Advances in Operator Theory (2024) 9:22
https://doi.org/10.1007/s43036-024-00315-y

Tusi Mathematical Research Group

ORIGINAL PAPER

KMS Dirichlet forms, coercivity and superbounded Markovian semigroups on von Neumann algebras

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Received: 10 March 2023 / Accepted: 30 December 2023
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Abstract
We introduce a construction of Dirichlet forms on von Neumann algebras M associated to any eigenvalue of the Araki modular Hamiltonian of a faithful normal non-tracial state, providing also conditions by which the associated Markovian semigroups are GNS symmetric. The structure of these Dirichlet forms is described in terms of spatial derivations. Coercivity bounds are proved and the spectral growth is derived. We introduce a regularizing property of positivity preserving semigroups (superboundedness) stronger than hypercontractivity, in terms of the symmetric embedding of M into its standard space $L^2(M)$ and the associated noncommutative $L^p(M)$ spaces. We prove superboundedness for a special class of positivity preserving semigroups and that some of them are dominated by the Markovian semigroups associated to the Dirichlet forms introduced above, for type I factors M. These tools are applied to a general construction of the quantum Ornstein–Uhlembeck semigroups of the Canonical Commutation Relations CCR and some of their non-perturbative deformations.

Keywords Non-commutative Dirichlet form · Superbounded Markov semigroup · Domination of forms and semigroups · Derivation · KMS state · Spectrum growth rate · Noncommutative $\mathbb{L}_p$ spaces · Quantum Ornstein–Uhlenbeck semigroup

Mathematics Subject Classification 47C15 · 46L57 · 47D06 · 47L90 · 7N60 · 46L51

Communicated by Uwe Franz.

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Published online: 29 February 2024

Birkhäuser

1 Introduction and description of the results

The structure of completely Dirichlet forms with respect to lower semicontinuous, faithful traces on von Neumann algebras is well understood in terms of closable derivations taking values in Hilbert bimodules (see [14] and the recent [38, 39]). However, for applications to Quantum Statistical Mechanics (see [4–7, 10, 11, 22, 23, 25, 27–30, 40]) and Quantum Probability (see [12, 26]) or to deal with general Compact Quantum Groups, is unavoidable to consider quadratic forms which are Markovian with respect to non-tracial states or weights. Concerning the structure of Dirichlet forms of GNS-symmetric Markovian semigroups, one is invited to consult the recent [38, 39]. In QSM, for example, the relevant states one wishes to consider are the KMS equilibria of time evolution automorphisms which are non-tracial at finite temperature. In the CQGs situation, on the other hand, the Haar state is a trace only for the special subclass of CQGs of Kac type. In several most studied CQGs the Haar state is not a tracial state, as for examples for the special unitary CQGs $SU_q(N)$. In this framework a detailed understanding has been found for the completely Dirichlet forms generating translation invariant completely Markovian semigroups of Levy quantum stochastic processes. The construction relies on the Schürmann cocycle associated to the generating functional of the process (see [13]).
On the other hand, a general construction of completely Dirichlet forms on the standard form of a σ-finite von Neumann algebra with respect to a faithful, normal state in the sense of [8, 9, 16–18], has been introduced in [23, 24, 35] and by Y.M. Park and his school (see [4, 5, 28–30]) with applications to QSM of bosons and fermions system and their quasi-free states. In this approach the Dirichlet forms depend upon the explicitly knowledge of the modular automorphisms group of the state.

In this work we formulate a general and natural construction of a completely Dirichlet form, Markovian with respect to a fixed normal, faithful state ω_0, associated to each non zero and not necessarily discrete eigenvalue of the Araki modular Hamiltonian $\ln \Delta_0$. Hence, by superposition, one has a malleable tool to construct completely Dirichlet forms and completely Markovian, modular symmetric, semigroups starting from the spectrum of the modular operator Δ_0 or its associated Araki modular Hamiltonian $\ln \Delta_0$. Compared to Park's approach, this has the advantage to avoid the explicit use of the modular automorphism group. The present method generalizes the construction of bounded Dirichlet form of [8] Proposition 5.3 and that of unbounded Dirichlet forms of [8] Proposition 5.4, removing the assumption of self-adjointness and affiliation to the centralizer for the coefficients.
The framework of the construction is that of Dirichlet forms and Markovian semigroups on standard forms $(M, L^2(M), L^2_+(M), J)$ of von Neumann algebras M as in [8] and related modular theory [1, 2, 6, 34, 36, 37]. In particular, we associate in Sect. 2, a one-parameter family of unbounded, J-real, non negative, densely defined, closed quadratic forms $(\mathcal{E}^\lambda_Y, \mathcal{F}^\lambda_Y)$ on $L^2(M)$ satisfying the first Beurling–Deny condition to each densely defined, closed operator $(Y, D(Y))$ affiliated to M, thus generating C_0-continuous, contractive semigroups on $L^2(M)$ which are positivity preserving (in the sense that they leave globally invariant the positive self-polar cone $L^2_+(M)$). Moreover, the quadratic form $(\mathcal{E}^\lambda_Y, \mathcal{F}^\lambda_Y)$ is Markovian with respect to the cyclic vector

$\xi_0 \in L^2(M)_+$ representing ω_0, in the strong sense that $\mathcal{E}_Y^\lambda[\xi_0] = 0$, if and only if ξ_0 lies in the domain both of Y and its adjoint Y^* and $\xi := Y\xi_0$ is an eigenvector of the modular operator Δ_0 associated to the non zero eigenvalue $\lambda > 0$. This construction applies, in particular, to any eigenvector ξ of any non zero eigenvalue of Δ_0.
Further, we investigate the fact that, by definitions, each $(\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$ is the quadratic form of an M-bimodule derivations $(d_Y^\lambda, D(d_Y^\lambda))$ on the standard bimodule $L^2(M)$. In particular we show that in the Markovian case both $(\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$ and $(d_Y^\lambda, D(d_Y^\lambda))$ are represented by the symmetric embedding on $L^2(M)$ of the unbounded, spatial derivations $\delta_Y := i[Y, \cdot]$ on M provided by the operator $(Y, D(Y))$ affiliated to M.
In the subsequent Sect. 3, we prove natural lower bounds for the Dirichlet form $(\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$ in terms of the quadratic forms of the affiliated operators $Y^*Y, YY^*, [Y, Y^*]$ and derive implications on the lower boundedness and discreteness of spectrum of $(\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$.
By the general theory, using the symmetric embeddings of the von Neumann algebra M into the standard Hilbert space $L^2(M)$ and the embedding of $L^2(M)$ into the predual space $M_* = L^1(M)$, provided by the modular theory of the state ω, completely Markovian semigroups T_t on $L^2(M)$ extend to completely (Markovian) contractive semigroups on M and on $L^1(M)$ (weak*-continuous in the former case and strongly continuous in the latter one).
In Sect. 4, we introduce an extra regularity property of positivity preserving semigroups called *superboundedness* as the boundedness of T_t from $L^2(M)$ to M for all $t > t_0$ and some $t_0 \geq 0$. In case $t_0 = 0$ we call this property *ultraboundedness*. We prove that superboundedness holds true with respect to a finite temperature Gibbs state $\omega(\cdot) = \mathrm{Tr}\,(\cdot e^{-\beta_0 H_0})/\mathrm{Tr}(e^{-\beta_0 H_0})$ on a type I_∞ factor M, for the semigroup generated by the generalized sum $H_0 \dot{+} JH_0J$ and that the property is stable with respect to domination of positivity preserving semigroups.
In Sect. 5 we apply the framework above to investigate a class of Dirichlet forms associated on a type I_∞ factor which are Markovian with respect to a Gibbs state of the Number Operator of a representation of the CCR algebra. The construction fully generalizes that of Quantum Ornstein–Uhlenbeck semigroups introduced in [12]. In particular we prove the subexponential spectral growth rate of the generator and the domination of the Markovian semigroup with respect to the semigroup generated by $H_0 \dot{+} JH_0J$ (this special class of semigroups is discussed in Appendix 7.1).
In Sect. 6 we apply the tools developed in the previous sections to construct Dirichlet forms associated with dynamics generated by deformations of the Number Operator.
In Appendix we represent the generators of a class of positivity preserving semigroups as generalized sums and we clarify *superboundedness* for abelian von Neumann algebras.

2 Dirichlet forms and derivations on von Neumann algebras standard forms

Let $(M, L^2(M), L^2_+(M), J)$ be a standard form of a σ-finite von Neumann algebra (for this subject and the related modular theory we refer to [6, 7, 36, 37]).

Let ω_0 be the faithful normal state on M represented by the cyclic vector $\xi_0 \in L^2_+(M)$ as

$$\omega_0(x) = (\xi_0 | x\xi_0)_{L^2(M)} \qquad x \in M.$$

The anti-linear, densely defined operator on $L^2(M)$ defined on the left Hilbert algebra by

$$M\xi_0 \ni x\xi_0 \mapsto x^*\xi_0 \qquad x \in M,$$

is closable. Its closure S_0 has a polar decomposition $S_0 = J\Delta_0^{1/2}$ where the anti-unitary part J is called the modular conjugation and $\Delta_0 := S_0^* S_0$ is a densely defined, self-adjoint, positive operator on $L^2(M)$, called the modular operator of ω_0, defining the modular automorphism group of M by $\sigma_t^{\omega_0}(x) := \Delta_0^{it} x \Delta_0^{-it}$ for $x \in M$ and $t \in \mathbb{R}$. On the w*-dense, involutive, sub-algebra of its analytic elements $M_0 \subseteq M$, the modular group can be extended to any $t \in \mathbb{C}$. For any $x, y \in M_0$ and $z, w \in \mathbb{C}$, this extension satisfies

$$\sigma_z^{\omega_0}(xy) = \sigma_z^{\omega_0}(x)\sigma_z^{\omega_0}(y), \quad \sigma_{z+w}^{\omega_0}(x) = \sigma_z^{\omega_0}(\sigma_w^{\omega_0}(x)), \quad \left(\sigma_z^{\omega_0}(x)\right)^* = \sigma_{\bar{z}}^{\omega_0}(x^*).$$

We will make use of the symmetric embedding of M into its standard Hilbert space $L^2(M)$:

$$i_0 : M \to L^2(M) \qquad i_0(x) := \Delta_0^{1/4} x\xi_0.$$

Among its properties we recall that it is weak*-continuous, injective with dense range and positivity preserving in the sense that $i_0(x) \in L^2(M)_+$ if and only if $x \in M_+$. Also it maps the closed and convex set of all $x \in M_+$ such that $0 \le x \le 1$ onto the closed and convex set of all $\xi \in L^2_+(M)$ such that $0 \le \xi \le \xi_0$. The projection of a J-real vector $\xi = J\xi \in L^2(M)$ onto the closed, convex set $\xi_0 - L^2_+(M)$ wil be denoted by $\xi \wedge \xi_0$.
A *Dirichlet form* [8] Definition 4.8 with respect to (M, ω_0) is a lower bounded and lower semicontinuous quadratic form

$$\mathcal{E} : L^2(M) \to (-\infty, +\infty],$$

with domain $\mathcal{F} := \{\xi \in L^2(M) : \mathcal{E}[\xi] < +\infty\}$, satisfying the properties

(i) $\mathcal{F}$ is dense in $L^2(M)$,
(ii) $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$ for all $\xi \in L^2(M)$ (reality),
(iii) $\mathcal{E}[\xi \wedge \xi_0] \le \mathcal{E}[\xi]$ for all $\xi = J\xi \in L^2(M)$, (Markovianity).

$(\mathcal{E}, \mathcal{F})$ is said to be a *completely Dirichlet form* if its ampliation on the algebra $(M \otimes M_n(\mathbb{C}), \omega_0 \otimes \mathrm{tr}_n)$ defined by

$$\mathcal{E}^n : L^2(M \otimes M_n(\mathbb{C}), \omega_0 \otimes \mathrm{tr}_n) \to [0, +\infty] \qquad \mathcal{E}^n\left[[\xi_{i,j}]_{i,j=1}^n\right] := \sum_{i,j=1}^n \mathcal{E}[\xi_{i,j}]$$

is a Dirichlet form for all $n \geq 1$ (tr_n denotes the tracial state on the matrix algebra $M_n(\mathbb{C})$).

A C_0-continuous, self-adjoint semigroup $\{T_t : t \geq 0\}$ on $L^2(M)$ is called

(i) *positivity preserving* if $T_t\xi \in L^2_+(M)$ for all $\xi \in L^2_+(M)$ and $t \geq 0$;
(ii) *Markovian with respect to* ω_0 if it is positivity preserving and for $\xi = J\xi \in L^2(M)$

$$0 \leq \xi \leq \xi_0 \quad \Longrightarrow \quad 0 \leq T_t\xi \leq \xi_0 \qquad t \geq 0;$$

(iii) *completely positive (resp. Markovian)* if the extensions $T_t^n := T_t \otimes I_n$ to $L^2(M \otimes M_n(\mathbb{C}), \omega_0 \otimes \mathrm{tr}_n)$ are positivity preserving (resp. Markovian) semigroups for all $n \geq 1$.

In [8] Definition 2.8, property ii) above, Markovianity, was indicated as sub-Markovianity.

As a result of the general theory, Dirichlet forms are automatically nonnegative and Markovian semigroups are automatically contractive see [8] Proposition 4.10 and Theorem 4.11.

Dirichlet forms $(\mathcal{E}, \mathcal{F})$ are in one-to-one correspondence with Markovian semigroups $\{T_t : t \geq 0\}$: the self-adjoint, positive operator $(H, D(H))$ associated to $(\mathcal{E}, \mathcal{F})$ by $\mathcal{E}[\xi] = \|\sqrt{H}\xi\|^2_{L^2(M)}$ for all $\xi \in \mathcal{F}$, being the semigroup generator $T_t = e^{-tH}$, $t \geq 0$.

C_0-continuous, self-adjoint, positivity preserving semigroups are in one-to-one correspondence with nonnegative, densely defined, real, lower semicontinuous quadratic forms satisfying the following *first Beurling–Deny condition* (weaker than Markovianity)

$$\xi = J\xi \in \mathcal{F} \quad \Rightarrow \quad \xi_\pm \in \mathcal{F} \qquad \text{and} \qquad \mathcal{E}(\xi_+|\xi_-) \leq 0,$$

equivalently stated (see [8] Proposition 4.5 and Theorem 4.7]) as

$$\xi = J\xi \in \mathcal{F} \quad \Rightarrow \quad |\xi| \in \mathcal{F} \qquad \text{and} \qquad \mathcal{E}[|\xi|] \leq \mathcal{E}[\xi],$$

On the other hand, the first Beurling–Deny condition and the *conservativeness* condition

$$\xi_0 \in \mathcal{F}, \qquad \mathcal{E}[\xi_0] = 0$$

together imply the Markovianity of closed forms $(\mathcal{E}, \mathcal{F})$ (see [8] Lemma 2.9 and Theorem 4.11).

2.1 Dirichlet forms associated to eigenvalues of the modular operators

The forthcoming construction of Dirichlet forms is based on the following well known fact (see [6] Proposition 2.5.9, [34, 37] page 19; see also [2] where von Neumann

algebras with states having the logarithmic of the modular operators with spectrum consisting only of isolated eigenvalues are characterized).
We recall that a densely defined, closed operator $(Y, D(Y))$ on $L^2(M)$ is affiliated to M if for any $z' \in M'$ and any $\xi \in D(Y)$ one has $z'D(Y) \subseteq D(Y)$ and $Y(z'\xi) = z'(Y\xi)$ or, equivalently, if and only if its graph $\mathcal{G}(Y) \subset L^2(M) \oplus L^2(M)$ is left globally invariant $(z' \oplus z')\mathcal{G}(Y) \subseteq \mathcal{G}(Y)$ under the action of $z' \oplus z' \in M' \oplus M'$, for any $z' \in M'$ (see [36]).
For any operator $(Y, D(Y))$ affiliated to M, the operator $j(Y) := JYJ$ is affiliated to M'.

Lemma 2.1 *For any $\xi \in D(S_0) = D(\Delta_0^{1/2})$ there exists a densely defined, closed operator $(Y, D(Y))$ affiliated to M such that*

(i) $\xi_0 \in D(Y) \cap D(Y^*)$,
(ii) $\xi = Y\xi_0$ and $S_0(\xi) = Y^*\xi_0$. iii) Among the operators $(Y, D(Y))$ with the properties i) and ii) above, there exists a minimal one $(\overline{Y_0}, D(\overline{Y_0}))$ obtained as the closure of the closable operator $(Y_0, D(Y_0))$ defined by

$$D(Y_0) := M'\xi_0, \qquad Y_0(y'\xi_0) := y'\xi.$$

Proof The operator $(Y_0, D(Y_0))$ is affiliated to M because the action of any $w' \in M'$ leaves globally invariant the domain $M'\xi_0$ and $w'Y_0(y'\xi_0) = w'y'\xi = Y_0(w'y'\xi_0)$ for any $y' \in M'$. The operator $(Y_0, D(Y_0))$ is closable because it is in duality with the densely defined operator $Z_0 : M'\xi_0 \to L^2(M)$ given by $Z_0(z'\xi_0) := z'S_0\xi$ in the sense that

$$\begin{aligned}\left(z'\xi_0 | Y_0\left(y'\xi_0\right)\right) &= \left(z'\xi_0 | y'\xi\right) = \left(y'^* z'\xi_0 | \xi\right) = \left(J\Delta_0^{-1/2} z'^* y'\xi_0 | \xi\right) \\ &= \left(J\xi | \Delta_0^{-1/2} z'^* y'\xi_0\right) = \left(z' J\Delta_0^{1/2}\xi | y'\xi_0\right) = \left(z' S_0\xi | y'\xi_0\right) \\ &= \left(Z_0\left(z'\xi_0\right) | y'\xi_0\right).\end{aligned}$$

Clearly by definition $Y_0\xi_0 = \xi$ and the calculation above implies $Y_0^*\xi_0 = S_0\xi$. If $(Y, D(Y))$ is a closed operator affiliated to M with properties (i) and (ii) above, then, as $\xi_0 \in D(Y)$, we have $y'\xi_0 \in D(Y)$ for all $y' \in M'$ so that $M'\xi_0 \subseteq D(Y)$ and $Y(y'\xi_0) = y'Y\xi_0 = y'\xi = Y_0(y'\xi_0)$, which shows that $(Y, D(Y))$ is a closed extension of $(\overline{Y_0}, D(\overline{Y_0}))$. □

This representation will be applied below to eigenvectors ξ (if any) of the modular operator.

Lemma 2.2 *Let $(Y, D(Y))$ be a densely defined, closed operator affiliated to M and $\mu, \nu \geq 0$. Then defining $d_Y^{\mu,\nu} : D(d_Y^{\mu,\nu}) \to L^2(M)$ as*

$$d_Y^{\mu,\nu} := i\left(\mu Y - \nu j\left(Y^*\right)\right) \qquad D\left(d_Y^{\mu,\nu}\right) := D(Y) \cap JD\left(Y^*\right),$$

it results that $(d_Y^{\mu,\nu}, D(d_Y^{\mu,\nu}))$ is a densely defined, closable operator on $L^2(M)$.

Proof Since $J^2 = I$, we have $D(j(Y^*)) = JD(Y^*)$ so that $d_Y^{\mu,\nu}$ is well defined on $D(d_Y^{\mu,\nu})$. By hypotheses, $j(Y^*)$ is densely defined, closed and affiliated to the commutant von Neumann algebra M'. Hence Y and $j(Y^*)$ strongly commute and the contraction semigroup $e^{-t|Y|} \circ e^{-t|j(Y^*)|} = e^{-t|j(Y^*)|} \circ e^{-t|Y|}$ with parameter $t \geq 0$ strongly converges to the identity operator on $L^2(M)$ as $t \to 0^+$. Since $d_Y^{\mu,\nu} \circ e^{-t|Y|} \circ e^{-t|j(Y^*)|} = i(\mu Y \circ e^{-t|Y|} \circ e^{-t|j(Y^*)|} - \nu e^{-t|Y|} \circ j(Y^*) \circ e^{-t|j(Y^*)|})$ is a bounded operator for any $t > 0$, we have that $D(d_Y^{\mu,\nu})$ is dense in $L^2(M)$.
To prove the statement concerning closability, observe that reasoning as above with Y^* in place of Y and $Y^{**} = Y$ in place of Y^*, we have that $\mu Y^* - \nu j(Y)$ is densely defined on $D(Y^*) \cap JD(Y)$. Moreover, since

$$\begin{aligned}
\left(d_Y^{\mu,\nu}\eta|\zeta\right) &= \left(i\left(\mu Y - \nu j\left(Y^*\right)\right)\eta|\zeta\right) \\
&= -i\mu\left(Y\eta|\zeta\right) + i\nu\left(j\left(Y^*\right)\eta|\zeta\right) \qquad \eta \in D\left(d_Y^{\mu,\nu}\right) := D(Y) \cap JD\left(Y^*\right) \\
&= -i\mu\left(\eta|Y^*\zeta\right) + i\nu\left(\eta|j(Y)\zeta\right) \qquad \zeta \in D\left(Y^*\right) \cap JD(Y) \\
&= \left(\eta| - i\left(\mu Y^* - \nu j(Y)\right)\zeta\right),
\end{aligned}$$

the adjoint of $(d_Y^{\mu,\nu}, D(d_Y^{\mu,\nu}))$ is an extension of $(-i(\mu Y^* - \nu j(Y)), D(Y^*) \cap JD(Y))$. It is thus densely defined and consequently $(d_Y^{\mu,\nu}, D(d_Y^{\mu,\nu}))$ is closable. □

Lemma 2.3 *Let $(Y, D(Y))$ be a densely defined, closed operator affiliated to M. Then the J-real part of the domain $D(Y)$ is invariant under the modulus map:*

$$\xi \in D(Y), \quad J\xi = \xi \qquad \Rightarrow \qquad |\xi| \in D(Y)$$

and $\|Y|\xi|\| = \|Y\xi\|$. In particular, if $\xi = \xi_+ - \xi_-$ is the polar decomposition of a J-real vector $\xi = J\xi \in D(Y)$, then $\xi_\pm = (|\xi| \pm \xi)/2 \in D(Y)$ and

$$\|Y\xi_\pm\| \leq \|Y\xi\|.$$

Proof Consider first the case where Y is bounded, and let $s'_\pm \in M'$ be the supports in M' of the positive and negative parts $\xi_\pm$ of a J-real $\xi \in L^2(M)$. Then $(Y\xi_+|Y\xi_-) = (Ys'_+\xi_+|Ys'_-\xi_-) = (s'_+Y\xi_+|s'_-Y\xi_-) = 0$ since $\xi_+ \perp \xi_-$ imply $s'_+s'_- = 0$, by [1] Theorem 4. Thus

$$\begin{aligned}
\|Y\xi\|^2 &= (Y\xi|Y\xi) = (Y\xi_+ - Y\xi_-|Y\xi_+ - Y\xi_-) \\
&= (Y\xi_+ + Y\xi_-|Y\xi_+ + Y\xi_-) = \||Y|\xi\|^2.
\end{aligned}$$

To deal with the general case, fix $\xi = J\xi \in D(Y) = D(|Y|)$ and consider the family of bounded operators $|Y|_\varepsilon := |Y|(I + \varepsilon|Y|)^{-1} \in M$ for $\varepsilon > 0$ as well as the spectral measure $E^{|Y|}$ of the self-adjoint operator $|Y|$. Applying the result concerning the bounded case, for all $\varepsilon > 0$ we have

$$\int_0^{+\infty} E^{|Y|}_{|\xi|,|\xi|}(\mathrm{d}\lambda)\frac{\lambda^2}{(1+\varepsilon\lambda)^2} = \||Y|_\varepsilon|\xi|\|^2 = \||Y|_\varepsilon\xi\|^2 = \int_0^{+\infty} E^{|Y|}_{\xi,\xi}(\mathrm{d}\lambda)\frac{\lambda^2}{(1+\varepsilon\lambda)^2}.$$

Letting $\varepsilon \downarrow 0$, by the Monotone Convergence Theorem we have $|\xi| \in D(|Y|) = D(Y)$ and $\|Y|\xi|\| = \|Y\xi\|$. □

Lemma 2.4 *Let $(Y, D(Y))$ be a densely defined, closed operator affiliated to M and $\mu, \nu \geq 0$. Let $\xi \in D(d_Y^{\mu,\nu}) := D(Y) \cap JD(Y^*)$ be a J-real vector with polar decomposition $\xi = \xi_+ - \xi_-$. Then $\xi_\pm \in D(Y) \cap D(Y^*)$ and*

$$\left(Y\xi_+ | j\left(Y^*\right)\xi_-\right) \geq 0.$$

Proof If $Y \in M$ the assertion is true because in that case $Y^* j(Y^*)$ is positivity preserving. To deal with the general case, let $Y = U|Y|$ be the polar decomposition of Y. By the previous Lemma 2.3, since $\xi = J\xi \in D(Y)$, we have $\xi_\pm \in D(Y) = D(|Y|)$. Since also Y^* is a densely defined, closed operator affiliated to M and, by assumption, $\xi = J\xi \in JJD(Y^*) = D(Y^*)$, again by Lemma 2.3 we have $\xi_- \in D(Y^*)$ too so that $(Y\xi_+|j(Y^*)\xi_-) = (|Y|\xi_+|j(|Y|)U^* j(U^*)\xi_-)$ (here we implicitly used the fact that $U^* j(U^*)\xi_- \in D(|Y|)$ since $\xi_- \in D(|Y|)$ by the previous Lemma 2.3, $U \in M$ and $j(|Y|)$ is affiliated to the commutant algebra M'). Since $U \in M$ so that $U^* j(U^*)$ is positivity preserving, it is enough to prove that $(|Y|\eta|j(|Y|)\eta') \geq 0$ for all $\eta, \eta' \in D(|Y|) \cap L^2_+(M)$. Since $|Y|$ and $j(|Y|)$ strongly commute, we can represent the value $(|Y|\eta|j(|Y|)\eta')$ as an integral over the product of the spectral measures of the two operators

$$\left(|Y|\eta|j(|Y|)\eta'\right) = \int_{[0,+\infty)^2} \left(E_\eta^{|Y|} \times E_{\eta'}^{j(|Y|)}\right)\left(\mathrm{d}\lambda, \mathrm{d}\lambda'\right)\lambda \cdot \lambda'.$$

Setting $f_\varepsilon(\lambda) := \lambda/(1+\varepsilon\lambda)$ we have $0 \leq f_\varepsilon(\lambda) \leq \lambda$ and $\lim_{\varepsilon\to 0} f_\varepsilon(\lambda) = \lambda$. By the Dominated Convergence Theorem we have

$$\begin{aligned}
\left(|Y|\eta|j(|Y|)\eta'\right) &= \int_{[0,+\infty)^2} \left(E_\eta^{|Y|} \times E_{\eta'}^{j(|Y|)}\right)\left(\mathrm{d}\lambda, \mathrm{d}\lambda'\right)\lambda \cdot \lambda' \\
&= \lim_{\varepsilon\to 0} \int_{[0,+\infty)^2} \left(E_\eta^{|Y|} \times E_{\eta'}^{j(|Y|)}\right)\left(d\lambda, d\lambda'\right) f_\varepsilon(\lambda) \cdot f_\varepsilon(\lambda') \\
&= \lim_{\varepsilon\to 0} \left(f_\varepsilon(|Y|)\eta|j\left(f_\varepsilon(|Y|)\right)\eta'\right) \\
&= \lim_{\varepsilon\to 0} \left(\eta|f_\varepsilon(|Y|)j\left(f_\varepsilon(|Y|)\right)\eta'\right) \geq 0
\end{aligned}$$

since $f_\varepsilon(|Y|) \in M$ and $f_\varepsilon(|Y|)j(f_\varepsilon(|Y|))$ is positivity preserving. □

Theorem 2.5 *Let $(Y, D(Y))$ be a densely defined, closed operator affiliated to M such that $\xi_0 \in D(Y) \cap D(Y^*)$ and $\mu, \nu > 0$. Then the quadratic form $\tilde{\mathcal{E}}_Y^{\mu,\nu} : \tilde{\mathcal{F}}_Y \to [0, +\infty)$ on $L^2(M)$*

$$\tilde{\mathcal{E}}_Y^{\mu,\nu}[\xi] := \left\|d_Y^{\mu,\nu}\xi\right\|^2_{L^2(M)} + \left\|d_{Y^*}^{\nu,\mu}\xi\right\|^2_{L^2(M)} \qquad \tilde{\mathcal{F}}_Y := D\left(d_Y^{\mu,\nu}\right) \cap D\left(d_{Y^*}^{\nu,\mu}\right)$$

is densely defined, closable, J-real (recall that $\|d_{Y^*}^{\nu,\mu}\xi\|^2_{L^2(M)} = \|d_Y^{\mu,\nu}J\xi\|^2_{L^2(M)}$) and satisfies the first Beurling–Deny condition

$$\xi = J\xi \in \tilde{\mathcal{F}}_Y \qquad \Rightarrow \qquad \xi_\pm \in \tilde{\mathcal{F}}_Y \quad \text{and} \quad \tilde{\mathcal{E}}_Y^{\mu,\nu}(\xi_+|\xi_-) \leq 0.$$

Its closure $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$ satisfies the first Beurling–Deny condition too and generates a contractive, positivity preserving semigroup $\{T_t : t \geq 0\}$. Moreover, $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$ is a conservative, in the sense that

$$\xi_0 \in \mathcal{F}_Y^{\mu,\nu}, \qquad \mathcal{E}_Y^{\mu,\nu}[\xi_0] = 0,$$

completely Dirichlet form with respect to (M, ω_0) and the associated completely Markovian semigroup is conservative, in the sense that

$$T_t\xi_0 = \xi_0 \qquad t \geq 0,$$

if and only if $Y\xi_0 \in L^2(M)$ is an eigenvector of the modular operator corresponding to the eigenvalue μ/ν

$$Y\xi_0 \in D\left(\Delta_0^{1/2}\right), \qquad \Delta_0^{1/2}Y\xi_0 = (\mu/\nu)Y\xi_0.$$

Proof Since $\tilde{\mathcal{F}}_Y = D(Y) \cap JD(Y^*) \cap D(Y^*) \cap JD(Y) = D(Y) \cap D(Y^*) \cap J(D(Y) \cap D(Y^*))$, we have $J\tilde{\mathcal{F}}_Y = \tilde{\mathcal{F}}_Y$ and $\xi_0 = J\xi_0 \in \tilde{\mathcal{F}}_Y$. Since $d_Y^{\mu,\nu}J\xi = i(\mu Y - \nu JY^*J)J\xi = i(\mu YJ\xi - \nu JY^*\xi) = iJ(\mu JYJ\xi - \nu Y^*\xi) = Ji(\nu Y^*\xi - \mu JYJ\xi) = Jd_{Y^*}^{\nu,\mu}\xi$ for all $\xi \in \tilde{\mathcal{F}}_Y$ and, exchanging the role of Y and Y^*, we have $d_{Y^*}^{\nu,\mu}J\xi = Jd_Y^{\mu,\nu}\xi$ too, we get

$$\begin{aligned}\tilde{\mathcal{E}}_Y^{\mu,\nu}[J\xi] &= \left\|d_Y^{\mu,\nu}J\xi\right\|^2 + \left\|d_{Y^*}^{\nu,\mu}J\xi\right\|^2 = \left\|Jd_{Y^*}^{\nu,\mu}\xi\right\|^2 + \left\|Jd_Y^{\mu,\nu}\xi\right\|^2 \\ &= \tilde{\mathcal{E}}_Y^{\mu,\nu}[\xi]\ \xi \in \tilde{\mathcal{F}}_Y,\end{aligned}$$

which proves that the quadratic form $(\tilde{\mathcal{E}}_Y^{\mu,\nu}, \tilde{\mathcal{F}}_Y)$ is J-real.
Consider now a J-real vector $\xi \in \tilde{\mathcal{F}}_Y$, its polar decomposition $\xi = \xi_+ - \xi_-$ with respect to the self-polar cone $L^2_+(M)$ and recall that, by definition, $|\xi| := \xi_+ + \xi_-$. By the previous lemma, $|\xi| \in \tilde{\mathcal{F}}_Y$ so that $\xi_\pm = (|\xi| \pm \xi)/2 \in \tilde{\mathcal{F}}_Y$. Then, if $s_\pm \in M$ (resp. $s'_\pm \in M'$) are the supports of $\xi_\pm$ in M (resp. M'), we have

$$\begin{aligned}\left(d_Y^{\mu,\nu}\xi_+|d_Y^{\mu,\nu}\xi_-\right) &= \left((\mu Y\xi_+ - \nu j(Y^*)\xi_+|(\mu Y\xi_- - \nu j(Y^*)\xi_-\right) \\ &= \mu^2(Ys'_+\xi_+|Ys'_-\xi_-) + \nu^2(j(Y^*)s_+\xi_+|j(Y^*)s_-\xi_-) \\ &\quad -\mu\nu\Big((Y\xi_+|j(Y^*)\xi_-) + (j(Y^*)\xi_+|Y\xi_-)\Big) \\ &= -\mu\nu\Big((Y\xi_+|j(Y^*)\xi_-) + (j(Y^*)\xi_+|Y\xi_-)\Big) \leq 0\end{aligned}$$

by Lemma 2.4. Since, analogously, $(d_{Y^*}^{\nu,\mu}\xi_+|d_{Y^*}^{\nu,\mu}\xi_-) \le 0$ we have $\tilde{\mathcal{E}}_Y^{\mu,\nu}(\xi_+|\xi_-) \le 0$ and consequently the first Beurling–Deny condition is satisfied by $(\tilde{\mathcal{E}}_Y^{\mu,\nu}, \tilde{\mathcal{F}}_Y)$

$$\xi = J\xi \in \tilde{\mathcal{F}}_Y \qquad \Rightarrow \qquad |\xi| \in \tilde{\mathcal{F}}_Y \qquad \text{and} \qquad \tilde{\mathcal{E}}_Y^{\mu,\nu}[|\xi|] \le \tilde{\mathcal{E}}_Y^{\mu,\nu}[\xi].$$

To establish the same condition for the closure $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$, we adapt the proof of [8] Proposition 5.1 (according the suggestions which there precede it).
On one hand, since J is an isometry for the graph norm of $(\tilde{\mathcal{E}}_Y^{\mu,\nu}, \tilde{\mathcal{F}}_Y)$ and $\tilde{\mathcal{F}}_Y$ is a core for $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$,then J is an isometry for the latter form too and the closure form is J-real.
On the other hand, let $\xi = J\xi \in \mathcal{F}_Y^{\mu,\nu}$ be a fixed J-real vector and let $\xi_n \in \tilde{\mathcal{F}}_Y$ be a sequence converging to it in the graph norm of $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$. Since the Hilbert projection $\eta \mapsto \eta_+$ of the J-real part of $L^2(M)$ onto the closed, convex cone $L^2_+(M)$ is norm continuous and $|\eta| = 2\eta_+ - \eta$, it follows that the modulus map $\eta \mapsto |\eta|$ is norm continuous too. Then, since the form $\mathcal{E}_Y^{\mu,\nu}$ is norm lower semicontinuous on $L^2(M)$, it follows that

$$\begin{aligned}\mathcal{E}_Y^{\mu,\nu}[|\xi|] &\le \liminf_n \mathcal{E}_Y^{\mu,\nu}[|\xi_n|] = \liminf_n \tilde{\mathcal{E}}_Y^{\mu,\nu}[|\xi_n|] \le \liminf_n \tilde{\mathcal{E}}_Y^{\mu,\nu}[\xi_n] \\ &= \liminf_n \mathcal{E}_Y^{\mu,\nu}[\xi_n] = \mathcal{E}_Y^{\mu,\nu}[\xi].\end{aligned}$$

The first Beurling–Deny condition is thus verified and, by [8] Proposition 4.10, it follows that the semigroup $\{T_t : t \ge 0\}$ has the desired properties.
Concerning the conservativeness property, notice that $\xi_0 \in \tilde{\mathcal{F}}_Y \subseteq \mathcal{F}_Y^{\mu,\nu}$. If $\mathcal{E}_Y^{\mu,\nu}[\xi_0] = 0$ then $\tilde{\mathcal{E}}_Y[\xi_0] = 0$ so that $d_Y^{\mu,\nu}\xi_0 = 0$ which implies $\mu Y\xi_0 = \nu j(Y^*)\xi_0 = \nu JY^*\xi_0$ and, for any $x \in M$, $\mu(x\xi_0|Y\xi_0) = \nu(x\xi_0|JY^*\xi_0) = \nu(Y^*\xi_0|Jx\xi_0) = \nu(Y^*\xi_0|JxJ\xi_0) = \nu(Jx^*J\xi_0|Y\xi_0) = \nu(Jx^*\xi_0|Y\xi_0) = \nu(\Delta_0^{1/2}x\xi_0|Y\xi_0)$ since Y is affiliated to M and $Jx^*J \in M'$. Setting $\lambda^2 := \mu/\nu$, this in turn implies $((\Delta_0^{1/2} - \lambda^2 I)x\xi_0|Y\xi_0) = 0$ for all $x \in M$ and, since $M\xi_0$ is a core for $\Delta_0^{1/2}$, it follows that $Y\xi_0 \in D(\Delta_0^{1/2})$ and $\Delta_0^{1/2}Y\xi_0 = (\mu/\nu)Y\xi_0$, i.e. $Y\xi_0$ is an eigenvalue of the modular operator with eigenvalue μ/ν.
On the other hand, if $Y\xi_0 \in D(\Delta_0^{1/2})$ and $\Delta_0^{1/2}Y\xi_0 = (\mu/\nu)Y\xi_0$, using the identities above, it follows that $d_Y^{\mu,\nu}\xi_0 = 0$ and $d_{Y^*}^{\nu,\mu}\xi_0 = d_Y^{\mu,\nu}J\xi_0 = d_Y^{\mu,\nu}\xi_0 = 0$ so that $\mathcal{E}_Y^{\mu,\nu}\xi_0 = \tilde{\mathcal{E}}_Y[\xi_0] = 0$, i.e. $(\mathcal{E}_Y^{\mu,\nu}, \mathcal{F}_Y^{\mu,\nu})$ is conservative. By [8] Proposition 4.10, given conservativness, the first Beurling–Deny property and Markovianity are equivalent for quadratic forms as well as the positivity preserving property and the Markovianity are equivalent for the associated semigroups.
Concerning the complete Markovianity of the Dirichlet form, we notice that for any $n \ge 1$, the ampliation $(\mathcal{E}_Y^\lambda)^n : L^2(M \otimes M_n(\mathbb{C}), \omega_0 \otimes \mathrm{tr}_n) \to [0, +\infty]$, defined as $(\mathcal{E}_Y^\lambda)^n[[\xi_{i,j}]_{i,j=1}^n] := \sum_{i,j=1}^n \mathcal{E}[\xi_{i,j}]$, has the same structure as $\mathcal{E}_Y^\lambda$. More precisely, a closed operator $Y^n := Y \otimes I_n$ is densely defined on $D(Y^n) := D(Y) \otimes_{\mathrm{alg}} L^2(M_n(\mathbb{C}), tr_n) \subset L^2(M \otimes M_n(\mathbb{C}), \omega_0 \otimes tr_n)$ and one may check that $(\mathcal{E}_Y^\lambda)^n = \mathcal{E}_{Y^n}^\lambda$.
If $\xi_0 \in D(Y) \cap D(Y^*)$ and $Y\xi_0 \in L^2(M)$ is an eigenvector of the modular operator $\Delta_0^{1/2}$ of the state ω_0 on M, corresponding to the eigenvalue μ/ν, then, denoting by $\zeta_n \in L^2(M_n(\mathbb{C}), tr_n)$ the unit vector representing the trace state, it easily verified that

$\xi_0 \otimes \zeta_n \in D(Y^n) \cap D((Y^n)^*)$ and that $Y^n(\xi_0 \otimes \zeta_n) = Y\xi_0 \otimes \zeta_n$ is an eigenvalue of the modular operator of the state $\omega_0 \otimes tr_n$ on $M \otimes M_n(\mathbb{C})$, corresponding to the same eigenvalue μ/ν. Applying the results obtained above to the form $(\mathcal{E}_Y^\lambda)^n$ in place of $\mathcal{E}_Y^\lambda$, we get its Markovianity for any $n \geq 1$ and complete Markovianity of the associated semigroup. □

Notation. If $\lambda^2 \in \mathrm{Sp}(\Delta_0^{1/2})\backslash\{0\}$ is a strictly positive eigenvalue of the modular operator and $\mu/\nu = \lambda^2$, then $d_Y^{\mu,\nu} = \sqrt{\mu\nu} d_Y^{\lambda,\lambda^{-1}}$, $d_{Y^*}^{\nu,\mu} = \sqrt{\mu\nu} d_{Y^*}^{\lambda^{-1},\lambda}$ and $\mathcal{E}_Y^{\mu,\nu} = \mu\nu \cdot \mathcal{E}_Y^{\lambda,\lambda^{-1}}$. Since now on, we will adopt the simplified notation

$$\mathcal{E}_Y^\lambda := \mathcal{E}_Y^{\lambda,\lambda^{-1}}.$$

Remark. To any eigenvector $\xi \in D(S_0)$ of $\Delta_0^{1/2}$ or, equivalently, of the Araki Hamiltonian $\ln \Delta_0$, we associate a completely Dirichlet form $\mathcal{E}_Y^\lambda$ choosing a densely defined, closed operator $(Y, D(Y))$ as in Lemma 2.1. For this choice there exists a canonical candidate, namely $(\overline{Y_0}, D(\overline{Y_0}))$. In general $(\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$ may depend upon the operator $(Y, D(Y))$ and not only upon the eigenvector $\xi = Y\xi_0$ it represents. The next result shows how this is connected to the GNS symmetry of the Markovian semigroup.

Theorem 2.6 (GNS symmetry) *Let $(Y, D(Y))$ be a densely defined, closed operator affiliated to M, $\mu, \nu > 0$ such that $\xi_0 \in D(Y) \cap D(Y^*)$ and $Y\xi_0 \in L^2(M)$ is an eigenvector of $(\Delta_0^{1/2}, D(\Delta_0^{1/2}))$ for the eigenvalue $\lambda^2 := \mu/\nu$. Then, for any $t \in \mathbb{R}$,*

(i) the densely defined, closed operator $(Y_t, D(Y_t)) := (\Delta_0^{it} Y \Delta_0^{-it}, \Delta_0^{it} D(Y))$, affiliated to M, verifies $\xi_0 \in D(Y_t) \cap D(Y_t^*)$, $Y_t\xi_0 = \lambda^{4ti} Y\xi_0 \in L^2(M)$ and $Y_t = \lambda^{4ti} Y$ on the subspace $M'\xi_0$;

(ii) $(\mathcal{E}_{Y_t}^\lambda, \mathcal{F}_{Y_t}^\lambda)$ is a Dirichlet form with respect to (M, ξ_0) coinciding with

$$\mathcal{F}_{Y_t}^\lambda = \Delta_0^{it}(\mathcal{F}_Y^\lambda) \qquad \mathcal{E}_{Y_t}^\lambda[\eta] = \mathcal{E}_Y^\lambda[\Delta_0^{-it}\eta].$$

If, moreover, $M'\xi_0 \subseteq D(Y)$ is a core for $(Y, D(Y))$, then, for any $t \in \mathbb{R}$, we have

(iii) $(Y_t, D(Y_t)) = (\lambda^{4it} \cdot Y, D(Y))$, for any $t \in \mathbb{R}$;

(iv) $(\mathcal{E}_{Y_t}^\lambda, \mathcal{F}_{Y_t}^\lambda) = (\mathcal{E}_Y^\lambda, \mathcal{F}_Y^\lambda)$, the associated Markovian semigroup is symmetric

$$(T_t(x\xi_0)|y\xi_0) = (x\xi_0|T_t(y\xi_0)) \qquad x, y \in M, \ t \geq 0$$

and, in particular, it commutes with $\{\Delta_0^{it} : t \in \mathbb{R}\}$;

v) The semigroup generated by $(\mathcal{E}_{\overline{Y_0}}^\lambda, \mathcal{F}_{\overline{Y_0}}^\lambda)$ is GNS symmetric (notations of Lemma 2.1).

Proof (i) Since, for any $t \in \mathbb{R}$, one has $\Delta_0^{it}\xi_0 = \xi_0$, it follows that $\xi_0 \in D(Y_t) \cap D(Y_t^*)$, $Y_t\xi_0 = \Delta_0^{it} Y\xi_0 = \lambda^{4ti} Y\xi_0 \in L^2(M)$ and $Y_t(z'\xi_0) = z'Y_t\xi_0 = \lambda^{4ti} \cdot z'Y\xi_0 = \lambda^{4ti} \cdot Y(z'\xi_0)$ for any $z' \in M'$; (ii) thus $Y_t\xi_0$ is an eigenvector of $(\Delta_0^{1/2}, D(\Delta_0^{1/2}))$ for the eigenvalue λ^2 and, by Theorem 2.5, $(\mathcal{E}_{Y_t}^\lambda, \mathcal{F}_{Y_t}^\lambda)$ is a well defined Dirichlet form. The displayed identity follows from the identities $d_{Y_t}^{\mu,\nu} = \Delta_0^{it} \circ d_Y^{\mu,\nu} \circ \Delta_0^{-it}$,

$d^{\mu,\nu}_{Y_t^*} = \Delta_0^{it} \circ d^{\mu,\nu}_{Y^*} \circ \Delta_0^{-it}$, valid, for any $t \in \mathbb{R}$, on $\tilde{\mathcal{F}}_{Y_t} = \tilde{\mathcal{F}}_Y$ and the fact that this space is a form core for $\eta \mapsto \mathcal{E}^\lambda_{Y_t}[\eta]$ and $\eta \mapsto \mathcal{E}^\lambda_Y[\Delta_0^{-it}\eta]$.
(iii) Since the core $M'\xi_0$ for $(Y, D(Y))$ is invariant under the group $\{\Delta_0^{it} : t \in \mathbb{R}\}$, it is a core also for $(Y_t, D(Y_t))$, for any fixed $t \in \mathbb{R}$. Since, by i), $Y_t = \lambda^{4ti} \cdot Y$ on this common core, we have $D(Y_t) = D(Y)$ and $Y_t = \lambda^{4ti} \cdot Y$, for any $t \in \mathbb{R}$; iv) since, by iii), $(\mathcal{E}^\lambda_{Y_t}, \mathcal{F}^\lambda_{Y_t}) = (\mathcal{E}^\lambda_Y, \mathcal{F}^\lambda_Y)$ for any $t \in \mathbb{R}$, ii) implies that $(\mathcal{E}^\lambda_Y, \mathcal{F}^\lambda_Y)$ is invariant under the unitary group $\{\Delta_0^{it} : t \in \mathbb{R}\}$ so that the Markovian semigroup it generates commutes with $\{\Delta_0^{it} : t \in \mathbb{R}\}$ and it is GNS symmetric by [8] Theorem 6.6; v) follows from iv) as, by definition, $M'\xi_0$ is a core for $(\overline{Y_0}, D(\overline{Y_0}))$. □

2.2 Representation of Dirichlet forms as square of commutators

In this section we show how to represent the Dirichlet forms on $L^2(M)$ constructed above, in terms of generalized commutators, i.e. unbounded spatial derivations on M. We recall that $(S_0, D(S_0))$ is an *unbounded* conjugation, i.e. anti-linear and idempotent on its domain. Thus S_0^2 is the identity operator on $D(S_0)$ or, more explicitly, that $\xi \in D(S_0)$ implies $S_0\xi \in D(S_0)$ and $S_0(S_0\xi) = \xi$. In other terms, the image of S_0 coincides with its domain and $S_0 = S_0^{-1}$ holds true as an identity between densely defined, closed operators. In terms of the polar decomposition $S_0 = J\Delta_0^{1/2}$ we have $J\Delta_0^{1/2} = \Delta_0^{-1/2}J$ as an identity between densely defined, closed operators. This means, in particular, that the modular conjugation exchanges domains as follows $JD(\Delta_0^{1/2}) = D(\Delta_0^{-1/2})$, $D(\Delta_0^{1/2}) = JD(\Delta_0^{-1/2})$. More in general, one has the intertwining relation $\bar{f}(\Delta_0^{-1}) = Jf(\Delta_0)J$ between closed operators valid for any Borel measurable function $f : [0, +\infty) \to \mathbb{C}$ (see Introduction to Chapter 10 in [36]). The relation, which is equivalent to $JD(\bar{f}(\Delta_0^{-1})) = D(f(\Delta_0))$ and $\bar{f}(\Delta_0^{-1})\xi = Jf(\Delta_0)J\xi$ for all $\xi \in D(\bar{f}(\Delta_0^{-1}))$, will be mostly used for power functions f.
Among its consequences, we will make use of the following:

(a) for any $\alpha \in \mathbb{R}$, the closed operator $J\Delta_0^\alpha$ is an unbounded conjugation on its domain $D(\Delta_0^\alpha)$;
(b) $S_0 = J\Delta_0^{1/2} = \Delta_0^{-1/4}J\Delta_0^{1/4}$ is an identity between densely defined, closed operators: in fact, $D(\Delta_0^{-1/4}J\Delta_0^{1/4}) := \{\xi \in D(\Delta_0^{1/4}) : J\Delta_0^{1/4}\xi \in D(\Delta_0^{-1/4})\}$ but since $D(\Delta_0^{-1/4}) = JD(\Delta_0^{1/4})$ one has $D(\Delta_0^{-1/4}J\Delta_0^{1/4}) = \{\xi \in D(\Delta_0^{1/4}) : \Delta_0^{1/4}\xi \in D(\Delta_0^{1/4})\} = D(\Delta_0^{1/2})$ and, for all $\xi \in D(\Delta_0^{1/2})$,

$$\left(\Delta_0^{-1/4}J\Delta_0^{1/4}\right)\xi = \left(\Delta_0^{-1/4}J\right)\Delta_0^{1/4}\xi = \left(J\Delta_0^{1/4}\right)\Delta_0^{1/4}\xi = J\Delta_0^{1/2}\xi;$$

(c) $(J\Delta_0^{1/4}, D(\Delta_0^{1/4}))$ is a closed extension of the densely defined operator $(\Delta_0^{1/4}S_0, D(S_0))$: in fact, the latter operator is well defined since $\zeta \in D(S_0)$ implies $S_0\zeta \in D(S_0) = D(\Delta_0^{1/2}) \subset D(\Delta_0^{1/4})$ and also $\Delta_0^{1/4}S_0\zeta = \Delta_0^{1/4}S_0^{-1}\zeta = \Delta_0^{1/4}\Delta_0^{-1/2}J\zeta = \Delta_0^{-1/4}J\zeta = J\Delta_0^{1/4}\zeta$;

(d) $(\Delta_0^{-1/4}J, D(\Delta_0^{1/4})) = (J\Delta_0^{1/4}, D(\Delta_0^{1/4}))$ is a closed extension of the densely defined operator $(S_0\Delta_0^{-1/4}, D(\Delta_0^{-1/4}) \cap D(\Delta_0^{1/4}))$: in fact $D(S_0\Delta_0^{-1/4}) := \{\zeta \in D(\Delta_0^{-1/4}) : \Delta_0^{-1/4}\zeta \in D(S_0)\} = \{\zeta \in D(\Delta_0^{-1/4}) : \Delta_0^{-1/4}\zeta \in D(\Delta_0^{1/2})\} = D(\Delta_0^{-1/4}) \cap D(\Delta_0^{1/4})$ and $J\Delta_0^{1/4}\zeta = J\Delta_0^{1/2}\Delta_0^{-1/4}\zeta = S_0\Delta_0^{-1/4}\zeta$ for all $\zeta \in D(\Delta_0^{-1/4}) \cap D(\Delta_0^{1/4})$;

(e) Let $M_0 \subseteq M$ be the involutive w*-dense sub-algebra of analytic vectors of the group σ^{ω_0}. For any $y \in M_0$, the operator $\Delta_0^{1/4}y\Delta_0^{-1/4}$ on $L^2(M)$ is densely defined on $i_0(M_0)$ and closable. Its closure is a bounded operator belonging to M, which coincides with the analytic extension of the map $\mathbb{R} \ni t \mapsto \sigma_t^{\omega_0}(y) \in M_0 \subset M$ evaluated at $t = -i/4$

$$\overline{\left(\Delta_0^{1/4}y\Delta_0^{-1/4}\right)} = \sigma_{-i/4}^{\omega_0}(y) \in M_0 \subset M$$

and $\sigma_{-i/4}^{\omega_0}(y)i_0(x) = i_0(yx)$ for all $x \in M_0$;

(f) by Proposition in Section 9.24 in [36], for any $y \in M_0$ and any $\alpha \in \mathbb{C}$ one has the important identity

$$D\left(\Delta_0^{\alpha}y\Delta_0^{-\alpha}\right) = D\left(\Delta_0^{-\alpha}\right)$$

and the boundedness of the operator $\Delta_0^{\alpha}y\Delta_0^{-\alpha}$ on $D(\Delta_0^{-\alpha})$. Since $JD(\Delta_0^{1/2}) = D(\Delta_0^{-1/2})$, the case $\alpha = 1/2$ implies that $D(S_0yS_0) = D(S_0)$ and the boundedness of the operator S_0yS_0 on $D(S_0)$;

g) the involutive sub-algebra $M_0' := JM_0J \subset M'$ coincides with the set of analytic vectors of the modular group of the commutant M' associated to the state determined by $\xi_0 \in L^2(M)$. The left Hilbert sub-algebra $M_0\xi_0 \subset M\xi_0 \subset L^2(M)$ is dense in $L^2(M)$ and it coincides with the symmetric embedding of the algebra of analytic elements

$$M_0\xi_0 = i_0(M_0),$$

as it results from the identity $i_0(y) = \sigma_{-i/4}^{\omega_0}(y)\xi_0$ valid for all $y \in M_0$. Also, $M_0\xi_0$ is J-invariant

$$Ji_0(y) = i_0(y^*) \qquad y \in M_0.$$

Lemma 2.7 *If $\eta \in D(S_0)$, the densely defined operator $(\mathcal{L}_\eta, D(\mathcal{L}_\eta))$ given by*

$$D(\mathcal{L}_\eta) := i_0(M_0) \ni i_0(y) \qquad \mathcal{L}_\eta i_0(y) := J\sigma_{-i/4}^{\omega_0}(y^*)J\eta$$

is closable since its adjoint is an extension of the densely defined operator $B : D(B) \to L^2(M)$

$$D(B) := M'\xi_0 \ni z'\xi_0 \qquad B\left(z'\xi_0\right) := z'S_0\eta.$$

The densely defined operator $(\mathcal{R}_\eta, D(\mathcal{R}_\eta))$ *given by*

$$D(\mathcal{R}_\eta) := i_0(M_0) \ni i_0(y) \qquad \mathcal{R}_\eta i_0(y) := \sigma^{\omega_0}_{-i/4}(y) J\eta$$

satisfies the relation $\mathcal{R}_\eta = J\mathcal{L}_\eta J$ *from which it follows that it is closable too.*

Proof Since $\xi_0 = J\xi_0 \in D(\Delta_0^{-1/4}) = D(\Delta_0^{1/4} y \Delta_0^{-1/4})$, $\Delta_0^{1/4} J = J\Delta_0^{-1/4}$, $y\xi_0 \in D(\Delta_0^{1/2})$ and $\Delta_0^{1/2} y\xi_0 \in D(\Delta_0^{-1/4})$, we have

$$\begin{aligned} J\sigma^{\omega_0}_{-i/4}(y^*)J\xi_0 &= J\left(\Delta_0^{1/4} y^* \Delta_0^{-1/4}\right)\xi_0 = J\Delta_0^{1/4} y^* \xi_0 \\ &= J\Delta_0^{1/4} J\Delta_0^{1/2} y\xi_0 = \Delta_0^{1/4} y\xi_0 = i_0(y). \end{aligned}$$

Since, moreover, $w'^*\xi_0 = \Delta_0^{1/2} J w'\xi_0$ for all $w \in M'$ and $J\sigma^{\omega_0}_{-i/4}(y^*)J \in M'$ for all $y \in M_0$, for $z' \in M'$ we have

$$\begin{aligned} \left(z'\xi_0 | \mathcal{L}_\eta i_0(y)\right) &= \left(z'\xi_0 | J\sigma^{\omega_0}_{-i/4}(y^*)J\eta\right) = \left(\left(J\sigma^{\omega_0}_{-i/4}(y^*)J\right)^* z'\xi_0 | \eta\right) \\ &= \left(\Delta_0^{1/2} J z'^* J\sigma^{\omega_0}_{-i/4}(y^*)J\xi_0 | \eta\right) = \left(J z'^* i_0(y) | \Delta_0^{1/2}\eta\right) \\ &= \left(J\Delta_0^{1/2}\eta | z'^* i_0(y)\right) = \left(z' S_0 \eta | i_0(y)\right) = \left(B\left(z'\xi_0\right) | i_0(y)\right). \end{aligned}$$

The relation between the operators $\mathcal{L}_\eta$ and $\mathcal{R}_\eta$ follows from the identities $i_0(y^*) = \Delta_0^{1/4} y^* \xi_0 = \Delta_0^{1/4} S_0(y\xi_0) = \Delta_0^{1/4}\Delta_0^{-1/2} J(y\xi_0) = \Delta_0^{-1/4} J(y\xi_0) = J\Delta_0^{1/4} y\xi_0 = J i_0(y)$ for all $y \in M_0$ and the fact that J is idempotent and it leaves $D(\mathcal{L}_\eta) = D(\mathcal{R}_\eta) = i_0(M_0)$ globally invariant: $J\mathcal{L}_\eta J i_0(y) = J\mathcal{L}_\eta i_0(y^*) = J J\sigma^{\omega}_{-i/4}(y) J\eta = \mathcal{R}_\eta i_0(y)$. □

Lemma 2.8 *Let* $\xi \in D(S_0)$ *and fix, by Lemma 2.1, a densely defined, closed operator* $(X, D(X))$ *affiliated to* M *such that*

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(X\xi_0) = X^*\xi_0.$$

Then the following properties hold true:

(i) the intersection of domains $D(X) \cap D(X^*)$ *contains* $M_0\xi_0$*;*
(ii) the images of $M_0\xi_0$ *under* $(X, D(X))$ *and* $(X^*, D(X^*))$ *are contained in* $D(S_0)$

$$X(y\xi_0) \in D(S_0), \quad X^*(y^*\xi_0) \in D(S_0), \qquad \textit{for all} \quad y \in M_0$$

and

$$S_0(Xy\xi_0) = y^* X^* \xi_0, \quad S_0(X^* y^* \xi_0) = yX\xi_0 \qquad \textit{for all} \quad y \in M_0;$$

Consider the densely defined operators on $L^2(M)$ *given by*

$$L_\xi i_0(y) := J(\Delta_0^{1/4} y^* \Delta_0^{-1/4}) J\Delta_0^{1/4}\xi \qquad i_0(y) \in i_0(M_0) =: D(L_\xi),$$

$$R_\xi i_0(y) := (\Delta_0^{1/4} y \Delta_0^{-1/4}) J \Delta_0^{1/4} S_0(\xi) \qquad i_0(y) \in i_0(M_0) =: D(R_\xi).$$

These are closable, by Lemma 2.7, since $L_\xi = \mathcal{L}_\eta$ *and* $R_\xi = \mathcal{R}_\eta$ *for* $\eta := \Delta_0^{1/4}\xi \in D(S_0)$ *and*

(iii) for any $y \in M_0$ *we have*

$$Xy\xi_0 \in D\left(\Delta_0^{1/4}\right), \qquad L_\xi i_0(y) = \Delta_0^{1/4} Xy\xi_0;$$

(iv) for any $y \in M_0$ *we have*

$$yX\xi_0 \in D\left(\Delta_0^{1/4}\right), \qquad R_\xi i_0(y) = \Delta_0^{1/4} yX\xi_0;$$

(v) $\overline{L_\xi}$ *is affiliated with* M, $\overline{R_\xi}$ *is affiliated with* M' *and* $\overline{R_\xi} = J\overline{L_\xi}J$*;*

(vi) the operator $\Delta_0^{1/4} X \Delta_0^{-1/4}$ *is well defined on* $i_0(M_0)$ *and there it coincides with* L_ξ*;*

(vii) the operator $J\Delta_0^{1/4} X^* \Delta_0^{-1/4} J$ *is well defined on* $i_0(M_0)$ *and there it coincides with* R_ξ*;*

(viii) If $\xi = X\xi_0$ *is an eigenvector of* $\Delta_0^{1/2}$ *corresponding to the eigenvalue* $\lambda^2 > 0$*, with* $\lambda > 0$*,*

$$\Delta_0^{1/2} X\xi_0 = \lambda^2 \cdot X\xi_0,$$

then $L_\xi = \lambda X$ *and* $R_\xi = \lambda^{-1} JX^*J$ *on* $i_0(M_0)$*.*

Proof (i) As X and X^* are affiliated to M and $\xi_0 \in D(X) \cap D(X^*)$, it follows that $M'\xi_0 \subset D(X) \cap D(X^*)$ and, a fortiori, that $M_0\xi_0 = M_0'\xi_0 \subset M'\xi_0 \subset D(X) \cap D(X^*)$. (ii) Since $M_0\xi_0$ is a core for $(S_0, D(S_0))$, there exists a sequence $x_n \in M_0$ such that $\|x_n\xi_0 - X\xi_0\| \to 0$ and $\|x_n^*\xi_0 - X^*\xi_0\| \to 0$. As mentioned at item f) of the introduction of the present section, since $y \in M_0$, the operator $S_0 y^* S_0$ is bounded on $D(S_0)$ and then on $M_0\xi_0 \subset D(S_0)$. Thus $x_n y\xi_0 \in D(S_0)$ is a Cauchy sequence in $L^2(M)$ as

$$\begin{aligned}\|x_n y\xi_0 - x_m y\xi_0\| &= \left\|S_0\left(y^* x_n^*\xi_0 - y^* x_m^*\xi_0\right)\right\| = \left\|S_0 y^* S_0\left(x_n\xi_0 - x_m\xi_0\right)\right\| \\ &\le \left\|S_0 y^* S_0\right\| \cdot \|x_n\xi_0 - x_m\xi_0\|.\end{aligned}$$

Analogously, $S_0(x_n y\xi_0) = y^* x_n^*\xi_0 \in L^2(M)$ is a Cauchy sequence too as

$$\|S_0\left(x_n y\xi_0\right) - S_0\left(x_m y\xi_0\right)\| = \left\|y^* x_n^*\xi_0 - y^* x_m^*\xi_0\right\| \le \|y^*\| \cdot \left\|x_n^*\xi_0 - x_m^*\xi_0\right\|.$$

Hence $x_n y\xi_0 \in D(S_0)$ is a Cauchy sequence in the graph norm of the closed operator $(S_0, D(S_0))$ and the image of $\eta := \lim_n x_n y\xi_0 \in D(S_0)$ is given by $S_0\eta = \lim_n y^* x_n^*\xi_0 = y^* X^*\xi_0$. Since X is affiliated to M, $X\xi_0 - x_n\xi_0 \in D(S_0)$ and $S_0 y^* S_0$ is bounded on $D(S_0)$, for $z' \in M'$ we have

$$\begin{aligned}\left(z'\xi_0|Xy\xi_0 - x_n y\xi_0\right) &= \left(X^*\xi_0 - x_n^*\xi_0|yz'^*\xi_0\right) = \left(y^* S_0\left(X\xi_0 - x_n\xi_0\right)|S_0^*(z'\xi_0)\right)\\ &= \left(\left(S_0 y^* S_0\right)\left(X\xi_0 - x_n\xi_0\right)|z'\xi_0)\right)\\ &\le \|S_0 y^* S_0\| \cdot \|X\xi_0 - x_n\xi_0\| \cdot \|z'\xi_0\|.\end{aligned}$$

By the density of $M'\xi_0$ in $L^2(M)$, it follows that $\|Xy\xi_0 - x_n y\xi_0\| \le \|S_0 y^* S_0\| \cdot \|X\xi_0 - x_n\xi_0\| \to 0$ as $n \to \infty$ and we have $Xy\xi_0 = \eta \in D(S_0)$ and $S_0(Xy\xi_0) = S_0(\eta) = y^* X^*\xi_0$ for any $y \in M_0$.

As S_0^2 is the identity operator on $D(S_0)$, from $S_0(X\xi_0) = X^*\xi_0$ it follows that $X^*\xi_0 \in D(S_0)$ and $S_0(X^*\xi_0) = X\xi_0$. Thus $(X^*, D(X^*))$ satisfies the same hypotheses as $(X, D(X))$ and the statements involving $(X^*, D(X^*))$ can be deduced from those involving $(X, D(X))$ proved above, by substitution and the fact that the sub-algebra M_0 is involutive.

The operator L_ξ is well defined since $\xi \in D(S_0) = D(\Delta_0^{1/2})$ implies $\Delta_0^{1/4}\xi \in D(\Delta_0^{1/4})$ and $J\Delta_0^{1/4}\xi \in JD(\Delta_0^{1/4}) = D(\Delta_0^{-1/4}) = D(\Delta_0^{1/4} y \Delta_0^{-1/4})$. Since $\xi \in D(S_0)$ implies $S_0\xi \in D(S_0)$, analogous relations imply that $J\Delta_0^{1/4} S_0\xi \in D(\Delta_0^{1/4} y \Delta_0^{-1/4})$ so that R_ξ is well defined too.

As first step to prove (iii), we show that $D(\Delta_0^{1/4})$ is a left M_0-module, i.e. $y\zeta \in D(\Delta_0^{1/4})$ for any $y \in M_0$ and $\zeta \in D(\Delta_0^{1/4})$ (a fact probably known in literature). Notice first that since $\sigma_{-i/4}^{\omega_0}(y) \in M_0$ we have

$$\sigma_{-i/4}^{\omega_0}(y)\xi_0 \in D\left(\Delta_0^{1/4}\right) = D\left(\Delta_0^{-1/4} y^* \Delta_0^{1/4}\right),$$

which means, in particular, that $y^*\Delta_0^{1/4}\sigma_{-i/4}^{\omega_0}(y)\xi_0 \in D(\Delta_0^{-1/4})$ and implies

$$\begin{aligned}|\sigma_{-i/4}^{\omega_0}(y)|^2\xi_0 &= (\sigma_{-i/4}^{\omega_0}(y))^*\sigma_{-i/4}^{\omega_0}(y)\xi_0 = \sigma_{i/4}^{\omega_0}(y^*)\sigma_{-i/4}^{\omega_0}(y)\xi_0\\ &= \overline{\Delta_0^{-1/4} y^* \Delta_0^{1/4}}\sigma_{-i/4}^{\omega_0}(y)\xi_0 = \Delta_0^{-1/4} y^* \Delta_0^{1/4}\sigma_{-i/4}^{\omega_0}(y)\xi_0\\ &= \Delta_0^{-1/4} y^* \Delta_0^{1/4}\sigma_{-i/4}^{\omega_0}(y)\Delta_0^{1/4}\xi_0\\ &= \Delta_0^{-1/4} y^* \sigma_{-i/4}^{\omega_0}(\sigma_{-i/4}^{\omega_0}(y))\xi_0\\ &= \Delta_0^{-1/4} y^* \sigma_{-i/2}^{\omega_0}(y)\xi_0\\ &= \Delta_0^{-1/4} y^* \Delta_0^{1/2} y\xi_0\\ &= \Delta_0^{-1/4} y^* J y^* J\xi_0.\end{aligned}$$

Since i_0 is positivity preserving, setting $c := \|\sigma_{-i/4}^{\omega_0}(y)\|^2$, we thus obtain the bound

$$y^* J y^* J\xi_0 = \Delta_0^{1/4}\left(|\sigma_{-i/4}^{\omega_0}(y)|^2\xi_0\right) = i_0\left(|\sigma_{-i/4}^{\omega_0}(y)|^2\right) \le c \cdot \xi_0.$$

Consider now a sequence $x_n\xi_0 \in M\xi_0$ converging to $\zeta \in D(\Delta_0^{1/4})$ in the graph norm of $(\Delta_0^{1/4}, D(\Delta_0^{1/4}))$. Then $\lim_n \|yx_n\xi_0 - y\zeta\| \le \|y\| \cdot \lim_n \|x_n\xi_0 - \zeta\| = 0$.

Since $\Delta_0^{1/4} x_n \xi_0 \in L^2(M)$ is a Cauchy sequence, we have $((x_n - x_m)J(x_n - x_m)J\xi_0|\xi_0) = (x_n\xi_0 - x_m\xi_0|J(x_n^* - x_m^*)J\xi_0) = (x_n\xi_0 - x_m\xi_0|\Delta_0^{1/2}(x_n - x_m)\xi_0) = \|\Delta_0^{1/4} x_n\xi_0 - \Delta_0^{1/4} x_m\xi_0\|^2 \to 0$ and, by analogous identities, the self-polarity of $L^2_+(M)$ and the bound above, we get $\|\Delta_0^{1/4} y x_n\xi_0 - \Delta_0^{1/4} y x_m\xi_0\|^2 = ((x_n - x_m)J(x_n - x_m)J\xi_0|y^*Jy^*J\xi_0) \leq c\cdot((x_n - x_m)J(x_n - x_m)J\xi_0|\xi_0) \to 0$. Thus $yx_n\xi_0 \in D(\Delta_0^{1/4})$ converges in the graph norm of $(\Delta_0^{1/4}, D(\Delta_0^{1/4}))$ to $y\zeta \in D(\Delta_0^{1/4})$. The arbitrariness of $y \in M_0$ and $\zeta \in D(\Delta_0^{1/4})$ implies that $D(\Delta_0^{1/4})$ is an M_0-module.
Coming back to the proof of (iii), notice that, by the identity $J\Delta_0^{1/4} = \Delta_0^{-1/4}J$, one has $J\Delta_0^{1/4} = (J\Delta_0^{1/4})^{-1}$ so that the closed operator $J\Delta_0^{1/4}$ is idempotent on its domain:

$$\left(J\Delta_0^{1/4}\right)\xi \in D\left(\Delta_0^{1/4}\right), \qquad \left(J\Delta_0^{1/4}\right)^2\zeta = \zeta, \qquad \forall\,\zeta \in D\left(\Delta_0^{1/4}\right).$$

Thus, for $y \in M_0$ and $\zeta \in D(\Delta_0^{1/4})$, we have $y^*\zeta = y^*(J\Delta_0^{1/4})^2\zeta$ and, since $y^* \in M_0$ implies $y^*\zeta \in D(\Delta_0^{1/4})$, we have $y^*\zeta = (J\Delta_0^{1/4})^2 y^*(J\Delta_0^{1/4})^2\zeta$ too. Applying this identity to $\zeta := J\Delta_0^{1/4}\eta$ for any $\eta \in D(\Delta_0^{1/4})$, we have $(J\Delta_0^{1/4})^{-1}y^*(J\Delta_0^{1/4})\eta = (J\Delta_0^{1/4})y^*\left(J\Delta_0^{1/4}\right)^{-1}\eta$, i.e.

$$\Delta_0^{-1/4}Jy^*J\Delta_0^{1/4}\eta = J\Delta_0^{1/4}y^*\Delta_0^{-1/4}J\eta \qquad \eta \in D(\Delta_0^{1/4}).$$

Since, by hypotheses, $X\xi_0 \in D(\Delta_0^{1/2})$, we may apply the identity to $\eta := \Delta_0^{1/4}X\xi_0 \in D(\Delta_0^{1/4})$, to get

$$\begin{aligned}\Delta_0^{-1/4}Jy^*J\Delta_0^{1/4}\left(\Delta_0^{1/4}X\xi_0\right) &= J\Delta_0^{1/4}y^*\Delta_0^{-1/4}J\Delta_0^{1/4}X\xi_0\\ &= J\sigma^{\omega}_{-i/4}(y^*)J\Delta_0^{1/4}\xi = L_\xi i_0(y).\end{aligned}$$

Since, by (ii), $Xy\xi_0 \in D(S_0) = D(\Delta_0^{1/2}) \subseteq D(\Delta_0^{1/4})$ and $S_0^{-1} = \Delta_0^{-1/2}J$, we have

$$Xy\xi_0 = S_0^{-1}(S_0(Xy\xi_0)) = S_0^{-1}(y^*X^*\xi_0) = \Delta_0^{-1/2}Jy^*J\Delta_0^{1/2}X\xi_0$$

and we conclude the proof of (iii) by

$$\Delta_0^{1/4}Xy\xi_0 = \Delta_0^{-1/4}Jy^*J\Delta_0^{1/4}\left(\Delta_0^{1/4}X\xi_0\right) = L_\xi i_0(y).$$

To prove (iv), notice first that, since S_0^2 is the identity operator on $D(S_0)$ and $X\xi_0 \in D(S_0)$, we have $X^*\xi_0 = S_0(X\xi_0) \in D(S_0) = D(\Delta_0^{1/2})$, $X\xi_0 = S_0(X^*\xi_0)$, $\Delta_0^{1/4}X^*\xi_0 \in D(\Delta_0^{1/4})$ and, for all $y \in M$, $J\Delta_0^{1/4}X^*\xi_0 \in D(\Delta_0^{-1/4}) = D(\Delta_0^{1/4}y\Delta_0^{-1/4})$. Since, as shown above, $yX\xi_0 \in D(\Delta_0^{1/4})$ for all $y \in M_0$ as

$X\xi_0 \in D(\Delta_0^{1/2}) \subset D(\Delta_0^{1/4})$ and $J\Delta_0^{1/2} = \Delta_0^{-1/4} J \Delta_0^{1/4}$ as closed operators, we have

$$\begin{aligned}\Delta_0^{1/4} y X\xi_0 &= \Delta_0^{1/4} y S_0(X^*\xi_0) = \Delta_0^{1/4} y J \Delta_0^{1/2} X^*\xi_0 = \Delta_0^{1/4} y \Delta_0^{-1/4} J \Delta_0^{1/4} S_0(\xi) \\ &= \sigma^{\omega_0}_{-i/4}(y) J \Delta_0^{1/4} S_0(\xi) = R_\xi i_0(y).\end{aligned}$$

To prove (v), i.e. that $\overline{L_\xi}$ is affiliated with M, let us start to notice that for $z' \in M_0'$ and $y \in M_0$, setting $z := J\Delta_0^{-1/4} z'^* \Delta_0^{1/4} J = \Delta_0^{1/4} J z'^* J \Delta_0^{-1/4} = \sigma^{\omega_0}_{-1/4}(Jz'^*J) \in M$, since $\Delta_0^{-1/4} J = J\Delta_0^{1/4}$ and $\Delta_0^{-1/2} z'\xi_0 \in D(\Delta_0^{1/2}) \subset D(\Delta_0^{1/4}) = D(J\Delta_0^{1/4})$, we have $z\xi_0 = J\Delta_0^{-1/4} z'^* \Delta_0^{1/4} J\xi_0 = J\Delta_0^{-1/4} z'^*\xi_0 = J\Delta_0^{-1/4} J \Delta_0^{-1/2} z'\xi_0 == JJ\Delta_0^{1/4}\Delta_0^{-1/2} z'\xi_0 = \Delta_0^{-1/4} z'\xi_0$, $z'\xi_0 = \Delta_0^{1/4} z\xi_0 = \sigma^{\omega_0}_{-i/4}(z)\xi_0$ and

$$\begin{aligned}z' i_0(y) &= z'\Delta_0^{1/4} y \Delta_0^{-1/4}\xi_0 = z'\sigma^{\omega_0}_{-i/4}(y)\xi_0 = \sigma^{\omega_0}_{-i/4}(y) z'\xi_0 \\ &= \sigma^{\omega_0}_{-i/4}(y)\sigma^{\omega_0}_{-i/4}(z)\xi_0 = \sigma^{\omega_0}_{-i/4}(yz)\xi_0 = i_0(yz)\end{aligned}$$

so that $z' i_0(y) \in M_0\xi_0 = D(L_\xi)$. Since $\sigma^{\omega_0}_{i/4}(z') \in M'$, $y \in M$, X is affiliated to M, $y\xi_0 \in D(X)$, $yz\xi_0 \in D(X)$ by i), using iii) we have

$$\begin{aligned}L_\xi z' i_0(y) = L_\xi i_0(yz) &= \Delta_0^{1/4} X y z \xi_0 \\ &= \Delta_0^{1/4} X y \Delta_0^{-1/4} z'\xi_0 \\ &= \Delta_0^{1/4} X y \Delta_0^{-1/4} z' \Delta_0^{1/4}\xi_0 \\ &= \Delta_0^{1/4} X y \sigma^{\omega_0}_{i/4}\left(z'\right)\xi_0 \\ &= \Delta_0^{1/4} \sigma^{\omega_0}_{i/4}\left(z'\right) X y \xi_0 \\ &= \Delta_0^{1/4} \sigma^{\omega_0}_{i/4}\left(z'\right) \Delta_0^{-1/4}\Delta_0^{1/4} X y \xi_0 \\ &= \sigma^{\omega_0}_{-i/4}\left(\sigma^{\omega_0}_{i/4}\left(z'\right)\right) L_\xi i_0(y) \\ &= z' L_\xi i_0(y).\end{aligned}$$

Since by Lemma 2.7 $\mathcal{R}_\eta = J\mathcal{L}_\eta J$, for $\eta \in D(S_0)$ we have $R_\xi = JL_\xi J$ for $\xi \in D(\Delta_0^{1/2})$. This is equivalent to $(J \oplus J)\mathcal{G}(R_\xi) = \mathcal{G}(L_\xi)$ and implies $(J \oplus J)\mathcal{G}(\overline{R_\xi}) = \mathcal{G}(\overline{L_\xi})$, i.e. $\overline{R_\xi} = J\overline{L_\xi}J$ as an identity between densely defined closed operators.
To prove (vi), notice that, by (i) we have $i_0(y) \in D(\Delta_0^{-1/4})$ and $\Delta_0^{-1/4} i_0(y) = y\xi_0 \in D(X)$ and by ii) we have that $X\Delta_0^{-1/4} i_0(y) = Xy\xi_0 \in D(S_0) \subseteq D(\Delta_0^{1/4})$. Hence $i_0(y) \in D(\Delta_0^{1/4} X \Delta_0^{-1/4})$ and $(\Delta_0^{1/4} X \Delta_0^{-1/4}) i_0(y) = \Delta_0^{1/4} X y\xi_0 = L_\xi i_0(y)$.
To prove (vii), notice that, since $y\xi_0 \in D(S_0)$ and $S_0 = \Delta_0^{-1/4} J \Delta_0^{1/4}$ on $D(S_0)$, we have $y^*\xi_0 = S_0(y\xi_0) = \Delta_0^{-1/4} J \Delta_0^{1/4} y\xi_0 = \Delta_0^{-1/4} J i_0(y)$. By i) and ii), $y^*\xi_0 \in D(X^*)$, $X^* y^*\xi_0 \in D(S_0)$ and $yX\xi_0 = S_0(X^* y^*\xi_0) \in D(S_0) \subset D(\Delta_0^{1/4})$ so that by (iv), $R_\xi i_0(y) = \Delta_0^{1/4} y X\xi_0 = (\Delta_0^{1/4} S_0)(X^* y^*\xi_0)$. Since $\Delta_0^{1/4} S_0 = J\Delta_0^{1/4}$ on $D(S_0)$, we have

$$R_\xi i_0(y) = \left(\Delta_0^{1/4} S_0\right)\left(X^* y^* \xi_0\right) = \left(J\Delta_0^{1/4}\right)\left(X^* y^* \xi_0\right) = \left(J\Delta_0^{1/4} X^* \Delta_0^{-1/4} J\right) i_0(y)$$

showing that $J\Delta_0^{1/4} X^* \Delta_0^{-1/4} J$ is densely defined on $i_0(M_0)$ and there it coincides with R_ξ.
To prove the first identity in (viii), notice that, by the Spectral Theorem, $\xi = X\xi_0$ is an eigenvector of $\Delta_0^{1/4}$ with eigenvalue $\lambda > 0$: $\Delta_0^{1/4} X\xi_0 = \lambda \cdot X\xi_0$. By the density of $M_0'\xi_0$ in $L^2(M)$ and for all $z' \in M_0'$ we then have

$$\begin{aligned}
\left(z'\xi_0 | L_\xi i_0(y)\right) &= \left(z'\xi_0 | \Delta_0^{1/4} X y \xi_0\right) \\
&= \left(\Delta_0^{1/4} z' \Delta_0^{-1/4} \xi_0 | X y \xi_0\right) \\
&= \left(\Delta_0^{1/4} z' \Delta_0^{-1/4} X^* \xi_0 | y \xi_0\right) \\
&= \left(\Delta_0^{1/4} z' \Delta_0^{-1/4} J \Delta_0^{1/2} X \xi_0 | y \xi_0\right) \\
&= \lambda^2 \cdot \left(\Delta_0^{1/4} z' \Delta_0^{-1/4} J X \xi_0 | y \xi_0\right) \\
&= \lambda^2 \cdot \left(\Delta_0^{1/4} z' J \Delta_0^{1/4} X \xi_0 | y \xi_0\right) \\
&= \lambda^2 \cdot \left(z' J \Delta_0^{1/4} X \xi_0 | \Delta_0^{1/4} y \xi_0\right) \\
&= \lambda^2 \cdot \left(z' J \Delta_0^{1/4} X \xi_0 | i_0(y)\right) \\
&= \lambda^3 \cdot \left(z' J X \xi_0 | i_0(y)\right) \\
&= \lambda \cdot \left(z' X^* \xi_0 | i_0(y)\right) \\
&= \left(z'\xi_0 | \lambda X i_0(y)\right).
\end{aligned}$$

To prove the second identity in (viii), we first need to show that the adjoint of the densely defined operator $(\Delta_0^{1/4} X \Delta_0^{-1/4}, i_0(M_0))$ (which is closable by (operator $(\Delta_0^{-1/4} X^* \Delta_0^{1/4}, M_0\xi_0)$. By ii), for $x \in M_0$ and since $i_0(x) = \sigma_{-i/4}^{\omega_0}(x)\xi_0 \in M_0\xi_0 \subset D(X^*)$, we have $JX^* i_0(x) = JX^* J J i_0(x) = JX^* J i_0(x^*) = JX^* J \sigma_{-i/4}^{\omega_0}(x^*)\xi_0$. Since JX^*J is affiliated to M', $\sigma_{-i/4}^{\omega_0}(x^*) \in M$ and by ii) $\sigma_{-1/4}^{\omega_0}(x^*)\xi_0 \in D(JX^*J)$, we have

$$\begin{aligned}
JX^* i_0(x) &= \sigma_{-i/4}^{\omega_0}(x^*) JX^* J\xi_0 = \sigma_{-i/4}^{\omega_0}(x^*) JX^*\xi_0 = \sigma_{-i/4}^{\omega_0}(x^*) J S_0(X\xi_0) \\
&= \sigma_{-i/4}^{\omega_0}(x^*) \Delta_0^{1/2}(X\xi_0).
\end{aligned}$$

The hypothesis that $X\xi_0$ is an eigenvalue of $\Delta_0^{1/2}$ then implies $JX^* i_0(x) = \lambda^2 \cdot \sigma_{-i/4}^{\omega_0}(x^*) X\xi_0$ which in turn, by ii), implies $JX^* i_0(x) = \lambda^2 \cdot S_0(X^* \sigma_{i/4}^{\omega_0}(x)\xi_0) \in D(S_0) = D(\Delta_0^{1/2}) \subset D(\Delta_0^{1/4})$ so that $X^* \Delta_0^{1/4} x\xi_0 = X^* i_0(x) \in JD(\Delta_0^{1/4}) = D(\Delta_0^{-1/4})$ and $x\xi_0 \in D(\Delta_0^{-1/4} X^* \Delta_0^{1/4})$. For all $y \in M_0$ we may then compute

$$\begin{aligned}\left(x\xi_0 \middle| \left(\Delta_0^{1/4} X \Delta_0^{-1/4}\right) i_0(y)\right) &= \left(\Delta_0^{1/4} x\xi_0 \middle| \left(X \Delta_0^{-1/4}\right) i_0(y)\right) \\ &= \left(X^* \Delta_0^{1/4} x\xi_0 | \Delta_0^{-1/4} i_0(y)\right) \\ &= \left(\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) x\xi_0 | i_0(y)\right)\end{aligned}$$

so that $x\xi_0 \in D((\Delta_0^{1/4} X \Delta_0^{-1/4})^*)$ and $(\Delta_0^{-1/4} X^* \Delta_0^{1/4}) x\xi_0 = (\Delta_0^{1/4} X \Delta_0^{-1/4})^* x\xi_0$. Since by vi) and the first identity proved above we have

$$\left(\Delta_0^{1/4} X \Delta_0^{-1/4}\right) x\xi_0 = L_\xi x\xi_0 = \lambda \cdot X x\xi_0$$

for all $x\xi_0 \in M_0\xi_0 = i_0(M_0) = D(L_\xi) \subset D(\Delta_0^{1/4} X \Delta_0^{-1/4}) \cap D(X)$, by i) we then have

$$\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) x\xi_0 = \lambda X^* x\xi_0 \qquad x\xi_0 \in M_0\xi_0 \subset D\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) \cap D(X^*).$$

To finalize the proof of the second identity in viii), rewrite the eigenvalue equation satisfied by $\xi = X\xi_0$ as $JX^*J\xi_0 = \Delta_0^{1/2} X\xi_0 = \lambda^2 X\xi_0$ so that, for all $y \in M_0$, we have

$$\begin{aligned}R_\xi i_0(y) &= \Delta_0^{1/4} y X\xi_0 \\ &= \lambda^{-2} \cdot \Delta_0^{1/4} y (JX^*J)\xi_0 \\ &= \lambda^{-2} \cdot \Delta_0^{1/4} (JX^*J) y\xi_0 \\ &= \lambda^{-2} \cdot J\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) J \Delta_0^{1/4} y\xi_0 \\ &= \lambda^{-2} \cdot J\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) J i_0(y) \\ &= \lambda^{-2} \cdot J\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) i_0(y^*) \\ &= \lambda^{-2} \cdot J\left(\Delta_0^{-1/4} X^* \Delta_0^{1/4}\right) \sigma_{-i/4}^{\omega_0}(y^*)\xi_0 \\ &= \lambda^{-2} \cdot JX^* \sigma_{-i/4}^{\omega_0}(y^*)\xi_0 \\ &= \lambda^{-2} \cdot JX^* J i_0(y).\end{aligned}$$

□

Lemma 2.9 *Let $\xi \in D(S_0)$ be eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda^2 > 0$ and fix a densely defined, closed operator $(X, D(X))$ affiliated to M such that*

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(X\xi_0) = X^*\xi_0.$$

Then $S_0\xi \in D(S_0) = D(\Delta_0^{1/2})$ is eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue λ^{-2}.

Proof On one hand we have $S_0\xi = J\Delta_0^{1/2}\xi = \lambda^2 \cdot J\xi$. On the other hand, since $JD(\Delta_0^{1/2}) = D(\Delta_0^{-1/2})$ and $S_0 = S_0^{-1}$ on $D(\Delta_0^{1/2}) = D(S_0) = D(S_0^{-1})$ we have $S_0\xi = S_0^{-1}\xi = \Delta_0^{-1/2}J\xi$ so that $\Delta_0^{1/2}S_0\xi = J\xi = \lambda^{-2} \cdot S_0\xi$. □

Combining the results obtained, we have

Corollary 2.10 *Let $\xi \in D(S_0)$ be an eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda^2 > 0$ and fix a densely defined, closed operator $(X, D(X))$ affiliated to M such that*

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(X\xi_0) = X^*\xi_0.$$

Then for all $y \in M_0$ we have

$$(L_\xi - R_\xi)i_0(y) = \Delta_0^{1/4}(Xy - yX)\xi_0 \tag{2.1}$$

and

$$(L_{S_0\xi} - R_{S_0\xi})i_0(y) = \Delta_0^{1/4}(X^*y - yX^*)\xi_0. \tag{2.2}$$

The commutator $[X, y] := Xy - yX$ is in general only densely defined if X is affiliated to M but, within the hypotheses assumed at the beginning of this section, the vector ξ_0 belongs to the domain of $[X, y]$ and its image $[X, y]\xi_0$ belongs to $D(\Delta_0^{1/4})$. This may justify the notation

$$i_0([X, y]) := \Delta_0^{1/4}(Xy - yX)\xi_0 \qquad y \in M_0.$$

In the following we will use the notation $j(X^*) := JX^*J$.
Next result shows that the symmetric embedding i_0 intertwines the unbounded spatial derivations δ_X, δ_{X^*} on M with the unbounded bimodule derivations d_X^λ, $d_{X^*}^{\lambda^{-1}}$ on $L^2(M)$.

Proposition 2.11 (Bimodule derivations and spatial derivations) *Let $\xi \in D(S_0)$ be eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda^2 > 0$ and fix a densely defined, closed operator $(X, D(X))$ affiliated to M such that*

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(X\xi_0) = X^*\xi_0.$$

Then, setting $d_X^\lambda := i(\lambda X - \lambda^{-1}j(X^*))$ and $d_{X^*}^{\lambda^{-1}} := i(\lambda^{-1}X^* - \lambda j(X))$, we have

$$d_X^\lambda i_0(y) = i_0(i[X, y]) \qquad d_{X^*}^{\lambda^{-1}} i_0(y) = i_0(i[X^*, y]) \qquad y \in M_0.$$

Otherwise stated, setting $\delta_X(y) := i[X, y]$ for any $y \in M_0$, on the *-algebra M_0 we have

$$d_X^\lambda \circ i_0 = i_0 \circ \delta_X \qquad d_{X^*}^{\lambda^{-1}} \circ i_0 = i_0 \circ \delta_{X^*}.$$

Proof For $y \in M_0$ we have $\sigma^{\omega_0}_{-i/4}(y) \in M_0$, $J\sigma^{\omega_0}_{-i/4}(y)J \in M_0'$ and

$$J\sigma^{\omega_0}_{-i/4}(y)J\xi_0 = J\sigma^{\omega_0}_{-i/4}(y)\xi_0 = J\Delta_0^{1/4}y\xi_0 = Ji_0(y).$$

Since X^* is affiliated with M and $\xi_0 \in D(X^*)$, we have $(J\sigma^{\omega_0}_{-i/4}(y)J)\xi_0 \in D(X^*)$ and

$$\begin{aligned} j(X^*)i_0(y) &= JX^*Ji_0(y) \\ &= JX^*\left(J\sigma^{\omega_0}_{-i/4}(y)J\right)\xi_0 \\ &= J\left(J\sigma^{\omega_0}_{-i/4}(y)J\right)X^*\xi_0 \\ &= \sigma^{\omega_0}_{-i/4}(y)JX^*\xi_0 \\ &= \sigma^{\omega_0}_{-i/4}(y)\Delta_0^{1/2}X\xi_0 \\ &= \Delta_0^{1/4}y\Delta_0^{1/4}X\xi_0 \\ &= \lambda \cdot \Delta_0^{1/4}yX\xi_0. \end{aligned}$$

Since $\lambda Xi_0(y) = L_\xi i_0(y) = \Delta_0^{1/4}Xy\xi_0$ we have too $d_X^\lambda i_0(y) := i(\lambda X - \lambda^{-1}j(X^*))i_0(y) = i\Delta_0^{1/4}(Xy\xi_0 - yX\xi_0) = i_0(i[X, y]) = i_0(\delta_X(y))$. The proof of the second identity is similar. □

Theorem 2.12 *Let $\xi \in D(S_0)$ be an eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda^2 > 0$ and $(X, D(X))$ a densely defined, closed operator affiliated to M such that*

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(\xi) = X^*\xi_0.$$

Then the completely Dirichlet form $(\mathcal{E}_X^\lambda, \mathcal{F}_X^\lambda)$ constructed above may be represented as

$$\begin{aligned} \mathcal{E}_X^\lambda[i_0(y)] &= \|i_0([X, y])\|^2_{L^2(M)} + \|i_0([X^*, y])\|^2_{L^2(M)} \qquad y \in M_0 \\ &= \|i_0(\delta_X(y))\|^2_{L^2(M)} + \|i_0(\delta_{X^*}(y))\|^2_{L^2(M)} \end{aligned}$$

on the $L^2(M)$-dense, J-invariant subspace $M_0\xi_0 = i_0(M) \subset \tilde{\mathcal{F}}_X \subset \mathcal{F}_X^\lambda$.

Remark 2.13 These results prove a fortiori that and under the stated assumptions, the form

$$i_0(y) \mapsto \lambda^2\|i_0([X, y])\|^2_{L^2(M)} + \|i_0([X^*, y])\|^2_{L^2(M)}$$

extends to a completely Dirichlet form on $L^2(M)$ with respect to the cyclic vector $\xi_0 \in L^2_+(M)$. If ξ_0 would be the vector representing a finite, normal, faithful trace state ω_0, this result would follow from the general theory relating completely Dirichlet

forms and closable bimodule derivations on von Neumann algebras with trace (see [14]).

3 Coercivity of Dirichlet forms

In this section we still keep the assumption that $\xi \in D(S_0)$ is an eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda^2 > 0$ (we still assume $\lambda > 0$) and $(X, D(X))$ a densely defined, closed operator affiliated to M such that

$$\xi_0 \in D(X) \cap D(X^*), \qquad \xi = X\xi_0, \quad S_0(\xi) = X^*\xi_0.$$

We prove below natural lower bounds on the Dirichlet form $(\mathcal{E}_X^\lambda, \mathcal{F}_X^\lambda)$ constructed in Sect. 2, which lead to coercivity. Recall that $(\mathcal{E}_X^\lambda, \mathcal{F}_X^\lambda)$ is defined as the closure of the densely defined, J-real, closable quadratic form $\tilde{\mathcal{E}}_X^\lambda : \tilde{\mathcal{F}}_X \to [0, +\infty)$ on $L^2(M)$ given by

$$\tilde{\mathcal{E}}_X^\lambda[\eta] := \left\| d_X^\lambda \eta \right\|_{L^2(M)}^2 + \left\| d_{X^*}^{\lambda^{-1}} \eta \right\|_{L^2(M)}^2 \qquad \eta \in \tilde{\mathcal{F}}_X = D(d_X^\lambda) \cap D(d_{X^*}^{\lambda^{-1}}),$$

where $d_X^\lambda := i(\lambda X - \lambda^{-1} j(X^*))$ and $d_{X^*}^{\lambda^{-1}} := i(\lambda^{-1} X^* - \lambda j(X))$ are defined on the domain

$$\tilde{\mathcal{F}}_X = D(X) \cap D(X^*) \cap J\left(D(X) \cap D(X^*)\right)$$

containing the $L^2(M)$-dense, J-invariant subspace $i_0(M_0) = M_0\xi_0 \subset \tilde{\mathcal{F}}_X$. Obviously $\tilde{\mathcal{F}}_X$ is a form core for $(\mathcal{E}_X^\lambda, \mathcal{F}_X^\lambda)$ and on it $\mathcal{E}_X^\lambda$ and $\tilde{\mathcal{E}}_X^\lambda$ coincide.
We start showing an alternative representation of the Dirichlet form.

Theorem 3.1 *The following representation holds true for the quadratic form $(\mathcal{E}_X^\lambda, \tilde{\mathcal{F}}_X)$:*

$$\begin{aligned} \mathcal{E}_X^\lambda[\eta] = \lambda^2 \left(\|X\eta\|^2 + \|XJ\eta\|^2 \right) + \lambda^{-2} \left(\|X^*\eta\|^2 + \|X^*J\eta\|^2 \right) \\ - 2\left[(X\eta | JX^*J\eta) + (X^*\eta | JXJ\eta) \right] \qquad \eta \in \tilde{\mathcal{F}}_X. \end{aligned} \tag{3.1}$$

Proof In the following, we repeatedly use the fact that if $N \subseteq B(h)$ is a von Neumann algebra acting on a Hilbert space h and $(A, D(A))$, $(B, D(B))$ are densely defined, closed operator on h affiliated to N and N', respectively, then

$$(A\eta | B\zeta) = \left(B^*\eta | A^*\zeta\right) \qquad \eta \in D(A) \cap D(B^*), \quad \zeta \in D(B) \cap D(A^*).$$

This identity follows directly if $B \in M'$ is bounded since then $B^* \in M'$ and $\eta \in D(A)$ implies $B^*\eta \in D(A)$ and $AB^*\eta = B^*A\eta$ so that $(A\eta | B\zeta) = (B^*A\eta | \zeta) = (AB^*\eta | \zeta) = (B^*\eta | A^*\zeta)$. In general we may approximate B weakly by $B_\varepsilon := B(I + \varepsilon|B|)^{-1} \in M'$ as $\varepsilon \downarrow 0$.

We start the proof of the result setting

$$d_X := i\left(X - j(X^*)\right), \qquad V_X^\lambda := i\left(1-\lambda^{-1}\right)\left(\lambda X + j(X^*)\right)$$

and using the splittings

$$d_X^\lambda = d_X + V_X^\lambda \qquad d_{X^*}^{\lambda^{-1}} = d_{X^*} + V_{X^*}^{\lambda^{-1}},$$

for any $\eta \in \tilde{\mathcal{F}}_X$ to have the representation

$$\begin{aligned}\mathcal{E}_X^\lambda[\eta] := \|d_X\eta\|^2 + \|d_{X^*}\eta\|^2 + \left\|V_X^\lambda\eta\right\|^2 + \left\|V_{X^*}^{\lambda^{-1}}\eta\right\|^2 + \\ \left(d_X\eta|V_X^\lambda\eta\right) + \left(V_X^\lambda\eta|d_X\eta\right) + \left(d_{X^*}\eta|V_{X^*}^{\lambda^{-1}}\eta\right) + \left(V_{X^*}^{\lambda^{-1}}\eta|d_{X^*}\eta\right).\end{aligned} \tag{3.2}$$

Since

$$\begin{aligned}\|d_X\eta\|^2 &= \left\|(X - j(X^*))\eta\right\|^2 = \|X\eta\|^2 + \|j(X^*)\eta\|^2 \\ &\quad - \left[(X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)\right], \\ \|d_{X^*}\eta\|^2 &= \|(X^* - j(X))\eta\|^2 = \|X^*\eta\|^2 + \|j(X)\eta\|^2 \\ &\quad - \left[(X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)\right]\end{aligned}$$

the sum of the first two addends in (3.2) equals

$$\begin{aligned}\|d_X\eta\|^2 + \|d_{X^*}\eta\|^2 = \|X\eta\|^2 + \|j(X^*)\eta\|^2 + \|X^*\eta\|^2 + \|j(X)\eta\|^2 \\ - 2[(X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)].\end{aligned} \tag{3.3}$$

Since also

$$\begin{aligned}\|V_X^\lambda\eta\|^2 &= (1-\lambda^{-1})^2((\lambda X + j(X^*))\eta|(\lambda X + j(X^*))\eta) \\ &= (1-\lambda^{-1})^2\big[\lambda^2\|X\eta\|^2 + \|j(X^*)\eta\|^2 + \\ &\quad \lambda((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)\big], \\ \|V_{X^*}^{\lambda^{-1}}\eta\|^2 &= (1-\lambda)^2((\lambda^{-1}X^* + j(X))\eta|(\lambda^{-1}X^* + j(X))\eta) \\ &= (1-\lambda)^2\big[\lambda^{-2}\|X^*\eta\|^2 + \|j(X)\eta\|^2 + \\ &\quad \lambda^{-1}((X^*\eta|j(X)\eta) + (X\eta|j(X^*)\eta)\big]\end{aligned}$$

the sum of the third and fourth addends in (3.2) equals

$$\begin{aligned}
&\|V_X^\lambda \eta\|^2 + \|V_{X^*}^{\lambda^{-1}} \eta\|^2 = \\
&(\lambda-1)^2(\|X\eta\|^2 + \|j(X)\eta\|^2) + (\lambda^{-1}-1)^2(\|X^*\eta\|^2 + \|j(X^*)\eta\|^2) + \\
&[(1-\lambda^{-1})^2\lambda + (1-\lambda)^2\lambda^{-1}]((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)) = \\
&(\lambda-1)^2(\|X\eta\|^2 + \|j(X)\eta\|^2) + (\lambda^{-1}-1)^2(\|X^*\eta\|^2 + \|j(X^*)\eta\|^2) + \\
&2(\lambda-1)^2\lambda^{-1}\big((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta)\big).
\end{aligned} \tag{3.4}$$

Since we have too

$$\begin{aligned}
&(d_X\eta|V_X^\lambda\eta) + (V_X^\lambda\eta|d_X\eta) = \\
&(1-\lambda^{-1})\big[((X-j(X^*))\eta|(\lambda X + j(X^*))\eta) + ((\lambda X + j(X^*))\eta|(X-j(X^*))\eta)\big] = \\
&(1-\lambda^{-1})\big[\lambda\|X\eta\|^2 + (X\eta|j(X^*)\eta) - \lambda(X^*\eta|j(X)\eta) - \|j(X^*)\eta\|^2 + \\
&\lambda\|X\eta\|^2 - \lambda(X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta) - \|j(X^*)\eta\|^2\big],
\end{aligned}$$

the sum of the fifth and sixth addends in (3.2) equals

$$\begin{aligned}
&(d_X\eta|V_X^\lambda\eta) + (V_X^\lambda\eta|d_X\eta) = \\
&(1-\lambda^{-1})\big[2\lambda\|X\eta\|^2 - 2\|j(X^*)\eta\|^2 + \\
&(1-\lambda)((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta))\big]
\end{aligned} \tag{3.5}$$

and, analogously, the sum of the seventh and eighth addends in (3.2) equals

$$\begin{aligned}
&\Big(d_{X^*}\eta|V_{X^*}^{\lambda^{-1}}\eta\Big) + \Big(V_{X^*}^{\lambda^{-1}}\eta|d_{X^*}\eta\Big) \\
&\quad = (1-\lambda)\big[2\lambda^{-1}\|X^*\eta\|^2 - 2\|j(X)\eta\|^2 \\
&\qquad + (1-\lambda^{-1})((X^*\eta|j(X)\eta) + (X\eta|j(X^*)\eta))\big].
\end{aligned} \tag{3.6}$$

By substitution of (3.6), (3.5) and (3.4) in (3.2) we obtain

$$\begin{aligned}
&\mathcal{E}_X^\lambda[\eta] - \big(\|d_X\eta\|^2 + \|d_{X^*}\eta\|^2\big) \\
&= (\lambda-1)^2(\|X\eta\|^2 + \|j(X)\eta\|^2) + (\lambda^{-1}-1)^2(\|X^*\eta\|^2 + \|j(X^*)\eta\|^2) \\
&\quad + 2(\lambda-1)^2\lambda^{-1}((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta) \\
&\quad + (1-\lambda^{-1})\big[2\lambda\|X\eta\|^2 - 2\|j(X^*)\eta\|^2 \\
&\quad + (1-\lambda)((X\eta|j(X^*)\eta) + (X^*\eta|j(X)\eta))\big] \\
&\quad + (1-\lambda)\big[2\lambda^{-1}\|X^*\eta\|^2 - 2\|j(X)\eta\|^2 \\
&\quad + (1-\lambda^{-1})((X^*\eta|j(X)\eta) + (X\eta|j(X^*)\eta))\big] \\
&= \big[(\lambda-1)^2 + 2(\lambda-1)\big](\|X\eta\|^2 + \|j(X)\eta\|^2) \\
&\quad + \big[(\lambda^{-1}-1)^2 + 2(\lambda^{-1}-1)\big](\|X^*\eta\|^2 + \|j(X^*)\eta\|^2)
\end{aligned}$$

$$
\begin{aligned}
&+\left[2(1-\lambda)(1-\lambda^{-1})+2(\lambda-1)^2\lambda^{-1}\right]((X\eta|j(X^*)\eta)+(X^*\eta|j(X)\eta))\\
&=(\lambda^2-1)(\|X\eta\|^2+\|j(X)\eta\|^2)+(\lambda^{-2}-1)(\|X^*\eta\|^2+\|j(X^*)\eta\|^2)\\
&+\left[2(1-\lambda)(\lambda-1)\lambda^{-1}+2(\lambda-1)^2\lambda^{-1}\right]((X\eta|j(X^*)\eta)+(X^*\eta|j(X)\eta))\\
&=(\lambda^2-1)(\|X\eta\|^2+\|j(X)\eta\|^2)+(\lambda^{-2}-1)(\|X^*\eta\|^2+\|j(X^*)\eta\|^2)\\
&+\left[-2(\lambda-1)^2\lambda^{-1}+2(\lambda-1)^2\lambda^{-1}\right]((X\eta|j(X^*)\eta)+(X^*\eta|j(X)\eta))\\
&=(\lambda^2-1)(\|X\eta\|^2+\|j(X)\eta\|^2)+(\lambda^{-2}-1)(\|X^*\eta\|^2+\|j(X^*)\eta\|^2)
\end{aligned}
$$

and then, by (3.3), we finally obtain (3.1) for any $\eta\in\tilde{\mathcal{F}}_X$

$$
\begin{aligned}
\mathcal{E}_X^\lambda[\eta]=&\left(\|d_X\eta\|^2+\|d_{X^*}\eta\|^2\right)+(\lambda^2-1)(\|X\eta\|^2+\|j(X)\eta\|^2)+\\
&(\lambda^{-2}-1)(\|X^*\eta\|^2+\|j(X^*)\eta\|^2)\\
&=\lambda^2(\|X\eta\|^2+\|XJ\eta\|^2)+\lambda^{-2}(\|X^*\eta\|^2+\|X^*J\eta\|^2)-\\
&2\left[(X\eta|JX^*J\eta)+(X^*\eta|JXJ\eta)\right].
\end{aligned}
$$

□

Corollary 3.2 (Lower bound) *The following lower bounds hold true for any $\varepsilon,\delta>0$ and any $\eta\in\tilde{\mathcal{F}}_X$*

$$
\begin{aligned}
\mathcal{E}_X^\lambda[\eta]\geq&\left(\lambda^2-\varepsilon^2\right)\|X\eta\|^2+\left(\lambda^2-\delta^{-2}\right)\|XJ\eta\|^2+(\lambda^{-2}-\delta^2)\|X^*\eta\|^2+\\
&(\lambda^{-2}-\varepsilon^{-2})\|X^*J\eta|\|^2.
\end{aligned}
\tag{3.7}
$$

In particular, for $\varepsilon=\delta=1$ and any $\eta\in\tilde{\mathcal{F}}_X$ we have

$$
\mathcal{E}_X^\lambda[\eta]\geq(\lambda^2-1)\left(\|X\eta\|^2+\|XJ\eta\|^2\right)+(\lambda^{-2}-1)\left(\|X^*\eta\|^2+\|X^*J\eta|\|^2\right) \tag{3.8}
$$

Proof The result follows from (3.1) and the identities, valid for $\varepsilon,\delta>0$,

$$
\begin{aligned}
\varepsilon^2\|X\eta\|^2+\varepsilon^{-2}\left\|j(X^*)\eta\right\|^2-\left[(X\eta|j(X^*)\eta)+(X^*\eta|j(X)\eta)\right]&=\|d_X^\varepsilon\eta\|^2\geq 0\\
\delta^2\left\|X^*\eta\right\|^2+\delta^{-2}\|j(X)\eta\|^2-\left[(X^*\eta|j(X)\eta)+(X\eta|j(X^*)\eta)\right]&=\|d_{X^*}^\delta\eta\|^2\geq 0.
\end{aligned}
$$

□

We address now the problem to find conditions on $(X,D(X))$ sufficient to guarantee that the lower bounds above are coercive for our Dirichlet form. By this we mean bounds in which the Dirichlet form dominates a quadratic form with a certain degree of discreteness of the spectrum such as existence and finite degeneracy of a ground state, spectral gaps or emptiness of essential spectrum. The conditions will be formulated in terms of relative smallness of the quadratic form of the self-commutator $[X,X^*]$ with respect to the quadratic form of X^*X and they will be exploited in Sect. 5 when M is a type I_∞ factor.

Let us denote by $(t_X, D(t_X))$ and $(t_{X^*}, D(t_X^*))$, the densely defined, positive, closed quadratic forms defined as

$$t_X[\eta] := \|X\eta\|^2 \qquad \eta \in D(t_X) := D(X),$$
$$t_{X^*}[\eta] := \|X^*\eta\|^2 \qquad \eta \in D(t_{X^*}) := D(X^*),$$

whose associated positive, self-adjoint operators are $(X^*X, D(X^*X))$ and $(XX^*, D(XX^*))$.
Consider also the quadratic form $(\tilde{q}_X^\lambda, D(\tilde{q}_X^\lambda))$ given by

$$\tilde{q}_X^\lambda[\eta] := (\lambda^2 - 1)\|X\eta\|^2 + (\lambda^{-2} - 1)\|X^*\eta\|^2 \qquad \eta \in D(\tilde{q}_X^\lambda) := D(X) \cap D(X^*).$$

By the densely defined quadratic form $(q_0, D(q_0))$ defined as

$$q_0[\eta] := t_{X^*}[\eta] - t_X[\eta] = \|X^*\eta\|^2 - \|X\eta\|^2 \qquad \eta \in D(q_0) := D(X) \cap D(X^*),$$

on $D(\tilde{q}_X^\lambda) = D(X) \cap D(X^*)$ we can write

$$\tilde{q}_X^\lambda = \left(\lambda - \lambda^{-1}\right)^2 \cdot t_X + \left(\lambda^{-2} - 1\right) \cdot q_0 = \left(\lambda - \lambda^{-1}\right)^2 \cdot t_{X^*} + \left(1 - \lambda^2\right) \cdot q_0$$

and regard $\tilde{q}_X^\lambda$ as a perturbation of a multiple of t_X or t_{X^*} by a multiple of q_0. Notice that q_0 is the form of the self-commutator $[X, X^*] = XX^* - X^*X$, at least on $D(X^*X) \cap D(XX^*)$.
Using the quadratic form $(\tilde{\mathcal{Q}}_X^\lambda, \tilde{\mathcal{F}}_X)$ given by

$$\tilde{\mathcal{Q}}_X^\lambda[\eta] := \tilde{q}_X^\lambda[\eta] + \tilde{q}_X^\lambda[J\eta] \qquad \eta \in \tilde{\mathcal{F}}_X = D(X) \cap D(X^*) \cap J(D(X) \cap D(X^*)),$$

the lower bound (3.8) can be written as

$$\tilde{\mathcal{Q}}_X^\lambda[\eta] \leq \tilde{\mathcal{E}}_X^\lambda[\eta] \qquad \eta \in \tilde{\mathcal{F}}_X. \tag{3.9}$$

Although $\tilde{\mathcal{Q}}_X^\lambda$ is densely defined, since $i_0(M_0) = M_0\xi_0 \subset \tilde{\mathcal{F}}_X$ by Lemma 2.8 ii), it is not necessarily lower bounded, closable or a proper functional.
For sake of clarity, we recall some definition we will use concerning lower bounded quadratic forms $(\mathcal{A}, D(\mathcal{A}))$, $(\mathcal{B}, D(\mathcal{B}))$ and their associated self-adjoint operators $(A, D(A))$, $(B, D(B))$ on a Hilbert space h (see [19]):

(i) $(\mathcal{A}, D(\mathcal{A}))$ is *ε-bounded w.r.t.* $(\mathcal{B}, D(\mathcal{B}))$ for $\varepsilon > 0$, if $D(\mathcal{B}) \subseteq D(\mathcal{A})$ and $\mathcal{A}[\xi] \leq \varepsilon \cdot \mathcal{B}[\xi] + b_\varepsilon \cdot \|\xi\|^2$ for some $b_\varepsilon \geq 0$ and all $\xi \in D(\mathcal{B})$; the infimum of all such ε is the *form bound* of $(\mathcal{A}, D(\mathcal{A}))$ w.r.t. $(\mathcal{B}, D(\mathcal{B}))$;
(ii) $(\mathcal{A}, D(\mathcal{A}))$ is *small (resp. infinitesimally small) w.r.t.* $(\mathcal{B}, D(\mathcal{B}))$ if its form bound is strictly less than one (resp. vanishes);
(iii) $(A, D(A))$ is *ε-bounded w.r.t.* $(B, D(B))$ for $\varepsilon > 0$, if $D(B) \subseteq D(A)$ and $\|A\xi\|^2 \leq \varepsilon \cdot \|B\xi\|^2 + b_\varepsilon \cdot \|\xi\|^2$ for some $b_\varepsilon \geq 0$ and all $\xi \in D(B)$; the infimum of all such ε is the *operator bound* of $(A, D(A))$ w.r.t. $(B, D(B))$;

(iv) $(A, D(A))$ is *small (resp. infinitesimally small) w.r.t.* $(B, D(B))$ if its operator bound is strictly less than one (resp. vanishes);
(v) $(\mathcal{A}, D(\mathcal{A}))$ is said *an infinitesimal perturbation of* $(\mathcal{B}, D(\mathcal{B}))$ if $D(\mathcal{B}) \subseteq D(\mathcal{A})$ and $(\mathcal{A} - \mathcal{B}, D(\mathcal{B}))$ is infinitesimally small w.r.t. $(\mathcal{B}, D(\mathcal{B}))$;
(vi) $(A, D(A))$ is said *infinitesimally perturbation of* $(B, D(B))$ if $D(B) \subseteq D(A)$ and $(A - B, D(B))$ is infinitesimally small with respect to $(B, D(B))$;
It is well known that (iii) implies (i), (iv) implies (ii) and (vi) implies (v);
(vii) $(A, D(A))$ has *purely discrete spectrum* if this is made by discrete eigenvalues only (isolated eigenvalues of finite degeneracy); by the Min-Max Theorem this holds true if and only if $(\mathcal{A}, D(\mathcal{A}))$ is *proper* in the sense that $\{\xi \in D(\mathcal{A}) : \|\xi\| \leq 1, \mathcal{A}[\xi] \leq 1\}$ is relatively compact in h.

Theorem 3.3 *(*Coercivity*) Assume* $(\tilde{q}^\lambda_X, D(X) \cap D(X^*))$ *to be lower bounded and closable, denote by* $(q^\lambda_X, D(q^\lambda_X))$ *its closure and by* $(Q^\lambda_X, D(Q^\lambda_X))$ *the associated lower bounded, self-adjoint operator. Then*

(i) $(\tilde{\mathcal{Q}}^\lambda_X, \tilde{\mathcal{F}}_X)$ *is lower bounded, closable and its closure* $(\mathcal{Q}^\lambda_X, D(\mathcal{Q}^\lambda_X))$ *bounds the Dirichlet form*

$$\mathcal{Q}^\lambda_X[\eta] \leq \mathcal{E}^\lambda_X[\eta] \qquad \eta \in \mathcal{F}^\lambda_X \subseteq D(\mathcal{Q}^\lambda_X); \tag{3.10}$$

if moreover, the self-adjoint operator associated to $(\mathcal{Q}^\lambda_X, D(\mathcal{Q}^\lambda_X))$ *has discrete spectrum, then the spectrum of the self-adjoint operator* $(H^\lambda_X, D(H^\lambda_X))$ *associated to* $(\mathcal{E}^\lambda_X, \mathcal{F}^\lambda_X)$ *is discrete too.*

(ii) $(Q^\lambda_X, D(Q^\lambda_X))$ *is affiliated to* M, $(j(Q^\lambda_X), JD(Q^\lambda_X))$ *is affiliated to* M' *and* $D(Q^\lambda_X) \cap JD(Q^\lambda_X)$ *is dense in* $L^2(M)$.
Assume now on $D(X) = D(X^*)$, $D(X^*X) = D(XX^*)$ *and the quadratic form* $(q_0, D(q_0))$ *to be infinitesimally small with respect to* $(t_X, D(t_X))$. *Then*

(iii) the form $(\tilde{q}^\lambda_X, D(X))$ *is lower bounded, closed and* $(Q^\lambda_X, D(Q^\lambda_X))$ *equals the Friedrichs extension of the lower bounded, densely defined, symmetric operator*

$$\begin{aligned} D(N^\lambda_X) &:= D(X^*X) = D(XX^*) \\ N^\lambda_X &:= (\lambda - \lambda^{-1})^2 \cdot X^*X + (\lambda^{-2} - 1) \cdot [X, X^*]; \end{aligned} \tag{3.11}$$

(iv) in particular, the conclusions in (iii) subsist if the self-commutator $([X, X^*], D(X^*X))$ *is infinitesimally small w.r.t.* $(X^*X, D(X^*X))$ *and in this case*

$$\left(Q^\lambda_X, D(Q^\lambda_X)\right) = \left(N^\lambda_X, D(X^*X)\right). \tag{3.12}$$

If moreover the spectrum of $(X^*X, D(X^*X))$ *is discrete, then the spectrum of the generator* $(H^\lambda_X, D(H^\lambda_X))$ *of the Dirichlet form is discrete too.*

Proof (i) Since $(\tilde{q}^\lambda_X, D(X) \cap D(X^*))$ is lower bounded and closable and J is isometric, the quadratic form $J(D(X) \cap D(X^*)) \ni \eta \mapsto \tilde{q}^\lambda_X[J\eta]$ is densely defined, lower bounded and closable too. This implies that $(\tilde{\mathcal{Q}}^\lambda_X, \tilde{\mathcal{F}}_X)$ is lower bounded and closable as a sum of forms sharing these same properties. The lower bound

(3.10) follows from (3.9) and the lower boundedness of $(\tilde{\mathcal{Q}}_X^\lambda, \tilde{\mathcal{F}}_X)$. The last assertion concerning discreteness of spectra follows from the Min-Max Theorem.

(ii) Since $(X, D(X))$ and $(X^*, D(X^*))$ are closed operators affiliated to M, it follows that for any unitary $u' \in M'$ we have $u'(D(X) \cap D(X^*)) \subset D(X) \cap D(X^*)$ and $\tilde{q}_X^\lambda[u'\eta] = \tilde{q}_X^\lambda[\eta]$ for any $\eta \in D(X) \cap D(X^*)$. By approximation, these invariance still hold true for the closure $(q_X^\lambda, D(q_X^\lambda))$ and implies that for all unitaries $u' \in M'$ and all $\eta \in D(Q_X^\lambda)$ one has $u'\eta \in D(Q_X^\lambda)$ and $Q_X^\lambda u'\eta = u'Q_X^\lambda \eta$. Hence $(Q_X^\lambda, D(Q_X^\lambda))$ is affiliated to M, $(j(Q_X^\lambda), D(j(Q_X^\lambda)))$ is affiliated to M', the operators strongly commute and have a common dense core.

(iii) Since $D(X) = D(X^*)$ and $(q_0, D(X) \cap D(X^*)) = (q_0, D(X))$ is infinitesimally small with respect to $(t_X, D(X))$, the sum $\tilde{q}_X^\lambda = (\lambda - \lambda^{-1})^2 \cdot t_X + (\lambda^{-2} - 1) \cdot q_0$ is lower bounded and closed since $(t_X, D(X))$ is lower bounded and closed.
Since $q_0 = t_{X^*} - t_X$ is infinitesimally small with respect to t_X on the common domain $D(X) = D(X^*)$, we have that t_{X^*} is relatively bounded with respect to t_X and that t_X is relatively bounded with respect to t_{X^*}. As $D(X^*X) = D(XX^*)$ by assumption, the symmetric operator $(N_X^\lambda, D(N_X^\lambda))$ is densely defined and lower bounded since its quadratic form is the restriction of the lower bounded form $(\tilde{q}_X^\lambda, D(X))$ to $D(X^*X)$, i.e. $(\eta | N_X^\lambda \eta) = \tilde{q}_X^\lambda[\eta]$ for all $\eta \in D(X^*X)$. Since $D(X^*X)$ is form core for $(t_X, D(X))$ and $(\tilde{q}_X^\lambda, D(X))$ is an infinitesimal perturbation of a multiple of it, $D(X^*X)$ is a form core for $(\tilde{q}_X^\lambda, D(X))$ too. Since, by definition, the Friedrichs extension of $(N_X^\lambda, D(X^*X))$ is the self-adjoint operator associated to the closure of its quadratic form $(\tilde{q}_X^\lambda, D(X^*X))$, it results that $(Q_X^\lambda, D(Q_X^\lambda))$ coincides with it.

iv) In this case the operator $(N_X^\lambda, D(N_X^\lambda))$ is an infinitesimal symmetric perturbation of a multiple of the self-adjoint operator $(X^*X, D(X^*X))$ and it is self-adjoint by the Kato-Rellich Theorem. Since it is also lower bounded, it has to coincides with its Friedrichs extension $(Q_X^\lambda, D(Q_X^\lambda))$.
To prove the last assertion, recall that the spectrum of a lower bounded self-adjoint operator is discrete if and only if its associated quadratic form is proper (see [15]). Now, by a general corollary of the Min-Max Theorem, if the spectrum of $(\lambda - \lambda^{-1})^2 X^*X$ is discrete, then the spectrum of N_X^λ is discrete too, as the latter operator is the sum of the former and the lower bounded self-adjoint operator $(\lambda^{-2} - 1)[X, X^*]$, all with domain $D(X^*X)$. Hence, the lower bounded, closed quadratic form $(\tilde{q}_X^\lambda, D(X))$ of $(N_X^\lambda, D(X^*X))$ is a proper functional and consequently the lower bounded, closed form $(\mathcal{Q}_X^\lambda, D(\mathcal{Q}_X^\lambda))$ is proper too, as a sum of proper functionals. The lower bound (3.10) then implies that the Dirichlet form is a proper functional.

□

4 Superboundedness of a class of semigroups on type I von Neumann algebras

In this section we introduce a further continuity property, called *superboundedness*, for positivity preserving semigroups on standard forms of σ-finite von Neumann algebras,

showing that the property is owned by a class of semigroups on type I_∞ factors. Also we show how this property persists under domination of positivity preserving semigroups. As usual, $i_0 : M \to L^2(M)$ denotes the symmetric embedding of a σ-finite von Neumann algebra M endowed with a faithful normal state $\omega_0 \in M_{*+}$ represented by $\xi_0 \in L^2_+(M)$.

Definition 4.1 (*Excessive vectors and superboundedness*)

(i) The vector $\xi_0 \in L^2_+(M)$ is (γ_0, t_0)-*excessive* or *excessive*, for some $\gamma_0, t_0 \geq 0$, with respect to a positivity preserving semigroup $\{T_t : t \geq 0\}$ on $L^2(M)$ if the maps $e^{-\gamma_0 t} T_t$ are Markovian w.r.t. ξ_0 for any $t > t_0$.
Markovian semigroups are just those for which ξ_0 is $(0, 0)$-excessive;

ii) a positivity preserving semigroup $\{T_t : t \geq 0\}$ is *superbounded* if for some $\gamma_0, t_0 \geq 0$

(a) $\xi_0 \in L^2_+(M)$ is (γ_0, t_0)-*excessive*,
(b) $T_t(L^2(M)) \subseteq i_0(M)$ for all $t > t_0$.

If we endow the subspace $i_0(M) \subseteq L^2(M)$ by the norm of the von Neumann algebra, i.e. $\|i_0(x)\|_M := \|x\|_M$ for $x \in M$, then superboundedness implies the boundedness of T_t as a map from $(L^2(M), \|\cdot\|_2)$ to $(i_0(M), \|\cdot\|_M)$ for all $t > t_0$. In fact, by the norm continuity of the symmetric embedding $i_0 : M \to L^2(M)$, the norm $\|\cdot\|_M$ is stronger than the Hilbert norm $\|\cdot\|_2$ so that the continuous maps $T_t : L^2(M) \to L^2(M)$ are closed when considered from the Hilbert space $L^2(M)$ to the Banach space $(i_0(M), \|\cdot\|_M)$ and, by the Closed Graph Theorem, they result to be bounded (notice that this involves only condition (b) in Definition 4.1).
We shall refer to part (b) of superboundedness writing $\|T_t\|_{L^2(M)\to M} < +\infty$ for all $t > t_0$ and to part b) of *supercontractivity* writing $\|T_t\|_{L^2(M)\to M} \leq 1$ for all $t > t_0$.
By the Markovianity of $e^{-\gamma_0 t} T_t$ required in (i), bounded, positivity preserving maps $S_t : M \to M$ satisfying the relations $i_0(S_t(x)) = T_t(i_0(x))$ for $x \in M$ are well defined and one has, for suitable scalars $b_t \geq 0$,

$$\|S_t\| \leq e^{\gamma_0 t}, \qquad \|S_t x\|_M \leq b_t \cdot \|i_0(x)\|_{L^2(M)} \qquad x \in M, \quad t > t_0.$$

Consider the noncommutative spaces $L^p(M, \omega_0)$ for $p \in [2, +\infty]$ defined by the symmetric embedding $i_0 : M \to L^2(M)$ (see [21]). By complex interpolation it follows that a superbounded semigroup is *hypercontractive* too in the sense that there exists $T_0 \geq 0$ such that T_t is bounded from $L^2(M)$ to $L^4(M, \omega_0)$ for $t > T_0$.

The following observation will be useful later on.

Lemma 4.2 *(*Superboundedness by domination*) Let $\{e^{-tG_0} : t \geq 0\}$ be a superbounded semigroup on $L^2(M)$ such that, for some $\gamma_0, t_0 \geq 0$*

$$\xi_0 \in L^2_+(M) \text{ is } (\gamma_0, t_0)\text{-excessive}.$$

Let $\{e^{-tG_1} : t \geq 0\}$ be a C_0-continuous, self-adjoint, positivity preserving semigroup such that, for some $\gamma_1, t_1 \geq 0$

$$\xi_0 \in L^2_+(M) \text{ is } (\gamma_1, t_1)\text{-excessive.}$$

If the semigroup $\{e^{-tG_1} : t \geq 0\}$ *is dominated by the semigroup* $\{e^{-tG_0} : t \geq 0\}$ *in the sense*

$$e^{-tG_1}\eta \leq e^{-tG_0}\eta \qquad \eta \in L^2_+(M), \quad t \geq 0, \tag{4.1}$$

then $\{e^{-tG_1} : t \geq 0\}$ *is superbounded with*

$$\left\| e^{-tG_1}\eta \right\|_M \leq \left\| e^{-tG_0} \right\|_{L^2(M)\to M} \cdot \|\eta\|_2, \qquad \eta \in L^2_+(M), \quad t > t_0 \vee t_1. \tag{4.2}$$

Proof The superboundedness of $\{e^{-tG_0} : t \geq 0\}$ and the domination (4.1) imply that $e^{-tG_1}(L^2_+(M)) \subset i_0(M_+)$ for any $t > t_0 \vee t_1$. Since $L^2_+(M)$ linearly generates $L^2(M)$, it follows that $e^{-tG_1}(L^2(M)) \subseteq i_0(M)$ for all $t > t_0 \vee t_1$ so that $\{e^{-tG_1} : t \geq 0\}$ is superbounded. The bound (4.2) follows from the domination (4.1) and the superboundedness of $\{e^{-tG_0} : t \geq 0\}$. □

4.1 A class of superbounded Markovian semigroups on a type I_∞ factor

Let h be a Hilbert space and consider the type I factor $M := B(h)$. Its (Hilbert-Schmidt) standard representation acts, by left composition, on the space $L^2(M) = L^2(h)$ of Hilbert-Schmidt operators on h, where the standard cone $L^2_+(M) = L^2_+(h)$ is that of operators in $L^2(h)$. The modular involution is given by the operator adjoint: $J\xi := \xi^*$ for $\xi \in L^2(h)$ and the right representation of $B(h)$ on $L^2(h)$ is given by right composition.
Let H_0 be a lower bounded, self-adjoint operator affiliated to $B(h)$ (i.e. any self-adjoint, lower bounded operator on h) and consider the strongly continuous semigroup on $L^2(h)$ given by

$$T_t\eta = e^{-tH_0} J\left(e^{-tH_0} J(\eta)\right) = e^{-tH_0} \circ \eta \circ e^{-tH_0} \qquad \eta \in L^2(h).$$

Its self-adjoint generator G_0 on $L^2(h)$, defined by $G_0(\xi) := \lim_{t\to 0} t^{-1}(\xi - T_t\xi)$ on the subspace $D(G_0) \subset L^2(h)$ for whose vectors the limit exists, coincides with the generalized sum $H_0 \dot{+} JH_0J$ (see [19]) of the closed operators H_0 and JH_0J, affiliated to the commuting von Neumann algebras given by the left and right representations of $B(h)$ on $L^2(h)$ (see Lemma 7.1 in Appendix). The operator H_0, resp. JH_0J, is considered here as acting on a suitable dense subspace of the Hilbert–Schmidt space $L^2(h)$ by left, resp. right, composition. For example, $G_0(\xi) = \overline{H_0 \circ \xi} + \overline{\xi \circ H_0} \in L^2(h)$ for those $\xi \in L^2(h)$ such that the operators $H_0 \circ \xi$ and $\xi \circ H_0$ are densely defined, closable and bounded on their domains and their closures are Hilbert-Schmidt operators. To ease notation, the operators $H_0 \circ \xi, \xi \circ H_0$ will be represented by the juxtaposition $H_0\xi, \xi H_0$ of the symbols of the operators H_0 and ξ so that, the formula above appears $G_0(\xi) = \overline{H_0\xi} + \overline{\xi H_0}$. For further details on Hilbert-Schmidt standard form we refer to [12] Section 2.

Lemma 4.3 *If H_0 has discrete spectrum* $\mathrm{Sp}(H_0) := \{\lambda_j : j \in \mathbb{N}\}$[1] *with the increasing eigenvalues written with repetitions according to the their multiplicity, then*

(i) *G_0 has discrete spectrum too given by* $\mathrm{Sp}(G_0) := \{\lambda_j + \lambda_k \in \mathbb{R} : (j,k) \in \mathbb{N}\times\mathbb{N}\}$*;*
(ii) *if $n_{H_0}(\lambda) := \natural\{j \in \mathbb{N} : \lambda_j \le \lambda\}$ is the eigenvalue counting function of H_0, then the eigenvalue counting function of G_0 is bounded by $n_{G_0}(\lambda) \le (n_{H_0}(\lambda - \lambda_0))^2$, $\lambda \in \mathbb{R}$.*

Proof Let $H_0 = \sum_{k=0}^{\infty} \lambda_k P_k$ be the spectral decomposition of H_0 as an operator acting on h. Then the spectral decomposition of G_0 is given by

$$G_0 = \sum_{j,k=0}^{\infty} (\lambda_j + \lambda_k) P_j J P_k J,$$

since $\{P_j J P_k J : j, k \ge 0\}$ is a complete family of mutually orthogonal projections acting on the standard Hilbert space $L^2(h)$ such that

$$\begin{aligned}(H_0 + J H_0 J) P_j J P_k J &= H_0 P_j J P_k J + P_j J H_0 P_k J = \lambda_j P_j J P_k J + \lambda_k P_j P_k J \\ &= (\lambda_j + \lambda_k) P_j J P_k J.\end{aligned}$$

Thus G_0 has the discrete spectrum indicated in the statement and since $\lambda_j + \lambda_k \le \lambda$ implies both $\lambda_j + \lambda_0 \le \lambda$ and $\lambda_0 + \lambda_k \le \lambda$, the bound $n_{G_0}(\lambda) \le n_N(\lambda - \lambda_0)^2$ holds true for $\lambda \in \mathbb{R}$. □

Suppose now the lower bounded, self-adjoint operator H_0 on h to have a discrete spectrum $\mathrm{Sp}(H_0) := \{\lambda_j : j \in \mathbb{N}\}$ such that, for some $\beta > 0$,

$$\mathrm{Tr}(e^{-\beta H_0}) = \sum_{k=0}^{\infty} e^{-\beta \lambda_k} < +\infty,$$

so that the Gibbs state on $B(h)$ with density matrix $\rho_\beta := e^{-\beta H_0}/\mathrm{Tr}(e^{-\beta H_0})$ is well defined

$$\omega_\beta(x) := \mathrm{Tr}(x \rho_\beta) \qquad x \in B(h)$$

and its representative positive vector is given by $\xi_0 := \rho_\beta^{1/2} \in L^2_+(h)$. Recall that in this case the symmetric embedding $i_0 : B(h) \to L^2(h)$ is given by $i_0(x) = \rho_\beta^{1/4} x \rho_\beta^{1/4}$ for $x \in B(h)$.

Theorem 4.4 *(i) The C_0-continuous, self-adjoint semigroup $\{e^{-tG_0} : t > 0\}$ is positive preserving and $\xi_0 := \rho_\beta^{1/2} \in L^2_+(h)$ is $(-2(\lambda_0 \wedge 0), 0)$-excessive;*

[1] $\mathbb{N} = \{0, 1, \cdots\}$

(ii) the semigroup $\{e^{-tG_0} : t > 0\}$ *is superbounded with*

$$\|e^{-tG_0}\|_{L^2(M)\to M} \le e^{-(2t-\beta/2)\lambda_0} \qquad t > \beta/4.$$

In particular, if $\lambda_0 \ge 0$*, the semigroup is Markovian and supercontractive.*

Proof Replacing H_0 with $H_0 + \beta^{-1}\ln \mathrm{Tr}(e^{-\beta H_0})$, we may just consider the case $\mathrm{Tr}(e^{-\beta H_0}) = 1$.

(i) If $\xi \in L^2_+(h)$, since e^{-tH_0} is self-adjoint, we have $e^{-tG_0}\xi = (e^{-tH_0})^*\xi e^{-tH_0} \in L^2_+(h)$ for any $t \ge 0$, showing that the semigroup is positivity preserving. Since $\lambda_0 \le H_0$ and $\beta > 0$, for any $t \ge 0$ we have

$$\begin{aligned} e^{-tG_0}\xi_0 &= e^{-tH_0}\rho_\beta^{1/2}e^{-tH_0} = e^{-\beta H_0/4}e^{-2tH_0}e^{-\beta H_0/4} \\ &\le e^{-2t\lambda_0}e^{-\beta H_0/4}e^{-\beta H_0/4} = e^{-2t\lambda_0}\xi_0 \end{aligned}$$

so that ξ_0 is $(-2(\lambda_0 \wedge 0), 0)$-excessive.

(ii) For $\xi \in L^2(h)$ and $t > \beta/4$ we have $x := e^{-(t-\beta/4)H_0}\xi e^{-(t-\beta/4)H_0} \in B(h)$ and

$$\begin{aligned} i_0(x) &= \rho_\beta^{1/4}x\rho_\beta^{1/4} = e^{-\beta H_0/4}e^{-(t-\beta/4)H_0}\xi e^{-(t-\beta/4)H_0}e^{-\beta H_0/4} \\ &= e^{-tH_0}\xi e^{-tH_0} = e^{-tG_0}\xi. \end{aligned}$$

Since for $t > \beta/4$ we have $\|e^{-(t-\beta/4)H_0}\|_{B(h)} \le e^{-(t-\beta/4)\lambda_0}$, we get

$$\begin{aligned} \|x\|_{B(h)} &\le \|x\|_{L^2(h)} = \|e^{-(t-\beta/4)H_0}\xi e^{-(t-\beta/4)H_0}\|_{L^2(h)} \\ &\le \|e^{-(t-\beta/4)H_0}\|_{B(h)}\|\xi\|_{L^2(h)}\|e^{-(t-\beta/4)H_0}\|_{B(h)} \\ &\le e^{-(2t-\beta/2)\lambda_0} \cdot \|\xi\|_{L^2(h)}. \end{aligned}$$

□

5 General quantum Ornstein–Uhlenbeck semigroups

In this section we apply the above framework to construct a family of Dirichlet forms and Markovian semigroups, a special case of which is the quantum Ornstein–Uhlenbeck semigroup studied in [12]. While in [12] we computed explicitly the spectrum of the generator and proved the Feller property with respect to the algebra of compact operators, here we prove, for each semigroups we construct, subexponential spectral growth rate and domination with respect to positivity preserving semigroups belonging to a natural related class (see Appendix 7.1).

On the Hilbert space $h := l^2(\mathbb{N})$, consider the C*-algebra of compact operators $\mathcal{K}(h)$. The Number Operator $(N, D(N))$, defined by the natural basis $e := \{e_k \in l^2(\mathbb{N}) :$

$k \in \mathbb{N}\}$ as

$$D(N) := \left\{ \sum_{k\in\mathbb{N}} c_k \cdot e_k : \sum_{k\in\mathbb{N}} k^2 \cdot |c_k|^2 < +\infty \right\} \qquad Ne_k := ke_k \qquad k \in \mathbb{N},$$

generates the C_0-continuous group of automorphisms $\alpha := \{\alpha_t \in \mathrm{Aut}(\mathcal{K}(h)) : t \in \mathbb{R}\}$

$$\alpha_t(B) := e^{itN} B e^{-itN} \qquad B \in \mathcal{K}(h), \quad t \in \mathbb{R}.$$

For any $\beta > 0$ there exists a unique (α, β)-KMS state ω_β, satisfying the KMS condition

$$\omega_\beta(A\alpha_{i\beta}(B)) = \omega_\beta(BA)$$

for α-analytic elements A, B, given by, in terms of the density matrix,

$$\rho_\beta := \left(1 - e^{-\beta}\right) e^{-\beta N} = \left(1 - e^{-\beta}\right) \sum_{k\in\mathbb{N}} e^{-\beta k} p_k, \quad \omega_\beta(A) := \mathrm{Tr}(A\rho_\beta), \quad A \in \mathcal{K}(h)$$

(p_k being the projection onto $\mathbb{C}e_k$). The von Neumann algebra M generated by the GNS representation of ω_β can be identified with $B(h)$ and the normal extension of ω_β on it is still given by the formula above for any $A \in B(h)$. The extension of the automorphisms group α to a C_0^*-continuous group on $B(h)$ is given by the same formula above on $\mathcal{K}(h)$.
In the Hilbert–Schmidt standard form of $M := B(h)$ described in Sect. 4.1, the cyclic and separating vector representing ω_β is given by

$$\xi_0 := \rho_\beta^{1/2} = \sqrt{1 - e^{-\beta}} e^{-\beta N/2} \in L^2_+(h).$$

The action of the Hilbert algebra unbounded conjugation operator S_0 on $L^2(h)$, characterized as $S_0(x\xi_0) := x^*\xi_0$ for $x \in B(h)$, can be identified on a suitable domain $D(S_0) \subset L^2(h)$ with

$$S_0(\eta) = \overline{\rho_\beta^{-1/2} \eta^* \rho_\beta^{1/2}}$$

and its polar decomposition $S_0 = J\Delta_0^{1/2}$ is provided by the modular operator

$$\Delta_0^{1/2}(\eta) = \overline{\rho_\beta^{1/2} \eta \rho_\beta^{-1/2}} = \overline{e^{-\beta N/2} \eta e^{\beta N/2}} \qquad \eta \in D(S_0).$$

The modular group of ω_β, satisfying the modular condition $\omega_\beta(A\sigma_{-i}^{\omega_\beta}(B)) = \omega_\beta(BA)$, for analytics elements A, B, is then given by $\sigma_t^{\omega_\beta} = \alpha_{-\beta t}$ for $t \in \mathbb{R}$. Regarding the Number Operator N as an operator affiliated to $B(h)$ in its normal representation on $L^2(h)$ (i.e. acting, on a suitable domain of the Hilbert–Schmidt operators, by left

composition), we have that the modular (Araki) Hamiltonian is given by the strong sum of the densely defined, self-adjoint operators N and $-JNJ$ (belonging to commuting von Neumann algebras)

$$-\ln \Delta_0 = \beta \overline{N - JNJ}$$

and its (discrete) spectrum is given by $\mathrm{Sp}(-\ln \Delta_0) = \beta\mathbb{Z}$. Consequently $\mathrm{Sp}(\Delta_0^{1/2}) = e^{\beta\mathbb{Z}/2}$ with uniform multiplicity one.
Let us consider the annihilation and creation operators $(A, D(A))$, $(A^*, D(A^*))$ on h, defined on the domain $D(A) := D(\sqrt{N}) =: D(A^*)$ as

$$Ae_0 := 0, \quad Ae_k := \sqrt{k}e_{k-1} \quad \text{if } k \geq 1, \qquad A^*e_k := \sqrt{k+1}e_{k+1} \qquad k \in \mathbb{N}.$$

They satisfy the Canonical Commutation Relations $AA^* = A^*A + I$, as closed operators defined on $D(N)$, and allow to represent the Number Operator as $N = A^*A$. All these operators and their functional calculi are understood as affiliated to $B(h)$ acting by left composition on operators belonging to the Hilbert-Schmidt class $L^2(h)$.
Let us consider the family of operators affiliated to $B(h)$

$$D(X_m) = D\left(N^{m/2}\right) \qquad X_m := (A^*)^m \qquad m \in \mathbb{N} \setminus \{0\}. \tag{5.1}$$

Lemma 5.1 *(i) For any $m \geq 1$ and $\lambda_m^2 := e^{-m\beta/2}$ we have*

$$\begin{aligned} &D(X_m) = D\left(X_m^*\right) = D\left(N^{m/2}\right), \quad D\left(X_m^* X_m\right) = D\left(X_m X_m^*\right) = D(N^m) \\ &X_m^* X_m = A^m (A^*)^m = (N+m)(N+m-1)\cdots(N+2)(N+1) \\ &X_m X_m^* = (A^*)^m A^m = N(N-1)(N-2)\cdots(N-(m-1)) \end{aligned} \tag{5.2}$$

and the self-commutator $([X_m, X_m^*], D(N^m))$ is a self-adjoint operator, infinitesimally small with respect to $(X_m^* X_m, D(N^m))$ and $(N^m, D(N^m))$;

(ii) $X_m \xi_0 = (A^*)^m \xi_0 \in L^2(h)$ is an eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue λ_m^2,
$X_m^* \xi_0 = (A)^m \xi_0 \in L^2(h)$ is an eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue λ_m^{-2}.

Proof (i) Formulae (5.2) follow by induction starting from the case $m = 1$. They show that the self-commutator is a polynomial in N of degree $(m-1)$ and this implies the remaining conclusion. (ii) Since $A^*e_k := \sqrt{k+1}e_{k+1}$ for $k \in \mathbb{N}$, we have $(A^*\xi_0)e_k = A^*(\rho_\beta^{1/2}(e_k)) = (1-e^{-\beta})^{1/2}e^{-\beta k/2}A^*e_k = (1-e^{-\beta})^{1/2}e^{-\beta k/2}\sqrt{k+1}e_{k+1}$ and then for, any $m \in \mathbb{N}$, we have too $((A^*)^m\xi_0)e_k = (1-e^{-\beta})^{1/2}e^{-\beta k/2}(A^*)^m e_k =$

$(1-e^{-\beta})^{1/2}e^{-\beta k/2}\sqrt{k+1}\cdots\sqrt{k+m}\,e_{k+m}$ so that

$$\begin{aligned}\left(\Delta_0^{1/2}\left((A^*)^m\xi_0\right)\right)e_k &= (\rho_\beta^{1/2}\left((A^*)^m\rho_\beta^{1/2}\right)\rho_\beta^{-1/2})e_k = \rho_\beta^{1/2}\left((A^*)^m e_k\right)\\ &= \sqrt{k+1}\cdots\sqrt{k+m}\,\rho_\beta^{1/2}e_{k+m}\\ &= \left(1-e^{-\beta}\right)^{1/2}e^{-\beta(k+m)/2}\sqrt{k+1}\cdots\sqrt{k+m}\,e_{k+m}\\ &= e^{-m\beta/2}\left((A^*)^m\xi_0\right)e_k.\end{aligned}$$

Hence $(A^*)^m\xi_0 \in L^2(h)$ is eigenvector of $\Delta_0^{1/2}$ corresponding to the eigenvalue $\lambda_m^2 := e^{-m\beta/2}$. The other series of eigenvalues follow from Lemma 2.9. □

We are now in position to apply Theorem 2.5 with $Y = X_m, \lambda = e^{-m\beta/4}, \xi_0 = \rho_\beta^{1/2} \in L^2(l^2(\mathbb{N}))$
and consider the Dirichlet form $(\mathcal{E}_{X_m}^{\lambda_m}, \mathcal{F}_{X_m}^{\lambda_m})$ on $L^2(l^2(\mathbb{N}))$ and its generator $(H_{X_m}^{\lambda_m}, D(H_{X_m}^{\lambda_m}))$.
The following result generalizes, in particular, some of those obtained in [12] for the quantum Ornstein–Uhlebeck semigroup, corresponding to the present parameter $m = 1$.

Theorem 5.2 *(*Spectral growth rate*) For $m \geq 1$ and $\lambda_m^2 := e^{-m\beta/2}$, the operator $(H_{X_m}^{\lambda_m}, D(H_{X_m}^{\lambda_m}))$ has discrete spectrum and subexponential spectral growth rate*

$$\mathrm{Tr}(e^{-tH_{X_m}^{\lambda_m}}) < +\infty \qquad t > 0.$$

Proof By Lemma 5.1 (i) above, the self-adjoint operator

$$N_{X_m}^{\lambda_m} := \left(\lambda_m - \lambda_m^{-1}\right)^2 X_m^* X_m + \left(\lambda_m^{-2} - 1\right)\left[X_m, X_m^*\right]$$

has, on its domain, the following explicit form

$$\begin{aligned}N_{X_m}^{\lambda_m} &= \left(\lambda_m^2 - 1\right)A^m(A^*)^m + \left(\lambda_m^{-2} - 1\right)(A^*)^m A^m\\ &= \left(\lambda_m^2 - 1\right)(N+1)\cdots(N+m) + \left(\lambda_m^{-2} - 1)N(N-1)\cdots(N-(m-1)\right)\\ &= \left(\lambda_m^2 + \lambda_m^{-2} - 2\right)N^m + p_{m-1}(N) = \left(\lambda_m - \lambda_m^{-1}\right)^2 N^m + p_{m-1}(N), \quad (5.3)\end{aligned}$$

where $p_{m-1} : \mathbb{R} \to \mathbb{R}$ is a suitable polynomial of degree $(m-1)$ with real coefficients. Since for any $\varepsilon \in (0,1)$ one has $b_m^\varepsilon := \inf_{s\geq 0}(\varepsilon(\lambda_m - \lambda_m^{-1})^2 s^m + p_{m-1}(s)) > -\infty$ and $(\lambda_m - \lambda_m^{-1})^2 = (2\sinh(m\beta/4))^2 > 0$, $(N_{X_m}^{\lambda_m}, D(N^m))$ is lower bounded, self-adjoint with subexponential spectral growth rate

$$\begin{aligned}\mathrm{Tr}\left(e^{-tN_{X_m}^{\lambda_m}}\right) &\leq e^{-tb_m^{\varepsilon}}\mathrm{Tr}\left(e^{-t(1-\varepsilon)\left(\lambda_m-\lambda_m^{-1}\right)^2 N^m}\right)\\ &= e^{-tb_m^{\varepsilon}}\sum_{k=1}^{\infty} e^{-t(1-\varepsilon)\left(\lambda_m-\lambda_m^{-1}\right)^2 k^m}, \qquad t>0,\end{aligned}$$

by [31] Proposition 1.2.15. Applying Lemma 4.3, these same properties (having discrete spectrum and sub-exponential spectral growth rate) hold true for the sum $N_{X_m}^{\lambda_m} \dot{+} j(N_{X_m}^{\lambda_m})$. Also, since $D(X_m) = D(X_m^*) = D(N^{m/2})$ and $D(X_m^* X_m) = D(X_m X_m^*) = D(N^m)$, by Theorem 3.3 (iii) and (iv) with the notations there introduced, we deduce spectrum discreteness and growth rate for $H_{X_m}^{\lambda_m}$ too:

$$\begin{aligned}\mathrm{Tr}(e^{-tH_{X_m}^{\lambda_m}}) &\leq \mathrm{Tr}\left(e^{-t(N_{X_m}^{\lambda_m} \dot{+} j(N_{X_m}^{\lambda_m})}\right) = \mathrm{Tr}\left(e^{-tN_{X_m}^{\lambda_m}} J e^{-tN_{X_m}^{\lambda_m}} J\right)\\ &= \left(\mathrm{Tr}(e^{-tN_{X_m}^{\lambda_m}})\right)^2, \qquad t>0.\end{aligned}$$

□

Theorem 5.3 (Domination) *For $m \geq 1$ and $\lambda_m^2 := e^{-m\beta/2}$, the following properties hold:*
the Markovian semigroup $\{e^{-tH_{X_m}^{\lambda_m}} : t \geq 0\}$, associated to the Dirichlet form $(\mathcal{E}_{X_m}^{\lambda_m}, \mathcal{F}_{X_m}^{\lambda_m})$ dominates the Markovian semigroup $\{e^{-tG_1} : t \geq 0\}$, generated by the closed, self-adjoint operator $(G_1, D(G_1))$ on $L^2(h)$ given by

$$\begin{aligned}D(G_1) &:= D(N^m) \cap J D(N^m)\\ G_1 &:= \left(\lambda_m^2 \cdot X_m^* X_m + \lambda_m^{-2} \cdot X_m X_m^*\right) \dot{+} j\left(\lambda_m^2 \cdot X_m^* X_m + \lambda_m^{-2} \cdot X_m X_m^*\right),\end{aligned}$$

which can be expressed as

$$e^{-tG_1}(\eta) = e^{-tB_m} j\left(e^{-tB_m}\right)(\eta) = e^{-tB_m}\eta e^{-tB_m} \qquad \eta \in L^2(M),$$

by the self-adjoint, positive operator $(B_m, D(B_m)) := (\lambda_m^2 \cdot X_m^* X_m + \lambda_m^{-2} \cdot X_m X_m^*, D(N^m))$, affiliated to $M := B(h)$ in its left action on $L^2(M) = L^2(h)$.

Proof Set $(q_0, D(q_0)) := (\mathcal{E}_{X_m}^{\lambda_m}, \mathcal{F}_{X_m}^{\lambda_m})$ and consider the forms $(q_1, D(q_1)), (w, D(w))$ given by $D(q_1) := \tilde{\mathcal{F}}_{X_m} =: D(w)$ and

$$\begin{aligned}q_1[\eta] &:= \lambda_m^2\left(\|X_m\eta\|^2 + \|X_m J\eta\|^2\right)\Big) + \lambda_m^{-2}\left(\|X_m^*\eta\|^2 + \|X_m^* J\eta\|^2\right),\\ w[\eta] &:= 2\left[\left(X_m\eta | J X_m^* J\eta\right) + \left(X_m^*\eta | J X_m J\eta\right)\right],\end{aligned}$$

so that on $D(q_1) := \tilde{\mathcal{F}}_{X_m}$, the representation (3.1) of the form $(\mathcal{E}^{\lambda_m}_{X_m}, \mathcal{F}^{\lambda_m}_{X_m})$ can be written as

$$q_1[\eta] = q_0[\eta] + w[\eta] \qquad \eta \in D(q_1).$$

As, by definition, $(q_0, D(q_0))$ is a Dirichlet form, its associated self-adjoint operator

$$(G_0, D(G_0)) := \left(H^{\lambda_m}_{X_m}, D\left(H^{\lambda_m}_{X_m}\right)\right)$$

generates a Markovian, hence a C_0-continuous, self-adjoint, positivity preserving, semigroup $\{e^{-tG_0} : t \geq 0\}$. Since, by definition (see statement and proof of Theorem 2.5) and (5.1), $\tilde{\mathcal{F}}_{X_m} = D(X_m) \cap D(X^*_m) \cap J(D(X_m) \cap D(X^*_m)) = D(N^{m/2}) \cap JD(N^{m/2})$, the quadratic form $(q_1, D(q_1))$ is closed and the associated self-adjoint operator is just $(G_1, D(G_1))$. Since $\{e^{-tG_1} : t \geq 0\}$ is positivity preserving (see Appendix 7.1), to apply Lemma 4.2, we exploit the characterization of domination between positivity preserving semigroups on standard forms of von Neumann algebras, established in [3] Theorem 3.1: the semigroup $\{e^{-tG_1} : t \geq 0\}$ is dominated by $\{e^{-tG_0} : t \geq 0\}$ if and only if each one of the following properties is verified:

(a) $D(q_1) \subseteq D(q_0)$,
(b) $q_0(\eta|\zeta) \leq q_1(\eta|\zeta)$ for all $\eta, \zeta \in D(q_1) \cap L^2_+(M)$,
(c) if $\eta \in D(q_0) \cap L^2_+(M)$, $\zeta \in D(q_1) \cap L^2_+(M)$ and $\eta \leq \zeta$, then $\eta \in D(q_1)$.

Condition a) holds true since $D(q_1) := \tilde{\mathcal{F}}_{X_m} \subseteq \mathcal{F}^{\lambda_m}_{X_m} =: D(q_0)$. To prove b), consider the set $C(e) \subseteq L^2(h)$ of all Hilbert-Schmidt operators which are finite linear combination of the partial isometries $\{e_j \otimes e^*_k : j, k \in \mathbb{N}\}$ of the natural basis $e := \{e_k \in h : k \in \mathbb{N}\}$ and set $C_+(e) := C(e) \cap L^2_+(h)$, $C_{\mathbb{R}}(e) := C(e) \cap L^2_{\mathbb{R}}(h)$, where $L^2_{\mathbb{R}}(h) = L^2_+(h) - L^2_+(h)$ is the self-adjoint part of $L^2(h)$. Since $\{e_j \otimes e^*_k : j, k \in \mathbb{N}\}$ is a Hilbert basis for $L^2(h)$, $C(e)$ is dense in $L^2(h)$. For $A \in L^2_{\mathbb{R}}(h)$ and $B \in C(e)$ we have $(B^* + B)/2 \in C_{\mathbb{R}}(e)$ and $\|A - (B^* + B)/2\|_2 \leq \|A - B\|_2$ so that $C_{\mathbb{R}}(e)$ is dense in $L^2_{\mathbb{R}}(h)$. For any $B \in C_{\mathbb{R}}(e)$ we have $B_+ \in C_+(e)$ since $C_{\mathbb{R}}(e) = \bigcup_{j\in\mathbb{N}} L^2_{\mathbb{R}}(h_j)$, where $h_j := \mathrm{Lin}\{e_k \in h : k = 0, \ldots, j\}$, and if $B \in L^2_{\mathbb{R}}(h_j)$ for some $j \in \mathbb{N}$, then $B_+ \in L^2_{\mathbb{R}}(h_j)$. Since the Hilbertian projection of $L^2_{\mathbb{R}}(h)$ onto $L^2_+(h)$ is a contraction, for any $A \in L^2_+(h)$ and $B \in C_{\mathbb{R}}(e)$ we have $\|A - B_+\|_2 = \|A_+ - B_+\|_2 \leq \|A - B\|_2$ showing that the cone $C_+(e)$ is dense in the positive cone $L^2_+(h)$.

It follows from Lemma 5.1 that $C(e)$ is a J-invariant core for $(X_m, D(X_m))$ and $(X^*_m, D(X^*_m))$ which is left globally invariant by both operators: $X_m(C(e)) \subseteq C(e)$, $X^*_m(C(e)) \subseteq C(e)$. Let P_j the finite rank projection on h with range h_j, for any $j \in \mathbb{N}$. Then if $\eta, \zeta \in C_+(e)$ then $X_m\zeta = P_jX_mP_k\zeta$ and $X^*_m\eta = P_jX^*_mP_k\eta$ for sufficiently large $j, k \in \mathbb{N}$. Since $P_{j+m}X_mP_j \in B(h)$, $(P_jX_mP_{j+m})J(P_jX_mP_{j+m})J$ is positivity preserving and we have

$$\left(X^*_m\eta|JX_mJ\zeta\right) = \left(\eta|(P_jX_mP_{j+m})\,J\left(P_jX_mP_{j+m})J\zeta\right) \geq 0.$$

By the core property, the positivity of $(X_m^*\eta|JX_mJ\zeta)$ extends to any $\eta,\zeta \in D(X_m^*)\cap JD(X_m)$ and an analogous reasoning shows that $(X_m\eta|JX_m^*J\zeta)\geq 0$ is true for any $\eta,\zeta\in D(X_m)\cap JD(X_m^*)$. Since $D(q_1)=\tilde{\mathcal{F}}_{X_m}$, altogether these properties allows to check b) as follows for $\eta,\zeta\in D(q_1)\cap L^2_+(M)$

$$q_1(\eta|\zeta)-q_0(\eta|\zeta)=w(\eta|\zeta)=2\left[\left(X_m\eta|JX_m^*J\zeta\right)+\left(X_m^*\eta|JX_mJ\zeta\right)\right].$$

To check c), since $D(q_1):=\tilde{\mathcal{F}}_{X_m}$ is core for $(\mathcal{E}^{\lambda_m}_{X_m},\mathcal{F}^{\lambda_m}_{X_m})$, let $\eta_n\in D(q_1)$ be a sequence such that

$$\lim_n\left(q_0[\eta_n-\eta]+\|\eta_n-\eta\|_2^2]\right)=0.$$

Let $\eta_n\wedge\zeta:=\mathrm{Proj}(\eta_n,\zeta-L^2_+(M))$ be the Hilbert projection of $\eta_n\in L^2_+(M)$ onto the closed and convex set $\zeta-L^2_+(M)\subset L^2_{\mathbb{R}}(M)$. Since, by Lemma 4.4 in [8], we have $\eta_n\wedge\zeta=\zeta\wedge\eta_n=\eta_n-(\zeta-\eta_n)_-$, the continuity of the Hilbert projections and the fact that $\eta\leq\zeta$, imply

$$\lim_n\|\eta-\eta_n\wedge\zeta\|_2=\lim_n\|\eta-\eta_n+(\zeta-\eta_n)_-\|_2=\|(\zeta-\eta)_-\|_2=0.$$

Since $\{e^{-tG_0}:t\geq 0\}$ and $\{e^{-tG_1}:t\geq 0\}$ are positivity preserving, by Proposition 4.5 iii) in [8] we have

$$\eta_n\wedge\zeta\in D(q_1),\qquad q_0[\eta_n\wedge\zeta]\leq q_0[\eta_n\wedge\zeta]+q_0[\eta_n\vee\zeta]\leq q_0[\eta_n]+q_0[\zeta].$$

Since $\eta_n\wedge\zeta,\zeta\in D(q_1)$ and, by definition, $\eta_n\wedge\zeta\leq\zeta$, we have also (using the property of the quadratic form w established in the proof of b)) $w[\eta_n\wedge\zeta]\leq w[\zeta]$ so that

$$q_1[\eta_n\wedge\zeta]=q_0[\eta_n\wedge\zeta]+w[\eta_n\wedge\zeta]\leq q_0[\eta_n]+q_0[\zeta]+w[\zeta]=q_0[\eta_n]+q_1[\zeta].$$

Since the quadratic form $(q_1,D(q_1))$ is closed on $L^2(M)$, it is lower semicontinuous when considered as a functional on $L^2(M)$ taking values in the extended positive half-line $[0,+\infty]$ and it is finite exactly on $D(q_1)$. We then have

$$q_1[\eta]\leq\liminf_n q_1[\eta_n\wedge\zeta]\leq\liminf_n\left(q_0[\eta_n]+q_1[\zeta]\right)=q_0[\eta]+q_1[\zeta]<+\infty$$

so that $\eta\in D(q_1)$. By [3] Theorem 3.1, $\{e^{-tG_1}:t\geq 0\}$ is dominated by $\{e^{-tG_0}:t\geq 0\}$: $e^{-tG_1}\eta\leq e^{-tH^{\lambda_m}_{X_m}}\eta$ for all $\eta\in L^2_+(M)$ and $t\geq 0$. Choosing $\eta:=\xi_0$ one has $e^{-tG_1}\xi_0\leq e^{-tH^{\lambda_m}_{X_m}}\xi_0\leq\xi_0$ for all $t\geq 0$ so that $\{e^{-tG_1}:t\geq 0\}$ is Markovian.

□

6 Dirichlet forms associated to deformations of the CCR relations

In this section we outline the construction of Dirichlet forms associated to deformations of annihilation and creation operators in the framework and notations of Sect. 5. To use the tools of Sect. 2 to this end, we need to represent eigenvectors of (isolated) eigenvalues of the Araki Hamiltonian as in Lemma 2.1.

6.1 Deformation of the CCR relations

Let $g : \mathbb{R} \to \mathbb{R}$ be a function vanishing on $(-\infty, 0]$, strictly increasing on $[0, +\infty)$ and satisfying, for $\beta > 0$ and $\ell \in \mathbb{N}$ to be fixed later,

$$\sum_{n=0}^{\infty} n^{\ell} e^{-\beta g(n)} < +\infty. \tag{6.1}$$

Consider the automorphisms group of the C^*-algebra of compact operators

$$\alpha_t(B) := e^{itg(N)} B e^{-itg(N)} \qquad t \in \mathbb{R}, \quad B \in \mathcal{K}(h)$$

whose Gibbs equilibrium state $\omega_\beta(\cdot) = \mathrm{Tr}(\cdot \rho_\beta)$ is represented by the density matrix $\rho_\beta := e^{-\beta g(N)}/Z(\beta)$ with partition function $Z(\beta) := \mathrm{Tr}(e^{-\beta g(N)})$. Let $\xi_0 := \rho_\beta^{1/2} \in L^2_+(h)$ be the cyclic vector giving rise to the modular group of the normal extension of ω_β to $B(h)$

$$\sigma_t^{\omega_\beta}(B) = \alpha_{-t\beta}(B) = e^{-it\beta g(N)} B e^{it\beta g(N)} \qquad B \in B(h), \quad t \in \mathbb{R}.$$

Then $\Delta_0^{it}(\eta) = \rho_\beta^{it} \eta \rho_\beta^{-it}$ for all $\eta \in L^2(h)$ and the Araki Hamiltonian is the strong sum

$$\ln \Delta_0 = -\beta \overline{g(N) - j(g(N))}.$$

Since for each $m, n \in \mathbb{N}$, $\nu_{m,n} := \beta(g(m) - g(n))$ is an eigenvalue of $\ln \Delta_0$ with eigenvector $e_m \otimes e_n^* \in L^2(h)$ and $\{e_m \otimes e_n^* : m, n \in \mathbb{N}\}$ is a Hilbert basis, the spectrum of $\ln \Delta_0$ is

$$sp(\ln \Delta_0) = \overline{\{\nu_{m,n} : m, n \in \mathbb{N}\}}.$$

All eigenvalues are isolated if, for example,

$$\liminf_{m > n \geq 0} \frac{g(m) - g(n)}{m - n} > 0.$$

Proposition 6.1 *Suppose $\nu := \nu_{m,n} \geq 0$ to be an isolated eigenvalue of the Araki Hamiltonian with $m \geq n$, set $\ell := m - n \in \mathbb{N}$ and let $f \in C_0^\infty(\mathbb{R})$ be a Schwartz*

function whose Fourier Transform[2] $\hat{f} \in C_0^\infty(\mathbb{R})$ *is supported by* $[\nu - \varepsilon, \nu + \varepsilon]$ *and is strictly positive on* $(\nu - \varepsilon, \nu + \varepsilon)$, *with* $\hat{f}(\nu) = 1$, *for*

$$0 < \varepsilon < \operatorname{dist}(\nu, sp(\ln \Delta_0) \setminus \{\nu\}).$$

Then, setting $k(t) := \beta(g(t + \ell)) - g(t))$ *and* $p(t) := \hat{f}(k(t))$ *for* $t \in \mathbb{R}$, *we have*

(i) $sp(k(N)) \subset sp(\ln \Delta_0)$ *and* $\nu \in sp(k(N))$ *is an isolated eigenvalue of* $k(N)$ *acting on* h;
(ii) $p(N)$ *is the spectral projection of* N *corresponding to the Borel set*

$$B := \left\{n' \in \mathbb{N} : g(m) - g(n) = g(n' + \ell) - g(n')\right\} \subseteq sp(N)$$

and $p(N - \ell \cdot I)$ *is the spectral projection of* N *corresponding to* $B + \ell \subseteq \mathbb{N}$;
iii) the densely defined, closed operator $(X, D(X))$ *on* h, *given by*

$$D(X) := D\left(N^{\ell/2}\right) \qquad X := p(N) \circ A^\ell, \tag{6.2}$$

where A *is the annihilation operator defined in Sect. 5, satisfies the relations*

$$\begin{aligned} XX^* &= (N+1)\cdots(N+\ell\cdot I)p(N) \\ X^*X &= N(N-I)\cdots(N-(\ell-1)\cdot I)p(N-\ell\cdot I) \\ [X, X^*] &= (N+1)\cdots(N+\ell\cdot I)p(N) - \\ &\quad N(N-I)\cdots(N-(\ell-1)\cdot I)p(N-\ell\cdot I); \end{aligned} \tag{6.3}$$

iv) if B *is unbounded,* $(X^*X, D(N^\ell))$ *and* $(XX^*, D(N^\ell))$ *are unbounded with discrete spectra;*
v) if B *and* $B + \ell$ *differ by a finite set, then* $([X, X^*], D(N^\ell))$ *is infinitesimally small with respect to* $(N^\ell, D(N^\ell))$;
vi) $\xi := X\xi_0 \in L^2(h)$ *is an eigenvector of* $\ln \Delta_0$ *with eigenvalue* ν:

$$(\ln \Delta_0)\xi = \nu \cdot \xi.$$

Proof i) follows from $sp(k(N)) = \overline{\{\nu_{n'+m-n,n'} : n' \in \mathbb{N}\}} \subset \overline{\{\nu_{m,n} : m, n \in \mathbb{N}\}} = sp(\ln \Delta_0)$; ii) follows from i), the assumption on ε and the Spectral Theorem; iii) by the CCR we have

$$NA = A(N - I), \qquad A^*N = (N - I)A^* \tag{6.4}$$

as identities among closed operators on their common domain $D(N^{3/2})$. By induction

$$(A^*)^\ell A^\ell = N(N-I)\cdots(N-(\ell-1)\cdot I), \qquad A^\ell(A^*)^\ell = (N+I)\cdots(N+\ell\cdot I)$$

[2] Fourier transform convention: $\hat{f}(s) := \int_{\mathbb{R}} dt f(t) e^{ist}$.

on the domain $D(N^\ell)$ so that, by (6.2), one gets the first relation (6.3)

$$XX^* = p(N)A^\ell(A^*)^\ell p(N) = (N+I)\cdots(N+\ell\cdot I)p(N).$$

Since, by (6.4), $p(N)A = Ap(N-I)$, by induction one obtains the second relation (6.3) $X^*X = (A^*)^\ell p(N)A^\ell = (A^*)^\ell A^\ell p(N-\ell\cdot I) = N(N-I)\cdots(N-(\ell-1)\cdot I)p(N-\ell\cdot I)$; the last relation (6.3) follows by difference; iv) follows from (6.3) and the fact that $N(N-I)\cdots(N-(\ell-1)\cdot I)$ and $(N+I)\cdots(N+\ell\cdot I)$ are polynomials; v) in this case $p(N)-p(N-\ell\cdot I)$ has finite rank and $(N+1)\cdots(N+\ell\cdot I)-N(N-I)\cdots(N-(\ell-1)\cdot I)$ is polynomial of degree at most $\ell-1$; vi) since $sp(\ln\Delta_0)\cap[\nu-\varepsilon,\nu+\varepsilon]=\{\nu\}$ and $\hat{f}(\nu)=1$, by the Spectral Theorem, the spectral projection P of $\ln\Delta_0$, corresponding to $\{\nu\}$, can be represented as

$$P = \hat{f}(\ln\Delta_0) = \int_{\mathbb{R}} \mathrm{d}t f(t)e^{it\ln\Delta_0} = \int_{\mathbb{R}} \mathrm{d}t f(t)\Delta_0^{it};$$

since $\xi_0\in D(A^\ell)=D(N^{\ell/2})$ by (6.1), by (6.4) we have

$$\begin{aligned}\Delta_0^{it}\left(A^\ell(\xi_0)\right) &= \rho_\beta^{it}\circ\left(A^\ell(\xi_0)\right)\circ\rho_\beta^{-it} = \rho_\beta^{it}A^\ell\rho_\beta^{1/2}\rho_\beta^{-it}\\ &= e^{-it\beta g(N)}e^{it\beta g(N+\ell\cdot I)}A^\ell\rho_\beta^{1/2} = e^{itk(N)}\left(A^\ell(\xi_0)\right).\end{aligned}$$

Hence, $P(A^\ell\xi_0) = \int_{\mathbb{R}} dtf(t)e^{itk(N)}(A^\ell(\xi_0)) = (\hat{f}(k(N))A^\ell)(\xi_0) = X(\xi_0) =: \xi$ does not vanish and it is an eigenvector of $\ln\Delta_0$ corresponding to the eigenvalue ν. □

Example (1) If $g(t)=t$ for any $t\in\mathbb{R}$, $B=\mathbb{N}$, $p(N)=I$, $X=A^\ell$ and we reproduce the "unperturbed" case treated in Theorem 5.2.

(2) If $g(t) := t+[t/2]$ for $t\geq 0$, $n\in 2\mathbb{N}$ is even and $m\in 1+2\mathbb{N}$ is odd, then $\ell\in 1+2\mathbb{N}$ is odd, $g(m)-g(n)=3\ell/2-1/2$ and $B=\{n'\in\mathbb{N}: g(m)-g(n)=g(n'+\ell)-g(n')\}=2\mathbb{N}$.

Remark 6.2 The canonical commutation relations CCR arise in the spectral analysis of the quantum harmonic oscillator, which can be considered the canonical quantization of the classical harmonic oscillator whose phase space is the plane $\mathbb{R}^2$. D. Shale and W. F. Stinespring constructed in [32] a quantum system which can be regarded as the quantization of a harmonic oscillator whose phase space is the hyperbolic plane $\mathbb{H}^2$ with a fixed negative constant curvature $k<0$. It can be also considered as a quantum harmonic oscillator with self-interaction, the coupling constant being proportional to the curvature. In their work the authors found that the dynamics is generated by an Hamiltonian $H=\hbar\omega N$ proportional to the Number Operator and that annihilation and creation operators are replaced by operators X and X^* satisfying a deformed CCR

$$[X,X^*] = \hbar\cdot I - k\hbar^2\cdot N.$$

A similar commutation relation is satisfied by $X := A^2$ where A is the annihilation operator

$$\begin{aligned}
&= \left[A^2, (A^*)^2\right] = A\left[A, (A^*)^2\right] + \left[A, (A^*)^2\right]A \\
&= A[A, A^*]A^* + AA^*[A, A^*] + [A, A^*]A^*A + A^*[A, A^*]A \\
&= AA^* + AA^* + A^*A + A^*A = 2I + 4N.
\end{aligned}$$

In reference to Sect. 5, $e^{-tH_{X_2}^{\lambda_2}}$, compared with the quantum Ornstein–Uhlenbeck semigroup $e^{-tH_{X_1}^{\lambda_1}}$ (see [12]), could be called *quantum Ornstein–Uhlenbeck hyperbolic semigroup*.

Acknowledgements The authors warmly thank the anonymous referees for their careful and scrupulous reading and suggestions.

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement. This work has been supported by Laboratorio Ypatia delle Scienze Matematiche I.N.D.A.M. Italy (LYSM) and G.N.A.M.P.A.-I.N.D.A.M. Italy 2023.

Data availability The datasets generated during the current study are available from the reasonable request.

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7 Appendix

7.1 Generators of a class of positivity preserving semigroups

Let $(A, D(A))$ be a lower bounded, self-adjoint operator affiliated to a von Neumann algebra M and consider the C_0-continuous, self-adjoint, *positivity preserving* semigroup on $L^2(M)$, defined by

$$T_t^A := e^{-tA} j\left(e^{-tA}\right) = e^{-tA} J e^{-tA} J \qquad t \geq 0.$$

If $(q_A, D(q_A))$ is the lower bounded, closed quadratic form of $(A, D(A))$, then the lower bounded, closed quadratic form of $(j(A)), JD(A))$ is given by $JD(q_A) \ni \eta \mapsto q_A[J\eta]$ and the quadratic form $(t_A, D(q_A) \cap JD(q_A))$ given by $t_A[\eta] := q_A[\eta] + q_A[J\eta]$ is lower bounded and closed as a sum of forms sharing these same properties.

Lemma 7.1 *The lower bounded, closed, quadratic form of the C_0-continuous, self-adjoint semigroup $\{T_t^A : t \geq 0\}$ is given by $(t_A, D(q_A) \cap JD(q_A))$ and the associated*

self-adjoint generator, i.e. the generalized sum $A\dot{+}j(A)$ (see [19]), is given by the closure $\overline{A+j(A)}$

$$T_t^A = e^{-tA} j\left(e^{-tA}\right) = e^{-t\left(A\dot{+}j(A)\right)} = e^{-t\overline{A+j(A)}} \qquad t \geq 0.$$

Proof Since $(q_A, D(q_A))$ is lower bounded, for $\eta \in L^2(M)$ the limit

$$\begin{aligned}&\lim_{t\to 0^+} t^{-1}\left[\left(\eta|(I-T_t^A)\eta\right)\right]\\ &= \lim_{t\to 0^+} t^{-1}\left[\left(\eta|(I-e^{-tA}\eta)\right) + \left(e^{-tA}\eta|J(I-e^{-tA}J\eta)\right)\right]\end{aligned}$$

exists in $\mathbb{R}$ if and only if both limits on the right-hand side exist in $\mathbb{R}$, i.e. if and only if $\eta \in D(q_A) \cap JD(q_A)$ and in this case $\lim_{t\to 0^+} t^{-1}[(\eta|(I-T_t^A)\eta)] = q_A[\eta] + q_A[J\eta] =: t_A[\eta]$. Hence the lower bounded, closed quadratic form of $\{T_t^A : t \geq 0\}$ is $(t_A, D(q_A) \cap JD(q_A))$ and this form is densely defined. As $(A, D(A))$ and $(j(A), JD(A))$ are affiliated to commuting von Neumann algebras, they strongly commute and the sum $(A+J(A), D(A)\cap JD(A))$ is densely defined, lower bounded, symmetric and essentially self-adjoint so that $A\dot{+}j(A) = \overline{A+j(A)}$. □

7.2 Superbounded semigroups on abelian atomic von Neumann algebras.

The von Neumann algebra $B(h)$ is atomic and this suggests to have a look at the superboundedness property in the abelian situation of atomic measured spaces.
Let (X, m) be a locally compact, second countable, Hausdorff space, endowed with a fully supported Borel measure. Consider a real valued function U such that $e^{-U} \in L^1(X, m)$ and define a probability measure by

$$m_U := e^{-U} m \Big/ \int_X e^{-U} \mathrm{d}m.$$

By the unit norm function $u_0 := e^{-U/2}/\|e^{-U/2}\|_{L^2(X,m)} \in L^2(X, m)$, one recovers the integral with respect to m_U by

$$\int_X v dm_U = (u_0|vu_0)_{L^2(X,m)},$$

one has the embedding $i_0 : L^\infty(X, m) \to L^2(X, m)$ $i_0(v) := vu_0$ with $\|i_0(w)\|_{L^2(X,m)} = \|w\|_{L^2(X,m_U)}$.
A C_0-continuous semigroup $T_t : L^2(X, m) \to L^2(X, m)$ is Markovian with respect to m_U (in the sense we are discussing in this work, i.e. the one introduced in [8]), if

$$0 \leq v \leq u_0 \quad \Rightarrow \quad 0 \leq T_t v \leq u_0 \qquad t \geq 0.$$

Such a semigroup induces a semigroup on the abelian von Neumann algebra $L^\infty(X,m)$ by

$$S_t : L^\infty(X,m) \to L^\infty(X,m) \qquad i_0(S_t u) = T_t(i_0(u)) \qquad u \in L^\infty(X,m),$$

which is Markovian in the usual sense

$$0 \le u \le 1 \quad \Rightarrow \quad 0 \le S_t u \le 1 \qquad t \ge 0.$$

The definition of superboundedness considered above on von Neumann algebras, in the commutative setting reduces to say that T_t is *superbounded with respect to* m_U if

$$T_t\left(L^2(X,m)\right) \subset i_0\left(L^\infty(X,m)\right) \qquad t > t_0$$

for some $t_0 \ge 0$ and

$$\|u\|_{L^\infty(X,m)} \le \|v\|_{L^2(X,m)}$$

whenever $T_t v = i_0(u)$ for $v \in L^2(X,m)$, $u \in L^\infty(X,m)$ and $t > t_0$. In other words, T_t is superbounded with respect to m_U, if the induced Markovian semigroup satisfies

$$\|S_t u\|_{L^\infty(X,m)} \le \|i_0(u)\|_{L^2(X,m)} = \|u\|_{L^2(X,m_U)} \qquad u \in L^\infty(X,m), \qquad t > t_0.$$

In case (X,m) is an atomic measured space, the classical definition of super or ultracontractivity typically trivializes (see [15] Section 2.1): this happens, for example, if m is the counting measure because of the contractive embedding $L^2(X,m) \subseteq L^\infty(X,m)$. Superboundedness however may still be non trivial.
Let (X,m) be a countable, atomic measured space and let $m = e^{-h} m_0$ for some function h and the counting measure m_0. To simplify notations, we assume that $\|e^{-U}\|_{L^1(X,m)} = 1$.
For a fixed nonnegative measurable function $V : X \to [0,+\infty)$ let us consider the semigroup

$$T_t : L^2(X,m) \to L^2(X,m) \qquad T_t v := e^{-tV} v \qquad t \ge 0$$

which is clearly Markovian with respect to the probability measure m_U.

Lemma 7.2 *The semigroup T_t is supercontractive with respect to m_U if and only if*

$$(U+h)_+/V \in L^\infty(X,m).$$

More precisely, T_t extends to a contraction from $L^2(X,m_U)$ to $L^2(X,m)$ if and only if

$$t \ge t_0 := \frac{1}{2}\|(U+h)_+/V\|_\infty.$$

In case m is the counting measure we have $t_0 = \|U/V\|_\infty/2$.

Proof On one hand, if t_0 is finite and $t \geq t_0$, the result follows from $\|S_t v\|_\infty = \|ve^{-tV}\|_\infty \leq \|ve^{-tV}\|_{L^2(X,m_0)} = \|ve^{(U+h)/2-tV}\|_{L^2(X,m_U)} \leq \|v\|_{L^2(X,m_U)}$. On the other hand, if $\|S_t v\|_\infty \leq \|v\|_{L^2(X,m_U)}$ for some $t_0 \geq 0$ and all $t \geq t_0$, choosing $v := 1_{\{x\}}$ for any $x \in X$, we have $t_0 \geq \|(U+h)_+/V\|_\infty/2$. □

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