ (see [4, 8, 43]) and if, in addition, $p = q$ then we get the *Lebesgue spaces* $L_p(R)$.

In what follows, we work with couples of Lebesgue spaces $(L_{p_0}(R), L_{p_1}(R))$, $(L_{q_0}(S), L_{q_1}(S))$ and operators $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$. We put

$$\beta(T_j) = \beta(T : L_{p_j}(R) \to L_{q_j}(S)), \quad j = 0, 1.$$

It is shown in [29, Corollary 5.3] that

$$L_{p,q;b}(R) = (L_1(R), L_\infty(R))_{\theta,q;b} \tag{4.1}$$

provided that $1 < p < \infty$, $0 < \theta < 1$, $1/p = 1 - \theta$, $0 < q \le \infty$ and $b \in SV(0, \infty)$.

As a consequence of Theorem 3.1 we can establish the following result for Lorentz–Karamata spaces.

Theorem 4.1 *Let (R, μ) and (S, ν) be non-atomic σ-finite measure spaces. Let $1 < p_0 < p_1 < \infty$, $1 < q_0 < q_1 < \infty$, $0 < q < \infty$ and let $b \in SV(0,\infty)$ satisfying (2.8) and (2.12). Put*

$$b_0(t) = b(t^{1/p_0-1/p_1})\left(\frac{1}{b(t^{1/p_0-1/p_1})^q}\int_t^\infty b(s^{1/p_0-1/p_1})^q\frac{ds}{s}\right)^{1/\min\{p_0,q\}}$$

and

$$b_1(t) = b(t^{1/q_0-1/q_1})\left(\frac{1}{b(t^{1/q_0-1/q_1})^q}\int_t^\infty b(s^{1/q_0-1/q_1})^q\frac{ds}{s}\right)^{1/\max\{q_0,q\}}.$$

If $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$ then

$$T : L_{p_0,q;b_0}(R) \to L_{q_0,q;b_1}(S) \text{ boundedly.}$$

Moreover, for $\beta(T) = \beta(T : L_{p_0,q;b_0}(R) \to L_{q_0,q;b_1}(S))$ we have

(a) $\beta(T) = 0$ if $\beta(T_0) = 0$,
(b) $\beta(T) \le C\beta(T_0)$ if $0 \le \beta(T_1) < \beta(T_0)$,
(c) $\beta(T) \le C \max\left\{\beta(T_0), \beta(T_0)\overline{b}\left(\beta(T_0)/\beta(T_1)\right)\right\}$ if $0 < \beta(T_0) \le \beta(T_1)$.

Here $C > 0$ is a constant independent of T.

Proof Let $0 < \theta_0 < \theta_1 < 1$ such that $1/p_j = 1 - \theta_j$, $j = 0, 1$. We have $L_{p_j}(R) = (L_1(R), L_\infty(R))_{\theta_j,p_j}$. Hence, we can use the reiteration formula of [29, Theorem 3.2] to work with the space $(L_{p_0}(R), L_{p_1}(R))_{0,q;b}$. Then, according to [36, Theorem 4.10] and (4.1), we obtain

$$L_{p_0,q;b_0}(R) = (L_1(R), L_\infty(R))_{\theta_0,q;b_0} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q;b}.$$

Similarly, but using now [36, Theorem 4.8] with $\eta_j = 1-1/q_j$, $j = 0, 1$, we get

$$(L_{q_0}(S), L_{q_1}(S))_{0,q;b} \hookrightarrow (L_1(S), L_\infty(S))_{\eta_0,q;b_1} = L_{q_0,q;b_1}(S).$$

Therefore, the result follows interpolating with parameters $0, q, b$ the couples $(L_{p_0}(R), L_{p_1}(R))$, $(L_{q_0}(S), L_{q_1}(S))$, applying Theorem 3.1 and having in mind the embeddings pointed out above. □

Subsequently, for $\tau \in \mathbb{R}$ and $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we put $\mathbb{A}+\tau = (\alpha_0+\tau, \alpha_\infty+\tau)$. Recall that $\alpha^+ = \max\{\alpha, 0\}$ for $\alpha \in \mathbb{R}$.

Remark 4.2 Let $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ such that $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$ and let $b(t) = (1 + |\log t|)^{\mathbb{A}}$. Then for the function b_0 in Theorem 4.1 we obtain

$$b_0(t) \approx b(t)\left(\frac{1}{b(t)^q}\int_t^\infty b(s)^q\frac{ds}{s}\right)^{1/\min\{p_0,q\}}$$

$$\approx (1+|\log t|)^{\mathbb{A}}(1+|\log t|)^{1/\min\{p_0,q\}}$$
$$= (1+|\log t|)^{\mathbb{A}+\frac{1}{\min\{p_0,q\}}}.$$

Similarly,

$$b_1(t) \approx (1+|\log t|)^{\mathbb{A}+\frac{1}{\max\{q_0,q\}}}.$$

Hence, we have that

$$L_{p_0,q;b_0}(R) = L_{p_0,q}(\log L)_{\mathbb{A}+\frac{1}{\min\{p_0,q\}}}(R)$$

and

$$L_{q_0,q;b_1}(S) = L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(S).$$

Moreover, by [14, Lemma 2.1] and [5, (2.6)], we have

$$\overline{b}(t) \le (1+|\log t|)^{(\alpha_0^+-\alpha_\infty,(-\alpha_0)^+)}.$$

Consequently, writing down Theorem 4.1 for this choice of b we recover a result of Besoy and Cobos (see [5, Corollary 3.13]), which is a quantitative version of a compactness result of Edmunds and Opic (see [26, Corollary 4] and also [20, Corollary 4.5]).

The following result refers to the case $1 < p_1 < p_0 < \infty$.

Theorem 4.3 *Let (R,μ) and (S,ν) be non-atomic σ-finite measure spaces. Let $1 < p_1 < p_0 < \infty$, $1 < q_0 < q_1 < \infty$, $0 < q < \infty$ and let $b \in SV(0,\infty)$ satisfying (2.8) and (2.12). Put*

$$\widetilde{b}_0(t) = b(t^{1/p_0-1/p_1})\left(\frac{1}{b(t^{1/p_0-1/p_1})^q}\int_0^t b(s^{1/p_0-1/p_1})^q\,\frac{ds}{s}\right)^{1/\min\{p_0,q\}}$$

and

$$b_1(t) = b(t^{1/q_0-1/q_1})\left(\frac{1}{b(t^{1/q_0-1/q_1})^q}\int_t^\infty b(s^{1/q_0-1/q_1})^q\,\frac{ds}{s}\right)^{1/\max\{q_0,q\}}.$$

If $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$ then

$$T : L_{p_0,q;\widetilde{b}_0}(R) \to L_{q_0,q;b_1}(S) \text{ boundedly.}$$

Moreover, for $\beta(T) = \beta(T : L_{p_0,q;\widetilde{b}_0}(R) \to L_{q_0,q;b_1}(S))$ we have

(a) $\beta(T) = 0$ if $\beta(T_0) = 0$,

(b) $\beta(T) \le C\beta(T_0)$ *if* $0 \le \beta(T_1) < \beta(T_0)$,
(c) $\beta(T) \le C \max\left\{\beta(T_0), \beta(T_0)\overline{b}\left(\beta(T_0)/\beta(T_1)\right)\right\}$ *if* $0 < \beta(T_0) \le \beta(T_1)$.

Here $C > 0$ *is a constant independent of* T.

Proof Consider the couple $(L_\infty(R), L_1(R))$. We have

$$L_{p_j}(R) = (L_\infty(R), L_1(R))_{\widetilde{\theta}_j, p_j} \quad \text{where } \widetilde{\theta}_j = \frac{1}{p_j}, \ \ j = 0, 1.$$

So $0 < \widetilde{\theta}_0 < \widetilde{\theta}_1 < 1$ and we still can use [29, Theorem 3.2] and [36, Theorem 4.10] to get that

$$(L_\infty(R), L_1(R))_{\widetilde{\theta}_0, q; u} \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q;b}$$

where

$$u(t) = b(t^{1/p_1 - 1/p_0}) \left(\frac{1}{b(t^{1/p_1-1/p_0})^q} \int_t^\infty b(s^{1/p_1-1/p_0})^q \frac{\mathrm{d}s}{s} \right)^{1/\min\{p_0, q\}}.$$

According to the relationship between the K-functionals of $(L_\infty(R), L_1(R))$ and $(L_1(R), L_\infty(R))$, making a change of variables and using (4.1), we obtain

$$\begin{aligned}
\|f\|_{(L_\infty(R), L_1(R))_{\widetilde{\theta}_0, q; u}} &= \left(\int_0^\infty [t^{-\widetilde{\theta}_0} u(t) K(t, f; L_\infty(R), L_1(R))]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \left(\int_0^\infty [t^{1-\widetilde{\theta}_0} \widetilde{b}_0(t^{-1}) K(t^{-1}, f; L_1(R), L_\infty(R))]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \left(\int_0^\infty [t^{\widetilde{\theta}_0 - 1} \widetilde{b}_0(t) K(t, f; L_1(R), L_\infty(R))]^q \frac{\mathrm{d}t}{t} \right)^{1/q} \\
&= \|f\|_{(L_1(R), L_\infty(R))_{1-\widetilde{\theta}_0, q; \widetilde{b}_0}} \approx \|f\|_{L_{p_0, q; \widetilde{b}_0}(R)}.
\end{aligned}$$

Therefore

$$L_{p_0, q; \widetilde{b}_0}(R) \hookrightarrow (L_{p_0}(R), L_{p_1}(R))_{0,q;b}.$$

Since the embedding

$$(L_{q_0}(S), L_{q_1}(S))_{0,q;b} \hookrightarrow L_{q_0, q; b_1}(S)$$

has been established in Theorem 4.1, we can conclude the result by interpolating with parameters $0, q, b$ and applying Theorem 3.1. □

Remark 4.4 If $b(t) = (1+|\log t|)^{\mathbb{A}}$ with $\mathbb{A} = (\alpha_0, \alpha_\infty)$ and $\alpha_\infty + 1/q < 0 < \alpha_0 + 1/q$, then

$$\widetilde{b}_0(t) \approx (1 + |\log t|)^{\widetilde{\mathbb{A}} + \frac{1}{\min\{p_0, q\}}}$$

where $\widetilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ and Theorem 4.3 gives estimates for the measure of non-compactness of

$$T : L_{p_0,q}(\log L)_{\widetilde{\mathbb{A}}+\frac{1}{\min\{p_0,q\}}}(R) \to L_{q_0,q}(\log L)_{\mathbb{A}+\frac{1}{\max\{q_0,q\}}}(R).$$

Proceeding similarly, but using [36, Theorem 4.8], we can derive results for $1 < q_1 < q_0 < \infty$.

We finish the paper with some results when the main information on T refers to the restriction from $L_{p_1}(R)$ into $L_{q_1}(S)$.

Theorem 4.5 *Let (R, μ) and (S, ν) be non-atomic σ-finite measure spaces. Let $1 < p_1 < p_0 < \infty$, $1 < q_1 < q_0 < \infty$, $0 < q < \infty$ and let $v \in SV(0, \infty)$ satisfying that*

$$\left(\int_0^1 v(t)^q \frac{dt}{t}\right)^{1/q} < \infty \text{ and } \left(\int_1^\infty v(t)^q \frac{dt}{t}\right)^{1/q} = \infty.$$

Put

$$v_0(t) = v(t^{1/p_0-1/p_1})\left(\frac{1}{v(t^{1/p_0-1/p_1})^q}\int_t^\infty v(s^{1/p_0-1/p_1})^q \frac{ds}{s}\right)^{1/\min\{p_1,q\}}$$

and

$$v_1(t) = v(t^{1/q_0-1/q_1})\left(\frac{1}{v(t^{1/q_0-1/q_1})^q}\int_t^\infty v(s^{1/q_0-1/q_1})^q \frac{ds}{s}\right)^{1/\max\{q_1,q\}}.$$

If $T \in \mathcal{L}((L_{p_0}(R), L_{p_1}(R)), (L_{q_0}(S), L_{q_1}(S)))$ then

$$T : L_{p_1,q;v_0}(R) \to L_{q_1,q;v_1}(S) \text{ boundedly.}$$

Furthermore, for $\beta(T) = \beta(T : L_{p_1,q;v_0}(R) \to L_{q_1,q;v_1}(S))$ we have

(a) $\beta(T) = 0$ if $\beta(T_1) = 0$,
(b) $\beta(T) \le C\beta(T_1)$ if $0 \le \beta(T_0) < \beta(T_1)$,
(c) $\beta(T) \le C \max\{\beta(T_1), \beta(T_1)\overline{v}(\beta(T_1)/\beta(T_0))\}$ if $0 < \beta(T_1) \le \beta(T_0)$.

Here $C > 0$ is a constant independent of T.

Proof According to (2.7), for any quasi-Banach couple (A_0, A_1) we have $(A_0, A_1)_{1,q;v} = (A_1, A_0)_{0,q;b}$ where $b(t) = v(1/t)$. We also have that

$$T \in \mathcal{L}((L_{p_1}(R), L_{p_0}(R)), (L_{q_1}(S), L_{q_0}(S))).$$

Hence, the wanted result follows by interpolating with parameters $0, q, b$ and applying Theorem 4.1. □

If $1 < p_0 < p_1 < \infty$ and/or $1 < q_0 < q_1 < \infty$ we can obtain similar results.

Remark 4.6 Let $\mathbb{A} = (\alpha_0, \alpha_\infty)$ with $\alpha_0 + 1/q < 0 < \alpha_\infty + 1/q$ and let $v(t) = (1 + |\log t|)^{\mathbb{A}}$. So, v satisfies the assumptions of Theorem 4.5. We have

$$v_0(t) \approx (1 + |\log t|)^{\widetilde{\mathbb{A}} + \frac{1}{\min\{p_1, q\}}},$$
$$v_1(t) \approx (1 + |\log t|)^{\widetilde{\mathbb{A}} + \frac{1}{\max\{q_1, q\}}}.$$

Moreover, by [5, (2.6)] we know that $\overline{v}(t) \leq (1+|\log t|)^{((-\alpha_\infty)^+, \alpha_\infty^+ - \alpha_0)}$. Writing down Theorem 4.5 for this choice of the parameters we obtain estimates for the measure of non-compactness of

$$T : L_{p_1,q}(\log L)_{\tilde{\mathbb{A}} + \frac{1}{\min\{p_1,q\}}}(R) \to L_{q_1,q}(\log L)_{\tilde{\mathbb{A}} + \frac{1}{\max\{q_1,q\}}}(R).$$

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