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ORIGINAL PAPER

Extensions of hermitian linear functionals

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Abstract

We study, from a quite general point of view, the family of all extensions of a positive hermitian linear functional ω, defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$, with the aim of finding extensions that behave regularly. The sole constraint the extensions we are dealing with are required to satisfy is that their domain is a subspace of $\overline{G(\omega)}$, the closure of the graph of ω (these are the so-called slight extensions). The main results are two. The first is having characterized those elements of $\mathfrak{A}$ for which we can find a positive hermitian slight extension of ω, giving the range of the possible values that the extension may assume on these elements; the second one is proving the existence of maximal positive hermitian slight extensions. We show as it is possible to apply these results in several contexts: Riemann integral, Infinite sums, and Dirac Delta.

Keywords Positive linear functionals · Topological *-algebras

Mathematics Subject Classification 46H05 · 46H35

1 Introduction

Let $\mathfrak{A}$ be a topological *-algebra (in general, without unit), with topology τ and continuous involution *, and let $\mathfrak{A}_0$ be a dense *-subalgebra of $\mathfrak{A}$. Given a positive hermitian linear functional ω on $\mathfrak{A}_0$ (see Definition 2.1 below) is it possible to extend ω to some

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elements of $\mathfrak{A}$? In this paper we will continue the analysis undertaken in [3–25] with the aim of finding extensions that behave regularly.

The previous problem may have, in some situations, easy solutions, namely when ω is τ-continuous or closable [3, 22] (see Definition below).

In 2010 Bongiorno and two of us (CT, ST) proposed [3] to use the notion of *slight* extension given in [10, ch.7 §36.7] for nonclosable linear maps for studying extensions of linear functionals, moving from the basic example of the Riemann integral

$$\omega_R(f) = \int_I f(x)\mathrm{d}x, \quad f \in C(I),$$

regarded as a linear functional on $\mathfrak{A}_0$, the *-algebra $C(I)$ of complex valued continuous functions on a compact interval $I \subseteq \mathbb{R}$, considered as a dense *-subalgebra of $\mathfrak{A}$, the *-algebra of Lebesgue measurable functions on I with the topology of the convergence in measure. The involution * is given by pointwise complex conjugation. As noted in [3], it is quite elementary to realize that ω_R is not closable and this fact is responsible of the existence of several procedures for extending the integral that, starting from the Lebesgue integral, cover an extensive literature.

Coming back to the general case, three important questions arise. The first is for which elements of $\mathfrak{A}$ it is possible to find a slight extension which is still positive hermitian. The second is, given a such element, what values can the extension assume on it. The third is whether, for each choice, we can find a maximal positive hermitian slight extension. We will give answers to all these questions.

As an example we will prove that, for each γ such that, $0 \leq \gamma \leq 1$, there exists a positive hermitian slight extension $\widehat{\omega}_\gamma$ of the Riemann integral ω_R on $C([0, 1])$, taking the value γ on the Dirichlet function. Moreover for each γ there exists $\breve{\omega}_\gamma$, a maximal positive hermitian slight extension of the Riemann integral, that assumes the value γ on the Dirichlet function.

We introduce the notion of *widely positive* extension: roughly speaking a positive hermitian slight extension $\widehat{\omega}$ is said widely positive if is not possible to extend it to other positive elements of the algebra $\mathfrak{A}$. Moreover we give the notion of positive regular slight extension that closely reminds the construction of the Lebesgue integral or Segal's construction of noncommutative integration [18]. We will prove that if ω admits a positively regular absolutely convergent slight extension, which is fully positive, then this extension is unique.

Using the developed ideas we find interesting results concerning Infinite sums and, finally, we show that the Dirac Delta can be studied in the light of the present approach.

Several authors have considered the extension problem for hermitian, positive or representable functionals in various settings and from different points of view [1, 2, 17, 19–21]. Their approach is nevertheless different from the one adopted here.

2 Preliminary definitions and facts

With the aim of making the paper independent we specialize and repeat to the case of positive hermitian linear functionals the notion of slight extension and give, without proving them, the basic properties.

If $\mathfrak{A}$ is an arbitrary *-algebra, we put

$$\mathfrak{A}_h = \{b \in \mathfrak{A} \,:\, b = b^*\}, \quad \mathcal{P}(\mathfrak{A}) = \left\{ \sum_{i=1}^{n} a_i^* a_i \,:\, a_i \in \mathfrak{A} \right\}.$$

Elements of $\mathfrak{A}_h$ are called *self-adjoint*; elements of $\mathcal{P}(\mathfrak{A})$ are called *positive*. Clearly, $\mathcal{P}(\mathfrak{A}) \subseteq \mathfrak{A}_h$.

We will adopt the following terminology.

Definition 2.1 A linear functional ω, defined on a subspace $D(\omega)$ of $\mathfrak{A}$, is called

hermitian if $a \in D(\omega) \Leftrightarrow a^* \in D(\omega)$ and $\omega(a^*) = \overline{\omega(a)}$, for every $a \in D(\omega)$;
positive if $\omega(b) \geq 0$, for every $b \in D(\omega) \cap \mathcal{P}(\mathfrak{A})$.

In all this paper ω is a positive hermitian linear functional defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$, with continuous involution *.

The functional ω is said to be closable if one of the two equivalent statements which follow is satisfied. Define

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbb{C}; a \in \mathfrak{A}_0\}$$

- If $a_\alpha \to 0$ w.r. to τ and $\omega(a_\alpha) \to \ell$, then $\ell = 0$.
- $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$.

In this case, the closure $\overline{\omega}$ is defined on

$$D(\overline{\omega}) = \{a \in \mathfrak{A} \,:\, \exists \{a_\alpha\} \subset \mathfrak{A}_0,\ a_\alpha \to a \text{ and } \omega(a_\alpha) \text{ is convergent } \},$$

by

$$\overline{\omega}(a) = \lim_\alpha \omega(a_\alpha), \quad a \in D(\overline{\omega}).$$

The closability of ω implies that $\overline{\omega}$ is well-defined. The functional $\overline{\omega}$ is linear and is the minimal closed extension of ω (i.e., $G_{\overline{\omega}}$ is closed). Moreover the τ-continuity of the involution and the hermiticity of ω on $\mathfrak{A}_0$, imply that $\overline{\omega}$ is hermitian.

Coming back to the general case, let $\mathcal{S}_\omega$ denote the collection of all subspaces H of $\mathfrak{A} \times \mathbb{C}$ such that

(g1) $G_\omega \subseteq H \subseteq \overline{G_\omega}$;
(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

If ω is nonclosable, i.e. $\overline{G_\omega}$ contains pairs $(0, \ell)$ with $\ell \neq 0$, then $\overline{G_\omega} \notin \mathcal{S}_\omega$. In this case to every $H \in \mathcal{S}_\omega$, there corresponds an extension ω_H, to be called a *slight* extension of ω, defined on

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}$$

by

$$\omega_H(a) = \ell,$$

where, from (g2), ℓ is the unique complex number such that $(a, \ell) \in H$.

Moreover, by applying Zorn's lemma to the family $\mathcal{S}_\omega$, one proves that ω *admits a maximal slight extension*.

Notation: in all this paper we call briefly an *extension* of ω any ω_H such that $H \in \mathcal{S}_\omega$. In the case ω is closable, $\overline{G_\omega} \in \mathcal{S}_\omega$, so the corresponding $\overline{\omega}$ is the unique maximal extension of ω.

Let

$$\mathcal{K}_\omega := \{a \in \mathfrak{A} : (a, \ell) \in \overline{G_\omega}, \text{ for some } \ell \in \mathbb{C}\}.$$

Given ω, for every $(a, \ell) \in \overline{G_\omega}$, there exists a net $\{a_\alpha\} \subset \mathfrak{A}_0$, such that $a_\alpha \xrightarrow{\tau} a$ and $\omega(a_\alpha) \to \ell$. The τ-continuity of the involution and the hermiticity of ω on $\mathfrak{A}_0$, imply that $a_\alpha^* \xrightarrow{\tau} a^*$ and $\omega(a_\alpha^*) = \overline{\omega(a_\alpha)} \to \overline{\ell}$. Hence, $(a, \ell) \in \overline{G_\omega}$ if and only if $(a^*, \overline{\ell}) \in \overline{G_\omega}$. Then $\mathcal{K}_\omega$ is a subspace of $\mathfrak{A}$ with the property that $a \in \mathcal{K}_\omega$ if and only if $a^* \in \mathcal{K}_\omega$.

The following proposition holds.

Proposition 2.2 *For every maximal extension $\breve{\omega}$ of ω, $D(\breve{\omega}) = \mathcal{K}_\omega$.* □

Remark 2.3 In other words, all maximal extensions have the same domain $\mathcal{K}_\omega$; thus, if an extension has $\mathcal{K}_\omega$ as domain, then it is maximal.

Like in the general case considered in [10, ch.7 §36.7], uniqueness is certainly not a characteristic of the extensions of ω; in fact

Proposition 2.4 *If ω is not closable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal extensions.* □

The previous statement relies on the following

Proposition 2.5 *Let ω be nonclosable. Then*

(i) *if $\exists m \in \mathbb{C}$ such that $(a, m) \in \overline{G_\omega}$, then $(a, \ell) \in \overline{G_\omega}$, for every $\ell \in \mathbb{C}$.*
(ii) $\overline{G_\omega} = \mathcal{K}_\omega \times \mathbb{C}$. □

3 Hermitian and positive extensions

If ω is nonclosable the extensions defined above are neither hermitian nor positive, in general. It is natural to begin with considering the problem of the existence of hermitian extensions.

3.1 Hermiticity

Let $\mathcal{H}_\omega$ denote the collection of all subspaces $H \in \mathcal{S}_\omega$ for which the following additional condition holds

(h3) $(a, \ell) \in H$ implies $(a^*, \overline{\ell}) \in H$.

From (g2) and (h3) it follows

(h4) $(a, \ell) \in H$ and $a = a^*$, implies ℓ is real.

Since ω is hermitian then $\mathcal{H}_\omega \neq \emptyset$ and $G_\omega \subseteq H \subseteq \overline{G_\omega}$, for every $H \in \mathcal{H}_\omega$; moreover, if ω is nonclosable then $\overline{G_\omega} \notin \mathcal{S}_\omega$, hence $\overline{G_\omega} \notin \mathcal{H}_\omega$.

To every $H \in \mathcal{H}_\omega$ there corresponds an extension ω_H of ω defined on

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}$$

by

$$\omega_H(a) = \ell,$$

where, from $(g2)$, ℓ is the unique complex number such that $(a, \ell) \in H$. Since the condition (h3) implies that $a \in D(\omega_H) \Leftrightarrow a^* \in D(\omega_H)$ and, by definition, $\omega_H(a^*) = \overline{l} = \overline{\omega_H(a)}$, therefore ω_H is hermitian.

We observe that $\mathfrak{A}_0 \subseteq D(\omega_H) \subseteq \mathfrak{A}$ as vector spaces.

Remark 3.1 As is well-known, an arbitrary element $a \in \mathfrak{A}$ can be written in a unique way as $a = b + ic$, with b and c self-adjoint elements. Since H is a vector space, $a \in D(\omega_H) \Leftrightarrow b, c \in D(\omega_H)$; moreover $\omega_H(a) = \omega_H(b + ic) = \omega_H(b) + i\omega_H(c)$, where, by $(h4)$, $\omega_H(b), \omega_H(c) \in \mathbb{R}$.

The following proposition was already given in [3], but the proof was incomplete. We give a new proof here.

Proposition 3.2 *The following statements hold*

(i) *Every ω admits a maximal hermitian extension.*
(ii) *Let $\breve{\omega}$ be a maximal hermitian extension of ω. Then $D(\breve{\omega}) = \mathcal{K}_\omega$.*

(iii) *If ω is nonclosable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal hermitian extension.*

Proof (i): $\mathcal{H}_\omega$ satisfies the assumptions of Zorn's lemma. Then it has a maximal element $\breve{H}$ that defines a maximal hermitian extension $\breve{\omega}$.

(ii): The result is obvious if ω is closable. Let now ω be nonclosable. As it is clear, for every hermitian extension $\breve{\omega}$, one has $D(\breve{\omega}) \subseteq \mathcal{K}_\omega$. Let, by contradiction, $a \in \mathcal{K}_\omega \backslash D(\breve{\omega})$. Then also $a^* \in \mathcal{K}_\omega \backslash D(\breve{\omega})$ since $a \in D(\omega_H) \Leftrightarrow a^* \in D(\omega_H)$. Now let $a = b + ic$ with $b = (a + a^*)/2$, $c = (a - a^*)/2i$, $b = b^*$, $c = c^*$. Since $a \in D(\omega_H) \Leftrightarrow b, c \in D(\omega_H)$, then at least one, between b or c, does not belong to $D(\omega_H)$. Let, without loss of generality, $b \in \mathcal{K}_\omega \backslash D(\breve{\omega})$. Since $b \in \mathcal{K}_\omega$, by Lemma 2.5, we can choose $\ell \in \mathbb{R}$ such that $(b, \ell) \in \overline{G_\omega}$. Consider $G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle$, where $\langle (b, \ell) \rangle$ denotes the subspace generated by (b, ℓ). We will prove that $G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle \in \mathcal{H}_\omega$ and this contradicts the maximality of $\breve{\omega}$. Thus we need to show that if $(a_1, \ell_1) \in G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle$ then $(a_1^*, \bar{\ell}_1) \in G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle$. Let $(a_1, \ell_1) = (d + \alpha b, m + \alpha \ell)$ with $(d, m) \in G_{\breve{\omega}}$, $\alpha \in \mathbb{C}$. Since $b = b^*$, $\ell = \bar{\ell}$, then

$$(a_1^*, \bar{\ell}_1) = ((d + \alpha\, b)^*, \overline{m + \alpha\, \ell}) = (d^* + \bar{\alpha}\, b, \bar{m} + \bar{\alpha}\, \ell) = (d^*, \bar{m}) + (\bar{\alpha}\, b, \bar{\alpha}\, \ell).$$

But $(d^*, \bar{m}) \in G_{\breve{\omega}}$ and $(\bar{\alpha}\, b, \bar{\alpha}\, \ell) \in \langle (b, \ell) \rangle$, so $(a_1^*, \bar{\ell}_1) \in G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle$.

(iii): If $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$ then, as we have seen in the previous proof of (ii), there exists $b = b^* \in \mathcal{K}_\omega \backslash \mathfrak{A}_0$, and moreover, $\forall \ell \in \mathbb{R}$, $G_\omega \oplus \langle (b, \ell) \rangle \in \mathcal{H}_\omega$. From this there exists a maximal hermitian extension $\breve{\omega}$ such that $G_{\breve{\omega}} \supseteq G_\omega \oplus \langle (b, \ell) \rangle \supsetneq G_\omega$. It is clear that, for different values of $\ell \in \mathbb{R}$, the corresponding maximal hermitian extensions are different. □

From (i) and (ii) of the previous proposition and from Remark 2.3, we obtain the following

Theorem 3.3 *Every ω admits a maximal hermitian extension $\breve{\omega}$ which is, at once, a maximal extension, so $D(\breve{\omega}) = \mathcal{K}_\omega$.* □

Corollary 3.4 *The Riemann integral ω_R on $C(I)$ admits a maximal hermitian extension which is, at once, a maximal extension defined on a subspace of the *-algebra of measurable functions on I.* □

Remark 3.5 If we impose a constraint to an extension and then we are looking for a maximal element, we will find, in general, a smaller domain. In our case the situation is very different: as we have shown a maximal hermitian extensions of ω is a maximal extension.

3.2 Positivity

Definition 3.6 Let $\widehat{\omega}$ be an extension of ω defined on the domain $D(\widehat{\omega})$ with $\mathfrak{A}_0 \subseteq D(\widehat{\omega}) \subseteq \mathfrak{A}$. Put

$$\mathcal{K}_\omega^+ := \mathcal{K}_\omega \cap \mathcal{P}(\mathfrak{A}), \tag{1}$$

we say that $\widehat{\omega}$ is *fully positive* if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}_\omega^+$.

Remark 3.7 Since the domain of a maximal extension $\breve{\omega}$ is $\mathcal{K}_\omega$, then we deduce that if a maximal extension $\breve{\omega}$ is positive, then $\breve{\omega}$ is fully positive.

Definition 3.8 Given ω, we define $\mathcal{P}_\omega$ as the collection of all subspaces $K \in \mathcal{H}_\omega$ satisfying the following additional condition

(p3) $(a, \ell) \in K$ and $a \in \mathcal{P}(\mathfrak{A})$, implies $\ell \geq 0$.

Since ω is positive, then $\mathcal{P}_\omega \neq \emptyset$ and $G_\omega \subseteq K \subseteq \overline{G_\omega}$ for every $K \in \mathcal{P}_\omega$.

To every $K \in \mathcal{P}_\omega$, there corresponds a hermitian extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} \,:\, (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell, \quad a \in D(\omega_K),$$

where, from $(g2)$, of Sect. 2, ℓ is the unique complex number such that $(a, \ell) \in K$. By (p3), ω_K is a positive hermitian extension of ω.

We observe that $\mathfrak{A}_0 \subseteq D(\omega_K) \subseteq \mathfrak{A}$ as vector spaces.

Since $\mathcal{P}_\omega$ satisfies the assumptions of Zorn's lemma, we have the following

Theorem 3.9 *Every ω admits a maximal positive hermitian extension.* □

Corollary 3.10 *The Riemann integral ω_R on $C(I)$ admits a maximal positive hermitian extension defined on a subspace of the *-algebra of measurable functions on I.* □

Definition 3.11 For $a, b \in \mathfrak{A}_h$, we define

$$a \leq b \Leftrightarrow b - a \in \mathcal{P}(\mathfrak{A}).$$

Remark 3.12 Let $\widehat{\omega}$ be a hermitian extension of ω, $a \in D(\widehat{\omega})$ and $c \in \mathfrak{A}_h$. If $b := \pm(a - c) \in \mathfrak{A}_h$, then $\widehat{\omega}(a) \in \mathbb{R}$. Indeed if $b \in \mathfrak{A}_h$, then $a = c \pm b \in \mathfrak{A}_h$ and so, by the hermiticity of $\widehat{\omega}$, $\widehat{\omega}(a) \in \mathbb{R}$. Moreover if $\widehat{\omega}$ is a positive hermitian extension of ω and $a, c \in D(\widehat{\omega}) \cap \mathfrak{A}_h$ with $a \geq c$, put $b := a - c$, then $b \in \mathcal{P}(\mathfrak{A}) \cap D(\widehat{\omega})$ so $\widehat{\omega}(a) = \widehat{\omega}(c) + \widehat{\omega}(b) \geq \widehat{\omega}(c)$.

With this in mind, if $\widehat{\omega}$ is a positive hermitian extension of ω and $c \in \mathfrak{A}_h$, we introduce the following notations that will use to characterize both the elements for which it is possible to find a positive hermitian extension and, given such an element, the values this extension may assume.

$$\mu_{c,\widehat{\omega}} := \inf\{\widehat{\omega}(a) : a \in D(\widehat{\omega}),\, a \geq c\},$$

where we put $\mu_{c,\widehat{\omega}} := +\infty$ if the set in the right hand side of the definition is the empty set;

$$\lambda_{c,\widehat{\omega}} := \sup\{\widehat{\omega}(a) : a \in D(\widehat{\omega}),\, a \leq c\}.$$

From Remark 3.12, we deduce the following

Proposition 3.13 *Let $\widehat{\omega}$ be a positive hermitian extension of ω. If $c \in D(\widehat{\omega}) \cap \mathfrak{A}_h$ then $\lambda_{c,\widehat{\omega}} = \mu_{c,\widehat{\omega}} = \widehat{\omega}(c)$.* □

Lemma 3.14 *Let $\widehat{\omega}$ be a positive hermitian extension of ω and let $c \in \mathcal{K}^+_\omega$. Then $0 \leq \lambda_{c,\widehat{\omega}} \leq \mu_{c,\widehat{\omega}}$.*

Proof Observing that $c \in \mathcal{P}(\mathfrak{A}) \subseteq \mathfrak{A}_h$, we start by proving that $\lambda_{c,\widehat{\omega}}, \mu_{c,\widehat{\omega}} \geq 0$. Since, by definitions of $\mu_{c,\widehat{\omega}}$, if $a \in D(\widehat{\omega})$ and $b := a - c \in \mathcal{P}(\mathfrak{A})$, we have $a = b + c \in \mathcal{P}(\mathfrak{A}) \cap D(\widehat{\omega})$, so $\widehat{\omega}(a) \geq 0$ and therefore $\mu_{c,\widehat{\omega}} \geq 0$. Now, $0 \in D(\widehat{\omega})$, $c - 0 \in \mathcal{P}(\mathfrak{A})$ and $\widehat{\omega}(0) = 0$, implies $\lambda_{c,\widehat{\omega}} \geq 0$.

Let us now prove that $\lambda_{c,\widehat{\omega}} \leq \mu_{c,\widehat{\omega}}$. Let, by contradiction, $\mu_{c,\widehat{\omega}} < \lambda_{c,\widehat{\omega}}$. Then, $\forall \nu : \mu_{c,\widehat{\omega}} < \nu < \lambda_{c,\widehat{\omega}}$, there exist $a_1, a_2 \in D(\widehat{\omega})$ such that $a_1 - c,\ c - a_2 \in \mathcal{P}(\mathfrak{A})$ and $\widehat{\omega}(a_1) < \nu < \widehat{\omega}(a_2)$; so $\widehat{\omega}(a_1) < \widehat{\omega}(a_2)$ and hence $\widehat{\omega}(a_1 - a_2) < 0$. Now $a_1 - c,\ c - a_2 \in \mathcal{P}(\mathfrak{A})$ hence $0 \leq \widehat{\omega}(a_1 - c) + \widehat{\omega}(c - a_2) = \widehat{\omega}(a_1 - c + c - a_2) = \widehat{\omega}(a_1 - a_2)$, so $\widehat{\omega}(a_1 - a_2) \geq 0$: a contradiction. □

Remark 3.15 Let ω_1, ω_2 be positive hermitian extensions of ω and let $c \in \mathcal{K}^+_\omega$. If $D(\omega_1) \subseteq D(\omega_2)$, then, by Lemma 3.14 we have

$$\lambda_{c,\omega} \leq \lambda_{c,\omega_1} \leq \lambda_{c,\omega_2} \leq \mu_{c,\omega_2} \leq \mu_{c,\omega_1} \leq \mu_{c,\omega}.$$

As we will see later (see Sect. 4.2) the fact that $c \in \mathcal{K}^+_\omega$, does not guarantee that $\lambda_{c,\omega}$ is finite; for this reason, we give the following

Definition 3.16 If $\widehat{\omega}$ is a positive hermitian extension of ω, we define

$$\mathcal{K}^\ddagger_{\widehat{\omega}} := \{c \in \mathcal{K}^+_\omega : \lambda_{c,\widehat{\omega}} \text{ is finite}\}. \tag{2}$$

From Remark 3.15 we have the following

Lemma 3.17 *Let $\widehat{\omega}$ be a positive hermitian extension of ω, then $\mathcal{K}^\ddagger_{\widehat{\omega}} \subseteq \mathcal{K}^\ddagger_\omega$.* □

Definition 3.18 Let $\widehat{\omega}$ be a positive hermitian extension of ω. We say that $\widehat{\omega}$ is *widely positive* if $\widehat{\omega}$ is positive and $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^\ddagger_\omega$.

An immediate consequence of Proposition 3.13 is the next lemma.

Lemma 3.19 *Let $\widehat{\omega}$ be a hermitian extension of ω. If $\widehat{\omega}$ is positive then $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) \subseteq \mathcal{K}^{\ddagger}_{\widehat{\omega}}$.* □

Corollary 3.20 *Let $\widehat{\omega}$ be a positive hermitian extension of ω. If $\widehat{\omega}$ is fully positive then $\mathcal{K}^{\ddagger}_{\widehat{\omega}} = \mathcal{K}^{+}_{\omega}$ and $\widehat{\omega}$ is widely positive. In particular if ω is fully positive then $\mathcal{K}^{\ddagger}_{\omega} = \mathcal{K}^{+}_{\omega}$.*

Proof If $\widehat{\omega}$ is fully positive then (by Lemma 3.19), we have $\mathcal{K}^{\ddagger}_{\widehat{\omega}} \subseteq \mathcal{K}^{+}_{\omega} = D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) \subseteq \mathcal{K}^{\ddagger}_{\widehat{\omega}}$, from which the statement follows. □

An important result which follows from the previous discussion, shows that the sole elements $c \in \mathcal{K}^{+}_{\omega} \backslash \mathfrak{A}_0$ for which we can find a positive hermitian extension of ω, are exactly those with finite $\lambda_{c,\omega}$. More exactly the following theorems hold.

Theorem 3.21 *Let $c \in \mathcal{K}^{+}_{\omega} \backslash \mathcal{K}^{\ddagger}_{\omega}$. Then there is no positive hermitian extension $\widehat{\omega}$ of ω such that $c \in D(\widehat{\omega})$.*

Proof Were $c \in D(\widehat{\omega})$ then, by Lemma 3.19 and Lemma 3.17, $c \in \mathcal{K}^{\ddagger}_{\widehat{\omega}} \subseteq \mathcal{K}^{\ddagger}_{\omega}$: a contradiction. □

Theorem 3.22 *Let ω be nonclosable, let $\widehat{\omega}$ be a positive hermitian extension of ω and let $c \in \mathcal{K}^{\ddagger}_{\widehat{\omega}}$ with $c \notin D(\widehat{\omega})$. Then, $\forall \gamma \in \mathbb{R} : \lambda_{c,\widehat{\omega}} \leq \gamma \leq \mu_{c,\widehat{\omega}}$, $G_{\widehat{\omega}} \subsetneq G_{\widehat{\omega}} \oplus \langle (c, \gamma) \rangle \in \mathcal{P}_{\omega}$.*

Proof We first show that, $\forall a \in D(\widehat{\omega})$, $\alpha \in \mathbb{R}$, such that, $a + \alpha c \in \mathcal{P}(\mathfrak{A})$, we have $\widehat{\omega}(a) + \alpha\gamma \geq 0$. The case $\alpha = 0$ is trivial, so we can distinguish two cases: $\alpha < 0$ and $\alpha > 0$.

Let $\alpha < 0$. Then $-1/\alpha(a + \alpha c) = -a/\alpha - c \in \mathcal{P}(\mathfrak{A})$, with $-a/\alpha \in D(\widehat{\omega})$. So, from the definition of $\mu_{c,\widehat{\omega}}$, $\widehat{\omega}(-a/\alpha) \geq \mu_{c,\widehat{\omega}} \geq \gamma$, that is $-1/\alpha\, \widehat{\omega}(a) - \gamma \geq 0$, and finally $\widehat{\omega}(a) + \alpha\gamma \geq 0$.

Let $\alpha > 0$. Then $a + \alpha c \in \mathcal{P}(\mathfrak{A})$ implies $1/\alpha\,(a + \alpha c) = c - (-a/\alpha) \in \mathcal{P}(\mathfrak{A})$, with $-a/\alpha \in D(\widehat{\omega})$. Then, by the definition of $\lambda_{c,\widehat{\omega}}$, $\widehat{\omega}(-a/\alpha) \leq \lambda_{c,\widehat{\omega}} \leq \gamma$. So $-1/\alpha\, \widehat{\omega}(a) \leq \gamma$ that is $\widehat{\omega}(a) + \alpha\gamma \geq 0$.

Now since ω is nonclosable and $c \in \mathcal{K}_{\omega}$, by Lemma 2.5, $(c, \gamma) \in \overline{G_{\omega}}$, and since $c = c^* \in \mathcal{K}_{\omega}$, $\gamma \in \mathbb{R}$, from the proof of (ii) in Proposition 3.2, we see that $K := G_{\widehat{\omega}} \oplus \langle (c, \gamma) \rangle \in \mathcal{H}_{\omega}$, thus $G_{\widehat{\omega}} \subsetneq K \subseteq \overline{G_{\omega}}$. Now if $(x, l) \in K$ then $x = y + \eta c$, with $y \in D(\widehat{\omega})$, $\eta = \alpha + i\beta \in \mathbb{C}$. If $x \in \mathcal{P}(\mathfrak{A})$ then $x = a + \alpha c$ with $a := (y + y^*)/2$. Indeed $x = x^* = y^* + \bar{\eta} c$, and $2x = (y + y^*) + 2\mathfrak{R}(\eta)c$ thus $x = (y + y^*)/2 + \mathfrak{R}(\eta)c$. Hence $(x, l) = (a + \alpha c, l)$, $a \in D(\widehat{\omega})$, $\alpha \in \mathbb{R}$. But $(a, \widehat{\omega}(a)) \in G_{\widehat{\omega}}$ and $\alpha(c, \gamma) \in \langle (c, \gamma) \rangle$, so $(a, \widehat{\omega}(a)) + \alpha(c, \gamma) = (x, \widehat{\omega}(a) + \alpha\gamma) \in K$ and then $(x, l) - (x, \widehat{\omega}(a) + \alpha\gamma) = (0, l - \widehat{\omega}(a) - \alpha\gamma) \in K$. Since $K \in \mathcal{H}_{\omega} \subseteq \mathcal{S}_{\omega}$, from (g2), $l = \widehat{\omega}(a) + \alpha\gamma$, so, by the first part of the proof, $l \geq 0$; hence $K \in \mathcal{P}_{\omega}$. □

Remark 3.23 We observe that if $\lambda_{c,\omega} = \mu_{c,\omega} < +\infty$, then the only value that a positive hermitian extension $\widehat{\omega}$ of ω may assume in c, is $\gamma := \lambda_{c,\omega}$.

The next theorem establishes that if ω is nonclosable, then any maximal positive hermitian extension $\breve{\omega}$ of ω, has "all possible" (see Lemma 3.19) positive elements in its domain.

Theorem 3.24 *Let ω be nonclosable and let $\breve{\omega}$ be a maximal positive hermitian extension of ω. Then $D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^{\ddagger}_{\breve{\omega}}$*

Proof By Lemma 3.19, we will just prove that $\mathcal{K}^{\ddagger}_{\breve{\omega}} \subseteq D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A})$. Let $c \in \mathcal{K}^{\ddagger}_{\breve{\omega}} \subseteq \mathcal{P}(\mathfrak{A})$, and suppose, by contradiction, that $c \notin D(\breve{\omega})$. Using the previous definitions of $\lambda_{c,\widehat{\omega}}$ and $\mu_{c,\widehat{\omega}}$ (with $\breve{\omega}$ instead of $\widehat{\omega}$), by Theorem 3.22, for all $\gamma \in \mathbb{R}$ such that $\lambda_{c,\widehat{\omega}} \leq \gamma \leq \mu_{c,\widehat{\omega}}$, $G_{\breve{\omega}} \subsetneq G_{\breve{\omega}} \oplus \langle (c,\gamma) \rangle \in \mathcal{P}_{\omega}$. Since $\breve{\omega}$ is a maximal positive hermitian extension of ω, this leads to a contradiction. □

Remark 3.25 The converse of Theorem 3.24 does not hold in general. It is sufficient to find a widely positive hermitian extension which is not a maximal positive hermitian extension: indeed, by Corollary 3.20, if $\widehat{\omega}$ is widely positive then $D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^{\ddagger}_{\widehat{\omega}}$. As we will see in section 4.2, Proposition 4.7, ω_1 is a widely positive extension of ω which is not a maximal positive hermitian extension: indeed ω_H is a positive hermitian extension of ω_1.

Corollary 3.26 *If there is $c \in \mathcal{K}^{\ddagger}_{\omega} \backslash \mathcal{P}(\mathfrak{A}_0)$ such that, $\lambda_{c,\omega} < \mu_{c,\omega}$, then ω admits infinitely many maximal positive hermitian extensions.*

Proof From Theorem 3.22 it follows that for each $\gamma \in \mathbb{R}$ such that $\lambda_{c,\omega} \leq \gamma \leq \mu_{c,\omega}$, $G_{\omega} \oplus \langle (c,\gamma) \rangle$ defines a positive hermitian extensions of ω. It is clear that different γ's give rise to different positive hermitian extensions, from which we obtain different maximal positive hermitian extensions of ω. □

Remark 3.27 Let c be the Dirichlet function,

$$c(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \backslash \mathbb{Q} \end{cases}.$$

Then, as we will see, $c \in \mathcal{K}^{\ddagger}_{\omega} \backslash \mathcal{P}(\mathfrak{A}_0)$, where $\mathfrak{A}_0 = C([0,1])$, and $\lambda_{c,\omega_R} = 0$, $\mu_{c,\omega_R} = 1$. The fact that $c \in \mathcal{K}^{+}_{\omega}$ is clear. Now let $a \in \mathfrak{A}_0$ with $a - c \in \mathcal{P}(\mathfrak{A})$ (as in the Introduction, $\mathfrak{A}$ is the *-algebra of measurable functions); then $a(x) \geq 1$, $\forall x \in [0,1] \cap \mathbb{Q}$. Thus, $a(x)$ continuous and $\mathbb{Q}$ dense in $\mathbb{R}$, imply $a(x) \geq 1$, $\forall x \in [0,1]$ and so, $\omega_R(a) \geq 1$. But $\omega(a) = 1$, if we take $a(x) = 1$, so $\mu_{c,\omega_R} = 1$. Analogously, $c - a \in \mathcal{P}(\mathfrak{A})$ implies $a(x) \leq 0$, $\forall x \in [0,1] \backslash \mathbb{Q}$, thus, as before, $a(x) \leq 0$, $\forall x \in [0,1]$ and so $\omega(a) \leq 0$. But $\omega(a) = 0$, if we take $a(x) = 0$, so $\lambda_{c,\omega_R} = 0$.

Theorem 3.28 *The Riemann integral ω_R on $C([0,1])$ admits infinitely many maximal positive hermitian extensions.*

Proof For the Dirichlet function c of Remark 3.27 we have $c \in \mathcal{K}_\omega^+ \setminus \mathcal{P}(\mathfrak{A}_0)$, $\lambda_{c,\omega_R} = 0$, $\mu_{c,\omega_R} = 1$. By Corollary 3.26, for each γ such that, $0 \leq \gamma \leq 1$, there exists $\breve{\omega}$, maximal positive hermitian extension of the Riemann integral, with $\breve{\omega}(c) = \gamma$. □

Remark 3.29 The previous theorem shows that, for each γ such that, $0 \leq \gamma \leq 1$, there exists a positive hermitian extension $\widehat{\omega}_\gamma$ of the Riemann integral ω_R on $C([0, 1])$, taking the value γ on the Dirichlet function c, despite of the Lebesgue integral of c being equal to 0. Obviously, $\widehat{\omega}$ is neither an extension of the Lebesgue integral, nor depends on it.

3.3 Absolutely convergent extensions

Now we will require the *-algebra $\mathfrak{A}$ to satisfy further conditions.

Definition 3.30 Let $\mathfrak{A}$ be a *-algebra. We say that $\mathfrak{A}$ has the property (D) if, for every $a \in \mathfrak{A}_h$, there exists a unique pair (a_+, a_-) of elements of $\mathfrak{A}$, with $a_+, a_- \in \mathcal{P}(\mathfrak{A})$ such that

(D1) $a = a_+ - a_-$;
(D2) $a_+ a_- = a_- a_+ = 0$;
(D3) $(\lambda a)_+ = \lambda a_+, \quad \forall a \in \mathfrak{A}_h, \lambda \in \mathbb{R}^+$;

Then we put

$$|a| := a_+ + a_-.$$

In what follows we suppose that $\mathfrak{A}$ has the property (D). In this case, one has:

$$|a| \in \mathcal{P}(\mathfrak{A}), \quad \forall a \in \mathfrak{A}_h.$$

Definition 3.31 A positive hermitian linear functional $\bar{\omega}$ defined on a subspace of $\mathfrak{A}$ is called *absolutely convergent* if for all $a \in D(\bar{\omega}) \cap \mathfrak{A}_h$, $a_+, a_- \in D(\bar{\omega})$, and so $|a| \in D(\bar{\omega})$.

From the last definition and from Remark 3.1, it follows that, if $\widehat{\omega}$ is an absolutely convergent extension of ω, then $\forall a \in D(\widehat{\omega})$, put $a = b + ic$, we have, $b_+, b_-, c_+, c_- \in D(\widehat{\omega})$.

Proposition 3.32 *Assume that $\widehat{\omega}$ is an absolutely convergent extension of ω. Then, $a = b + ic \in D(\widehat{\omega})$ if and only if $|b|$, $|c|$, b_+, $c_+ \in D(\widehat{\omega})$. Moreover if $a \in \mathfrak{A}_h$, then $|\widehat{\omega}(a)| \leq \widehat{\omega}(|a|)$.*

Proof If $a \in D(\widehat{\omega})$ then, by hypothesis, $b_+, b_-, c_+, c_- \in D(\widehat{\omega})$ thus $|b| = b_+ + b_- \in D(\widehat{\omega})$ and $|c| = c_+ + c_- \in D(\widehat{\omega})$. Conversely if $|b|$, $|c|$, b_+, $c_+ \in D(\widehat{\omega})$, then, easily, $b_-, c_- \in D(\widehat{\omega})$ and so $a = (b_+ - b_-) + i(c_+ - c_-) \in D(\widehat{\omega})$.

Now, for $a \in D(\widehat{\omega}) \cap \mathfrak{A}_h$,

$$\begin{aligned}|\widehat{\omega}(a)| = |\widehat{\omega}(a_+ - a_-)| = |\widehat{\omega}(a_+) - \widehat{\omega}(a_-)| \leq |\widehat{\omega}(a_+)| + |\widehat{\omega}(a_-)|\\ = \widehat{\omega}(a_+) + \widehat{\omega}(a_-) = \widehat{\omega}(a_+ + a_-) = \widehat{\omega}(|a|)\end{aligned}$$

□

The following theorem states that the domain of an absolutely convergent extension is determined by its positive elements.

Theorem 3.33 *If ω admits an absolutely convergent extension $\widehat{\omega}$, then $D(\widehat{\omega})$ is* span $\{D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})\}$.

Proof Since $\widehat{\omega}$ is an absolutely convergent extension, if $a = b + ic \in D(\widehat{\omega})$, then $b_+, b_-, c_+, c_- \in D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})$. This implies that

$$a = (b_+ - b_-) + i(c_+ - c_-) \in \text{ span } \{D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})\}.$$

On the other hand,

$$\text{span}\{D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})\} \subseteq \text{span}\{D(\widehat{\omega})\} = D(\widehat{\omega}).$$

□

Corollary 3.34 *If ω admits an absolutely convergent extension $\widehat{\omega}$ which is widely positive, then $D(\widehat{\omega})$ is* span $\{\mathcal{K}_\omega^\ddagger\}$. □

Let $\mathcal{P}_\omega$ be the family of subspaces of $\mathfrak{A} \times \mathbb{C}$ considered in Sect. 3.2, $K \in \mathcal{P}_\omega$, $(a, \ell) \in K$, and let ω_K be the positive hermitian extension of ω corresponding to K. If ω_K is absolutely convergent and $a \in \mathfrak{A}_h$, then the following conditions hold:

(1) $\exists\, \ell_1, \ell_2 \geq 0$ such that $(a_+, \ell_1), (a_-, \ell_2) \in K$,
(2) $\ell = \ell_1 - \ell_2$

Indeed, since ω_K is absolutely convergent, then $a_+, a_- \in D(\omega_K) \cap \mathcal{P}(\mathfrak{A})$, whence the (1); moreover, $(a_+, \ell_1), (a_-, \ell_2) \in K$ implies $(a_+, \ell_1) - (a_-, \ell_2) = (a_+ - a_-, \ell_1 - \ell_2) = (a, \ell_1 - \ell_2) \in K$, whence the (2).

Then we are induced to give the following

Definition 3.35 Let ω be absolutely convergent. We define $\mathcal{AC}_\omega$ the subfamily of $\mathcal{P}_\omega$, whose elements K satisfy the following additional condition:

($\mathrm{p_{ac}}$) $(a, \ell) \in K$, with $a \in \mathfrak{A}_h$, implies that $\exists\, \ell_1, \ell_2 \geq 0$ such that, $(a_+, \ell_1), (a_-, \ell_2) \in K$.

Since K is a vector space verifying condition ($g2$) of Sect. 2, then $\underline{\ell} = \ell_1 - \ell_2$, and since ω is absolutely convergent, then $\mathcal{AC}_\omega \neq \emptyset$ and $G_\omega \subseteq K \subseteq \overline{G_\omega}$, for every $K \in \mathcal{AC}_\omega$. To every $K \in \mathcal{AC}_\omega$, there corresponds a hermitian extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} : (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell, \quad a \in D(\omega_K),$$

where, from ($g2$), of Sect. 2, ℓ is the unique complex number such that $(a, \ell) \in K$. By (p_{ac}), ω_K is an absolutely convergent extension of ω.

We observe that $\mathfrak{A}_0 \subseteq D(\omega_K) \subseteq \mathfrak{A}$ as vector spaces.

Theorem 3.36 *If ω is absolutely convergent, then ω has a maximal absolutely convergent extension.*

Proof The family $\mathcal{AC}_\omega$ satisfies the assumptions of Zorn's lemma, hence it has a maximal element to which there corresponds a maximal absolutely convergent extension. □

Proposition 3.37 *Let $\breve{\omega}$ be a maximal positive hermitian extension of ω. If $\breve{\omega}$ is absolutely convergent, then $D(\breve{\omega})$ is* span $\{\mathcal{K}^\ddagger_\omega\}$.

Proof If $\breve{\omega}$ is a maximal positive hermitian extension, then, by Theorem 3.24, $\breve{\omega}$ is widely positive, so the statement follows by the previous Corollary 3.34. □

Now we state the following important theorem and corollary.

Theorem 3.38 *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is widely positive then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

Proof Suppose, by contradiction, that $\breve{\omega}$ is not maximal. Then, by the construction of a maximal absolutely convergent extension, given in Theorem 3.36, starting from $\breve{\omega}$, we can find a maximal absolutely convergent extension of ω, say $\widehat{\omega}$, which is a proper extension of $\breve{\omega}$: $D(\breve{\omega}) \subsetneq D(\widehat{\omega})$. Now by Theorem 3.33, Lemma 3.19 and Corollary 3.34, $D(\widehat{\omega}) =$ span $\{D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})\} \subseteq$ span $\{\mathcal{K}^\ddagger_\omega\} = D(\breve{\omega})$: a contradiction. □

By using Corollary 3.20 we have the following

Corollary 3.39 *Let $\breve{\omega}$ be an absolutely convergent extension of ω. If $\breve{\omega}$ is fully positive then $\breve{\omega}$ is a maximal absolutely convergent extension of ω.*

3.4 Positively regular extensions

In this section we will construct a particular extension $\widehat{\omega}$ of a hermitian positive linear functional ω, following essentially the model of the construction of the Lebesgue integral.

We recall that for $a, b \in \mathfrak{A}_h$, we have defined

$$a \leq b \Leftrightarrow b - a \in \mathcal{P}(\mathfrak{A}).$$

Then we have:

$$0 \leq |a|, \qquad a \leq |a|, \qquad -a \leq |a|, \quad \forall a \in \mathfrak{A}_h.$$

Indeed $|a| - 0 \in \mathcal{P}(\mathfrak{A})$, $|a| - a = 2a_- \in \mathcal{P}(\mathfrak{A})$ and $|a| + a = 2a_+ \in \mathcal{P}(\mathfrak{A})$.

Definition 3.40 An extension $\widehat{\omega}$ of ω is said to be *positively regular* if

$$\widehat{\omega}(a) = \sup\{\omega(b) \,:\, 0 \leq b \leq a, \quad b \in \mathcal{P}(\mathfrak{A}_0)\}, \quad \forall a \in D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}).$$

We observe that, by definition, a positively regular extension is positive.

To obtain a positively regular extension $\widehat{\omega}$ of ω, we start from the next

Definition 3.41 Let ω be absolutely convergent and positively regular. For $a \in \mathfrak{A}$, let $a = a_1 - a_2 + i(a_3 - a_4)$, the unique writing of a, with $a_i \in \mathcal{P}(\mathfrak{A})$, $1 \leq i \leq 4$. Then we define $\mathcal{PR}_\omega$ as the subfamily of $\mathcal{AC}_\omega$, whose elements K satisfy the following additional condition:

($\mathrm{p_{pr}}$) $(a, \ell) \in K$, implies $\exists\, \ell_i \geq 0,\ 1 \leq i \leq 4$, such that

(1) $(a_i, \ell_i) \in K$,
(2) $\ell_i = \sup\{\omega(b) \,:\, 0 \leq b \leq a_i, \quad b \in \mathcal{P}(\mathfrak{A}_0)\}$.

We observe that, as before, since ω is absolutely convergent and positively regular, then $\mathcal{PR}_\omega \neq \emptyset$ and, for every $K \in \mathcal{PR}_\omega$, $G_\omega \subseteq K \subseteq \overline{G_\omega}$. Moreover, since K is a vector space verifying condition ($g2$) of Sect. 2, then $\ell = \ell_1 - \ell_2 + i(\ell_3 - \ell_4)$.

To every $K \in \mathcal{PR}_\omega$ there corresponds an extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} \,:\, (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell,$$

where, from ($g2$) of Sect. 2, ℓ is the unique complex number such that $(a, \ell) \in K$. By definition, ω_K is a positively regular absolutely convergent extension of ω.

Again we observe that $\mathfrak{A}_0 \subseteq D(\omega_K) \subseteq \mathfrak{A}$ as vector spaces.

Invoking Zorn's Lemma we have the following

Theorem 3.42 *If ω is absolutely convergent and positively regular, then ω has a maximal positively regular absolutely convergent extension.*

Now we state the following

Theorem 3.43 *If ω admits a positively regular absolutely convergent extension $\widehat{\omega}$, which is fully positive, then this extension is unique.*

Proof Let ω' be another absolutely convergent, positively regular and fully positive extension of ω. As ω' and $\widehat{\omega}$ are fully positive, then $D(\omega') \cap \mathcal{P}(\mathfrak{A}) = D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}^+_\omega$ and, being ω' and $\widehat{\omega}$ absolutely convergent, by Proposition 3.33, $D(\omega') = D(\widehat{\omega}) = \operatorname{span}\{\mathcal{K}^+_\omega\}$. Now, if $b \in \mathcal{K}^+_\omega$, then

$$\omega'(b) = \sup\{\omega(c) : c \in \mathcal{P}(\mathfrak{A}_0),\ 0 \le c \le b\} = \widehat{\omega}(b).$$

Finally, $\forall a \in D(\omega') = D(\widehat{\omega})$, let $a = (b_+ - b_-) + i(c_+ - c_-)$, the unique writing of a, with $b_+, b_-, c_+, c_- \in \mathcal{K}^+_\omega$.

$$\begin{aligned}\omega'(a) &= \omega'(b_+ - b_- + i(c_+ - c_-)) = \omega'(b_+) - \omega'(b_-) + i(\omega'(c_+) - \omega'(c_-))\\ &= \widehat{\omega}(b_+) - \widehat{\omega}(b_-) + i(\widehat{\omega}(c_+) - \widehat{\omega}(c_-)) = \widehat{\omega}(b_+ - b_- + i(c_+ - c_-))\\ &= \widehat{\omega}(a).\end{aligned}$$

□

Since $\breve{\omega}$ maximal positive hermitian implies $\breve{\omega}$ fully positive, from the previous Theorem 3.43 and from the Theorem 3.42 we deduce the following result.

Corollary 3.44 *If ω admits an absolutely convergent positively regular extension $\breve{\omega}$, which is a maximal positive hermitian extension, then this extension is unique.* □

4 Three simple examples

We give now some easy examples, without going into the details of the proofs.

4.1 Example: the Lebesgue integral

The Henstock–Kurzweil integral is an extension of the Lebesgue integral and it is possible to verify it applying the abstract method developed in this section.

We use all the conventions given in the introduction and let, $\omega := \omega_R$, be the Riemann integral on I. Then the Lebesgue integral on I is a positive hermitian extension of ω.

In this case, there exist many possible extensions of the Lebesgue integral. We consider in what follows the Henstock–Kurzweil (HK) integral. The fact that the HK integral includes the Lebesgue integral was proved by Henstock [7, 8].

In [3] it was proved that the HK integral is not a maximal positive hermitian extension of the Riemann integral. From Theorem 3.9 we have the following

Theorem 4.1 *There exists a maximal positive hermitian extension of the Henstock–Kurzweil integral.*

Moreover, the approach proposed here allows us to give a theoretical proof of the existence of a maximal hermitian positive slight extension for the Henstock–Kurzweil integral opening the challenge of finding it explicitly.

4.2 Example: infinite sums

Let $\mathfrak{S}$ denote the complex vector space of all infinite sequences of complex numbers. $\mathfrak{S}$ is a *-algebra if the product $\mathbf{a} \cdot \mathbf{b}$ of two sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, $k \geq 1$, is defined component-wise and the involution by $\mathbf{a}^* = (\overline{a_k})$. Let us endow $\mathfrak{S}$ with the topology defined by the set of seminorms

$$p_k(\mathbf{a}) = |a_k|, \quad \mathbf{a} = (a_k) \in \mathfrak{S}.$$

Let $\mathfrak{S}_0$ denote the *-subalgebra of $\mathfrak{S}$ consisting of all *finite* sequences in the sense that $\mathbf{a} = (a_k) \in \mathfrak{S}_0$ if, and only if, there exists $N \in \mathbb{N}$ such that $a_k = 0$ if $k > N$. We define

$$\omega(\mathbf{a}) = \sum_{k=1}^{\infty} a_k, \quad \mathbf{a} = (a_k) \in \mathfrak{S}_0.$$

The symbol of series is only *graphic* since all sums are finite.

This functional, which is obviously positive hermitian, is nonclosable. To see this let us consider the sequence of sequences $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$ with, for $n \geq 1$, $a_{n,k} := \delta_{n,k}$ (the Kronecker delta).

For fixed k, clearly $\lim_{n\to\infty} a_{n,k} = 0$. Hence $\mathbf{a}_n \to \mathbf{0}$ as $n \to \infty$ and, applying ω, we get

$$\omega(\mathbf{a}_n) = \omega((a_{n,k})) = 1, \quad \forall n \geq 1.$$

We observe that any convergent series which converges to $l \in \mathbf{C}$, can be "rewritten" as a sequence of sequences $(\mathbf{a}_n) \subseteq \mathfrak{S}_0$, with $\mathbf{a}_n \to \mathbf{0}$ and $\omega(\mathbf{a}_n) \to l$, as $n \to \infty$.

Indeed, given the series $c_1 + c_2 + c_3 \ldots$, we define $(\mathbf{a}_n) = ((a_{n,k}))$, for $n \geq 1$, as follows:

$$a_{n,k} := \begin{cases} c_{n+1-k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Clearly $((a_{n,k})) \subseteq \mathfrak{S}_0$ and $\omega((a_{n,k})) = c_1 + c_2 + \cdots + c_n$. Since the series is convergent, for fixed k, $a_{n,k} \to 0$ as $n \to \infty$ and, finally, $\omega((a_{n,k})) \to l$ as $n \to \infty$.

The next proposition shows that in this case $\mathcal{K}_\omega$ is not a proper subset of the algebra.

Proposition 4.2 *Let $\mathfrak{S}$ and ω be as above. Then $\mathfrak{S}_0$ is a dense subalgebra of $\mathfrak{S}$ with $\mathcal{K}_\omega = \mathfrak{S}$.*

Proof We will prove, that for any $\mathbf{c} = (c_k) \in \mathfrak{S}$ and $l \in \mathbb{C}$, there exists a sequence of sequences $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$, such that $(\mathbf{a}_n) \to \mathbf{c}$ and $\omega((\mathbf{a}_n)) \to l$, as $n \to \infty$.

Fixed $l \in \mathbb{C}$, for each $n \geq 1$ we define, the element $a_{n,k}$, $k \geq 1$, of the sequence $((a_{n,k}))$ as following:

$$a_{n,k} := \begin{cases} c_k & \text{if } 1 \leq k \leq n, \\ l - \sum_{i=1}^{n} c_i & \text{if } k = n+1, \\ 0 & \text{if } k > n+1. \end{cases}$$

Then, for each $1 \leq k \leq n$, $c_k - a_{n,k} = 0$, so $\mathbf{a}_n \to \mathbf{c}$ as $n \to \infty$. Since $\omega((a_{n,k})) = l, \forall n \geq 1$, then $\omega((a_{n,k})) \to l$ as $n \to \infty$, hence, by definition, $\mathbf{c} \in \mathcal{K}_\omega$, and therefore $\mathcal{K}_\omega = \mathfrak{S}$. □

Then, from Theorem 3.3, we have the following

Proposition 4.3 *The functional ω admits a maximal hermitian extension $\breve{\omega}$ which is a maximal extension with $D(\breve{\omega}) = \mathfrak{S}$.* □

As discussed above, there exists infinitely many extensions of ω, the procedure of taking the limit of the partial sums $s_n := a_1 + a_2 + \cdots + a_n$, being just one of them. This is historically a very well known fact which dates back to the Grandi series

$$1 - 1 + 1 - 1 + \cdots .$$

Grandi asserted that this infinite sum is equal to $\frac{1}{2}$. Now we know that this can be obtained via Ramanujan sums. But also, more elementary, considering the following extensions of ω.

4.2.1 Hölder summation

The first (historically rigorous) extension of ω, due to A.L. Cauchy, is the following.

Definition 4.4 Given $\mathbf{a} = (a_k) \in \mathfrak{S}$, we put

$$s_n := a_1 + a_2 + \cdots + a_n, \quad \forall n \geq 1.$$

We define the hermitian extension ω_1 with domain

$$D(\omega_1) := \{\mathbf{a} \in \mathfrak{S} : \lim_{n\to\infty} s_n, \text{ exists finite}\},$$

by

$$\omega_1(\mathbf{a}) := \lim_{n\to\infty} s_n.$$

Another possible extension of ω, that indeed is a hermitian extension of ω_1, is the Hölder summation.

Given a sequence $\mathbf{a} = (a_k) \in \mathfrak{S}_0$, define by induction:

$$\begin{aligned} H_n^0 =& a_1 + a_2 + \cdots + a_n, \\ H_n^{h+1} =& \frac{1}{n} \sum_{t=1}^{n} H_t^h. \end{aligned}$$

We define an extension ω_H with domain

$$D(\omega_H) := \{\mathbf{a} \in \mathfrak{S} : \lim_{n\to\infty} H_n^{\bar{h}}, \text{ exists finite for some } \bar{h} \in \mathbb{N}\},$$

by

$$\omega_H(\mathbf{a}) := \lim_{n\to\infty} H_n^{\bar{h}}.$$

Since, by the Stolz–Cesàro theorem, the existence of $\lim_{n\to\infty} H_n^{\bar{h}}$, for some $\bar{h} \in \mathbb{N}$, implies that

$$\lim_{n\to\infty} H_n^{\bar{h}+1} = \lim_{n\to\infty} H_n^{\bar{h}},$$

$\omega_H(\mathbf{a})$ is well defined.

It is easy to see that ω_H is a positive hermitian extension of ω_1.

We observe that if we impose, in defining $D(\omega_H)$, $\bar{h} = 0$, we obtain ω_1. If we impose $\bar{h} \le 1$ we obtain the so called Cesàro summation. Moreover a direct calculation shows that the Cesàro sum of Grandi's series is defined and its value is

$$\omega_H((1, -1, 1, -1, \ldots)) = \frac{1}{2},$$

whereas $(1, -1, 1, -1, \ldots) \notin D(\omega_1)$.

4.2.2 Abel summation

Given a sequence $\mathbf{a} = (a_k)$, we define an extension ω_A of ω with domain

$$D(\omega_A) := \{\mathbf{a} \in \mathfrak{S} : \lim_{x\to 0^+} \lim_{N\to\infty} \sum_{k=1}^{N} e^{-kx} a_k \text{ exists finite}\},$$

defined by

$$\omega_A(\mathbf{a}) = \lim_{x\to 0^+} \lim_{N\to\infty} \sum_{k=1}^{N} e^{-kx} a_k$$

Clearly $\omega_A(\mathbf{a}) = a_1 + a_2 + \cdots + a_N$, if $(\mathbf{a}) \in \mathfrak{S}_0$ and it easy to see that ω_A is a positive hermitian extension of ω.

Now take $\mathbf{a} = ((-1)^{k+1})$. Then we obtain

$$\sum_{k=1}^{N} e^{-kx}(-1)^{k+1} = \frac{e^{-x} + (-1)^{N+1} e^{-(N+1)x}}{1 + e^{-x}}$$

which for every $x > 0$ converges, in the usual sense, to

$$\frac{e^{-x}}{1 + e^{-x}}.$$

Taking the limit for $x \to 0^+$, we get

$$\omega_A((1, -1, 1, -1, \dots)) = \frac{1}{2}.$$

Indeed it is well known that the Abel summation is a generalization of the Hölder summation or still better, in light of our approach, the Abel summation is a positive hermitian extension of the Hölder summation.

4.2.3 Positive hermitian extensions

As we have seen, Hölder summation and Abel summation are both positive hermitian extensions of ω and, moreover they are extension of ω_1. At this point the following question arises: is the classical definition of the sum of a series really natural? The answer is yes, if we are looking to positive hermitian extensions of ω.

We start with the following corollary which is a direct consequence of Proposition 4.2.

Corollary 4.5 *Given the algebra $\mathfrak{S}$, we have $\mathcal{K}_\omega^+ = \mathcal{P}(\mathfrak{S})$.* □

With the next proposition given $\mathbf{c} \in \mathcal{K}_\omega^+ \setminus D(\omega)$, we find explicitly $\lambda_{\mathbf{c},\omega}$ and $\mu_{\mathbf{c},\omega}$.

Proposition 4.6 *Let $\mathbf{c} = (c_k) \in \mathcal{K}_\omega^+$ with $\mathbf{c} \notin D(\omega)$. Then $\lambda_{\mathbf{c},\omega} = \lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n)$ and $\mu_{\mathbf{c},\omega} = +\infty$.*

Proof Since $\mathbf{c} \in \mathcal{K}_\omega^+$, then $\mathbf{c} \in \mathcal{P}(\mathfrak{S})$ so $c_k \geq 0, \forall k \geq 1$. Hence there exists $S := \lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n) \leq +\infty$. We recall that $\lambda_{\mathbf{c},\omega} := \{\sup \omega(\mathbf{a}) : \mathbf{a} \in \mathfrak{S}_0, \mathbf{a} \leq \mathbf{c}\}$. Let us consider the sequence $(\mathbf{b}_n) = ((b_{n,k}))$ with, for $k \geq 1$,

$$b_{n,k} := \begin{cases} c_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

It is clear that $\forall n \geq 1$, $(b_{n,k}) \in \mathfrak{S}_0$, $(b_{n,k}) \leq \mathbf{c}$ and $\omega((b_{n,k})) = c_1 + c_2 + \cdots + c_n$. Now we observe that if $\mathbf{a} \in \mathfrak{S}_0$, then there exists $\bar{k}(\mathbf{a}) \geq 0$ such that $a_k = 0, \forall k > \bar{k}$, and, if $\mathbf{a} \leq \mathbf{c}$, then $a_k \leq b_{\bar{k}(\mathbf{a}),k}$, $\forall k \geq 1$. Thus $\forall \mathbf{a} \in \mathfrak{S}_0$, with $\mathbf{a} \leq \mathbf{c}$, $\omega(\mathbf{a}) \leq \omega((b_{\bar{k}(\mathbf{a}),k})) = c_1 + c_2 + \cdots c_{\bar{k}(\mathbf{a})}$. Since $c_k \geq 0, \forall k \geq 1$, then $\omega(\mathbf{a}) \leq \lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n) = S$, from which $\lambda_{\mathbf{c},\omega} \leq S$. Finally, as $\sup_n\{\omega((b_{n,k}))\} = \lim_{n\to\infty} \omega((b_{n,k})) = S$, then $\lambda_{\mathbf{c},\omega} = S$. Now since $\mu_{\mathbf{c},\omega} := \inf\{\omega(\mathbf{a}) : \mathbf{a} \in \mathfrak{S}_0, \mathbf{a} \geq \mathbf{c}\}$, it is clear that if $\mathbf{c} \notin D(\omega)$, then $c_k > 0$ for infinitely many k so, the set in the right hand side of the definition of $\mu_{\mathbf{c},\omega}$ is the empty set and, by definition, $\mu_{\mathbf{c},\omega} = +\infty$ as required from the statement of the theorem. □

Proposition 4.7 *Given ω, $\mathcal{K}^{\ddagger}_{\omega} = D(\omega_1) \cap \mathcal{P}(\mathfrak{S})$ so ω_1 is widely positive.*

Proof From Corollary 4.5 and Proposition 4.6 we have $\mathcal{K}^{\ddagger}_{\omega} := \{\mathbf{c} \in \mathcal{K}^{+}_{\omega} : \lambda_{\mathbf{c},\omega}$ is finite$\} = \{\mathbf{c} \in \mathcal{P}(\mathfrak{S}) : \lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n)$ is finite$\} = D(\omega_1) \cap \mathcal{P}(\mathfrak{S})$. □

Remark 4.8 Since $D(\omega_1) \cap \mathcal{P}(\mathfrak{S}) \subsetneq \mathcal{K}^{+}_{\omega}$ then ω_1 is an example of a widely positive extension of ω that is not a fully positive extension. Moreover since $\mathcal{K}_{\omega} = \mathfrak{S}$, there exists a maximal hermitian extension $\breve{\omega}$ with $D(\breve{\omega}) = \mathfrak{S}$. If instead $\breve{\omega}$ is a maximal positive hermitian extension of ω then, by Theorem 3.24, Lemma 3.17 and Proposition 4.7, $D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{S}) = \mathcal{K}^{\ddagger}_{\breve{\omega}} \subseteq K^{\ddagger}_{\omega} \subsetneq \mathcal{K}^{+}_{\omega} = \mathcal{P}(\mathfrak{S})$, so $\breve{\omega}$ is never fully positive. Hence there are no maximal positive hermitian extension of ω that are fully positive.

Now it seems interesting to us to show another example in which $\mathcal{K}_{\omega}$ coincides with the entire algebra $\mathfrak{A}$. Starting with a subalgebra of $\mathfrak{S}$ and changing the topology with a finer one, we will find a new topological *-algebra $\mathfrak{S}_1$. Then, taking the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$, we will obtain the required algebra $\mathfrak{A} \subseteq \mathfrak{S}_1$.

Let us consider the subalgebra $\mathfrak{S}_1 \subseteq \mathfrak{S}$ of all bounded sequences, endowed with the norm

$$\|x\|_{\infty} = \sup_k |x_k|.$$

Then $\mathfrak{S}_1$ is a topological *-algebra with $\mathfrak{S}_0 \subseteq \mathfrak{S}_1$.

Now we find the closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$.

Proposition 4.9 *The closure of $\mathfrak{S}_0$ in $\mathfrak{S}_1$ is the set $\mathfrak{A} := \{(c_k) \in \mathfrak{S}_1 : |c_k| \to 0$ as $k \to \infty\}$.*

Proof We first show that $\overline{\mathfrak{S}_0} \subseteq \mathfrak{A}$.

If $\mathbf{c} = (c_k) \in \overline{\mathfrak{S}_0}$, then there exists $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$, such that $(\mathbf{a}_n) \to \mathbf{c}$ as $n \to \infty$. Since $(a_{n,k}) \in \mathfrak{S}_0$, then there exists $\bar{k}(n)$ such that $a_{n,k} = 0$, $\forall k > \bar{k}(n)$ and, moreover,

$$\forall \varepsilon > 0,\ \exists\, n_{\varepsilon} \text{ such that}, \forall n > n_{\varepsilon},\ \sup_k |a_{n,k} - c_k| < \varepsilon.$$

So, $\forall k > \bar{k}(n_{\varepsilon})$, $|c_k| < \varepsilon$, and the required inclusion holds.

Now let $\mathbf{c} = (c_k) \in \mathfrak{A}$. We will show that there exists a sequence $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$ such that $(\mathbf{a}_n) \to \mathbf{c}$ as $n \to \infty$.

Define

$$a_{n,k} := \begin{cases} c_k & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n \end{cases}$$

It is clear that $\forall n \ge 1$, $(a_{n,k}) \in \mathfrak{S}_0$; moreover, fixed $\varepsilon > 0$, for $(c_k) \in \mathfrak{A}$, there exists $n_\varepsilon \ge 1$ such that $|c_k| < \varepsilon$, $\forall n \ge n_\varepsilon$. So, $\forall n \ge n_\varepsilon$,

$$\sup_k |c_k - a_{n,k}| < \varepsilon,$$

showing that $(\mathbf{a}_n) \to \mathbf{c}$ as $n \to \infty$. □

It is evident that $\mathfrak{A}$ is an algebra, hence ω is a positive hermitian linear functional defined on $\mathfrak{S}_0$, a dense *-subalgebra of the topological *-algebra $\mathfrak{A}$.

Again the functional ω is nonclosable. To see this let us consider the sequence $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$ with,

$$a_{n,k} := \begin{cases} 1/n & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Since $\lim_{n\to\infty} 1/n = 0$ then $\mathbf{a}_n \to 0$ as $n \to \infty$, while $\omega(\mathbf{a}_n) = 1$, $\forall n \ge 1$.

With the next proposition we will find $\mathcal{K}_\omega$.

Proposition 4.10 *Let $\mathfrak{A}$, $\mathfrak{S}_0$ and ω be as above. Then $\mathcal{K}_\omega = \mathfrak{A}$.*

Proof We will prove, that for any $\mathbf{c} = (c_k) \in \mathfrak{A}$ and $l \in \mathbb{C}$, there exists a sequence of sequences $(\mathbf{a}_n) = ((a_{n,k})) \subseteq \mathfrak{S}_0$, such that $(\mathbf{a}_n) \to \mathbf{c}$ and $\omega((\mathbf{a}_n)) \to l$, as $n \to \infty$.

Fixed $\varepsilon > 0$, by definition of $\mathfrak{A}$, there exists $k_1 \ge 1$ such that $|c_k| < \varepsilon/2$, $\forall k > k_1$. Let $z_1 := \sum_{k=1}^{k_1} c_k$, $z := l - z_1$ and let $m \ge 1$ such that $|z/m| < \varepsilon/2$.

Define

$$b_{\varepsilon,k} := \begin{cases} c_k & \text{if } 1 \le k \le k_1, \\ z/m & \text{if } k_1 < k \le k_1 + m, \\ 0 & \text{if } k > k_1 + m \end{cases}$$

It is clear that $(b_{\varepsilon,k}) \in \mathfrak{S}_0$ and, since

$$\sup_{k_1 < k \le k_1 + m} |c_k - b_{\varepsilon,k}| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

then, by definition of $(b_{\varepsilon,k})$, $\sup_k |c_k - b_{\varepsilon,k}| < \varepsilon$; furthermore $\omega((b_{\varepsilon,k})) = l$. Taking $\varepsilon(n) = 1/n$, $n \ge 1$, and defining $(a_n, k) := (b_{\varepsilon(n),k})$ one has $((a_{n,k})) \to \mathbf{c}$ and $\omega((a_{n,k})) \to l$, as $n \to \infty$. □

Substituting the algebra $\mathfrak{S}$ by $\mathfrak{A}$, we can prove, without substantial changes, the following results.

Corollary 4.11 *Given the algebra* $\mathfrak{A}$, *we have* $\mathcal{K}_\omega^+ = \mathcal{P}(\mathfrak{A})$. □

Proposition 4.12 *Let* $\mathbf{c} = (c_k) \in \mathcal{K}_\omega^+$ *with* $\mathbf{c} \notin D(\omega)$. *Then* $\lambda_{\mathbf{c},\omega} = \lim_{n\to\infty}(c_1 + c_2 + \cdots + c_n)$ *and* $\mu_{\mathbf{c},\omega} = +\infty$. □

Proposition 4.13 *Given* ω, $\mathcal{K}_\omega^\ddagger = D(\omega_1) \cap \mathcal{P}(\mathfrak{A})$ *so* ω_1 *is widely positive.* □

The Remark 4.8 may be summarized as follows.

Remark 4.14 Even in the algebra $\mathfrak{A}$, ω_1 is an example of a widely positive extension of ω that is not fully positive; moreover there exists a maximal hermitian extension $\breve{\omega}$ of ω with $D(\breve{\omega}) = \mathfrak{A}$, and there are no maximal positive hermitian extension of ω that are fully positive.

4.3 Example: the Dirac delta

Let us consider the Banach convolution algebra $L^1(\mathbb{R})$ with its usual norm, involution and multiplication; i.e, for $f, g \in L^1(\mathbb{R})$

$$\|f\| = \int_{\mathbb{R}} |f(x)| \mathrm{d}x$$
$$f^*(x) = \overline{f(-x)}$$
$$(f \star g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$$

Let $C_c(\mathbb{R})$ denote the *-algebra of continuous functions with compact support. Then, $C_c(\mathbb{R})$ is a *-subalgebra of $L^1(\mathbb{R})$.

For $f \in C_c(\mathbb{R})$ define

$$\omega(f) = f(0).$$

ω is a positive linear functional on $C_c(\mathbb{R})$. Indeed, if $f \in C_c(\mathbb{R})$, we have

$$(f^* \star f)(x) = \int_{\mathbb{R}} \overline{f(y-x)} f(y)\mathrm{d}y.$$

Thus,

$$\omega(f^* \star f) = (f^* \star f)(0) = \int_{\mathbb{R}} |f(y)|^2 \mathrm{d}y \geq 0.$$

In order to extend ω to some subspace of $L^1(\mathbb{R})$ we start from the identity

$$f(0) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x)\mathrm{d}x, \quad \forall f \in C_c(\mathbb{R}).$$

While the left hand side could be meaningless for $f \in L^1(\mathbb{R})$, the right hand side could produce a finite number for certain $f \in L^1(\mathbb{R})$.

Consider the sequence of functions

$$f_n(x) = \begin{cases} 1 - n|x| & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f_n \in C_c(\mathbb{R})$ and $\|f_n\| = \frac{1}{n} \to 0$, while $f_n(0) = 1$, for all $n \in \mathbb{N}$. Hence ω is nonclosable in $L^1(\mathbb{R})$.

Let us put

$$D(\widehat{\omega}) := \left\{ f \in L^1(\mathbb{R}) : \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x) \mathrm{d}x \text{ exists in } \mathbb{C} \right\}$$

and define

$$\widehat{\omega}(f) := \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x) \mathrm{d}x, \quad f \in D(\widehat{\omega}).$$

First, we observe that $D(\widehat{\omega}) \subsetneq L^1(\mathbb{R})$. Indeed, the function

$$f(x) = \begin{cases} \frac{1}{\sqrt{|x|}} & \text{if } \in [-1, 1] \backslash \{0\} \\ 0 & \text{otherwise} \end{cases}$$

obviously belongs to $L^1(\mathbb{R})$; but

$$\lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x) \mathrm{d}x = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \cdot 4\sqrt{\epsilon} = \infty.$$

So we have the following

Remark 4.15 The linear functional $\widehat{\omega}$ is a positive hermitian extension of ω.

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