



Nuclear global spaces of ultradifferentiable functions in the matrix weighted setting

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Abstract

We prove that the Hermite functions are an absolute Schauder basis for many global weighted spaces of ultradifferentiable functions in the matrix weighted setting and we determine also the corresponding coefficient spaces, thus extending the previous work by Langenbruch. As a consequence, we give very general conditions for these spaces to be nuclear. In particular, we obtain the corresponding results for spaces defined by weight functions.

Keywords Weight matrices · Ultradifferentiable functions · Sequence spaces · Nuclear spaces

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1 Introduction

The systematic study of nuclear locally convex spaces began in 1951 with the fundamental dissertation of Grothendieck [20] to classify those infinite dimensional locally convex spaces which are not normed, suitable for mathematical analysis. Among the properties of a nuclear space, the existence of a Schwartz kernel for a continuous linear operator on the space is of crucial importance for the theory of linear partial differential operators. In our setting of ultradifferentiable functions, this fact helps, for instance, to study the behaviour (propagation of singularities or wave front sets) of a differential or pseudodifferential operator when acting on a distribution. See, for example, [1, 7, 16, 17, 33, 38] and the references therein.

Since the middle of the last century, several authors have studied the topological structure of global spaces of ultradifferentiable functions and, in particular, when the spaces are nuclear. See [31], or the book [19]. More recently, the first three authors in [9] used the isomorphism established by Langenbruch [28] between global spaces of ultradifferentiable functions in the sense of Gel'fand and Shilov [18] and some sequence spaces to see that under the condition that appears in [11, Corollary 16(3)] on the weight function ω (as in [12]) the space $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ of rapidly decreasing ultradifferentiable functions of Beurling type in the sense of Björck [3] is nuclear. However, there was the restriction that the powers of the logarithm were not allowed as admissible weight functions. Later, the authors of the present work proved in [10] that $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ is nuclear for any weight function satisfying $\log(t) = O(\omega(t))$ and $\omega(t) = o(t)$ as t tends to infinity. The techniques used in [10] come especially from the field of time–frequency analysis and a mixture of ideas from [7, 21, 22, 38]. In both [9] and [10], we use (different) isomorphisms between that space $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ and some sequence space and prove that $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ is nuclear by an application of the Grothendieck–Pietsch criterion [32, Theorem 28.15]. Very recently, Debrouwere, Neyt and Vindas [14, 15] (cf. [27] for related results about local spaces), using different techniques have extended our previous results in a very general framework. In [14], they characterize when mixed spaces of Björck [3] of Beurling type or of Roumieu type are nuclear under very mild conditions on the weight functions. In [15], using weight matrices in the sense of [37], the same authors characterize the nuclearity of generalized Gel'fand–Shilov classes which extend their previous work [14] and treat also many other mixed classes defined by sequences.

The aim of the present paper is twofold. On the one hand, we extend the work of Langenbruch [28] to the matrix weighted setting in the sense of [37, 40]. In particular, we prove that the Hermite functions are a Schauder basis of many global weighted spaces of ultradifferentiable functions. Moreover, we determine the coefficient spaces corresponding to this Hermite expansion (Theorem 1). These results are applied to spaces defined by weight functions $\mathcal{S}_{[\omega]}(\mathbb{R}^d)$, being $[\omega] = (\omega)$ (Beurling setting) or $[\omega] = \{\omega\}$ (Roumieu setting). Hence, we extend part of the previous work of Aubry [2] to the several variables case. As a consequence we are able to generalize our previous study [9, 10] about the nuclearity of the space $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ to global spaces of ultradifferentiable functions defined by

weight matrices (Corollary 2). An application to particular matrices gives that $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ is nuclear when $\omega(t) = o(t^2)$ as t tends to infinity. Similarly, we also prove the analogous result for the Roumieu setting, namely that $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ is nuclear when $\omega(t) = O(t^2)$ as t tends to infinity (see Theorem 6 for both results). For weights of the form $\omega(t) = \log^\beta(1 + t)$ with $\beta > 1$, our results hold and, hence, we generalize the results of [28] to spaces that could not be treated there since, as is easily deduced from [11, Example 20], $\mathcal{S}_{\{(M_p)_p\}}(\mathbb{R}) \neq \mathcal{S}_{\{\omega\}}(\mathbb{R})$ for any sequence of positive numbers $(M_p)_{p \in \mathbb{N}}$ in the sense of [26] (see Remark 4). We do not treat here the classical case $\omega(t) = \log(1 + t)$, for which $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$, the Schwartz class, because in this case infinitely many entries of our weight matrices are not well defined. However, the results presented here are already well known for the Schwartz class.

The classes of functions treated in [15] are in general different from ours. In fact, here we consider spaces of functions f that are bounded in the following sense: for some (or any) $h > 0$, there is $C > 0$ such that for all $x \in \mathbb{R}^d$ and every multi-indices α and β , we have

$$|x^\alpha \partial^\beta f(x)| \leq Ch^{|\alpha+\beta|} M_{\alpha+\beta} \tag{A}$$

And we pass to the matrix setting for the multi-sequence $(M_\alpha)_\alpha$, i.e. we make M_α^λ depend also on a parameter $\lambda > 0$ (see the precise definition in the next section). In [15], the authors consider spaces of functions f bounded in the following sense: there is $C > 0$ such that for all $x \in \mathbb{R}^d$ and every multi-index β they have

$$|w(x) \partial^\beta f(x)| \leq CM_\beta \tag{B}$$

where w is a positive continuous function. They pass to the matrix setting by making M_β^λ and w^λ depend on the same parameter $\lambda > 0$. Hence, taking unions (Roumieu setting) or intersections (Beurling setting) in λ in the situation (A) gives different classes of functions than in the situation (B) in general. On the other hand, it is a very difficult problem to determine when the classes treated in this work are non-trivial, a question not considered in [14, 15]. We characterize in a very general way (Propositions 2 and 3) when the Hermite functions are contained in our classes and this fact is closely related to classes being non-trivial. Indeed, we can deduce from our results that, in the Beurling setting, the space $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ contains the Hermite functions if and only if $\omega(t) = o(t^2)$ as t tends to infinity (Corollary 3). However, it is not difficult to see from the uncertainty principle [23, Theorem] that $\mathcal{S}_{(\omega)}(\mathbb{R}^d) = \{0\}$ when $t^2 = O(\omega(t))$ as t tends to infinity. In the same way, in the Roumieu case, the space $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ contains the Hermite functions if and only if $\omega(t) = O(t^2)$ as t tends to infinity (Corollary 3), but again from [23, Theorem] we can deduce $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d) = \{0\}$ when $t^2 = o(\omega(t))$ as t tends to infinity. For more information on the uncertainty principle for $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ where, as stated above, $[\omega] = (\omega)$ or $\{\omega\}$, see the nice introduction to the paper of Aubry [2] and the references therein. Moreover, our classes are well adapted for Fourier transform (Corollary 1). We should also mention that throughout this paper we assume, on the multi-sequence $(M_\alpha)_\alpha$, that $(M_\alpha)^{1/|\alpha|}$ tends to infinity when $|\alpha|$ tends to infinity, which is stronger than the

condition $\inf_{\alpha \in \mathbb{N}_0^d} (M_\alpha/M_0)^{1/|\alpha|} > 0$ considered in [26, Def. 3.1] (for the one-dimensional case). The reason is that it is not clear how the results read when the associated function is infinite (see Remark 1).

The paper is organized as follows: in the next section, we give some necessary definitions, in Sect. 3 we introduce the classes under study in the matrix weighted setting and establish the analogous conditions to [28] to determine in Sect. 4 when the Hermite functions belong to our classes. In Sect. 5, we introduce the suitable matrix sequence spaces and prove that they are isomorphic to our classes, which is the fundamental tool to see that our spaces are nuclear. We finally apply these results to the particular case of spaces defined by weight functions in Sect. 6.

2 Preliminaries

In what follows, for given $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, we are setting $|t|_\infty := \max_{1 \leq j \leq d} |t_j|$. We briefly recall from [26] those basic notions about sequences $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$, for $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, that we need in what follows. A sequence $(M_p)_p$ is called *normalized* if $M_0 = 1$. For a normalized sequence $\mathbf{M} = (M_p)_p$, the *associated function* is denoted by

$$\omega_{\mathbf{M}}(t) = \sup_{p \in \mathbb{N}_0} \log \frac{|t|^p}{M_p}, \quad t \in \mathbb{R}. \tag{2.1}$$

We say that $(M_p)_p$ satisfies the *logarithmic convexity condition* (M1) of [26] if

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbb{N}. \tag{2.2}$$

The following lemma is well known (see Lemmas 2.0.6 and 2.0.4 of [39] for a proof).

Lemma 1 *Let $(M_p)_{p \in \mathbb{N}_0}$ be a normalized sequence satisfying (2.2). Then*

- (a) $M_j M_k \leq M_{j+k}$ for all $j, k \in \mathbb{N}_0$;
- (b) $p \mapsto (M_p)^{1/p}$ is increasing;
- (c) $\liminf_{p \rightarrow +\infty} (M_p)^{1/p} > 0$.

From Lemma 1(c) and [26, Prop. 3.2], we have that a normalized sequence $\mathbf{M} = (M_p)_p$ satisfies (2.2) if and only if

$$M_p = \sup_{t > 0} \frac{t^p}{\exp \omega_{\mathbf{M}}(t)}, \quad p \in \mathbb{N}_0. \tag{2.3}$$

We say that $(M_p)_p$ satisfies the *stability under differential operators condition* (M2)' of [26] if

$$\exists A, H \geq 1 \forall p \in \mathbb{N}_0 : M_{p+1} \leq AH^p M_p, \tag{2.4}$$

and $(M_p)_p$ satisfies the stronger *moderate growth* condition (M2) of [26] if

$$\exists A \geq 1 \forall p, q \in \mathbb{N}_0 : M_{p+q} \leq A^{p+q} M_p M_q. \tag{2.5}$$

The following lemma extends [26, Proposition 3.4] for two sequences. We give the proof for the convenience of the reader.

Lemma 2 *Let $\mathbf{M} = (M_p)_{p \in \mathbb{N}_0}$ and $\mathbf{N} = (N_p)_{p \in \mathbb{N}_0}$ be two normalized sequences satisfying (2.2). Then the following conditions are equivalent:*

- (i) $\exists A \geq 1 \forall p \in \mathbb{N}_0 : M_{p+1} \leq A^{p+1} N_p.$
- (ii) $\exists A \geq 1, B > 0 \forall t > 0 : \omega_{\mathbf{N}}(t) + \log t \leq \omega_{\mathbf{M}}(At) + B.$

Proof If (i) is satisfied, then, for all $t > 0$,

$$te^{\omega_{\mathbf{N}}(t)} = t \sup_{p \in \mathbb{N}_0} \frac{t^p}{N_p} \leq \sup_{p \in \mathbb{N}_0} \frac{(At)^{p+1}}{M_{p+1}} \leq \sup_{p \in \mathbb{N}_0} \frac{(At)^p}{M_p} = e^{\omega_{\mathbf{M}}(At)}.$$

Conversely, if (ii) holds, then, by (2.3),

$$\begin{aligned} N_p &= \sup_{t>0} \frac{t^p}{\exp \omega_{\mathbf{N}}(t)} \geq \sup_{t>0} \frac{t^{p+1}}{e^B \exp \omega_{\mathbf{M}}(At)} \\ &= e^{-B} \sup_{s>0} \frac{(s/A)^{p+1}}{\exp \omega_{\mathbf{M}}(s)} = \frac{e^{-B}}{A^{p+1}} M_{p+1}. \end{aligned}$$

□

Now, we consider sequences $\mathbf{M} = (M_{\alpha})_{\alpha \in \mathbb{N}_0^d}$ of positive real numbers for multi-indices $\alpha \in \mathbb{N}_0^d$. As in the one-dimensional case, we say that $(M_{\alpha})_{\alpha \in \mathbb{N}_0^d}$ is normalized if $M_0 = 1$. We recall condition (3.7) of [28]

$$\exists A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_{\alpha} M_{\beta} \leq A^{|\alpha+\beta|} M_{\alpha+\beta}. \tag{2.6}$$

Condition (2.4) takes in this setting the form (see [28, (2.1)])

$$\exists A \geq 1 \forall \alpha \in \mathbb{N}_0^d, 1 \leq j \leq d : M_{\alpha+e_j} \leq A^{|\alpha|+1} M_{\alpha}, \tag{2.7}$$

and (2.5) turns into

$$\exists A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_{\alpha+\beta} \leq A^{|\alpha+\beta|} M_{\alpha} M_{\beta}. \tag{2.8}$$

Now, for $t \in \mathbb{R}^d$, we denote

$$\mathbb{N}_{0,t}^d := \{ \alpha \in \mathbb{N}_0^d : \alpha_j = 0 \text{ if } t_j = 0, j = 1, \dots, d \}. \tag{2.9}$$

The associated weight function of a normalized $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$ is given by

$$\omega_{\mathbf{M}}(t) = \sup_{\alpha \in \mathbb{N}_{0,t}^d} \log \frac{|t^\alpha|}{M_\alpha}, \quad t \in \mathbb{R}^d,$$

where by convention $0^0 := 1$. Note that for a normalized sequence we have $\omega_{\mathbf{M}}(0) = 0$.

Remark 1 As it has already been pointed out in the geometric construction in [30, Chap. I] for the one dimensional weight function (see (2.1)), we have that $\omega_{\mathbf{M}}(t) < +\infty$ for all $t \in \mathbb{R}^d$ if and only if $\lim_{|\alpha| \rightarrow \infty} (M_\alpha)^{1/|\alpha|} = +\infty$.

First, assume that $\omega_{\mathbf{M}}(t) < +\infty$ for all $t \in \mathbb{R}^d$. Hence for all $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ satisfying $t_{\min} := \min_{1 \leq j \leq d} |t_j| \geq 1$ we have that there exists some C (depending only on t) such that $\log \frac{|t^\alpha|}{M_\alpha} \leq C$ for all $\alpha \in \mathbb{N}_0^d$. So $t_{\min}^{|\alpha|} \leq |t^\alpha| \leq t_1^{\alpha_1} \dots t_d^{\alpha_d} = |t^\alpha| \leq e^C M_\alpha$ for all $\alpha \in \mathbb{N}_0^d$ and now let $t_{\min} \rightarrow +\infty$.

Conversely, let $\lim_{|\alpha| \rightarrow \infty} (M_\alpha)^{1/|\alpha|} = \infty$ and so for any $A > 0$ large, we can find some $C > 0$ large enough such that $A^{|\alpha|} \leq CM_\alpha$. Since $|t^\alpha| \leq |t|^{|\alpha|}$ for all $t \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}_0^d$, we see that for any given $t \in \mathbb{R}^d$ we get $\frac{|t^\alpha|}{M_\alpha} \leq \frac{|t|^{|\alpha|}}{M_\alpha} \leq C$ for some $C > 0$ and all $\alpha \in \mathbb{N}_0^d$.

Lemma 3 Let $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$. Then, for all $h > 0$ and $\alpha \in \mathbb{N}_0^d$,

$$M_\alpha h^{|\alpha|} \geq \sup_{t \in \mathbb{R}^d} |t^\alpha| e^{-\omega_{\mathbf{M}}(t/h)}. \tag{2.10}$$

Proof Fix $\alpha \in \mathbb{N}_0^d$ and $h > 0$; we write $\mathbb{R}_\alpha^d := \{t \in \mathbb{R}^d : t_j \neq 0 \text{ for } \alpha_j \neq 0, j = 1, \dots, d\}$. Then for $t \in \mathbb{R}^d \setminus \mathbb{R}_\alpha^d$, we have $t^\alpha = 0$, and so it is enough to prove that

$$M_\alpha h^\alpha \geq \sup_{t \in \mathbb{R}_\alpha^d} |t^\alpha| e^{-\omega_{\mathbf{M}}(t/h)}. \tag{2.11}$$

We have

$$\begin{aligned} \frac{1}{\sup_{t \in \mathbb{R}_\alpha^d} |t^\alpha| e^{-\omega_{\mathbf{M}}(t/h)}} &= \inf_{t \in \mathbb{R}_\alpha^d} \frac{e^{\omega_{\mathbf{M}}(t/h)}}{|t^\alpha|} = \inf_{t \in \mathbb{R}_\alpha^d} \frac{\exp \sup_{\beta \in \mathbb{N}_{0,t}^d} \log \frac{|\left(\frac{t}{h}\right)^\beta|}{M_\beta}}{|t^\alpha|} \\ &= \inf_{t \in \mathbb{R}_\alpha^d} \frac{1}{|t^\alpha|} \sup_{\beta \in \mathbb{N}_{0,t}^d} \frac{|\left(\frac{t}{h}\right)^\beta|}{M_\beta}; \end{aligned}$$

observe that $\alpha \in \mathbb{N}_{0,t}^d$ and so, choosing $\beta = \alpha$, we get

$$\frac{1}{\sup_{t \in \mathbb{R}_\alpha^d} |t^\alpha| e^{-\omega_{\mathbf{M}}(t/h)}} \geq \inf_{t \in \mathbb{R}_\alpha^d} \frac{|t^\alpha|}{|t^\alpha| h^{|\alpha|} M_\alpha} = \frac{1}{h^{|\alpha|} M_\alpha},$$

which proves (2.11), and then the proof is complete.

Note that if $\omega_{\mathbf{M}}(t/h) = +\infty$, then (2.10) is clear and so we could restrict in the estimates above to all $t \in \mathbb{R}_\alpha^d$ such that $\omega_{\mathbf{M}}(t/h)$ is finite. \square

In the following, we use two normalized sequences as above $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$ and $\mathbf{N} = (N_\alpha)_{\alpha \in \mathbb{N}_0^d}$ and we compare them in the sense:

$$\mathbf{M} \leq \mathbf{N} \quad \text{if} \quad M_\alpha \leq N_\alpha, \quad \alpha \in \mathbb{N}_0^d.$$

This clearly implies

$$\omega_{\mathbf{N}}(t) \leq \omega_{\mathbf{M}}(t), \quad t \in \mathbb{R}^d.$$

In [28], Langenbruch uses his condition (1.2) to prove that the Hermite functions belong to the spaces considered there. In the present paper we need, for the same reason, a mixed condition that involves two sequences:

$$\exists H, C, B > 0 \forall \alpha, \beta \in \mathbb{N}_0^d : \quad \alpha^{\alpha/2} M_\beta \leq BC^{|\alpha|} H^{|\alpha+\beta|} N_{\alpha+\beta}. \quad (2.12)$$

Remark 2 Condition (2.12) yields that $\lim_{|\alpha| \rightarrow \infty} (N_\alpha)^{1/|\alpha|} = +\infty$. Indeed, since by convention $0^0 = 1$ and by definition $\alpha^{\alpha/2} := \alpha_1^{\alpha_1/2} \dots \alpha_d^{\alpha_d/2}$, from (2.12) with $\beta = 0$ we get (recall $|\alpha|_\infty := \max_{1 \leq j \leq d} \alpha_j$) that

$$\begin{aligned} N_\alpha^{1/|\alpha|} &\geq B^{-\frac{1}{|\alpha|}} C^{-1} H^{-1} (\alpha^{\alpha/2})^{1/|\alpha|} = B^{-\frac{1}{|\alpha|}} C^{-1} H^{-1} (\alpha_1^{\alpha_1/2} \dots \alpha_d^{\alpha_d/2})^{1/|\alpha|} \\ &\geq B^{-\frac{1}{|\alpha|}} C^{-1} H^{-1} (|\alpha|_\infty^{|\alpha|/2})^{1/|\alpha|} \geq B^{-\frac{1}{|\alpha|}} C^{-1} H^{-1} \left(\frac{|\alpha|}{d} \right)^{\frac{1}{2d}} \rightarrow +\infty. \end{aligned}$$

3 Global ultradifferentiable functions in the matrix weighted setting

In this section, we consider matrices of normalized sequences $(M_\alpha^{(\lambda)})_{\lambda > 0, \alpha \in \mathbb{N}_0^d}$ of real positive numbers:

$$\begin{aligned} \mathcal{M} := \{(\mathbf{M}^{(\lambda)})_{\lambda > 0} : \mathbf{M}^{(\lambda)} = (M_\alpha^{(\lambda)})_{\alpha \in \mathbb{N}_0^d}, M_0^{(\lambda)} = 1, \\ \mathbf{M}^{(\lambda)} \leq \mathbf{M}^{(\kappa)} \text{ for all } 0 < \lambda \leq \kappa\}. \end{aligned} \quad (3.1)$$

We call \mathcal{M} a *weight matrix* and consider matrix weighted global ultradifferentiable functions of Roumieu type defined as follows (from now on $\|\cdot\|_\infty$ denotes the supremum norm): first, for a given normalized sequence \mathbf{M} , we set

$$\mathcal{S}_{\{M\}} := \left\{ f \in C^\infty(\mathbb{R}^d) : \exists C, h > 0, \|f\|_{\infty, M, h} := \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}} \leq C \right\},$$

$$\mathcal{S}_{(M)} := \{f \in C^\infty(\mathbb{R}^d) : \forall h > 0 \exists C_h > 0, \|f\|_{\infty, M, h} \leq C_h\},$$

endowed with the inductive limit topology in the Roumieu setting (which may be thought countable if we take $h \in \mathbb{N}$) and with the projective limit topology in the Beurling setting (countable for $h^{-1} \in \mathbb{N}$). Next, we define the matrix type spaces as follows:

$$\mathcal{S}_{\{\mathcal{M}\}} := \bigcup_{\lambda > 0} \mathcal{S}_{\{M^{(\lambda)}\}} = \{f \in C^\infty(\mathbb{R}^d) : \exists C, h, \lambda > 0, \|f\|_{\infty, M^{(\lambda)}, h} \leq C\},$$

$$\begin{aligned} \mathcal{S}_{(\mathcal{M})} &:= \bigcap_{\lambda > 0} \mathcal{S}_{(M^{(\lambda)})} \\ &= \{f \in C^\infty(\mathbb{R}^d) : \forall h, \lambda > 0 \exists C_{\lambda, h} > 0, \|f\|_{\infty, M^{(\lambda)}, h} \leq C_{\lambda, h}\}, \end{aligned}$$

again endowed with the inductive limit topology in the Roumieu setting (which may be thought countable if we take $\lambda, h \in \mathbb{N}$) and endowed with the projective limit topology in the Beurling setting (countable for $\lambda^{-1}, h^{-1} \in \mathbb{N}$).

Now we consider different conditions on the weight matrices that we use following the lines of [28]. The next basic condition extends (1.2) of [28] in the Roumieu case and is needed to show that the Hermite functions belong to $\mathcal{S}_{\{\mathcal{M}\}}$ (see Proposition 3):

$$\begin{aligned} \forall \lambda > 0 \exists \kappa \geq \lambda, B, C, H > 0 \forall \alpha, \beta \in \mathbb{N}_0^d : \\ \alpha^{\alpha/2} M_\beta^{(\lambda)} \leq BC^{|\alpha|} H^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa)}. \end{aligned} \tag{3.2}$$

The analogous condition to (3.2) in the Beurling case, which is needed to show that the Hermite functions belong to $\mathcal{S}_{(\mathcal{M})}$ is the following (see Proposition 3):

$$\begin{aligned} \forall \lambda > 0 \exists 0 < \kappa \leq \lambda, H > 0 \forall C > 0 \exists B > 0 \forall \alpha, \beta \in \mathbb{N}_0^d : \\ \alpha^{\alpha/2} M_\beta^{(\kappa)} \leq BC^{|\alpha|} H^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}. \end{aligned} \tag{3.3}$$

Remark 3 Similarly, as commented in Remark 2 for (2.12), property (3.2) (property (3.3)) yields that $\lim_{|\alpha| \rightarrow \infty} (M_\alpha^{(\kappa)})^{1/|\alpha|} = +\infty$ for some $\kappa > 0$, and hence for all $\kappa' \geq \kappa$ ($\lim_{|\alpha| \rightarrow \infty} (M_\alpha^{(\lambda)})^{1/|\alpha|} = +\infty$ for all $\lambda > 0$).

We also need to extend condition (3.7) of [28] to the matrix weighted setting. First, we state it in the Roumieu case:

$$\forall \lambda > 0 \exists \kappa \geq \lambda, A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_\alpha^{(\lambda)} M_\beta^{(\lambda)} \leq A^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa)}; \tag{3.4}$$

and in the Beurling case:

$$\forall \lambda > 0 \exists 0 < \kappa \leq \lambda, A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_\alpha^{(\kappa)} M_\beta^{(\kappa)} \leq A^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}. \tag{3.5}$$

The extensions of condition (2.7) (*mixed derivation closedness properties*) for a weight matrix \mathcal{M} in the Roumieu and Beurling cases read as follows:

$$\forall \lambda > 0 \exists \kappa \geq \lambda, A \geq 1 \forall \alpha \in \mathbb{N}_0^d, 1 \leq j \leq d : M_{\alpha+e_j}^{(\lambda)} \leq A^{|\alpha|+1} M_\alpha^{(\kappa)}, \tag{3.6}$$

$$\forall \lambda > 0 \exists 0 < \kappa \leq \lambda, A \geq 1 \forall \alpha \in \mathbb{N}_0^d, 1 \leq j \leq d : M_{\alpha+e_j}^{(\kappa)} \leq A^{|\alpha|+1} M_\alpha^{(\lambda)}. \tag{3.7}$$

The following conditions generalize (2.8) to the weight matrix setting:

$$\forall \lambda > 0 \exists \kappa \geq \lambda, A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_{\alpha+\beta}^{(\lambda)} \leq A^{|\alpha+\beta|} M_\alpha^{(\kappa)} M_\beta^{(\kappa)}, \tag{3.8}$$

$$\forall \lambda > 0 \exists 0 < \kappa \leq \lambda, A \geq 1 \forall \alpha, \beta \in \mathbb{N}_0^d : M_{\alpha+\beta}^{(\kappa)} \leq A^{|\alpha+\beta|} M_\alpha^{(\lambda)} M_\beta^{(\lambda)}. \tag{3.9}$$

It is immediate that for any given matrix \mathcal{M} satisfying (3.8) and (3.4) we can replace in the definition of $\mathcal{S}_{\{\mathcal{M}\}}$ the seminorm $\|\cdot\|_{\infty, \mathcal{M}^{(\lambda)}, h}$ by

$$\sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_\alpha^{(\lambda)} M_\beta^{(\lambda)}}.$$

We have an analogous statement for the class $\mathcal{S}_{(\mathcal{M})}$ under (3.9) and (3.5). When we define the spaces $\mathcal{S}_{\{\mathcal{M}\}}$ or $\mathcal{S}_{(\mathcal{M})}$ with the weighted L^2 norms treated below in (3.17), the similar property holds.

Lemma 4 *Let \mathcal{M} be a weight matrix as defined in (3.1).*

If (3.6) holds, then

$$\forall \lambda > 0 \exists \kappa \geq \lambda, B_1, B_2 \geq 1 \forall t \in \mathbb{R}^d : \tag{3.10}$$

$$(1 + |t|)^{2(d+1)} \exp \omega_{\mathcal{M}^{(\kappa)}}(t) \leq B_1 \exp \omega_{\mathcal{M}^{(\lambda)}}(B_2 t).$$

If (3.7) holds, then

$$\forall \lambda > 0 \exists 0 < \kappa \leq \lambda, B_1, B_2 \geq 1 \forall t \in \mathbb{R}^d : \tag{3.11}$$

$$(1 + |t|)^{2(d+1)} \exp \omega_{\mathcal{M}^{(\lambda)}}(t) \leq B_1 \exp \omega_{\mathcal{M}^{(\kappa)}}(B_2 t).$$

Proof First, we consider the Roumieu case. By $2(d+1)$ iterated applications of (3.6), we find $\kappa_{2d+2} \geq \kappa_{2d+1} \geq \dots \geq \kappa_1 \geq \lambda > 0$ and $A_1, \dots, A_{2d+2} \geq 1$ such that, for all $\alpha \in \mathbb{N}_0^d$ and $1 \leq j \leq d$,

$$\begin{aligned}
 M_{\alpha+2(d+1)e_j}^{(\lambda)} &\leq A_1^{|\alpha|+2d+2} M_{\alpha+(2d+1)e_j}^{(\kappa_1)} \\
 &\leq A_1^{|\alpha|+2d+2} A_2^{|\alpha|+2d+1} M_{\alpha+2de_j}^{(\kappa_2)} \\
 &\leq \dots \leq A_1^{|\alpha|+2d+2} A_2^{|\alpha|+2d+1} \dots A_{2d+2}^{|\alpha|+1} M_{\alpha}^{(\kappa_{2d+2})} \\
 &\leq A^{|\alpha|+2d+2} M_{\alpha}^{(\kappa)}
 \end{aligned} \tag{3.12}$$

for $A := (\max\{A_1, \dots, A_{2d+2}\})^{2d+2}$ and $\kappa := \kappa_{2d+2}$.

Now, we have for $|t|_{\infty} \geq 1$,

$$\begin{aligned}
 (1 + |t|)^{2(d+1)} &= \sum_{j=0}^{2(d+1)} \binom{2d+2}{j} |t|^j \leq \sum_{j=0}^{2(d+1)} \binom{2d+2}{j} (\sqrt{d}|t|_{\infty})^j \\
 &\leq d^{d+1} |t|_{\infty}^{2(d+1)} \sum_{j=0}^{2(d+1)} \binom{2d+2}{j} = (4d)^{d+1} |t|_{\infty}^{2(d+1)}
 \end{aligned}$$

since $|t| = \sqrt{t_1^2 + \dots + t_d^2} \leq \sqrt{d}|t|_{\infty}$. Therefore, by the definition of the associated weight function, choosing $\kappa \geq \lambda > 0$ as in (3.12), we have, assuming $|t|_{\infty} = t_j$ for some $1 \leq j \leq d$ and $|t|_{\infty} \geq 1$:

$$\begin{aligned}
 (1 + |t|)^{2(d+1)} \exp \omega_{\mathbf{M}^{(\kappa)}}(t) &\leq (4d)^{d+1} |t_j|^{2(d+1)} \sup_{\alpha \in \mathbb{N}_0^d} \frac{|t^\alpha|}{M_{\alpha}^{(\kappa)}} \\
 &\leq (4d)^{d+1} \sup_{\alpha \in \mathbb{N}_0^d} \frac{|(At)^{\alpha+2(d+1)e_j}|}{M_{\alpha+2(d+1)e_j}^{(\lambda)}} \\
 &\leq (4d)^{d+1} \sup_{\beta \in \mathbb{N}_0^d} \frac{|(At)^{\beta}|}{M_{\beta}^{(\lambda)}} = (4d)^{d+1} \exp \omega_{\mathbf{M}^{(\lambda)}}(At).
 \end{aligned}$$

On the other hand, if $t \in \mathbb{R}^d$ with $|t|_{\infty} \leq 1$, then $|t| \leq \sqrt{d}$ and hence, for κ as in (3.12),

$$(1 + |t|)^{2(d+1)} \exp \omega_{\mathbf{M}^{(\kappa)}}(t) \leq C_{\lambda} \leq C_{\lambda} \exp \omega_{\mathbf{M}^{(\lambda)}}(At),$$

with C_{λ} depending on λ since κ depends on λ .

We have thus proved (3.10) with $B_1 = \max\{(4d)^{d+1}, C_{\lambda}\}$ and $B_2 = A$.

In the Beurling case, by $2(d+1)$ iterated applications of (3.7), we find $0 < \kappa_{2d+2} \leq \kappa_{2d+1} \leq \dots \leq \kappa_1 \leq \lambda$ and $A_1, \dots, A_{2d+2} \geq 1$ such that

$$\begin{aligned}
 M_{\alpha}^{(\lambda)} &\geq A_1^{-|\alpha|-1} M_{\alpha+e_j}^{(\kappa_1)} \geq A_1^{-|\alpha|-1} A_2^{-|\alpha|-2} M_{\alpha+2e_j}^{(\kappa_2)} \\
 &\dots \geq A_1^{-|\alpha|-1} A_2^{-|\alpha|-2} \dots A_{2d+2}^{-|\alpha|-2d-2} M_{\alpha+2(d+1)e_j}^{(\kappa_{2d+2})} \geq A^{-|\alpha|-2d-2} M_{\alpha+2(d+1)e_j}^{(\kappa)},
 \end{aligned} \tag{3.13}$$

for $A := (\max\{A_1, \dots, A_{2d+2}\})^{2d+2}$ and $\kappa := \kappa_{2d+2}$. Then we proceed as in the Roumieu case and prove that

$$(1 + |t|)^{2(d+1)} \exp \omega_{\mathbf{M}(\lambda)}(t) \leq B'_1 \exp \omega_{\mathbf{M}(\kappa)}(At),$$

for $B'_1 := \max\{(4d)^{d+1}, \max_{|t| \leq \sqrt{d}}(1 + |t|)^{2(d+1)} \exp \omega_{\mathbf{M}(\lambda)}(t)\}$. □

Lemma 5 *Let \mathcal{M} be a weight matrix that satisfies (3.7). Then*

$$\forall \lambda > 0, N \in \mathbb{N} \exists 0 < \kappa \leq \lambda, A, B \geq 1 \forall t \in \mathbb{R}^d \setminus \{0\} : \tag{3.14}$$

$$\omega_{\mathbf{M}(\lambda)}(t) + N \log |t| \leq \omega_{\mathbf{M}(\kappa)}(At) + B.$$

Let \mathcal{M} be a weight matrix that satisfies (3.6). Then

$$\forall \lambda > 0, N \in \mathbb{N} \exists \kappa \geq \lambda, A, B \geq 1 \forall t \in \mathbb{R}^d \setminus \{0\} : \tag{3.15}$$

$$\omega_{\mathbf{M}(\kappa)}(t) + N \log |t| \leq \omega_{\mathbf{M}(\lambda)}(At) + B.$$

Proof If $t \in \mathbb{R}^d \setminus \{0\}$, then by the definition of the associated weight function, for $1 \leq j \leq d$ such that $|t|_\infty = t_j$,

$$\begin{aligned} |t|^N \exp \omega_{\mathbf{M}(\lambda)}(t) &\leq (\sqrt{d}|t|_\infty)^N \exp \omega_{\mathbf{M}(\lambda)}(t) = d^{N/2} |t_j|^N \exp \omega_{\mathbf{M}(\lambda)}(t) \\ &= d^{N/2} |t^{N e_j}| \sup_{\alpha \in \mathbb{N}_{0,r}^d} \frac{|t^\alpha|}{M_\alpha^{(\lambda)}} = d^{N/2} \sup_{\alpha \in \mathbb{N}_{0,r}^d} \frac{|t^{\alpha + N e_j}|}{M_\alpha^{(\lambda)}}, \end{aligned} \tag{3.16}$$

where $\mathbb{N}_{0,r}^d$ is defined by (2.9). This estimate is valid for any given index $\lambda > 0$.

In the Beurling case, by N iterated applications of (3.7) we find $\kappa_N \leq \kappa_{N-1} \leq \dots \leq \kappa_1 \leq \lambda$ and $A_1, \dots, A_N \geq 1$ such that, for $A := (\max\{A_1, \dots, A_N\})^N$ and $\kappa := \kappa_N$, we have, proceeding as in (3.13), $M_{\alpha + N e_j}^{(\kappa)} \leq A^{|\alpha| + N} M_\alpha^{(\lambda)}$. Therefore,

$$|t|^N \exp \omega_{\mathbf{M}(\lambda)}(t) \leq d^{N/2} \sup_{\alpha \in \mathbb{N}_{0,r}^d} \frac{|(At)^{\alpha + N e_j}|}{M_{\alpha + N e_j}^{(\kappa)}} \leq d^{N/2} \exp \omega_{\mathbf{M}(\kappa)}(At),$$

and we conclude that (3.14) is satisfied for $B := \max\{\frac{N}{2} \log d, 1\}$.

In the Roumieu case, we make N iterated applications of (3.6) and we find indices $\kappa := \kappa_N \geq \kappa_{N-1} \geq \dots \geq \kappa_1 \geq \lambda$ and $A_1, \dots, A_N \geq 1$ such that, for $A := (\max\{A_1, \dots, A_N\})^N$ and $\kappa = \kappa_N$, as in (3.12) we have that $M_{\alpha + N e_j}^{(\lambda)} \leq A^{|\alpha| + N} M_\alpha^{(\kappa)}$ and hence from (3.16):

$$\begin{aligned} |t|^N \exp \omega_{\mathbf{M}(\lambda)}(t) &\leq d^{N/2} \sup_{\alpha \in \mathbb{N}_{0,r}^d} \frac{|t^{\alpha + N e_j}|}{M_\alpha^{(\lambda)}} \leq d^{N/2} \sup_{\alpha \in \mathbb{N}_{0,r}^d} \frac{|(At)^{\alpha + N e_j}|}{M_{\alpha + N e_j}^{(\kappa)}} \\ &\leq d^{N/2} \exp \omega_{\mathbf{M}(\kappa)}(At), \end{aligned}$$

so that (3.15) is satisfied with $B = \max\{\frac{N}{2} \log d, 1\}$. □

Now, we consider the different system of seminorms

$$\|f\|_{2, \mathbf{M}^{(\lambda), h}} := \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_2}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}}, \quad \lambda, h > 0, \tag{3.17}$$

on $\mathcal{S}_{(\mathcal{M})}$ and $\mathcal{S}_{\{\mathcal{M}\}}$, where $\|\cdot\|_2$ is the L^2 norm. Under suitable conditions on the weight matrix \mathcal{M} , it turns out to be equivalent to the previous one given by sup norms, as we prove in the following:

Proposition 1 *Let \mathcal{M} be a weight matrix as defined in (3.1) that satisfies (3.3) and (3.7) ((3.2) and (3.6)). Then the system of seminorms $\|\cdot\|_{\infty, \mathbf{M}^{(\lambda), h}}$ in $\mathcal{S}_{(\mathcal{M})}$ ($\mathcal{S}_{\{\mathcal{M}\}}$) is equivalent to the system of seminorms $\|\cdot\|_{2, \mathbf{M}^{(\lambda), h}}$. More precisely, in the Beurling case we have the following two conditions for every $f \in C^\infty(\mathbb{R}^d)$:*

$$\begin{aligned} \exists C_1 > 0 \forall \lambda, h > 0 \exists \kappa > 0, \tilde{h} = \tilde{h}_{\lambda, h} > 0 : \\ \|f\|_{2, \mathbf{M}^{(\lambda), h}} \leq C_1 \|f\|_{\infty, \mathbf{M}^{(\kappa), \tilde{h}}}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \forall \lambda, h > 0 \exists \tilde{\kappa} > 0, C_{\lambda, h} > 0, \tilde{h} = \tilde{h}_{\lambda, h} > 0 : \\ \|f\|_{\infty, \mathbf{M}^{(\lambda), h}} \leq C_{\lambda, h} \|f\|_{2, \mathbf{M}^{(\tilde{\kappa}), \tilde{h}}}; \end{aligned} \tag{3.19}$$

in the Roumieu case we have the following two conditions, for every $f \in C^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \forall \lambda, h > 0 \exists C_{\lambda, h} > 0, \exists \kappa \geq \lambda, \tilde{h} > 0 : \\ \|f\|_{2, \mathbf{M}^{(\kappa), \tilde{h}}} \leq C_{\lambda, h} \|f\|_{\infty, \mathbf{M}^{(\lambda), h}}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \forall \lambda, h > 0 \exists C_{\lambda, h} > 0, \tilde{\kappa} > 0, \tilde{h} > 0 : \\ \|f\|_{\infty, \mathbf{M}^{(\tilde{\kappa}), \tilde{h}}} \leq C_{\lambda, h} \|f\|_{2, \mathbf{M}^{(\lambda), h}}. \end{aligned} \tag{3.21}$$

Proof Let $f \in C^\infty(\mathbb{R}^d)$. Then for $C_1 = (\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^{d+1}} dx)^{1/2}$, we have

$$\|x^\alpha \partial^\beta f\|_2 \leq C_1 \|(1 + |x|^2)^{\frac{d+1}{2}} x^\alpha \partial^\beta f(x)\|_\infty.$$

If $|x|_\infty \leq 1$, then

$$(1 + |x|^2)^{\frac{d+1}{2}} \leq (1 + d|x|_\infty^2)^{\frac{d+1}{2}} \leq (1 + d)^{\frac{d+1}{2}}.$$

On the other hand, if $|x|_\infty \geq 1$ then

$$(1 + |x|^2)^{\frac{d+1}{2}} \leq (|x|_\infty^2 + |x|^2)^{\frac{d+1}{2}} \leq (|x|_\infty^2 + d|x|_\infty^2)^{\frac{d+1}{2}} \leq (d + 1)^{\frac{d+1}{2}} |x|_\infty^{d+1}.$$

Therefore, for any fixed $x \in \mathbb{R}^d$, being $|x|_\infty = |x_j|$ for some $1 \leq j \leq d$, we have

$$|(1 + |x|^2)^{\frac{d+1}{2}} x^\alpha| \leq (d + 1)^{\frac{d+1}{2}} \max\{|x^{\alpha+(d+1)e_j}|, |x^\alpha|\}$$

and hence

$$\|x^\alpha \partial^\beta f\|_2 \leq C_1(d + 1)^{\frac{d+1}{2}} \max\{\|x^{\alpha+(d+1)e_1} \partial^\beta f\|_\infty, \|x^{\alpha+(d+1)e_2} \partial^\beta f\|_\infty, \dots, \|x^{\alpha+(d+1)e_d} \partial^\beta f\|_\infty, \|x^\alpha \partial^\beta f\|_\infty\}. \tag{3.22}$$

Now, we consider separately the Beurling and Roumieu cases. In the Beurling case, for every $\lambda, h > 0$, we first estimate $\|x^{\alpha+(d+1)e_j} \partial^\beta f\|_{2, \mathbf{M}^{(\lambda)}, h}$ to use (3.22). By $(d + 1)$ iterated applications of (3.7) there exist $0 < \kappa := \kappa_{d+1} \leq \kappa_d \leq \dots \leq \kappa_1 \leq \lambda$ and $A_1, \dots, A_{d+1} \geq 1$ (A_j depending on λ) such that, proceeding as in (3.13), we obtain $M_{\alpha+\beta+(d+1)e_j}^{(\kappa)} \leq A_\lambda^{|\alpha+\beta|+d+1} M_{\alpha+\beta}^{(\lambda)}$ for $A_\lambda = (\max\{A_1, \dots, A_{d+1}\})^{d+1} \geq 1$. Hence, we deduce

$$\frac{\|x^{\alpha+(d+1)e_j} \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}} \leq \frac{\|x^{\alpha+(d+1)e_j} \partial^\beta f\|_\infty}{h^{|\alpha+\beta|+d+1} M_{\alpha+\beta+(d+1)e_j}^{(\kappa)}} \cdot h^{d+1} A_\lambda^{|\alpha+\beta|+d+1}.$$

Therefore, from (3.22) and the fact that $\mathbf{M}^{(\kappa)} \leq \mathbf{M}^{(\lambda)}$, we have for every $\lambda, h > 0$,

$$\frac{\|x^\alpha \partial^\beta f\|_2}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}} \leq C_1(d + 1)^{\frac{d+1}{2}} \max \left\{ \frac{\|x^{\alpha+(d+1)e_1} \partial^\beta f\|_\infty}{h^{|\alpha+\beta|+d+1} M_{\alpha+\beta+(d+1)e_1}^{(\kappa)}} h^{d+1} A_\lambda^{|\alpha+\beta|+d+1}, \dots, \frac{\|x^{\alpha+(d+1)e_d} \partial^\beta f\|_\infty}{h^{|\alpha+\beta|+d+1} M_{\alpha+\beta+(d+1)e_d}^{(\kappa)}} h^{d+1} A_\lambda^{|\alpha+\beta|+d+1}, \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa)}} \right\}. \tag{3.23}$$

If $h \geq 1$ then $h^{d+1} A_\lambda^{|\alpha+\beta|+d+1} \leq (h A_\lambda)^{|\alpha+\beta|+d+1}$.

If $0 < h < 1$ then $h^{d+1} A_\lambda^{|\alpha+\beta|+d+1} \leq A_\lambda^{|\alpha+\beta|+d+1}$. Hence, for

$$\tilde{h} := \begin{cases} \min \left\{ \frac{1}{A_\lambda}, h \right\} = \frac{1}{A_\lambda}, & \text{if } h \geq 1, \\ \min \left\{ \frac{h}{A_\lambda}, h \right\} = \frac{h}{A_\lambda}, & \text{if } 0 < h < 1, \end{cases}$$

we obtain

$$\|f\|_{2, \mathbf{M}^{(\lambda)}, h} \leq C_1(d + 1)^{\frac{d+1}{2}} \|f\|_{\infty, \mathbf{M}^{(\kappa)}, \tilde{h}}.$$

This shows (3.18).

Now, since $\delta! \leq \delta_1^{\delta_1} \dots \delta_d^{\delta_d} = \delta^\delta$, we have $\frac{\alpha!}{(\alpha-\delta)!} \leq \binom{\alpha}{\delta} \delta! \leq 2^{|\alpha|} \delta^\delta$. So it follows by Leibniz's rule and [28, formula (2.3)] that, for some $C_2 > 0$,

$$\begin{aligned}
 \|x^\alpha \partial^\beta f\|_\infty &\leq C_2 \sup_{|\gamma|_\infty \leq 2d+2} \|\partial^\gamma (x^\alpha \partial^\beta f)\|_2 \\
 &\leq C_2 \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \|(\partial^\delta x^\alpha) \partial^{\beta+\gamma-\delta} f\|_2 \\
 &\leq C_2 \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\substack{\delta \leq \gamma \\ \delta \leq \alpha}} \binom{\gamma}{\delta} 2^{|\alpha|} \delta^\delta \|x^{\alpha-\delta} \partial^{\beta+\gamma-\delta} f\|_2.
 \end{aligned}
 \tag{3.24}$$

On the other hand, by $|\gamma|$ iterated applications of (3.7), there exist $0 < \kappa := \kappa_{|\gamma|} \leq \kappa_{|\gamma|-1} \leq \dots \leq \kappa_1 \leq \lambda$ and $A_1, \dots, A_{|\gamma|} \geq 1$ such that, for $A_\lambda := \max_{|\gamma|_\infty \leq 2d+2} (\max\{A_1, \dots, A_{|\gamma|}\})^{|\gamma|}$, we have $M_{\alpha+\beta+\gamma}^{(\kappa)} \leq A_\lambda^{|\alpha+\beta+\gamma|} M_{\alpha+\beta}^{(\lambda)}$. By (3.3), there exist $0 < \tilde{\kappa} \leq \kappa$ and $H > 0$ such that for all $C > 0$ there is $B > 0$ so that

$$\begin{aligned}
 \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}} &\leq C_2 \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\substack{\delta \leq \gamma \\ \delta \leq \alpha}} \binom{\gamma}{\delta} \frac{\|x^{\alpha-\delta} \partial^{\beta+\gamma-\delta} f\|_2}{h^{|\alpha+\beta+\gamma-2\delta|} M_{\alpha+\beta+\gamma-2\delta}^{(\tilde{\kappa})}} \cdot h^{|\gamma-2\delta|} \\
 &\quad \cdot 2^{|\alpha|} A_\lambda^{|\alpha+\beta+\gamma|} B C^{|\alpha+\beta+\gamma|} H^{|\alpha+\beta+\gamma|}.
 \end{aligned}$$

Observe that $\tilde{\kappa}$ may depend on γ . From (3.1) we can consider in the previous estimates, instead of $\tilde{\kappa}$, the minimum of all these $\tilde{\kappa}$ for $|\gamma|_\infty \leq 2d + 2$, so that we can finally choose $\tilde{\kappa}$ independent of γ . Since $|\gamma| \leq d|\gamma|_\infty \leq 2d(d + 1)$ we have

$$\begin{aligned}
 \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}} &\leq C_2 B (2CHA_\lambda)^{4d(d+1)} \\
 &\quad \cdot \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\substack{\delta \leq \gamma \\ \delta \leq \alpha}} \binom{\gamma}{\delta} \frac{\|x^{\alpha-\delta} \partial^{\beta+\gamma-\delta} f\|_2}{h^{|\alpha+\beta+\gamma-2\delta|} M_{\alpha+\beta+\gamma-2\delta}^{(\tilde{\kappa})}} \\
 &\quad \cdot (2HA_\lambda)^{|\alpha+\beta+\gamma-2\delta|} h^{|\gamma-2\delta|}.
 \end{aligned}$$

Now, if $h \geq 1$, then $h^{|\gamma-2\delta|} \leq h^{|\alpha+\beta+\gamma-2\delta|}$. And if $0 < h < 1$, then $h^{|\gamma-2\delta|} \leq 1$ when $|\gamma - 2\delta| \geq 0$ and $h^{|\gamma-2\delta|} \leq h^{-|\gamma|} \leq h^{-2d(d+1)}$ when $|\gamma - 2\delta| < 0$. For

$$\tilde{h} = \begin{cases} \frac{1}{2HA_\lambda} & \text{if } h \geq 1 \\ \frac{h}{2HA_\lambda} & \text{if } 0 < h < 1, \end{cases}
 \tag{3.25}$$

taking into account that

$$\sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \leq d^{|\gamma|} \leq d^{2d(d+1)},$$

we finally have that for all $\lambda, h > 0$ there exist $\tilde{\kappa}, C_{\lambda,h} > 0$ and $\tilde{h} > 0$, such that

$$\|f\|_{\infty, \mathbf{M}^{(\lambda), h}} \leq C_{\lambda, h} \|f\|_{2, \mathbf{M}^{(\tilde{\kappa}), \tilde{h}}} \tag{3.26}$$

Since neither H nor A_λ are depending on h , we have $\tilde{h} \rightarrow 0$ as $h \rightarrow 0$. This shows (3.19) and concludes the proof in the Beurling case.

Let us now consider the Roumieu case. In (3.22), for any given λ by $(d + 1)$ iterated applications of (3.6), we obtain $\kappa := \kappa_{d+1} \geq \kappa_d \geq \dots \geq \kappa_1 \geq \lambda > 0$ and $A_1, \dots, A_{d+1} \geq 1$ such that, for $A_\lambda := (\max\{A_1, \dots, A_{d+1}\})^{d+1}$, we have $M_{\alpha+\beta+(d+1)e_j}^{(\lambda)} \leq A_\lambda^{|\alpha+\beta|+d+1} M_{\alpha+\beta}^{(\kappa)}$. Then from (3.22) and the fact that $M_{\alpha+\beta}^{(\kappa)} \geq M_{\alpha+\beta}^{(\lambda)}$ we obtain, given a fixed $h > 0$, for $\tilde{h} := \max\{hA_\lambda, 1\}$,

$$\begin{aligned} \frac{\|x^\alpha \partial^\beta f\|_2}{M_{\alpha+\beta}^{(\kappa)}} &\leq C_1(d+1)^{\frac{d+1}{2}} \max \left\{ \frac{\|x^\alpha \partial^\beta f\|_\infty}{M_{\alpha+\beta}^{(\kappa)}}, \frac{\|x^{\alpha+(d+1)e_j} \partial^\beta f\|_\infty}{M_{\alpha+\beta+(d+1)e_j}^{(\lambda)}} A_\lambda^{|\alpha+\beta|+d+1} \right\} \\ &\leq C_1(d+1)^{\frac{d+1}{2}} \tilde{h}^{|\alpha+\beta|+d+1} \\ &\quad \cdot \max \left\{ \frac{\|x^\alpha \partial^\beta f\|_\infty}{h^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}}, \frac{\|x^{\alpha+(d+1)e_j} \partial^\beta f\|_\infty}{h^{|\alpha+\beta|+d+1} M_{\alpha+\beta+(d+1)e_j}^{(\lambda)}} \right\}. \end{aligned}$$

Hence, dividing by $\tilde{h}^{|\alpha+\beta|}$,

$$\frac{\|x^\alpha \partial^\beta f\|_2}{\tilde{h}^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa)}} \leq C_1(d+1)^{\frac{d+1}{2}} \tilde{h}^{d+1} \|f\|_{\infty, \mathbf{M}^{(\lambda), h}};$$

then (3.20) is proved, with $C_{\lambda, h} = C_1(d+1)^{\frac{d+1}{2}} \tilde{h}^{d+1}$ (observe that \tilde{h} depends on h and λ).

Now, given any $\lambda > 0$ consider $\kappa \geq \lambda > 0$ and $B, C, H > 0$ as in (3.2). Then, by $|\gamma|$ iterated applications of (3.6), there exist $\tilde{\kappa} := \kappa_{|\gamma|} \geq \dots \geq \kappa_1 \geq \kappa \geq \lambda$ and $A_1, \dots, A_{|\gamma|} \geq 1$ such that, for $A_\lambda := (\max\{A_1, \dots, A_{|\gamma|}\})^{|\gamma|}$, $M_{\alpha+\beta+\gamma}^{(\kappa)} \leq A_\lambda^{|\alpha+\beta+\gamma|} M_{\alpha+\beta}^{(\tilde{\kappa})}$. So, from (3.24) with $h = 1$ and $\tilde{\kappa}$ instead of λ , applying (3.2) and proceeding as before, we get

$$\begin{aligned} \frac{\|x^\alpha \partial^\beta f\|_\infty}{M_{\alpha+\beta}^{(\tilde{\kappa})}} &\leq C_2 B C^{4d(d+1)} \\ &\quad \cdot \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\substack{\delta \leq \gamma \\ \delta \leq \alpha}} \binom{\gamma}{\delta} (2A_\lambda H)^{|\alpha+\beta+\gamma|} \frac{\|x^{\alpha-\delta} \partial^{\beta+\gamma-\delta} f\|_2}{M_{\alpha+\beta+\gamma-2\delta}^{(\lambda)}}. \end{aligned} \tag{3.27}$$

Since for every $h > 0$ and $\alpha, \beta, \gamma, \delta$ as above

$$\frac{\|x^{\alpha-\delta} \partial^{\beta+\gamma-\delta} f\|_2}{h^{|\alpha+\beta+\gamma-2\delta|} M_{\alpha+\beta+\gamma-2\delta}^{(\lambda)}} \leq \|f\|_{2, \mathbf{M}^{(\lambda), h}},$$

dividing (3.27) by $(2A_\lambda H h)^{|\alpha+\beta|}$ we obtain

$$\frac{\|x^\alpha \partial^\beta f\|_\infty}{(2A_\lambda Hh)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\tilde{\kappa})}} \leq \|f\|_{2, \mathbf{M}^{(\lambda), h}} C_2 BC^{4d(d+1)} \cdot \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} (2A_\lambda Hh)^{|\gamma|} h^{-|2\delta|}.$$

Taking the sup on α and β in the left-hand side, we then get (3.21) with $\tilde{h} = 2A_\lambda Hh$ and

$$C_{\lambda, h} = C_2 BC^{4d(d+1)} \sup_{|\gamma|_\infty \leq 2d+2} \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} (2A_\lambda Hh)^{|\gamma|} h^{-|2\delta|}.$$

□

We observe that in (3.18) the constant C_1 is fixed (it depends only on the dimension d), and moreover, we only need (3.7) to prove it. On the other hand, to obtain (3.19) we consider (3.7) and (3.3). In the Roumieu case, we just need (3.6) to prove (3.20), while for the proof of (3.21) we use (3.2) to choose $\kappa \geq \lambda$ and then (3.6) to get $\tilde{\kappa} \geq \kappa$.

4 Hermite functions: properties in the matrix setting

We recall the definition of the *Hermite functions* H_γ for $\gamma \in \mathbb{N}_0^d$:

$$H_\gamma(x) := (2^{|\gamma|} \gamma! \pi^{d/2})^{-1/2} h_\gamma(x) \exp\left(-\sum_{j=1}^d \frac{x_j^2}{2}\right), \quad x \in \mathbb{R}^d,$$

where h_γ are the Hermite polynomials

$$h_\gamma(x) := (-1)^{|\gamma|} \exp\left(\sum_{j=1}^d x_j^2\right) \cdot \partial^\gamma \exp\left(-\sum_{j=1}^d x_j^2\right), \quad x \in \mathbb{R}^d.$$

As in [28] we consider, for $f \in C^\infty(\mathbb{R}^d)$, the operators

$$A_{\pm, i}(f) := \mp \partial_{x_i} f + x_i f, \quad 1 \leq i \leq d,$$

$$A_\pm^\alpha(f) := \prod_{i=1}^d A_{\pm, i}^{\alpha_i}(f), \quad \alpha \in \mathbb{N}_0^d,$$

with $A_{\pm, i}^0 := \text{id}$.

By [32, Example 29.5(2)], setting $H_\beta = 0$ if $\beta_j = -1$ for some $1 \leq j \leq d$, we have

$$A_{-j}(H_\gamma) = \sqrt{2\gamma_j} H_{\gamma-e_j}, \quad \gamma \in \mathbb{N}_0^d.$$

It follows that, for $\alpha, \gamma \in \mathbb{N}_0^d$,

$$A_-^\alpha(H_{\gamma+\alpha}) = \prod_{1 \leq j \leq d} A_{-j}^{\alpha_j}(H_{\gamma+\alpha}) = \prod_{1 \leq j \leq d} (\sqrt{2\gamma_j})^{\alpha_j} H_\gamma = \sqrt{2^{|\alpha|}} \gamma^\alpha H_\gamma. \tag{4.1}$$

We also recall the following two lemmas from [28]:

Lemma 6 *Let $f \in C^\infty(\mathbb{R}^d)$. Then, for all $\gamma \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,*

$$(A_+^\gamma f)(x) = \sum_{\alpha+\beta \leq \gamma} C_{\alpha,\beta}(\gamma) x^\alpha \partial^\beta f(x),$$

for some coefficients $C_{\alpha,\beta}(\gamma)$ satisfying

$$|C_{\alpha,\beta}(\gamma)| \leq 3^{|\gamma|} \left(\frac{\gamma!}{(\alpha + \beta)!} \right)^{1/2}, \quad \alpha, \beta, \gamma \in \mathbb{N}_0^d.$$

Lemma 7 *For all $\alpha, \beta, \gamma \in \mathbb{N}_0^d$*

$$\|x^\alpha \partial^\beta H_\gamma\|_2 \leq 2^{\frac{|\alpha+\beta|}{2}} \left(\frac{(\alpha + \beta + \gamma)!}{\gamma!} \right)^{1/2}.$$

We can generalize Lemma 3.1(b) of [28] in the following way:

Lemma 8 *Let $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$ and $\mathbf{N} = (N_\alpha)_{\alpha \in \mathbb{N}_0^d}$ be two sequences satisfying (2.12) for some $C, B, H > 0$. Assume that $f \in C^\infty(\mathbb{R}^d)$ satisfies, for some $C_1 > 0$ and for the same constant C as in (2.12),*

$$\|f\|_{2,\mathbf{M},C} = \sup_{\alpha,\beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_2}{C^{|\alpha+\beta|} M_{\alpha+\beta}} \leq C_1. \tag{4.2}$$

Then

$$\|A_+^\gamma f\|_2 \leq C_1 B e^{d/2} (9\sqrt{2}HC)^{|\gamma|} N_\gamma, \quad \gamma \in \mathbb{N}_0^d.$$

Proof By Stirling's inequality $e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n$ for any $n \in \mathbb{N}$. Hence, by Lemma 6 and assumption (4.2), we have

$$\begin{aligned}
 \|A_{+}^{\gamma}f\|_2 &\leq \sum_{\alpha+\beta\leq\gamma} |C_{\alpha,\beta}(\gamma)| \cdot \|x^{\alpha}\partial^{\beta}f\|_2 \\
 &\leq C_1 3^{|\gamma|} \sum_{\alpha+\beta\leq\gamma} \binom{\gamma}{\alpha+\beta}^{1/2} (\gamma-\alpha-\beta)!^{1/2} C^{|\alpha+\beta|} M_{\alpha+\beta} \\
 &\leq C_1 3^{|\gamma|} \sum_{\alpha+\beta\leq\gamma} \binom{\gamma}{\alpha+\beta}^{1/2} e^{d/2} \left(\prod_{j=1}^d (\gamma_j - \alpha_j - \beta_j)^{1/2} \right) \\
 &\quad \cdot \frac{(\gamma-\alpha-\beta)^{\frac{\gamma-\alpha-\beta}{2}}}{\exp\left\{\frac{|\gamma-\alpha-\beta|}{2}\right\}} C^{|\alpha+\beta|} M_{\alpha+\beta} \\
 &\leq C_1 3^{|\gamma|} 2^{|\gamma|/2} e^{d/2} \sum_{\alpha+\beta\leq\gamma} \binom{\gamma}{\alpha+\beta} (\gamma-\alpha-\beta)^{\frac{\gamma-\alpha-\beta}{2}} C^{|\alpha+\beta|} M_{\alpha+\beta}.
 \end{aligned}$$

Applying now (2.12) and $\sum_{\alpha+\beta\leq\gamma} \binom{\gamma}{\alpha+\beta} \leq 3^{|\gamma|}$ (by [28, pg 274]), we get

$$\begin{aligned}
 \|A_{+}^{\gamma}f\|_2 &\leq C_1 e^{d/2} (3\sqrt{2})^{|\gamma|} \sum_{\alpha+\beta\leq\gamma} \binom{\gamma}{\alpha+\beta} B C^{|\gamma-\alpha-\beta|} H^{|\gamma|} N_{\gamma} C^{|\alpha+\beta|} \\
 &\leq C_1 B e^{d/2} (9\sqrt{2})^{|\gamma|} (CH)^{|\gamma|} N_{\gamma}.
 \end{aligned}$$

□

As a corollary, we immediately have the following:

Lemma 9 *Let \mathcal{M} be a weight matrix satisfying (3.2) and assume that $f \in C^{\infty}(\mathbb{R}^d)$ satisfies, for some $\lambda, C_1 > 0$*

$$\|f\|_{2,\mathcal{M}^{(\lambda)},C} \leq C_1 \tag{4.3}$$

for the constant C of (3.2). Then

$$\|A_{+}^{\gamma}f\|_2 \leq C_1 B e^{d/2} (9\sqrt{2}HC)^{|\gamma|} M_{\gamma}^{(\kappa)}, \quad \forall \gamma \in \mathbb{N}_0^d,$$

with κ, B, H, C as in (3.2).

If \mathcal{M} satisfies (3.3) and if, for some $\lambda > 0$, $f \in C^{\infty}(\mathbb{R}^d)$ satisfies

$$\|f\|_{2,\mathcal{M}^{(\lambda)},C} \leq C_1$$

for the constant $\kappa \leq \lambda$ of (3.3) and for some $C, C_1 > 0$, then

$$\|A_{+}^{\gamma}f\|_2 \leq C_1 B e^{d/2} (9\sqrt{2}HC)^{|\gamma|} M_{\gamma}^{(\lambda)}, \quad \forall \gamma \in \mathbb{N}_0^d,$$

where $H = H(\lambda)$ and $B = B(C, \lambda)$ are given by (3.3).

The following lemma generalizes [28, Lemma 3.2(b)].

Lemma 10 Let $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$ and $\mathbf{N} = (N_\alpha)_{\alpha \in \mathbb{N}_0^d}$ be two weight sequences satisfying (2.12). Then

$$\|H_\gamma\|_{2, \mathbf{N}, 2HC} = \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta H_\gamma\|_2}{(2HC)^{|\alpha+\beta|} N_{\alpha+\beta}} \leq B e^{\omega_{\mathbf{M}}(\gamma^{1/2}/C)}, \quad \forall \gamma \in \mathbb{N}_0^d,$$

where $\gamma^{1/2} := (\gamma_1^{1/2}, \dots, \gamma_d^{1/2})$ and $B, C, H > 0$ are the constants in (2.12).

Proof For $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ we set

$$J := \{j \in \mathbb{N} : 1 \leq j \leq d, \alpha_j + \beta_j \leq \gamma_j\}$$

$$J^c := \{j \in \mathbb{N} : 1 \leq j \leq d, \alpha_j + \beta_j > \gamma_j\}.$$

Then for any $\delta \in \mathbb{N}^d$ we denote

$$\delta_J := \sum_{j \in J} \delta_j e_j, \quad \delta_{J^c} := \sum_{j \in J^c} \delta_j e_j,$$

so that $\delta = \delta_J + \delta_{J^c}$. By Lemma 7 and (2.12), we have

$$\begin{aligned} \|x^\alpha \partial^\beta H_\gamma\|_2 &\leq 2^{\frac{|\alpha+\beta|}{2}} \left(\frac{(\alpha + \beta + \gamma)!}{\gamma!} \right)^{1/2} \leq 2^{\frac{|\alpha+\beta|}{2}} (\alpha + \beta + \gamma)^{\frac{\alpha+\beta}{2}} \\ &\leq 2^{|\alpha+\beta|} (\alpha_{J^c} + \beta_{J^c})^{\frac{\alpha_{J^c} + \beta_{J^c}}{2}} \gamma_J^{\frac{\alpha_J + \beta_J}{2}} \\ &\leq B(2HC)^{|\alpha+\beta|} N_{\alpha+\beta} \gamma_J^{\frac{\alpha_J + \beta_J}{2}} \frac{1}{M_{\alpha_J + \beta_J} C^{|\alpha_J + \beta_J|}}. \end{aligned} \tag{4.4}$$

Now, since α_j has the j th entry equal to α_j for $j \in J$ and 0 for $j \in J^c$,

$$\gamma_J^{\frac{\alpha_J + \beta_J}{2}} = \prod_{j \in J} \gamma_j^{\frac{\alpha_j + \beta_j}{2}} = \prod_{j \in J} \gamma_j^{\frac{\alpha_j + \beta_j}{2}} \prod_{j \in J^c} \gamma_j^0 = \gamma^{\frac{\alpha_J + \beta_J}{2}}. \tag{4.5}$$

Moreover, by Lemma 3,

$$M_{\alpha_J + \beta_J} C^{|\alpha_J + \beta_J|} \geq \sup_{t \in \mathbb{R}^d} |t^{\alpha_J + \beta_J} e^{-\omega_{\mathbf{M}}(t/C)}| \geq \gamma^{\frac{\alpha_J + \beta_J}{2}} e^{-\omega_{\mathbf{M}}(\gamma^{1/2}/C)}, \tag{4.6}$$

taking $t = \gamma^{1/2}$.

If we replace (4.5) and (4.6) in (4.4) we finally get

$$\|x^\alpha \partial^\beta H_\gamma\|_2 \leq B(2HC)^{|\alpha+\beta|} N_{\alpha+\beta} e^{\omega_{\mathbf{M}}(\gamma^{1/2}/C)}.$$

□

Proposition 2 Let \mathcal{M} be a weight matrix that satisfies (3.2) and (3.6) ((3.3) and (3.7)). Then $H_\gamma \in \mathcal{S}_{\{\mathcal{M}\}}(H_\gamma \in \mathcal{S}_{(\mathcal{M})})$ for all $\gamma \in \mathbb{N}_0^d$.

Proof By Lemma 10, if (3.2) is satisfied, we have

$$\begin{aligned} &\forall \lambda > 0 \exists \kappa \geq \lambda, \exists B, C, H > 0 : \\ &\|x^\alpha \partial^\beta H_\gamma\|_2 \leq B(2HC)^{|\alpha+\beta|} M_{|\alpha+\beta|}^{(\kappa)} e^{\omega_{M(\kappa)}(\gamma^{1/2}/C)}. \end{aligned}$$

Hence, $H_\gamma \in \mathcal{S}_{\{\mathcal{M}\}}$ by Proposition 1. Similarly, in the Beurling case, if (3.3) is satisfied, we obtain

$$\begin{aligned} &\forall \lambda > 0 \exists 0 < \kappa \leq \lambda, \exists H > 0 : \forall C > 0 \exists B > 0 : \\ &\|x^\alpha \partial^\beta H_\gamma\|_2 \leq B(2HC)^{|\alpha+\beta|} M_{|\alpha+\beta|}^{(\lambda)} e^{\omega_{M(\kappa)}(\gamma^{1/2}/C)}. \end{aligned}$$

So $H_\gamma \in \mathcal{S}_{\{\mathcal{M}\}}$ by Proposition 1. □

The next result gives information about the non-triviality of the classes $\mathcal{S}_{\{\mathcal{M}\}}$ and $\mathcal{S}_{\{\mathcal{M}\}}$. Indeed, we characterize when the Hermite functions H_γ are contained in such classes.

Proposition 3 *Let \mathcal{M} be a weight matrix that satisfies (3.6), (3.4); then the following are equivalent:*

- (a) $\exists \lambda > 0 \exists C, C_1 > 0 : \alpha^{\alpha/2} \leq C_1 C^{|\alpha|} M_\alpha^{(\lambda)}, \quad \forall \alpha \in \mathbb{N}_0^d;$
- (b) \mathcal{M} satisfies (3.2);
- (c) $H_\gamma \in \mathcal{S}_{\{\mathcal{M}\}}$ for all $\gamma \in \mathbb{N}_0^d.$

If \mathcal{M} satisfies (3.7), (3.5), then the following are equivalent:

- (a)' $\forall \lambda, C > 0 \exists C_1 > 0 : \alpha^{\alpha/2} \leq C_1 C^{|\alpha|} M_\alpha^{(\lambda)}, \quad \forall \alpha \in \mathbb{N}_0^d;$
- (b)' \mathcal{M} satisfies (3.3);
- (c)' $H_\gamma \in \mathcal{S}_{\{\mathcal{M}\}}$ for all $\gamma \in \mathbb{N}_0^d.$

Proof The implications (b) \Rightarrow (c) and (b)' \Rightarrow (c)' follow from Proposition 2. To see (a) \Rightarrow (b), fix an arbitrary $\mu > 0$ and λ as in (a). We have

$$\alpha^{\alpha/2} M_\beta^{(\mu)} \leq C_1 C^{|\alpha|} M_\alpha^{(\lambda)} M_\beta^{(\mu)}.$$

So, for $\nu = \max\{\lambda, \mu\}$, by (3.1) and (3.4), there exists $\kappa \geq \nu$ and $A \geq 1$ such that

$$\alpha^{\alpha/2} M_\beta^{(\mu)} \leq C_1 C^{|\alpha|} M_\alpha^{(\nu)} M_\beta^{(\nu)} \leq C_1 C^{|\alpha|} A^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa)}, \quad \alpha, \beta \in \mathbb{N}_0^d.$$

Now, we prove (a)' \Rightarrow (b)'. For any given $\lambda > 0$, let $0 < \kappa \leq \lambda$ and $A \geq 1$ such that (3.5) holds. By (a)' applied to this κ , there is, for any $C > 0$, some $C_1 > 0$ depending on κ and C such that

$$\alpha^{\alpha/2} M_\beta^{(\kappa)} \leq C_1 C^{|\alpha|} M_\alpha^{(\kappa)} M_\beta^{(\kappa)} \leq C_1 C^{|\alpha|} A^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}, \quad \alpha, \beta \in \mathbb{N}_0^d.$$

If (c) holds, in particular, $H_0 \in \mathcal{S}_{\{\mathcal{M}\}}$. Hence, there exist some $C, h > 0$ and $\lambda > 0$ such that $\|x^\alpha H_0\|_\infty \leq Ch^{|\alpha|} M_\alpha^{(\lambda)}$ for all $\alpha \in \mathbb{N}_0^d$. Taking $x = \alpha^{1/2}$, $\alpha \in \mathbb{N}_0^d$ arbitrary, yields

$$|\alpha^{\alpha/2} H_0(\alpha^{1/2})| = \frac{1}{\pi^{d/4}} \alpha_1^{\alpha_1/2} e^{-\alpha_1/2} \dots \alpha_d^{\alpha_d/2} e^{-\alpha_d/2} = \frac{1}{\pi^{d/4}} \alpha^{\alpha/2} e^{-|\alpha|/2}.$$

Hence, $\alpha^{\alpha/2} \pi^{-d/4} e^{-|\alpha|/2} \leq \|x^\alpha H_0\|_\infty \leq Ch^{|\alpha|} M_\alpha^{(\lambda)}$ for all $\alpha \in \mathbb{N}_0^d$, which shows (a).

The Beurling case (c)' \Rightarrow (a)' is analogous since, for any given λ and $h > 0$, there exists $C_{\lambda,h} > 0$ such that $\|x^\alpha H_0\|_\infty \leq C_{\lambda,h} h^{|\alpha|} M_\alpha^{(\lambda)}$ for all $\alpha \in \mathbb{N}_0^d$. \square

5 Matrix sequence spaces

Let us consider, for $\mathbf{M} = (M_\alpha)_{\alpha \in \mathbb{N}_0^d}$, the following sequence spaces in the Roumieu and the Beurling cases:

$$\Lambda_{\{\mathbf{M}\}} := \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \exists h > 0, \|\mathbf{c}\|_{\mathbf{M},h} := \sup_{\alpha \in \mathbb{N}_0^d} |c_\alpha| e^{\omega_{\mathbf{M}}(\alpha^{1/2}/h)} < +\infty\},$$

$$\Lambda_{(\mathbf{M})} := \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall h > 0, \|\mathbf{c}\|_{\mathbf{M},h} < +\infty\}.$$

Since $h \mapsto \omega_{\mathbf{M}}(\alpha^{1/2}/h)$ is decreasing we can also write

$$\Lambda_{\{\mathbf{M}\}} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \exists j \in \mathbb{N}, \|\mathbf{c}\|_{\mathbf{M},j} < +\infty\},$$

$$\Lambda_{(\mathbf{M})} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall j \in \mathbb{N}, \|\mathbf{c}\|_{\mathbf{M},1/j} < +\infty\}.$$

Now, for a weight matrix \mathcal{M} as in (3.1), we denote

$$\Lambda_{\{\mathcal{M}\}} := \bigcup_{\lambda > 0} \Lambda_{\{\mathbf{M}^{(\lambda)}\}} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \exists \lambda, h > 0, \|\mathbf{c}\|_{\mathbf{M}^{(\lambda)},h} < +\infty\},$$

$$\Lambda_{(\mathcal{M})} := \bigcap_{\lambda > 0} \Lambda_{(\mathbf{M}^{(\lambda)})} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall \lambda, h > 0, \|\mathbf{c}\|_{\mathbf{M}^{(\lambda)},h} < +\infty\}.$$

Since $\mathbf{M}^{(\lambda)} \leq \mathbf{M}^{(\kappa)}$ for $0 < \lambda \leq \kappa$ by assumption, then $\omega_{\mathbf{M}^{(\lambda)}} \geq \omega_{\mathbf{M}^{(\kappa)}}$. Moreover, $h \mapsto e^{\omega_{\mathbf{M}^{(\kappa)}}(\alpha^{1/2}/h)}$ is decreasing for all $\kappa > 0$, $\alpha \in \mathbb{N}_0^d$. It follows that we can write $\Lambda_{\{\mathcal{M}\}}$ ($\Lambda_{(\mathcal{M})}$) as inductive (projective) limit:

$$\Lambda_{\{\mathcal{M}\}} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \exists j \in \mathbb{N}, \|\mathbf{c}\|_{\mathbf{M}^{(j)},j} < +\infty\}, \tag{5.1}$$

$$\Lambda_{(\mathcal{M})} = \{\mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall j \in \mathbb{N}, \|\mathbf{c}\|_{\mathbf{M}^{(j)},1/j} < +\infty\}. \tag{5.2}$$

Note that by Remark 1, it seems natural to require that $\lim_{|\alpha| \rightarrow \infty} (M_\alpha)^{1/|\alpha|} = +\infty$ for the definition of $\Lambda_{\{\mathbf{M}\}}$ and $\Lambda_{(\mathbf{M})}$. In fact, otherwise $\omega_{\mathbf{M}}(t) = +\infty$ for all large $t \in \mathbb{R}^d$

and we get $\Lambda_{\{M\}} = \{0\}$ and $\Lambda_{\{M\}}$ consisting of sequences having only finitely many values $\neq 0$.

However, in our next main result, by Remark 3 and assumption (3.2) ((3.3), respectively), we have the warranty of the finiteness of all associated weight functions under consideration.

Theorem 1 *Let \mathcal{M} be a weight matrix satisfying (3.2) and (3.6). Then the Hermite functions are an absolute Schauder basis in $\mathcal{S}_{\{\mathcal{M}\}}$ and*

$$T : \mathcal{S}_{\{\mathcal{M}\}} \longrightarrow \Lambda_{\{\mathcal{M}\}}$$

$$f \longmapsto (\xi_\gamma(f))_{\gamma \in \mathbb{N}_0^d} := \left(\int_{\mathbb{R}^d} f(x) H_\gamma(x) dx \right)_{\gamma \in \mathbb{N}_0^d}$$

defines an isomorphism.

If \mathcal{M} satisfies (3.3) and (3.7), then the Hermite functions are an absolute Schauder basis in $\mathcal{S}_{\{\mathcal{M}\}}$ and the above defined operator $T : \mathcal{S}_{\{\mathcal{M}\}} \rightarrow \Lambda_{\{\mathcal{M}\}}$ is an isomorphism.

Proof By Proposition 1 we can assume that $\mathcal{S}_{\{\mathcal{M}\}}$ and $\mathcal{S}_{\{\mathcal{M}\}}$ are defined by L^2 norms. First, we consider the Roumieu case. If $f \in \mathcal{S}_{\{\mathcal{M}\}}$, there exist $\lambda, C, C_1 > 0$ such that

$$\|f\|_{2, \mathbf{M}^{(\lambda), C}} =: C_1 < +\infty.$$

By (4.1) and Lemma 9, there exist $\kappa \geq \lambda, B, C, H > 0$ such that for all $\gamma, \alpha \in \mathbb{N}_0^d$, since $\|H_\gamma\|_2 = 1$ for all $\gamma \in \mathbb{N}_0^d$, we have

$$\begin{aligned} |\xi_\gamma(f)|^2 \gamma^\alpha &= |\langle f, H_\gamma \rangle|^2 \gamma^\alpha \leq |\langle f, \sqrt{2^{|\alpha|}} \gamma^\alpha H_\gamma \rangle|^2 = |\langle f, A_-^\alpha(H_{\gamma+\alpha}) \rangle|^2 \\ &= |\langle A_+^\alpha(f), H_{\gamma+\alpha} \rangle|^2 \leq \|A_+^\alpha(f)\|_2^2 \|H_{\gamma+\alpha}\|_2^2 \\ &\leq C_1^2 B^2 e^d (9\sqrt{2}HC)^{2|\alpha|} (M_\alpha^{(\kappa)})^2. \end{aligned}$$

Therefore, by definition of the associated weight function, and using the notation of (2.9), since $|(\gamma^{1/2})^\alpha| = |\gamma_1^{\alpha_1/2} \dots \gamma_d^{\alpha_d/2}| = (\gamma^\alpha)^{1/2}$, we obtain

$$|\xi_\gamma(f)| e^{\omega_{\mathbf{M}^{(\kappa)}}(\gamma^{1/2}/(9\sqrt{2}HC))} = \sup_{\alpha \in \mathbb{N}_{0, \gamma}^d} \frac{|\xi_\gamma(f)| \left| \left(\frac{\gamma^{1/2}}{9\sqrt{2}HC} \right)^\alpha \right|}{M_\alpha^{(\kappa)}} \leq C_1 B e^{d/2}.$$

Hence, $(\xi_\gamma(f))_\gamma \in \Lambda_{\{\mathcal{M}\}}$ and, more precisely, there exist $\kappa \geq \lambda, H, C > 0$ and $B \geq 1$ such that

$$\|(\xi_\gamma(f))_\gamma\|_{\mathbf{M}^{(\kappa), 9\sqrt{2}HC}} \leq B e^{d/2} \|f\|_{2, \mathbf{M}^{(\lambda), C}}. \tag{5.3}$$

This proves that T is continuous in the Roumieu case [32, Proposition 24.7].

On the other hand, given $\mathbf{c} = (c_\gamma)_{\gamma \in \mathbb{N}_0^d} \in \Lambda_{\{\mathcal{M}\}}$, let $\lambda, C^* > 0$ such that

$$\sup_{\gamma \in \mathbb{N}_0^d} |c_\gamma| e^{\omega_{M(\lambda)}(\gamma^{1/2}/C^*)} = \|c\|_{M(\lambda), C^*} =: C_1^* < +\infty.$$

By Lemma 4, there exist $\kappa \geq \lambda$ and $B_1, B_2 \geq 1$ such that

$$e^{-\omega_{M(\lambda)}(B_2 t) + \omega_{M(\kappa)}(t)} \leq B_1(1 + |t|)^{-2(d+1)}, \quad t \in \mathbb{R}^d.$$

Then, by (3.2), there exist $\kappa' \geq \kappa$ and $B, C, H > 0$ with $C \geq B_2 C^*$, such that, by Lemma 10,

$$\begin{aligned} |c_\gamma| \cdot \|x^\alpha \partial^\beta H_\gamma\|_2 &\leq |c_\gamma| (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} B e^{\omega_{M(\kappa)}(\gamma^{1/2}/C)} \\ &\leq C_1^* B (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} e^{-\omega_{M(\lambda)}(\gamma^{1/2}/C^*) + \omega_{M(\kappa)}(\gamma^{1/2}/(B_2 C^*))} \\ &\leq C_1^* B B_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} \left(1 + \left| \frac{\gamma^{1/2}}{B_2 C^*} \right| \right)^{-2(d+1)}. \end{aligned} \tag{5.4}$$

Since here $|\gamma^{1/2}|$ denotes the Euclidean norm of the multi-index $\gamma^{1/2}$, we have

$$|\gamma^{1/2}|^{2(d+1)} = (\gamma_1 + \dots + \gamma_d)^{d+1} \geq |\gamma|^{d+1}. \tag{5.5}$$

Hence,

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}_0^d} |c_\gamma| \cdot \|x^\alpha \partial^\beta H_\gamma\|_2 &\leq C_1^* B B_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} \sum_{\gamma \in \mathbb{N}_0^d} \frac{1}{\left(1 + \left| \frac{\gamma^{1/2}}{B_2 C^*} \right| \right)^{2(d+1)}} \\ &\leq C_1^* B B_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} \sum_{\gamma \in \mathbb{N}_0^d} \frac{1}{1 + \left| \frac{\gamma^{1/2}}{B_2 C^*} \right|^{2(d+1)}} \\ &= C_1^* B B_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\kappa')} \sum_{\gamma \in \mathbb{N}_0^d} \frac{(B_2 C^*)^{2(d+1)}}{(B_2 C^*)^{2(d+1)} + |\gamma|^{d+1}}. \end{aligned}$$

Hence,

$$\left\| \sum_{\gamma \in \mathbb{N}_0^d} c_\gamma H_\gamma \right\|_{2, M(\kappa'), 2HC} \leq C_1^* B B_1 \tilde{C} = B B_1 \tilde{C} \|c\|_{M(\lambda), C^*}, \tag{5.6}$$

for $\tilde{C} = \sum_{\gamma \in \mathbb{N}_0^d} (B_2 C^*)^{2(d+1)} / ((B_2 C^*)^{2(d+1)} + |\gamma|^{d+1}) < +\infty$. This shows that T^{-1} is continuous and, moreover, that $(H_\gamma)_\gamma$ is an absolute Schauder basis in $\mathcal{S}_{\{\mathcal{M}\}}$.

Let now $f \in \mathcal{S}_{\{\mathcal{M}\}}$ and $\lambda, C > 0$ be given. We consider $0 < \kappa \leq \lambda, H, B > 0$ as in (3.3) (with κ and H depending only on λ) and we set

$$C_1 := \|f\|_{2, M(\kappa), C} < +\infty.$$

By Lemma 8, we have

$$\|A_+^\alpha f\|_2 \leq C_1 B e^{d/2} (9\sqrt{2}HC)^{|\alpha|} M_\alpha^{(\lambda)}, \quad \alpha \in \mathbb{N}_0^d.$$

Hence, proceeding as in the Roumieu case, we deduce that, for all $\lambda, C > 0$, there exist $0 < \kappa \leq \lambda$ and $B, H > 0$ such that

$$\|(\xi_\gamma(f))_\gamma\|_{\mathbf{M}^{(\lambda),9\sqrt{2}HC}} \leq B e^{d/2} \|f\|_{2,\mathbf{M}^{(\kappa),C}}. \tag{5.7}$$

This shows that T is continuous in the Beurling case.

Now, if $\mathbf{c} = (c_\gamma)_{\gamma \in \mathbb{N}_0^d} \in \Lambda_{(\mathcal{M})}$, then by (3.3) and Lemma 10, for all $\lambda, C > 0$ there exist $0 < \kappa \leq \lambda$, and $H, B > 0$ (with κ and H depending only on λ) such that

$$\|x^\alpha \partial^\beta H_\gamma\|_2 \leq (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)} B e^{\omega_{\mathbf{M}^{(\kappa)}}(\gamma^{1/2}/C)}.$$

By Lemma 4, there exist $0 < \kappa' \leq \kappa$ and $B_1, B_2 \geq 1$ such that

$$e^{-\omega_{\mathbf{M}^{(\kappa')}}(B_2 t) + \omega_{\mathbf{M}^{(\kappa)}}(t)} \leq B_1 (1 + |t|)^{-2(d+1)}, \quad t \in \mathbb{R}^d.$$

Since $\mathbf{c} \in \Lambda_{(\mathcal{M})}$, we have

$$\sup_{\gamma \in \mathbb{N}_0^d} |c_\gamma| e^{\omega_{\mathbf{M}^{(\kappa')}}(B_2 \gamma^{1/2}/C)} = \|\mathbf{c}\|_{\mathbf{M}^{(\kappa'),C/B_2}} =: C_1 < +\infty.$$

Therefore, arguing as in the Roumieu case,

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}_0^d} |c_\gamma| \cdot \|x^\alpha \partial^\beta H_\gamma\|_2 &\leq C_1 B (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)} \\ &\cdot \sum_{\gamma \in \mathbb{N}_0^d} e^{-\omega_{\mathbf{M}^{(\kappa')}}(B_2 \gamma^{1/2}/C) + \omega_{\mathbf{M}^{(\kappa)}}(\gamma^{1/2}/C)} \\ &\leq C_1 B B_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)} \sum_{\gamma \in \mathbb{N}_0^d} \frac{1}{(1 + |\gamma^{1/2}/C|)^{2(d+1)}} \\ &\leq \tilde{B} C_1 (2HC)^{|\alpha+\beta|} M_{\alpha+\beta}^{(\lambda)}, \end{aligned}$$

for $\tilde{B} = B B_1 \sum_{\gamma \in \mathbb{N}_0^d} C^{2(d+1)} / (C^{2(d+1)} + |\gamma|^{d+1}) < +\infty$. For all $\lambda, h > 0$, there exist then $\kappa' \leq \lambda$ and $\tilde{h} = h / (2HB_2) > 0$ such that

$$\left\| \sum_{\gamma \in \mathbb{N}_0^d} c_\gamma H_\gamma \right\|_{2,\mathbf{M}^{(\lambda),h}} \leq \tilde{B} \|\mathbf{c}\|_{\mathbf{M}^{(\kappa'),\tilde{h}}}. \tag{5.8}$$

This shows that T^{-1} is continuous on $\mathcal{S}_{(\mathcal{M})}$ and that $(H_\gamma)_\gamma$ is an absolute Schauder basis in $\mathcal{S}_{(\mathcal{M})}$, which finishes the proof. \square

As in [28, Corollary 3.6], we also have that the Fourier transform is well adapted to our spaces and it is an isomorphism:

Corollary 1 *Let \mathcal{M} be a weight matrix satisfying (3.2) and (3.6) ((3.3) and (3.7)). Then the Fourier transform is an isomorphism in $\mathcal{S}_{\{\mathcal{M}\}}$ ($\mathcal{S}_{(\mathcal{M})}$).*

Now, we prove that the spaces of sequences are nuclear.

Theorem 2 Let $\mathcal{M} = (M_\alpha^{(\lambda)})_{\lambda>0, \alpha \in \mathbb{N}_0^d}$ be a weight matrix satisfying (3.7). Then $\Lambda_{(\mathcal{M})}$ is nuclear.

Proof By (5.2) and [32, Prop. 28.16] (see also [10, Theorem 3.1] for a self-contained proof in the case of countable lattices), the sequence space $\Lambda_{(\mathcal{M})}$ is nuclear if and only if

$$\forall j \in \mathbb{N} \exists \ell \in \mathbb{N}, \ell > j : \sum_{\gamma \in \mathbb{N}_0^d} e^{\omega_{\mathbf{M}(1/j)}(j\gamma^{1/2}) - \omega_{\mathbf{M}(1/\ell)}(\ell\gamma^{1/2})} < +\infty. \tag{5.9}$$

Moreover, by Lemma 5, condition (3.14) is satisfied. We can thus proceed as in the proof of Theorem 1 of [9] to prove that (3.14) implies that the series in (5.9) converges, and hence $\Lambda_{(\mathcal{M})}$ is nuclear. To this aim, we fix an index $\lambda > 0$ and $N \in \mathbb{N}$ with $N > 2d$ and remark that if the inequality (3.14) holds for $\lambda = 1/j$ and $\kappa \leq \lambda$, then it holds also if, instead of κ , we put $\kappa' = 1/h$ with $h \in \mathbb{N}, h > [\frac{1}{\kappa}] + 1$, since $\mathbf{M}^{(\kappa')} \leq \mathbf{M}^{(\kappa)}$ for $\kappa' \leq \kappa$ and hence $\omega_{\mathbf{M}^{(\kappa)}} \leq \omega_{\mathbf{M}^{(\kappa)'}}$. Then for $\ell \geq Ah$ (so that $\ell \geq Aj$ and $\ell \geq h > j$ and note that the constant A is also depending on the chosen N):

$$\begin{aligned} \sum_{\gamma \in \mathbb{N}_0^d} e^{\omega_{\mathbf{M}(1/j)}(j\gamma^{1/2}) - \omega_{\mathbf{M}(1/\ell)}(\ell\gamma^{1/2})} &\leq \sum_{\gamma \in \mathbb{N}_0^d \setminus \{0\}} e^{\omega_{\mathbf{M}(1/j)}(j\gamma^{1/2}) - \omega_{\mathbf{M}(1/h)}(Aj\gamma^{1/2})} + 1 \\ &\leq \sum_{\gamma \in \mathbb{N}_0^d \setminus \{0\}} e^{-N \log |\gamma|^{1/2} + B} + 1 = e^B j^{-N} \sum_{\gamma \in \mathbb{N}_0^d \setminus \{0\}} \frac{1}{|\gamma|^{N/2}} + 1 < +\infty, \end{aligned}$$

by our choice of $N > 2d$. □

Concerning the Roumieu case, we have the following result.

Theorem 3 Let $\mathcal{M} = (M_\alpha^{(\lambda)})_{\lambda>0, \alpha \in \mathbb{N}_0^d}$ be a weight matrix satisfying (3.6). Then $\Lambda_{\{\mathcal{M}\}}$ is nuclear.

Proof For

$$a_{\alpha j} := e^{-\omega_{\mathbf{M}^{(j)}}(\alpha^{1/2}/j)},$$

we consider the matrices

$$A := (a_{\alpha j})_{\alpha \in \mathbb{N}_0^d, j \in \mathbb{N}}, \quad V := (v_{\alpha j})_{\alpha \in \mathbb{N}_0^d, j \in \mathbb{N}} \text{ with } v_{\alpha j} = a_{\alpha j}^{-1}.$$

We observe that A is a Köthe matrix since its entries are strictly positive and $a_{\alpha j} \leq a_{\alpha, j+1}$ for every $j \in \mathbb{N}$. We consider now the space

$$\lambda_{(\mathcal{M})} := \{ \mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall j \in \mathbb{N}, \sum_{\alpha \in \mathbb{N}_0^d} |c_\alpha| a_{\alpha j} < \infty \}.$$

Since $\mathbb{N}_0^d = \cup_{m \in \mathbb{N}} I_m$ with $I_m = \{ \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \}$ and $v_{\alpha j} > 0$ for every α and j , we have that the matrix V satisfies the condition (D) of [5] (see also [4]). From [4,

Theorem 18(1)], we have that $\lambda_{(\mathcal{M})}$ is distinguished, and then, from [4, Corollary 8(f)] and (5.1) we get

$$(\lambda_{(\mathcal{M})})'_b = \Lambda_{\{\mathcal{M}\}}.$$

Since a Fréchet space is nuclear if and only if its dual is nuclear [34, pg. 78], it is enough to prove that $\lambda_{(\mathcal{M})}$ is nuclear; from [10, Theorem 3.1] this is true if and only if

$$\forall k \in \mathbb{N} \exists m \in \mathbb{N}, m > k : \sum_{\gamma \in \mathbb{N}_0^d} e^{\omega_{\mathbf{M}^{(m)}}(\gamma^{1/2}/m) - \omega_{\mathbf{M}^{(k)}}(\gamma^{1/2}/k)} < +\infty. \tag{5.10}$$

By Lemma 5, we can now use (3.15) with $\lambda = k$ and with a fixed $N > 2d$; since $\omega_{\mathbf{M}^{(m)}}(t) \leq \omega_{\mathbf{M}^{(k)}}(t)$ for every $m \geq k$, we can replace in (3.15) κ by $m = \max\{\kappa, Ak\}$, obtaining that for every $k \in \mathbb{N}$ there exists $m \geq k$ such that

$$\omega_{\mathbf{M}^{(m)}}\left(\frac{\gamma^{1/2}}{m}\right) + N \log \left| \frac{\gamma^{1/2}}{m} \right| \leq \omega_{\mathbf{M}^{(k)}}\left(A \frac{\gamma^{1/2}}{m}\right) + B,$$

for every $\gamma \neq 0$. Since $A \leq m/k$ we obtain

$$e^{\omega_{\mathbf{M}^{(m)}}(\gamma^{1/2}/m) - \omega_{\mathbf{M}^{(k)}}(\gamma^{1/2}/k)} \leq e^B m^N \frac{1}{|\gamma^{1/2}|^N} \leq e^B m^N \frac{1}{|\gamma|^{N/2}},$$

for $\gamma \neq 0$; since $N > 2d$ we have that (5.10) holds, and estimating as in Theorem 2, we get the conclusion. □

Corollary 2 *If $\mathcal{M} = (M_\alpha^{(\lambda)})_{\lambda>0, \alpha \in \mathbb{N}_0^d}$ is a weight matrix satisfying (3.3) and (3.7), then the space $\mathcal{S}_{(\mathcal{M})}$ is nuclear. If \mathcal{M} satisfies (3.2) and (3.6), then the space $\mathcal{S}_{\{\mathcal{M}\}}$ is nuclear.*

Proof The Beurling case follows from Theorems 1 and 2, and the Roumieu case follows from Theorems 1 and 3. □

Proposition 4 *Let $\mathcal{M} = (M_p^{(\lambda)})_{\lambda>0, p \in \mathbb{N}_0}$ be a weight matrix (with $d = 1$), such that each sequence $\mathbf{M}^{(\lambda)}$ satisfies (2.2) and $\lim_{p \rightarrow \infty} (M_p)^{1/p} = +\infty$. Assume, moreover, that*

$$\mu_p^{(\lambda)} := \frac{M_p^{(\lambda)}}{M_{p-1}^{(\lambda)}}, \quad p \in \mathbb{N},$$

satisfies $\mu^{(\lambda)} \leq \mu^{(\kappa)}$ for all $0 < \lambda \leq \kappa$ and $\mu_0^{(\lambda)} = 1$ for all $\lambda > 0$. Then the following conditions are equivalent:

- (a) $\forall j \in \mathbb{N} \exists \ell \in \mathbb{N}, \ell > j : \sum_{k=1}^{+\infty} e^{\omega_{\mathbf{M}^{(1/\ell)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2})} < +\infty;$
- (b) $\forall \lambda > 0 \exists 0 < \kappa < \lambda, A \geq 1 \forall p \in \mathbb{N} : M_{p+1}^{(\kappa)} \leq A^{p+1} M_p^{(\lambda)}.$

Proof If condition (b) is satisfied, then (3.7) is satisfied and hence also condition (a), as we already saw in the proof of Theorem 2.

Let us now assume condition (a) and prove (b). To this aim let us first remark that

$$k \mapsto \omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) \tag{5.11}$$

is decreasing. Indeed,

$$\begin{aligned} \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) - \omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) &= (\omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(jk^{1/2})) \\ &\quad + (\omega_{\mathbf{M}^{(1/\ell)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/j)}}(jk^{1/2})) \\ &=: \omega_1 + \omega_2. \end{aligned}$$

The first difference $\omega_1 = \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(jk^{1/2})$ is increasing since by definition $t \mapsto \omega_{\mathbf{M}^{(1/\ell)}}(e^t)$ is convex (see the proof of Theorem 1 in [9] for the implication that the convexity implies that ω_1 is increasing).

To prove that also the second difference ω_2 is increasing, we set

$$\Sigma_{\mathbf{M}^{(a)}}(t) := \#\{p \in \mathbb{N} : \mu_p^{(a)} \leq t\}$$

and remark that, by (2.2) (see [26, formula (3.11)]),

$$\omega_{\mathbf{M}^{(a)}}(t) = \int_0^t \frac{\Sigma_{\mathbf{M}^{(a)}}(s)}{s} ds.$$

Then

$$\omega_{\mathbf{M}^{(1/\ell)}}(t) - \omega_{\mathbf{M}^{(1/j)}}(t) = \int_0^t \frac{\Sigma_{\mathbf{M}^{(1/\ell)}}(s) - \Sigma_{\mathbf{M}^{(1/j)}}(s)}{s} ds$$

is an increasing function of t since

$$\Sigma_{\mathbf{M}^{(1/\ell)}}(s) \geq \Sigma_{\mathbf{M}^{(1/j)}}(s), \quad \ell > j,$$

by the assumption $\mu_p^{(1/\ell)} \leq \mu_p^{(1/j)}$ for $\ell > j$.

Therefore, ω_1 and ω_2 are increasing and we have thus proved that (5.11) is decreasing. This condition together with assumption (a) implies that

$$\lim_{k \rightarrow +\infty} k e^{\omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2})} = 0.$$

There exists then $A \geq 1$ such that

$$\sup_{k \in \mathbb{N}} k e^{\omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2})} \leq A$$

and hence, for all $k \in \mathbb{N}$,

$$\omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) - \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) \leq -\log k + \log A \leq -\log(jk^{1/2}) + \log(jA).$$

Choosing, for every $t \geq 1$, the smallest $k \in \mathbb{N}$ such that $jk^{1/2} \in [t, (j + 1)t]$, we finally have

$$\begin{aligned}
 \omega_{\mathbf{M}^{(1/j)}}(t) + \log t &\leq \omega_{\mathbf{M}^{(1/j)}}(jk^{1/2}) + \log(jk^{1/2}) \\
 &\leq \omega_{\mathbf{M}^{(1/\ell)}}(\ell k^{1/2}) + \log(jA) \\
 &\leq \omega_{\mathbf{M}^{(1/\ell)}}\left(\frac{\ell}{j}(j+1)t\right) + \log(jA).
 \end{aligned}
 \tag{5.12}$$

Since (5.12) is trivial for $0 < t \leq 1$, we have proved that condition (ii) of Lemma 2 is satisfied for $\mathbf{N} = \mathbf{M}^{(1/j)}$ and $\mathbf{M} = \mathbf{M}^{(1/\ell)}$ and hence, from (i) of Lemma 2, there exists $\tilde{A} \geq 1$ such that

$$M_{p+1}^{(1/\ell)} \leq \tilde{A}^{p+1} M_p^{(1/j)}, \quad \forall p \in \mathbb{N}_0.$$

Then, for all $\lambda > 0$, choosing $j \in \mathbb{N}$ so that $\frac{1}{j} \leq \lambda$, there exists $\kappa = \frac{1}{\ell} < \frac{1}{j} \leq \lambda$ such that condition (b) holds. □

Proposition 4 yields now the following result.

Theorem 4 *Let $\mathcal{M} = (M_p^{(\lambda)})_{\lambda>0, p \in \mathbb{N}_0}$ be a weight matrix as in Proposition 4. Then the space $\Lambda_{\{\mathcal{M}\}}$ is nuclear if and only if condition (3.7) is satisfied.*

Proof It follows from Theorem 2 and, in particular, (5.9). □

Theorem 5 *Let $\mathcal{M} = (M_p^{(\lambda)})_{\lambda>0, p \in \mathbb{N}_0}$ be a weight matrix as in Proposition 4. Then the space $\Lambda_{\{\mathcal{M}\}}$ is nuclear if and only if condition (3.6) is satisfied.*

Proof By the proof of Theorem 3 we have that $\Lambda_{\{\mathcal{M}\}}$ is nuclear if and only if (5.10) is satisfied, and this is equivalent to (3.6) since, analogously as in Proposition 4, the following two conditions are equivalent:

- (a)' $\forall j \in \mathbb{N} \exists \ell \in \mathbb{N}, \ell > j : \sum_{k=1}^{+\infty} e^{\omega_{\mathbf{M}^{(\ell)}}(k^{1/2}/\ell) - \omega_{\mathbf{M}^{(j)}}(k^{1/2}/j)} < +\infty,$
- (b)' $\forall \lambda > 0 \exists \kappa > \lambda, A \geq 1 \forall p \in \mathbb{N} : M_{p+1}^{(\lambda)} \leq A^{p+1} M_p^{(\kappa)}.$

Indeed, (b)' implies (3.6) and hence (a)', i.e. (5.10), in the one-dimensional case, by the proof of Theorem 3.

Conversely, if (a)' holds then for every fixed $j \in \mathbb{N}$, and $\ell > j$ as in (a)', there exists $A > \ell$ such that

$$\sup_{k \in \mathbb{N}} k e^{\omega_{\mathbf{M}^{(\ell)}}(k^{1/2}/\ell) - \omega_{\mathbf{M}^{(j)}}(k^{1/2}/j)} \leq A$$

since

$$k \mapsto \omega_{\mathbf{M}^{(\ell)}}(k^{1/2}/\ell) - \omega_{\mathbf{M}^{(j)}}(k^{1/2}/j)$$

is decreasing, similarly as in the proof of Proposition 4. Then, for all $k \in \mathbb{N}$,

$$\omega_{\mathbf{M}^{(\ell)}}(k^{1/2}/\ell) - \omega_{\mathbf{M}^{(j)}}(k^{1/2}/j) \leq -\log k + \log A \leq -\log(k^{1/2}/\ell) + \log(A/\ell).$$

If $t \geq 1$ we can choose a smallest $k \in \mathbb{N}$ such that $k^{1/2}/\ell \in [t, (1 + \frac{1}{\ell})t]$ and obtain that

$$\begin{aligned} \omega_{\mathbf{M}^{(\ell)}}(t) + \log t &\leq \omega_{\mathbf{M}^{(\ell)}}(k^{1/2}/\ell) + \log(k^{1/2}/\ell) \\ &\leq \omega_{\mathbf{M}^{(j)}}(k^{1/2}/j) + \log(A/\ell) \\ &\leq \omega_{\mathbf{M}^{(j)}}\left(\frac{\ell}{j}\left(1 + \frac{1}{\ell}\right)t\right) + \log(A/\ell). \end{aligned} \tag{5.13}$$

Since (5.13) is trivial for $0 < t \leq 1$, we have that

$$\omega_{\mathbf{M}^{(\ell)}}(t) + \log t \leq \omega_{\mathbf{M}^{(j)}}(At) + B, \quad \forall t > 0,$$

for $A = \frac{\ell}{j}\left(1 + \frac{1}{\ell}\right) \geq 1$ and $B = \log(A/\ell) > 0$. By Lemma 2 with $\mathbf{M} = \mathbf{M}^{(j)}$ and $\mathbf{N} = \mathbf{M}^{(\ell)}$, for every $\lambda > 0$ we can choose $j \in \mathbb{N}$, $j \geq \lambda$ so that $(b)'$ is satisfied for $\kappa = \ell > j \geq \lambda$. □

6 Rapidly decreasing ultradifferentiable functions

We shall now consider weight functions ω defined as below:

Definition 1 A *weight function* is a continuous increasing function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ such that

- (α) $\exists L \geq 1 \forall t \geq 0 : \omega(2t) \leq L(\omega(t) + 1)$;
- (β) $\omega(t) = O(t^2)$ as $t \rightarrow +\infty$;
- (γ) $\log t = o(\omega(t))$ as $t \rightarrow +\infty$;
- (δ) $\varphi_\omega(t) := \omega(e^t)$ is convex on $[0, +\infty)$.

Then we define $\omega(t) := \omega(|t|)$ if $t \in \mathbb{R}^d$.

It is not restrictive to assume $\omega|_{[0,1]} \equiv 0$. As usual, we define the Young conjugate φ_ω^* of φ_ω by

$$\varphi_\omega^*(s) := \sup_{t \geq 0} \{ts - \varphi_\omega(t)\},$$

which is an increasing convex function such that $\varphi_\omega^{**} = \varphi_\omega$ and $\varphi^*(s)/s$ is increasing [12, 24]. We remark that condition (β) and the stronger condition $\omega(t) = o(t^2)$ as t tends to infinity are needed in the Roumieu and Beurling cases for Corollary 5 and Theorem 6. On the other hand, condition (γ) guarantees that φ_ω^* is finite, so that, from the properties of φ_ω^* (see [12] or [8, Lemma A.1]) we easily obtain (cf. [37]):

Lemma 11 Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a weight function as in Definition 1, and set

$$W_\alpha^{(\lambda)} := e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha|)}, \quad \lambda > 0, \alpha \in \mathbb{N}_0^d. \tag{6.1}$$

Then $W_\alpha^{(\lambda)} \in \mathbb{R}$ and the weight matrix

$$\mathcal{M}_\omega := (\mathbf{W}^{(\lambda)})_{\lambda>0} = (W_\alpha^{(\lambda)})_{\lambda>0, \alpha \in \mathbb{N}_0^d} \tag{6.2}$$

satisfies the following properties:

- (i) $W_0^{(\lambda)} = 1, \lambda > 0$;
- (ii) $(W_\alpha^{(\lambda)})^2 \leq W_{\alpha-e_i}^{(\lambda)} W_{\alpha+e_i}^{(\lambda)}, \lambda > 0, \alpha \in \mathbb{N}_0^d$ with $\alpha_i \neq 0$, and $i = 1, \dots, d$;
- (iii) $\mathbf{W}^{(\kappa)} \leq \mathbf{W}^{(\lambda)}, 0 < \kappa \leq \lambda$;
- (iv) $W_{\alpha+\beta}^{(\lambda)} \leq W_\alpha^{(2\lambda)} W_\beta^{(2\lambda)}, \lambda > 0, \alpha, \beta \in \mathbb{N}_0^d$;
- (v) $\forall h > 0 \exists A \geq 1 \forall \lambda > 0 \exists D \geq 1 \forall \alpha \in \mathbb{N}_0^d : h^{|\alpha|} W_\alpha^{(\lambda)} \leq D W_\alpha^{(A\lambda)}$;
- (vi) Both conditions (3.6) and (3.7) are valid;
- (vii) Conditions (3.4) and (3.5) are satisfied for $\kappa = \lambda$ and $A = 1$.

Proof Let us first remark that condition (γ) of Definition 1 ensures that $W_\alpha^{(\lambda)} \in \mathbb{R}$ for all $\lambda > 0$ and $\alpha \in \mathbb{N}_0^d$. Condition (i) is trivial since $\varphi_\omega^*(0) = 0$. Condition (ii) follows from the convexity of φ_ω^* :

$$e^{\frac{2}{\lambda} \varphi_\omega^*(\lambda|\alpha|)} = e^{\frac{2}{\lambda} \varphi_\omega^*\left(\frac{\lambda(|\alpha|-1)+\lambda(|\alpha|+1)}{2}\right)} \leq e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha-e_i|)} e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha+e_i|)}.$$

The monotonicity property (iii) is clear since $\varphi_\omega^*(s)/s$ is increasing. Properties (iv), (v) and (vii) follow from [8, Lemma A.1]. Indeed, from [8, Lemma A.1(ix)]

$$e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha+\beta|)} \leq e^{\frac{1}{2\lambda} \varphi_\omega^*(2\lambda|\alpha|)} + e^{\frac{1}{2\lambda} \varphi_\omega^*(2\lambda|\beta|)}.$$

From [8, Lemma A.1(iv)] with $A = L^2 + L$ and $B = L^2$, where L is the constant of condition (α) of Definition 1,

$$h^{|\alpha|} e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha|)} \leq \Lambda_{h,\lambda} e^{\frac{1}{\lambda'} \varphi_\omega^*(\lambda'|\alpha|)}$$

for all $\lambda' \geq \lambda B^{\lceil \log h + 1 \rceil}$ and $\Lambda_{h,\lambda} := e^{\frac{1}{\lambda} \left(1 + \frac{1}{L}\right) \lceil \log h + 1 \rceil}$. From [8, Lemma A.1(ii)]

$$e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha|)} + e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\beta|)} \leq e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha+\beta|)}.$$

Finally, (vi) is an immediate consequence of (iv). □

Let us now define the spaces of rapidly decreasing ω -ultradifferentiable functions, in the Roumieu case

$$\begin{aligned} \mathcal{S}_{\{\omega\}}(\mathbb{R}^d) &:= \{f \in C^\infty(\mathbb{R}^d) : \exists \lambda > 0, C > 0 \text{ s.t.} \\ &\quad \sup_{\alpha, \beta \in \mathbb{N}_0^d} \|x^\alpha \partial^\beta f\|_\infty e^{-\frac{1}{\lambda} \varphi_\omega^*(\lambda|\alpha+\beta|)} \leq C\} \\ &= \{f \in C^\infty(\mathbb{R}^d) : \exists \lambda > 0, C > 0 \text{ s.t.} \\ &\quad \|f\|_{\infty, \mathbf{W}^{(\lambda)}} := \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_\infty}{W_{\alpha+\beta}^{(\lambda)}} \leq C\}, \end{aligned}$$

and in the Beurling case

$$\mathcal{S}_{(\omega)}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) : \forall \lambda > 0 \exists C_\lambda > 0 : \|f\|_{\infty, \mathbf{W}^{(\lambda)}} \leq C_\lambda\}.$$

From Lemma 11(iv) and (vii) (see also [6, Thm. 4.8]):

$$\mathcal{S}_{\{\omega\}}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \exists \lambda > 0, C > 0 : \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_\infty}{W_\alpha^{(\lambda)} W_\beta^{(\lambda)}} \leq C\}$$

and

$$\mathcal{S}_{(\omega)}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \forall \lambda > 0 \exists C_\lambda > 0 : \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{\|x^\alpha \partial^\beta f\|_\infty}{W_\alpha^{(\lambda)} W_\beta^{(\lambda)}} \leq C_\lambda\}.$$

We refer to [6, 8, 22] for more equivalent seminorms on $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$, if $\omega(t) = o(t^2)$.

We can also insert $h^{|\alpha+\beta|}$ at the denominator (for some $h > 0$ in the Roumieu case and for all $h > 0$ in the Beurling case) by Lemma 11(v). In particular, we have the following

Proposition 5 *Let ω be a weight function and \mathcal{M}_ω the weight matrix defined in (6.1), (6.2). We have $\mathcal{S}_{\{\mathcal{M}_\omega\}}(\mathbb{R}^d) = \mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ and $\mathcal{S}_{(\mathcal{M}_\omega)}(\mathbb{R}^d) = \mathcal{S}_{(\omega)}(\mathbb{R}^d)$ and the equalities are also topological.*

Remark 4 We observe that for the weight function $\omega(t) = \log^s(1+t)$, for some $s > 1$, we have that $\mathcal{S}_{(\omega)}(\mathbb{R})$ never equals $\mathcal{S}_{(M_p)}(\mathbb{R})$ for any sequence $(M_p)_{p \in \mathbb{N}_0}$. Hence, $\mathcal{S}_{(\omega)}(\mathbb{R})$ cannot be defined with sequences as in [28] when (M_p) satisfies (M0), (M1) and (M2)' (see [11] for the definition of (M0); (M1) and (M2)' are recalled in (2.2) and (2.4)).

Indeed, by [11, Example 20], $\mathcal{E}_{(\omega)}(\mathbb{R}) \neq \mathcal{E}_{(M_p)}(\mathbb{R})$ for any sequence (M_p) as considered just above, where $\mathcal{E}_{(\omega)}(\mathbb{R})$ and $\mathcal{E}_{(M_p)}(\mathbb{R})$ are the spaces of ultradifferentiable functions defined by weights and sequences (for the definitions see [11]). We fix a sequence (M_p) and prove that $\mathcal{S}_{(\omega)}(\mathbb{R}) \neq \mathcal{S}_{(M_p)}(\mathbb{R})$. Clearly, we can assume that (M_p) is non-quasianalytic since the weight ω is non-quasianalytic. In particular, (M_p) satisfies (M0) (see [11], condition (M3)', and use also (M1)). If $f \in \mathcal{E}_{(M_p)}(\mathbb{R}) \setminus \mathcal{E}_{(\omega)}(\mathbb{R})$, then there are a compact set $K \subseteq \mathbb{R}$ and $m \in \mathbb{N}$ such that

$$\sup_{j \in \mathbb{N}_0} \sup_{x \in K} |f^{(j)}(x)| e^{-m\varphi_\omega^*\left(\frac{j}{m}\right)} = +\infty.$$

Hence,

$$\forall n \in \mathbb{N} \exists x_n \in K, j_n \in \mathbb{N} \text{ such that } |f^{(j_n)}(x_n)| \geq n e^{m\varphi_\omega^*\left(\frac{j_n}{m}\right)}.$$

Since K is compact we can assume that the sequence (x_n) converges to some $x_0 \in K$. Let $\varphi \in \mathcal{D}_{(M_p)}(\mathbb{R})$ (the space of functions in $\mathcal{E}_{(M_p)}(\mathbb{R})$ with compact support) with $\varphi \equiv 1$ in a neighbourhood of x_0 . Then $g = f\varphi \in \mathcal{D}_{(M_p)}(\mathbb{R}) \subseteq \mathcal{S}_{(M_p)}(\mathbb{R})$ but, for n sufficiently large,

$$\frac{|g^{(j_n)}(x_n)|}{e^{m\varphi_\omega^*\left(\frac{j_n}{m}\right)}} = \frac{|f^{(j_n)}(x_n)|}{e^{m\varphi_\omega^*\left(\frac{j_n}{m}\right)}} \geq n \longrightarrow +\infty,$$

and hence $g \notin \mathcal{S}_{(\omega)}(\mathbb{R})$ (see the definition of $\mathcal{S}_{(\omega)}(\mathbb{R})$ above).

Analogously, for $f \in \mathcal{E}_{(\omega)}(\mathbb{R}) \setminus \mathcal{E}_{(M_p)}(\mathbb{R})$ we can construct $g \in \mathcal{S}_{(\omega)}(\mathbb{R}) \setminus \mathcal{S}_{(M_p)}(\mathbb{R})$.

The same arguments are valid for the Roumieu case and for dimension bigger than one (considering always isotropic classes).

The following Lemma was proved in dimension 1 in [25, Lemma 2.5]; here we give a version of it in dimension d .

Lemma 12 *Let ω be a weight function. Then there exists a constant $B > 0$ and, for every $\lambda > 0$, there exists $C_\lambda > 0$, such that*

$$\lambda \omega_{\mathbf{W}^{(\lambda)}}(t) \leq \omega(t) \leq B \lambda \omega_{\mathbf{W}^{(\lambda)}}(t) + C_\lambda, \quad t \in \mathbb{R}^d. \tag{6.3}$$

Proof For $t = 0$ the thesis is trivial, so we can consider $t \neq 0$. Since $|t^\alpha| \leq |t|^{|\alpha|}$ for every multi-index α , we have

$$\begin{aligned} \lambda \omega_{\mathbf{W}^{(\lambda)}}(t) &= \lambda \sup_{\alpha \in \mathbb{N}_{0,r}^d} \log \frac{|t^\alpha|}{e^{\varphi_\omega^*(\lambda|\alpha|)/\lambda}} \leq \sup_{\alpha \in \mathbb{N}_{0,r}^d} \{ \lambda |\alpha| \log |t| - \varphi_\omega^*(\lambda|\alpha|) \} \\ &\leq \varphi_\omega^{**}(\log |t|) = \omega(t), \end{aligned}$$

so the first inequality of (6.3) is proved. Now, similar to [37, proof of Lemma 5.7], we can prove that, for every $t \in \mathbb{R}^d$ such that $|t| \geq e^{\varphi_\omega^*(\lambda)/\lambda}$,

$$\omega(t) \leq 2 \sup_{M \in \mathbb{N}_0} \{ \lambda M \log |t| - \varphi_\omega^*(\lambda M) \}. \tag{6.4}$$

Observe now that for every $t \in \mathbb{R}^d$, we have $|t| \leq \sqrt{d}|t|_\infty \leq d|t|_\infty$. Then by [8, Remark 2.2(iii)],

$$\omega(t) \leq \omega(d|t|_\infty) \leq D_d(\omega(|t|_\infty) + 1), \tag{6.5}$$

for $D_d = L + L^2 + \dots + L^{d-1}$, where L is the constant of condition (α) in Definition 1.

Fix now $t \in \mathbb{R}^d$ with $|t| \geq e^{\varphi_\omega^*(\lambda)/\lambda}$ and let j_0 be such that $|t|_\infty = |t_{j_0}|$; for every $M \in \mathbb{N}_0$, we then write $\alpha_M := Me_{j_0}$. We then have $|t|_\infty^M = |t^{\alpha_M}|$, and so by (6.4) we obtain

$$\begin{aligned} \omega(|t|_\infty) &= \omega(|t_{j_0}|) \leq 2\lambda \sup_{M \in \mathbb{N}_0} \log \frac{|t^{\alpha_M}|}{e^{\varphi_\omega^*(\lambda|\alpha_M)/\lambda}} \\ &\leq 2\lambda \sup_{\alpha \in \mathbb{N}_{0,r}^d} \log \frac{|t^\alpha|}{e^{\varphi_\omega^*(\lambda|\alpha)/\lambda}} = 2\lambda \omega_{\mathbf{W}(\lambda)}(t), \end{aligned}$$

since $\alpha_M \in \mathbb{N}_{0,r}^d$ due to the fact that $t_{j_0} \neq 0$ (we are in fact considering $t \in \mathbb{R}^d$ such that $|t| \geq e^{\varphi_\omega^*(\lambda)/\lambda}$). By (6.5) we then obtain

$$\omega(t) \leq 2\lambda D_d \omega_{\mathbf{W}(\lambda)}(t) + D_d$$

for $|t| \geq e^{\varphi_\omega^*(\lambda)/\lambda}$. Then the second inequality of (6.3) holds for

$$B = 2D_d \quad \text{and} \quad C_\lambda = D_d + \sup_{|t| \geq e^{\varphi_\omega^*(\lambda)/\lambda}} \omega(t).$$

□

Lemma 13 *Let ω be a weight function and consider the weight matrix \mathcal{M}_ω as defined in (6.1), (6.2). Then for $r > 0$:*

(a) $\omega(t) = O(t^{1/r})$ as $t \rightarrow +\infty$ if and only if

$$\forall \lambda > 0 \exists C, D \geq 1 \forall \alpha \in \mathbb{N}^d : \alpha^{r\alpha} \leq CD^{|\alpha|} W_\alpha^{(\lambda)}; \tag{6.6}$$

(b) $\omega(t) = o(t^{1/r})$ as $t \rightarrow +\infty$ if and only if

$$\forall \lambda, D > 0 \exists C \geq 1 \forall \alpha \in \mathbb{N}^d : \alpha^{r\alpha} \leq CD^{|\alpha|} W_\alpha^{(\lambda)}. \tag{6.7}$$

Moreover, in the conditions above we can replace “ $\forall \lambda$ ” by “ $\exists \lambda$ ”.

Proof We only consider the case “ $\forall \lambda$ ” since the proof for the case “ $\exists \lambda$ ” is analogous.

(a): If $\omega(t) = O(t^{1/r})$ as $t \rightarrow +\infty$, there exists $c \geq 1$ such that

$$\omega(t) \leq ct^{1/r} + c, \quad t \geq 0,$$

and hence

$$\varphi_\omega(y) = \omega(e^y) \leq ce^{y/r} + c, \quad y \geq 0.$$

Then

$$\begin{aligned} \varphi_\omega^*(x) &= \sup_{y \geq 0} \{xy - \varphi_\omega(y)\} \geq \sup_{y \geq 0} \{xy - ce^{y/r}\} - c \\ &= xr \left(\log \frac{xr}{c} - 1 \right) - c, \quad \text{if } x \geq \frac{c}{r}. \end{aligned} \tag{6.8}$$

Therefore, for every $\lambda > 0$ and $j \in \mathbb{N}$ with $j \geq \frac{c}{r\lambda}$, choosing $x = \lambda j$ and multiplying by $1/\lambda$ in (6.8), we have

$$\frac{1}{\lambda} \varphi_\omega^*(\lambda j) \geq jr \left(\log \frac{\lambda jr}{c} - 1 \right) - \frac{c}{\lambda} = \log j^{jr} + jr \log \frac{\lambda r}{ec} - \frac{c}{\lambda}$$

and hence, for $j \geq \frac{c}{r\lambda}$,

$$j^{jr} \leq e^{\frac{1}{\lambda} \varphi_\omega^*(\lambda j)} \left(\frac{ec}{\lambda r} \right)^{jr} e^{\frac{c}{\lambda}} \leq \tilde{C}_\lambda D_\lambda^j \tilde{W}_j^{(\lambda)} \tag{6.9}$$

for $\tilde{C}_\lambda = e^{c/\lambda}$, $D_\lambda = \max \left\{ \left(\frac{ec}{\lambda r} \right)^r, 1 \right\}$, and $\tilde{W}_j^{(\lambda)} = e^{\varphi_\omega^*(\lambda j)/\lambda}$. Enlarging the constants $\tilde{C}_\lambda, D_\lambda$ we have (6.9) for all $j \in \mathbb{N}$. Then

$$\alpha^{r\alpha} = \alpha_1^{r\alpha_1} \dots \alpha_d^{r\alpha_d} \leq \tilde{C}_\lambda D_\lambda^{\alpha_1} \tilde{W}_{\alpha_1}^{(\lambda)} \dots \tilde{C}_\lambda D_\lambda^{\alpha_d} \tilde{W}_{\alpha_d}^{(\lambda)},$$

and so we obtain (6.6) for $C = \tilde{C}_\lambda^d$ in view of Lemma 11(vii).

Conversely, if (6.6) holds then, by definition of associated function we obtain, for $z \in \mathbb{R}^d$,

$$\begin{aligned} \omega_{\mathbf{W}^{(\lambda)}}(z) &= \sup_{\alpha \in \mathbb{N}_{0,z}^d} \log \frac{|z^\alpha|}{W_\alpha^{(\lambda)}} \leq \sup_{\alpha \in \mathbb{N}_{0,z}^d} \log |z^\alpha| \frac{CD^{|\alpha|}}{\alpha^{r\alpha}} \\ &\leq \sup_{\alpha \in \mathbb{N}_{0,z}^d} \left(\log C + \sum_{j=1}^d \log \frac{(|z_j|D)^{\alpha_j}}{\alpha_j^{r\alpha_j}} \right). \end{aligned}$$

Consider now j such that $z_j \neq 0$ (otherwise the corresponding addend in the previous sum is 0). A simple computation shows that

$$\sup_{\alpha_j \in \mathbb{N}} \log \frac{(|z_j|D)^{\alpha_j}}{\alpha_j^{r\alpha_j}} \leq \sup_{s>0} \log \frac{(|z_j|D)^s}{s^{rs}} \leq \frac{r}{e} (|z_j|D)^{1/r}.$$

We then have

$$\omega_{\mathbf{W}^{(\lambda)}}(z) \leq \log C + \sum_{j=1}^d \frac{r}{e} (|z_j|D)^{1/r} \leq \log C + \frac{dr}{e} (|z|D)^{1/r}. \tag{6.10}$$

By Lemma 12, we have $\omega(z) = \omega(|z|) = O(|z|^{1/r})$ as $|z| \rightarrow +\infty$ for $z \in \mathbb{R}^d$, which is equivalent to $\omega(t) = O(t^{1/r})$ as $t \rightarrow +\infty$ for $t \in \mathbb{R}$.

(b): If $\omega(t) = o(t^{1/r})$ as $t \rightarrow +\infty$, then for every $D > 0$ there exists $c > 0$ such that

$$\omega(t) \leq Dt^{1/r} + c, \quad t \geq 0.$$

Proceeding as in (a) we have

$$\varphi_{\omega}^*(x) \geq xr \left(\log \frac{xr}{D} - 1 \right) - c, \quad \text{for } x \geq \frac{D}{r},$$

and hence

$$\alpha^{r\alpha} \leq e^{c/\lambda} \left(\frac{eD}{\lambda r} \right)^{r|\alpha|} W_{\alpha}^{(\lambda)}$$

and (6.7) is satisfied by the arbitrariness of $D > 0$.

Conversely, if (6.7) holds then, proceeding as in (a), we have that for every $\lambda, D > 0$ there exists $C > 0$ such that (6.10) is valid and, therefore, by Lemma 12, $\omega(z) = o(|z|^{1/r})$ as $|z| \rightarrow +\infty$ for $z \in \mathbb{R}^d$, or, equivalently, $\omega(t) = o(t^{1/r})$ as $t \rightarrow +\infty$. □

Corollary 3 *Let ω be a weight function. We have*

- (a) *The Hermite functions belong to $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ if and only if $\omega(t) = O(t^2)$ as $t \rightarrow +\infty$.*
- (b) *The Hermite functions belong to $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ if and only if $\omega(t) = o(t^2)$ as $t \rightarrow +\infty$.*

Proof By Lemmas 13 and 11 and Proposition 3, $\omega(t) = O(t^2)$ as $t \rightarrow +\infty$ if and only if \mathcal{M}_{ω} satisfies (3.2) if and only if the space $\mathcal{S}_{\{\mathcal{M}_{\omega}\}}(\mathbb{R}^d)$ contains the Hermite functions; while $\omega(t) = o(t^2)$ as $t \rightarrow +\infty$ if and only if \mathcal{M}_{ω} satisfies (3.3) if and only if $\mathcal{S}_{(\mathcal{M}_{\omega})}(\mathbb{R}^d)$ contains the Hermite functions. □

At this point, some considerations are worthy to be expressed. Among all classes of ultradifferentiable functions defined by global estimates, and in particular classes of rapidly decreasing ultradifferentiable functions, the Gel'fand–Shilov spaces $\mathcal{S}_s(\mathbb{R}^d)$ (Roumieu) and $\Sigma_s(\mathbb{R}^d)$ (Beurling) have been largely investigated. If $s \geq 1/2$, the space $\mathcal{S}_s(\mathbb{R}^d)$ consists of those smooth functions f such that there is $C > 0$ for which for any $\alpha, \beta \in \mathbb{N}_0^d$ and any $x \in \mathbb{R}^d$ we have $|x^{\alpha} \partial^{\beta} f(x)| \leq C^{|\alpha+\beta|+1} |\alpha + \beta|!^s$. While $\Sigma_s(\mathbb{R}^d)$ is the space of all the smooth functions f such that for each $C > 0$ there is $D > 0$ such that for any $\alpha, \beta \in \mathbb{N}_0^d$ and any $x \in \mathbb{R}^d$ we have $|x^{\alpha} \partial^{\beta} f(x)| \leq D C^{|\alpha+\beta|} |\alpha + \beta|!^s$. For $s > 1$, they correspond to the Schwartz class in the context of Gevery classes (Roumieu and Beurling). In this setting the value $s = 1/2$ is critical since $\mathcal{S}_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s \geq 1/2$, while $\Sigma_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s > 1/2$ (see [35]). Under the above conditions the Hermite functions constitute a basis for the Gel'fand–Shilov spaces $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$. In fact, $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are the subspaces of $\mathcal{S}(\mathbb{R}^d)$ consisting of those functions f that can be expressed through Hermite expansions with coefficients c_{α} satisfying

$$|c_{\alpha}(f)| \leq ce^{-r|\alpha|^{1/2s}}$$

for some $c, r > 0$ (for every $r > 0$), as was shown by Zhang [41] (see also [13, 29, 36]). The critical exponent $s = 1/2$ for the Gel'fand–Shilov spaces is closely related

to condition $\omega(t) = O(t^2)$ as $t \rightarrow \infty$ for the space $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$, as we can see in the following corollary, which is a consequence of Lemma 13 applied to $r = 1/2$. On the other hand, we observe that the inclusion $\Sigma_{1/2}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ trivially holds since $\Sigma_{1/2}(\mathbb{R}^d) = \{0\}$.

Corollary 4 *Let ω be a weight function. If $\omega(t) = O(t^2)$ as $t \rightarrow \infty$, then the Gel'fand–Shilov space $\mathcal{S}_{1/2}(\mathbb{R}^d)$ is continuously embedded in $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$.*

Proof We consider the weight function $\omega_2(t) = \max(0, t^2 - 1)$ and its corresponding weighted matrix as defined in (6.1), i.e. $\mathcal{M}_{\omega_2} = (W_\alpha^{(\lambda)})_{\lambda > 0, \alpha \in \mathbb{N}_0^d}$, where $W_\alpha^{(\lambda)} := e^{\frac{1}{\lambda} \varphi_{\omega_2}^*(\lambda|\alpha|)}$. A straightforward calculation and Stirling’s formula show that there are two constants $A, C > 0$ such that for each $\lambda > 0$, there are $B_\lambda, D_\lambda > 0$ satisfying that for any $\alpha \in \mathbb{N}_0^d$, we have

$$B_\lambda A^{|\alpha|} \lambda^{|\alpha|/2} |\alpha|!^{1/2} \leq W_\alpha^{(\lambda)} \leq D_\lambda C^{|\alpha|} \lambda^{|\alpha|/2} |\alpha|!^{1/2}.$$

This gives immediately that $\mathcal{S}_{1/2}(\mathbb{R}^d) = \mathcal{S}_{\{\mathcal{M}_{\omega_2}\}}$. Now, by an application of Proposition 5 we get $\mathcal{S}_{1/2}(\mathbb{R}^d) = \mathcal{S}_{\{\omega_2\}}(\mathbb{R}^d)$. On the other hand, by Lemma 13 applied in the particular case $r = 1/2$ (and again Proposition 5 and Stirling’s formula) we have

$$\mathcal{S}_{1/2}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{M}_\omega\}} = \mathcal{S}_{\{\omega\}}(\mathbb{R}^d),$$

for every weight function ω such that $\omega(t) = O(t^2)$ as $t \rightarrow \infty$, which concludes the proof. □

For a weight function ω we now consider the sequence spaces

$$\Lambda_{\{\omega\}} := \{ \mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \exists j \in \mathbb{N}, \|\mathbf{c}\|_{\omega, j} := \sup_{\alpha \in \mathbb{N}_0^d} |c_\alpha| e^{\frac{1}{j} \omega(\alpha^{1/2})} < +\infty \},$$

$$\Lambda_{(\omega)} := \{ \mathbf{c} = (c_\alpha) \in \mathbb{C}^{\mathbb{N}_0^d} : \forall j \in \mathbb{N}, \|\mathbf{c}\|_{\omega, 1/j} = \sup_{\alpha \in \mathbb{N}_0^d} |c_\alpha| e^{j \omega(\alpha^{1/2})} < +\infty \}.$$

Proposition 6 *Let ω be a weight function and \mathcal{M}_ω the weight matrix defined by (6.1), (6.2). Then $\Lambda_{\{\omega\}} = \Lambda_{\{\mathcal{M}_\omega\}}$ and $\Lambda_{(\omega)} = \Lambda_{(\mathcal{M}_\omega)}$ and the equalities are also topological.*

Proof From Lemma 12 with $\lambda = j$ (and taking $B \in \mathbb{N}$), we have

$$e^{\frac{1}{Bj} \omega\left(\frac{\alpha^{1/2}}{Bj}\right)} \leq e^{\omega_{\mathbf{w}(j)}\left(\frac{\alpha^{1/2}}{Bj}\right) + \frac{C_j}{Bj}} \leq e^{\frac{C_j}{Bj}} e^{\omega_{\mathbf{w}(j)}(\alpha^{1/2}/j)}$$

and, conversely, $e^{\omega_{\mathbf{w}(j)}(\alpha^{1/2}/j)} \leq e^{\frac{1}{j} \omega(\alpha^{1/2})}$. This proves the Roumieu case. Taking $\lambda = 1/j$ we prove analogously the Beurling case. □

We now easily deduce the following consequence of Theorem 1.

Corollary 5 *Let ω be a weight function. The Hermite functions are an absolute Schauder basis in $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ and*

$$\begin{aligned} T : \mathcal{S}_{\{\omega\}}(\mathbb{R}^d) &\longrightarrow \Lambda_{\{\omega\}} \\ f &\longmapsto (\xi_\gamma(f))_{\gamma \in \mathbb{N}_0} \end{aligned}$$

defines an isomorphism.

If moreover $\omega(t) = o(t^2)$ as $t \rightarrow +\infty$, then the Hermite functions are an absolute Schauder basis in $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ and

$$T : \mathcal{S}_{(\omega)}(\mathbb{R}^d) \longrightarrow \Lambda_{(\omega)}$$

as defined above is also an isomorphism.

We finally have

Theorem 6 *If ω is a weight function, then $\mathcal{S}_{\{\omega\}}(\mathbb{R}^d)$ is nuclear. If, moreover, $\omega(t) = o(t^2)$ as $t \rightarrow +\infty$, then $\mathcal{S}_{(\omega)}(\mathbb{R}^d)$ is nuclear.*

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