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Spectral of the nonlinear weighted superposition operator on Fock spaces

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Abstract
We follow several approaches in nonlinear spectral theory and determine the various spectral forms for the nonlinear weighted superposition operator on Fock spaces. The results show that most of the forms introduced so far coincide and contain singeltons. The classical, asymptotic, and connected eigenvalues, and some numerical ranges of the operator are also identified. We further prove that the operator is both linear and odd asymptotically with respect to the pointwise multiplication operator on the spaces.

Keywords Fock space · Spectrum · Superposition · Eigenvalue · Numerical range

Mathematics Subject Classification Primary 47H30 · Secondary 46E15 · 46E20

1 Introduction

The superposition operator is a typical example of nonlinear operators which plays a significant role in the study of nonlinear functional analysis, differential and integral equations [2, 3, 21]. We may recall its definition. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be metric spaces defined over a given domain G, and ψ be a function on G. The superposition operator $S_\psi : \mathcal{H}_1 \to \mathcal{H}_2$ is defined by $S_\psi f = \psi \circ f$ for all f in $\mathcal{H}_1$. If u is a holomorphic functions on G as well, then the weighted superposition operator $S_{(u,\psi)} : \mathcal{H}_1 \to \mathcal{H}_2$

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is defined by $S_{(u,\psi)} f = M_u \cdot \psi \circ f$, where $M_u f = uf$ is the pointwise multiplication operator of symbol u. The operator $S_{(u,\psi)}$ reduces to S_ψ and M_u, respectively, when $u = 1$ and $\psi(z) = z$.

Though the superposition operator is known to have a long history in the context of real-valued functions [2], there have been only some studies about its actions on spaces of analytic functions defined over the unit disc [1, 4, 5, 7, 9, 10]. In 2022, the second author took the study further and investigated some of the operators basic analytical structures on Fock spaces defined over the whole complex plane $\mathbb{C}$ [17]. Let $\mathcal{H}(\mathbb{C})$ denote the set of entire functions on $\mathbb{C}$. Then the Fock space $\mathcal{F}_p$ is the space of all f in $\mathcal{H}(\mathbb{C})$ for which

$$\|f\|_p := \left(\frac{p}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z) \right)^{\frac{1}{p}} < \infty$$

for $0 < p < \infty$, where A is the usual Lebesgue area measure on $\mathbb{C}$, and for $p = \infty$

$$\|f\|_\infty := \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty.$$

For easier further referencing, we now state the result from [17] which identifies the weighted superposition operators acting between the Fock spaces.

Theorem 1.1 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $0 < p, q \leq \infty$.*

1. *If $p \leq q$, then the following statements are equivalent.*

    (a) *$S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$;*
    (b) *Either $\psi(z) = az + b$ for some constants a and b in $\mathbb{C}$ and u is a constant or ψ is a constant and u belongs to $\mathcal{F}_q$;*
    (c) *$S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;*
    (d) *$S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is globally Lipschitz continuous.*

2. *If $p > q$, then the following statements are also equivalent.*

    (a) *$S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$;*
    (b) *ψ is a constant and u belongs to $\mathcal{F}_q$;*
    (c) *$S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;*
    (d) *$S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is globally Lipschitz continuous.*

3. *The map $S_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ cannot be compact for all pairs of p and q.*

The result shows except in the case when ψ is a constant, every weighted superposition operator on Fock spaces is a superposition operator since $S_{(u,\psi)} f = \alpha a f + b\alpha = S_\Omega f$, where $\Omega(z) = a\alpha z + b\alpha$.

One of the main challenges in dealing with nonlinear theory is the lack of reasonable definitions to concepts that can be applied to a wide range of operators. Reasonable definition here is in the sense that it should reduce to the familiar property in case of linear operators and attempts to preserve or share some of the useful linear structures. Recently, we continued the line of research in [17] and studied a number of concepts

related to the topological and dynamical characteristics of $S_{(u,\psi)}$ on Fock spaces. Our results showed that in several cases, the natural extensions of definitions for nonlinear from linear fail to hold with the operator $S_{(u,\psi)}$ [11]. Thus, this operator may serve as a good example for illustrating basic structural variations in the theories of linear and nonlinear operators; which also motivates us to study the operator further.

2 The various spectral forms of $S_{(u,\psi)}$ on Fock spaces

The notion of spectrum for nonlinear operators has been a subject of extensive studies in nonlinear analysis. Several attempts have been made to define the notion, but none of the definitions proposed so far has been all encompassing as to the extent of the spectral theory of linear operators. Simple examples show the spectrum would fail to have even basic properties like being closed, bounded or nonempty. For a detailed account of the subject, we refer readers to the monograph [3].

In this work, we plan to determine and compare the various spectral forms of weighted superposition operator on Fock spaces using several proposed definitions. We provide the various forms in Theorem 2.1, Theorem 2.2, Proposition 2.3, Proposition 2.4, and Theorem 2.5. We further describe the classical eigenvalues in Proposition 2.6, the asymptotic eigenvalues in Corollary 2.7, Connected eigenvalues in Corollary 2.8, and some numerical ranges of the operator in Theorem 2.10. We show, in Corollary 2.9 and Corollary 2.11, that the nonlinear weighted superposition operator is both linear and odd asymptotically with respect to the pointwise multiplication operator on the spaces.

We begin with a spectrum whose definition goes back to Kachurovskij [15] in 1969. Let T be a Lipschitz continuous operator on a Banach space $\mathcal{H}$. The Kachurovskij resolvent set of T is given by

$$\rho_K(T) = \left\{\lambda \in \mathbb{K} : \lambda I - T \text{ is bijective and } R(\lambda; T) \text{ is Lipschitz continuous}\right\},$$

where I denotes the identity operator, and $R(\lambda; T) = (\lambda I - T)^{-1}$ is the resolvent operator of T at λ, and $\mathbb{K}$ refers to the scalar set $\mathbb{R}$ or $\mathbb{C}$ which we will simply write $\mathbb{C}$ in the rest of the manuscript. The complement of the set, $\sigma_K(T) = \mathbb{C}\backslash\rho_K(T)$, is called the Kachurovskij spectrum of the operator. In the case of a bounded linear operator, this gives the usual definition of the spectrum. The Kachurovskij spectrum is always compact but it can be empty for some operators.

In 1969, J. Neuberger [18] suggested also another approach through which the corresponding spectrum is always nonempty when the operator acts on complex Banach spaces. The Neuberger spectrum for an operator T is defined by $\sigma_N(T) = \mathbb{C}\backslash\rho_N(T)$, where

$$\rho_N(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective and } R(\lambda; T) \in C^1(\mathcal{H})\right\}, \qquad (2.1)$$

and $C^1(\mathcal{H})$ is the set of all continuously Fréchet differentiable operators on $\mathcal{H}$. Recall that T is called Fréchet differentiable at point x_0 in $\mathcal{H}$ if there exists a bounded linear operator L on $\mathcal{H}$ such that

$$\lim_{\|h\| \to 0} \frac{\|T(x_0 + h) - T(x_0) - L(h)\|}{\|h\|} = 0. \tag{2.2}$$

The operator L is called the Fréchet derivative of T at x_0. On the other hand, unlike the theory of linear operators, the Neuberger spectrum may not be closed or bounded [3]. This spectrum plays vital role in solvability of certain operator equations and eigenvalue problems [18].

In 1977, Rhodius [19] provided another definition of spectrum for a continuous operator T on $\mathcal{H}$. The Rhodius resolvent set, $\rho_R(T)$, consists of all complex numbers λ for which $T - \lambda I$ is bijective and $R(\lambda; T)$ is continuous as well. Then, the set $\sigma_R(T) = \mathbb{C} \backslash \rho_R(T)$ is called the Rhodius spectrum of T. For the case of bounded linear operators, this again agrees with the usual definition of the spectrum.

Another definition was introduced by Dörfner in 1997 for a continuous operator T as $\sigma_D(T) = \mathbb{C} \backslash \rho_D(T)$, where $\rho_D(T)$ refers to the Dörfner resolvent set given by

$$\rho_D(T) = \left\{ \lambda \in \mathbb{C} : \lambda I - T \text{ is bijective and } \sup_{\|x\| \neq 0} \frac{\|R(\lambda; T)x\|}{\|x\|} < \infty \right\}. \tag{2.3}$$

Note that both the Rhodius and Dörfner spectra could be empty as well. However, if the space is infinite dimensional and the operator is compact, then zero belongs to the Rhodius spectrum. Furthermore, the Rhodius spectrum is neither closed nor bounded.

The main difference among the various definitions considered above is the requirements imposed on the resolvent operator $R(\lambda; T)$, whether it is Lipschitz continuous, Fréchet differentiable, continuous, or linearly bounded. All the spectra mentioned above have the following forms for $S_{(u,\psi)}$ on Fock spaces.

Theorem 2.1 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $1 \leq p \leq \infty$. Let $S_{(u,\psi)}$ be continuous on $\mathcal{F}_p$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then,*

$$(i) \quad \sigma_R(S_{(u,\psi)}) = \sigma_K(S_{(u,\psi)}) = \sigma_N(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.4}$$

$$(ii) \qquad \sigma_D(S_{(u,\psi)}) = \begin{cases} \{a\alpha\}, & b = 0 \\ \mathbb{C}, & b \neq 0. \end{cases}$$

Notice that in contrary to the dichotomy whether $a = 0$ or $a \neq 0$ in the hypothesis of the theorem, we find it easier to merge the two cases and interpret the set on the right-hand side of (2.4) as a set containing only zero whenever $a = 0$. The same interpretation applies in the rest of the manuscript as needed. As the result shows three of the spectral forms for $S_{(u,\psi)}$ coincide and contain a single element. When $b = 0$

in which case the operator becomes linear, the Dörfner spectrum coincides with the other three as well.

Proof (i) Clearly, if $a = 0$, then $S_{(u,\psi)}f - \lambda f = ub - \lambda f$ is bijective for all $\lambda \neq 0$. The operator $R(\lambda; S_{(u,\psi)})f = (ub - f)/\lambda$ is also Lipschitz continuous since for any f and g in $\mathcal{F}_p$

$$\|R(\lambda; S_{(u,\psi)})f - R(\lambda; S_{(u,\psi)})g\|_p = \left\| \frac{ub - f}{\lambda} - \frac{ub - g}{\lambda} \right\|_p = \frac{\|f - g\|_p}{|\lambda|}.$$

Therefore, the claim for $\sigma_R(S_{(u,\psi)})$ and $\sigma_K(S_{(u,\psi)})$ holds in this case. On the other hand, for $a \neq 0$,

$$S_{(u,\psi)}f - \lambda f = \alpha a f + \alpha b - \lambda f = (a\alpha - \lambda)f + \alpha b$$

is bijective only when $a\alpha \neq \lambda$. If not, it fails to be injective. In this case,

$$R(\lambda; S_{(u,\psi)})f = \frac{f}{a\alpha - \lambda} - \frac{b\alpha}{a\alpha - \lambda} = S_\Upsilon f, \qquad (2.5)$$

where $\Upsilon(z) = (a\alpha - \lambda)^{-1}z - b\alpha(a\alpha - \lambda)^{-1}$. Thus, $R(\lambda; S_{(u,\psi)})$ is itself a superposition operator and by Theorem 1.1, it is Lipschitz continuous and hence the assertion for $\sigma_R(S_{(u,\psi)})$ and $\sigma_K(S_{(u,\psi)})$ in (2.4) holds again.

Next, we consider $\sigma_N(S_{(u,\psi)})$. If $a = 0$, then for any f in $\mathcal{F}_p$ and $\lambda \neq 0$,

$$R(\lambda; S_{(u,\psi)})(f + h) - R(\lambda; S_{(u,\psi)})f - Lh = -\frac{h}{\lambda} - Lh.$$

Therefore, (2.2) holds with $L = M_{-\lambda^{-1}}$, where $M_{-\lambda^{-1}}$ is the multiplication operator with symbol $-\lambda^{-1}$ on Fock spaces.

We may now set $\psi(z) = az + b$, $u = \alpha$, and λ in $\mathbb{C}$ such that $a\alpha \neq \lambda$. Then for each f in $\mathcal{F}_p$,

$$R(\lambda; S_{(u,\psi)})(f + h) - R(\lambda; S_{(u,\psi)})f - Lh = \frac{h}{a\alpha - \lambda} - Lh.$$

This shows (2.2) holds again with $L = M_{(\lambda - a\alpha)^{-1}}$. Hence, we arrive at the claim for $\sigma_N(S_{(u,\psi)})$ as well.

(ii) To compute $\sigma_D(S_{(u,\psi)})$, let us first assume $a \neq 0$. From above, $\lambda I - S_{(u,\psi)}$ is bijective only when $\lambda \neq a\alpha$. We proceed to check the linearly bounded condition for the resolvent operator in (2.3). Using (2.5),

$$\sup_{\|f\|_p \neq 0} \frac{\|R(\lambda; S_{(u,\psi)})f\|_p}{\|f\|_p} = |a\alpha - \lambda|^{-1} \sup_{\|f\|_p \neq 0} \frac{\|f - b\alpha\|_p}{\|f\|_p} < \infty$$

only if $b = 0$. If not, consider the sequence $f_n = 1/n$ to see that the above supremum diverges. Observe that $b = 0$ implies the operator $S_{(u,\psi)}$ is linear and hence $\sigma_D(S_{(u,\psi)})$ coincides with the linear spectrum in this case.

Similarly, if $a = 0$ and $\lambda \neq 0$, then

$$\sup_{\|f\|_p \neq 0} \frac{\|R(\lambda; S_{(u,\psi)})f\|_p}{\|f\|_p} = \sup_{\|f\|_p \neq 0} \frac{\|f - ub\|_p}{|\lambda|\|f\|_p} < \infty \qquad (2.6)$$

holds only when $b = 0$. Note also that both a and b cannot be zero since ψ is non-zero.

□

2.1 The Furi–Martelli–Vignoli spectrum(FMV spectrum) of $S_{(u,\psi)}$

The four spectra discussed in the preceding section are built on some familiar requirements for spectrum of bounded linear operators. Now, we consider another type of spectrum for continuous operators which was introduced in 1978 by Furi, Martelli, and Vignoli [14]. This spectrum is constructed differently than the familiar approach and has found a sufficiently large varieties of interesting applications [14]. To define the spectrum, we need some preparations. For a Banach space $\mathcal{H}$, denote by $B_\epsilon(\mathcal{H})$ the closed ball with center at 0 and radius $\epsilon > 0$. For a bounded set M in $\mathcal{H}$, its Kuratowski measure of noncompactness is defined by

$$\gamma(M) = \inf \left\{ \epsilon : \epsilon > 0, \text{ M has a finite } \epsilon - \text{net in } \mathcal{H} \right\},$$

where a finite ϵ-net refers to a finite set $\{z_1, z_2, z_3, ..., z_n\}$ in $\mathcal{H}$ for which M is covered by the union of the sets $z_j + B_\epsilon(\mathcal{H})$, $j = 1, 2, ..., n$. Notice that $\gamma(M) = 0$ if and only if the closure of M is compact. If $\mathcal{H}$ is finite dimensional, then $\gamma(M) = 0$ for any bounded subset M of $\mathcal{H}$.

For a continuous operator T on $\mathcal{H}$, recall that the upper and lower measures of noncompactness are defined by

$$[T]_A := \inf \left\{ k : \gamma(T(M)) \leq k\gamma(M) \right\} \text{ and}$$

$$[T]_{non} := \sup\{k : k > 0, \gamma(T(M)) \geq k\gamma(M)\} = \inf_{\infty > \gamma(M) > 0} \frac{\gamma(T(M))}{\gamma(M)}, \qquad (2.7)$$

where the last equality in (2.7) makes sense when $\mathcal{H}$ is infinite dimensional. In finite-dimensional spaces, as all bounded sets are precompact, and so there exists no set M satisfying $0 < \gamma(M) < \infty$.

Now, the operator T is called stably solvable, if for any given compact operator S on $\mathcal{H}$ with the asymptotic property

$$[S]_Q := \limsup_{\|x\| \to \infty} \frac{\|Sx\|}{\|x\|} = 0, \qquad (2.8)$$

the equation $Tx = Sx$ has a solution in $\mathcal{H}$. A stably solvable operator is always surjective, and in fact in the case of linear operators, stable solvability reduces to surjectivity but not conversely; see [2, p.130] for a counterexample. Thus, solvability

is simply a notion in nonlinear theory that generalizes surjectivity in linear theory. We call T is FMV-regular if it is stably solvable, $[T]_{non} > 0$, and

$$[T]_{qua} := \liminf_{\|x\| \to \infty} \frac{\|Tx\|}{\|x\|} > 0. \tag{2.9}$$

A prototype example of an FMV-regular operator is the identity map I. The set $\sigma_{FMV}(T) = \mathbb{C} \backslash \rho_{FMV}(T)$ is the FMV-spectrum of T, where

$$\rho_{FMV}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is FMV-regular}\} \tag{2.10}$$

is its resolvent set. Unlike most of the spectra discussed above, the FMV-spectrum is always closed but may fail to be bounded [14]. Note that setting $\sigma_\delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not stable solvable}\}$, $\sigma_{qua}(T) = \{\lambda \in \mathbb{C} : [\lambda I - T]_{qua} = 0\}$, and $\sigma_{non}(T) = \{\lambda \in \mathbb{C} : [\lambda I - T]_{non} = 0\}$, we may decompose the spectrum as

$$\sigma_{FMV}(T) = \sigma_\delta(T) \cup \sigma_{qua}(T) \cup \sigma_{non}(T).$$

This decomposition is not necessarily disjoint as will be shown later with the weighted superposition operator on Fock spaces. If one extends the interpretation from the theory of linear operators, the subspectrums $\sigma_\delta(T)$ and $\sigma_q(T) \cup \sigma_a(T)$ are respectively called the defect spectrum and approximate point spectrum of T. For a linear operator, the FMV-spectrum again reduces to the usual spectrum. Our next main result identifies the FMV-spectrum of $S_{(u,\psi)}$ on Fock spaces.

Theorem 2.2 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $1 \leq p \leq \infty$. Let $S_{(u,\psi)}$ be continuous on $\mathcal{F}_p$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then,*

$$\sigma_{FMV}(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.11}$$

Even if the FMV-spectrum is modeled not as such based on some requirements from linear spectral theory, the result shows $\sigma_{FMV}(S_{(u,\psi)})$ again contain only a single element and coincides with most of the spectral forms in Theorem 2.1.

Proof Depending on the values of a, we consider two different cases.

Case 1: Assume $a \neq 0$. Then, $\lambda f - S_{(u,\psi)} f = \lambda f - a\alpha f - \alpha b$ is surjective if and only if $\lambda \neq a\alpha$. We proceed to show that all λ in $\mathbb{C}$ such that $\lambda \neq a\alpha$ satisfy the condition for stable solvability of $\lambda I - S_{(u,\psi)}$. Let S be a compact operator on $\mathcal{F}_p$ satisfying (2.8). Considering the equation

$$\lambda f - S_{(u,\psi)} f = Sf,$$

and rewriting it further, we have

$$\left(-\frac{S}{\lambda - a\alpha} + I\right) f = \frac{\alpha b}{\lambda - a\alpha}. \tag{2.12}$$

Now for each $\lambda \neq a\alpha$, the operator $-\frac{S}{\lambda - a\alpha}$ is compact. By Fredholm Alternative Theorem [16, Theorem 3.4.24], it follows the operator

$$-\frac{S}{\lambda - a\alpha} + I$$

is invertible. Thus, the equation in (2.12) is solvable in $\mathcal{F}_p$, namely that

$$f = \left(-\frac{S}{\lambda - a\alpha} + I\right)^{-1}\left(\frac{\alpha b}{\lambda - a\alpha}\right).$$

Therefore,

$$\sigma_\delta(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.13}$$

We now check condition (2.9) and compute

$$[\lambda I - S_{(u,\psi)}]_{qua} = \liminf_{\|f\|_p \to \infty} \frac{\|(\lambda - a\alpha)f - \alpha b\|_p}{\|f\|_p}$$
$$\leq \liminf_{\|f\|_p \to \infty}\left(|\lambda - a\alpha| + \frac{|\alpha b|}{\|f\|_p}\right) = |a\alpha - \lambda|. \tag{2.14}$$

On the other hand,

$$[\lambda I - S_{(u,\psi)}]_{qua} = \liminf_{\|f\|_p \to \infty} \frac{\|(\lambda - a\alpha)f - \alpha b\|_p}{\|f\|_p}$$
$$\geq \liminf_{\|f\|_p \to \infty}\left||\lambda - a\alpha| - \frac{|\alpha b|}{\|f\|_p}\right| = |a\alpha - \lambda|. \tag{2.15}$$

Now, by (2.14) and (2.15),

$$[\lambda I - S_{(u,\psi)}]_{qua} = |a\alpha - \lambda| > 0$$

if and only if $a\alpha \neq \lambda$, and hence

$$\sigma_{qua}(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.16}$$

Next, we compute $\sigma_{non}(S_{(u,\psi)})$. Let M be a bounded set and $f_i + B_\epsilon(\mathcal{F}_p)$ for $i = 1, 2, 3, ..., n$ be a finite ϵ-net for M. Then, for each f in M, there exists $h_f \in B_\epsilon(\mathcal{F}_p)$ and f_j in $\{f_1, f_2, ..., f_n\}$ such that $f = f_j + h_f$. It follows that

$$\lambda f - S_{(u,\psi)}f = \lambda f - a\alpha f - b\alpha = (\lambda - a\alpha)(f_j + h_f) - b\alpha$$
$$= ((\lambda - a\alpha)f_j - b\alpha) - (\lambda - a\alpha)h_f.$$

Since $(\lambda - a\alpha)h_f$ belongs to the disc $B_{|\lambda - a\alpha|\epsilon}(\mathcal{F}_p)$, the set

$$\left\{(\lambda - a\alpha)f_i - b\alpha, i = 1, 2, ..., n\right\}$$

is a finite $|\lambda - a\alpha|\epsilon$-net for the set $S_{(u,\psi)}(M)$, and hence

$$\gamma\big((\lambda I - S_{(u,\psi)})(M)\big) \leq |\lambda - a\alpha|\epsilon. \tag{2.17}$$

Since (2.17) holds for every ϵ-net for M, taking the infimum over ϵ gives

$$\gamma\big((\lambda I - S_{(u,\psi)})(M)\big) \leq |\lambda - a\alpha|\gamma(M). \tag{2.18}$$

We proceed to show the relation in (2.18) is in fact an equality. To this end, let $g_i + B_\epsilon(\mathcal{F}_p)$ for $i = 1, 2, 3, ..., n$ be a finite ϵ-net for $S_{(u,\psi)}(M)$. Then for f in M,

$$\lambda f - S_{(u,\psi)}f = \lambda f - a\alpha f - b\alpha = (\lambda - a\alpha)f - b\alpha = g_j + t_f$$

for some $t_f \in B_\epsilon(\mathcal{F}_p)$ and g_j in $\{g_1, g_2, ..., g_n\}$. Solving the equation for f

$$f = \frac{g_j + b\alpha}{\lambda - a\alpha} + \frac{t_f}{\lambda - a\alpha}.$$

Since $t_f/(\lambda - a\alpha)$ belongs to the disc $B_{\epsilon|\lambda - a\alpha|^{-1}}(\mathcal{F}_p)$, the sets

$$\left\{\frac{g_j + b\alpha}{\lambda - a\alpha}, i = 1, 2, ..., n\right\}$$

is a finite $\epsilon|\lambda - a\alpha|^{-1}$-net for M. Consequently,

$$\gamma(M) \leq \epsilon|\lambda - a\alpha|^{-1}$$

and hence $\gamma\big((\lambda I - S_{(u,\psi)})(M)\big) \geq |\lambda - a\alpha|\gamma(M)$ after taking the infimum with respect to all possible ϵ. This together with (2.18) imply

$$\gamma\big((\lambda I - S_{(u,\psi)})(M)\big) = |\lambda - a\alpha|\gamma(M).$$

Therefore,

$$[\lambda I - S_{(u,\psi)}]_{non} = \inf_{\infty > \gamma(M) > 0} \frac{\gamma\big((\lambda I - S_{(u,\psi)})(M)\big)}{\gamma(M)} = |\lambda - a\alpha|,$$

and hence

$$\sigma_{non}(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.19}$$

Now, the assertion in the theorem follows from (2.13), (2.16) and (2.19).

Case 2: Assume $a = 0$. The proof of this case is a simple variant of the first case. We only have to replace α by u and $\lambda - a\alpha$ by λ and run the same argument to arrive at

$$\sigma_\delta(S_{(u,\psi)}) = \sigma_{qua}(S_{(u,\psi)}) = \sigma_{non}(S_{(u,\psi)}) = \{0\},$$

and completes the proof.   □

2.2 The Feng-spectrum of $S_{(u,\psi)}$

Unlike the linear case, the spectra considered above do not necessarily contain the classical eigenvalues of a given nonlinear operator. In 1977, another spectrum related to the FMV-spectrum was introduced by W. Feng [12] applying k-epi mapping theory aiming that the spectrum should contain all the eigenvalues. The FMV-spectrum takes into account the asymptotic properties of the operator while the Feng-spectrum considers the operators global structure as defined below. Denote by $\mathcal{B}(\mathcal{H})$ the family of all open, bounded, and connected subsets Ω of a Banach space $\mathcal{H}$ containing the zero vector. A continuous operator $T : \overline{\Omega} \to \mathcal{H}$ is called k-epi on $\overline{\Omega}$ for $k \geq 0$ if $T \neq 0$ on the boundary $\partial\Omega$, and for all operators $G : \overline{\Omega} \to \mathcal{H}$ satisfying $[G]_A \leq k$ and $G(x) = 0$ on $\overline{\Omega}$, the coincidence equation $Tx = Gx$ has a solution x in Ω. We call T simply epi on $\overline{\Omega}$ if the operator G above is compact. Now for $\Omega \in \mathcal{B}(\mathcal{H})$, we may set

$$V_\Omega(T) = \inf \left\{ k \geq 0 : T \text{ is not k-epi on } \overline{\Omega} \right\}$$

and

$$T_{vk} = \inf_{\Omega \in \mathcal{B}(\mathcal{H})} V_\Omega(T). \tag{2.20}$$

The problem of finding a map T which is not k-epi for any $k > 0$ was solved by Furi [13] in 2002. Furi showed that the map $x \to \|x\|x$ satisfies this property over any infinite dimensional Banach space H.

Now, the operator T on $\mathcal{H}$ is Feng regular(F-regular) if T_{vk}, $[T]_{non}$, and $[T]_{bin}$ are all positive, where

$$[T]_{bin} := \inf_{0 \neq x \in \mathcal{H}} \frac{\|Tx\|}{\|x\|}. \tag{2.21}$$

In this case, the set

$$\rho_F(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is F-regular}\} \tag{2.22}$$

is the Feng-resolvent of T, where its complement $\sigma_F(T) = \mathbb{C} \backslash \rho_F(T)$ is called the Feng-spectrum of T. The spectrum can be also decomposed as

$$\sigma_F(T) = \sigma_{vk}(T) \cup \sigma_{bin}(T) \cup \sigma_{non}(T),$$

where $\sigma_{vk}(T) = \{\lambda \in \mathbb{C} : [\lambda I - T]_v = 0\}$, $\sigma_{bin}(T) = \{\lambda \in \mathbb{C} : [\lambda I - T]_{bin} = 0\}$, and $\sigma_{non}(T) = \{\lambda \in \mathbb{C} : [\lambda I - T]_{non} = 0\}$. As in the case of FMV-spectrum, such decomposition need not be disjoint.

For a linear operator, this precisely gives the familiar spectrum again. Like the FMV-spectrum, the Feng-spectrum is always closed but may be unbounded. For the operator $S_{(u,\psi)}$ on Fock spaces, we prove the following.

Proposition 2.3 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $1 \leq p \leq \infty$. Let $S_{(u,\psi)}$ be continuous on $\mathcal{F}_p$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then,*

$$\sigma_F(S_{(u,\psi)}) = \begin{cases} \{a\alpha\}, & b = 0 \\ \mathbb{C}, & b \neq 0. \end{cases} \qquad (2.23)$$

Proof Note that if the given operator does not vanish at the origin, as explained by W. Feng in [12], the spectrum becomes the whole complex plane. In our case, $S_{(u,\psi)}$ fails to vanish at the origin only when $b \neq 0$ and hence the corresponding case in (2.23) holds.

On the other hand, if $b = 0$, then by [12, Theorem 5.2]

$$\sigma_{FMV}(S_{(u,\psi)}) \subseteq \sigma_F(S_{(u,\psi)}) \subseteq \sigma_K(S_{(u,\psi)}). \qquad (2.24)$$

Then, the claim follows after an application of Theorem 2.1 and Theorem 2.2. This case can be alternatively deduced from the fact that the operator is linear when $b = 0$. Then, the spectrum is the set containing all points λ in $\mathbb{C}$ for which $\lambda I - S_{(u,\psi)}$ fails to be bijective. $\qquad \square$

2.3 Remark

We remark that the various forms of spectra studied above for $S_{(u,\psi)}$ can be also considered when the operator acts between two different Fock spaces $\mathcal{F}_p$ and $\mathcal{F}_q$ for $1 \leq p, q \leq \infty$. Suppose the operator $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$. If $p < q$ and f in $\mathcal{F}_p$,

$$\lambda f - S_{(u,\psi)} f = \begin{cases} (\alpha a - \lambda) f + b\alpha, a \neq 0 \\ \lambda f - ub, & a = 0, \end{cases}$$

belongs to $\mathcal{F}_p$ for all λ in $\mathbb{C}$. Since the inclusion $\mathcal{F}_p \subset \mathcal{F}_q$ is proper in this case [23, Theorem 2.10], the operator $\lambda I - S_{(u,\psi)}$ is not surjective for any λ. Therefore,

$$\sigma_R(S_{(u,\psi)}) = \sigma_K(S_{(u,\psi)}) = \sigma_N(S_{(u,\psi)}) = \sigma_D(S_{(u,\psi)})$$
$$= \sigma_{FMV}(S_{(u,\psi)}) = \sigma_F(S_{(u,\psi)}) = \mathbb{C}.$$

On the other hand, if $p > q$, then by Theorem 1.1 the operator $S_{(u,\psi)}$ maps $\mathcal{F}_p$ into $\mathcal{F}_q$ only when $a = 0$ and u belongs to $\mathcal{F}_q$. Since

$$\lambda f - S_{(u,\psi)} f = \lambda f - ub, \tag{2.25}$$

using the proper inclusion of $\mathcal{F}_q$ in $\mathcal{F}_p$, we can find a function f in $\mathcal{F}_p$ for which the expressions in (2.25) fails to belong to $\mathcal{F}_q$ unless λ is zero. Thus, the operator $\lambda I - S_{(u,\psi)}$ does not exist.

2.4 The Väth phantom for $S_{(u,\psi)}$

In 2001, another kind of spectrum for nonlinear operators was introduced by Väth [20] which finds lots of interesting applications. This spectrum is based on the local structure of operators unlike the FMV and Feng which, respectively, are based on the asymptotic and global structures. To define the notation, let us first provide some technical preliminary concepts and notations. For $\Omega \in \mathcal{B}(\mathcal{H})$, we call a bounded $T : \overline{\Omega} \to \mathcal{H}$ is strictly epi on Ω if it is k-epi on $\overline{\Omega}$ for some $k > 0$ and

$$\inf_{x \in \partial\Omega} \|Tx\| > 0. \tag{2.26}$$

The map T is called properly epi on Ω if T is zero- epi on Ω and $\gamma(T|\overline{\Omega}) > 0$.

Now, T is V-regular on $\mathcal{H}$ if there exists some $\Omega \in \mathcal{B}(\mathcal{H})$ such that T is strictly epi on $\overline{\Omega}$, and V-regular if there exists some $\Omega \in \mathcal{B}(\mathcal{H})$ for which T is properly epi on $\overline{\Omega}$. Now, the phantom, $\phi(T)$, and large phantom, $\Phi(T)$, are, respectively, defined by

$$\phi(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not v-regular}\} \tag{2.27}$$

and

$$\Phi(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not V-regular}\}. \tag{2.28}$$

Said differently, a scalar λ belongs to $\phi(T)$ if and only if $\lambda I - T$ fails to be strictly epi on any set Ω in $\mathcal{B}(\mathcal{H})$. On the other hand, λ belongs to $\Phi(T)$ if and only if for any set Ω in $\mathcal{B}(\mathcal{H})$ either $[(\lambda I - T)|_{\overline{\Omega}}]_{non} = 0$ or $\lambda I - T$ is not epi on $\overline{\Omega}$.

Both of the spectra in (2.27) and (2.28) are closed and coincide with usual spectrum for linear operators. Furthermore, by [20]

$$\phi(T) \subseteq \Phi(T) \subseteq \sigma_{FMV}(T). \tag{2.29}$$

Proposition 2.4 *Let* $\psi, u \in \mathcal{H}(\mathbb{C})$ *be nonzero and* $1 \le p \le \infty$. *Let* $S_{(u,\psi)}$ *be continuous on* $\mathcal{F}_p$ *and hence either* $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$ *and* $u = \alpha$ *is a constant or* $\psi = b$ *is a constant and* u *belongs to* $\mathcal{F}_p$. *Then,*

$$\phi(S_{(u,\psi)}) = \Phi(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.30}$$

Proof Using the same covering argument used as in the proof of Theorem 2.2, it is straightforward to see that

$$[(\lambda I - S_{(u,\psi)})|_{\overline{\Omega}}]_{non} = \begin{cases} |\lambda - a\alpha|, & a \ne 0 \\ |\lambda|, & a = 0. \end{cases}$$

for any Ω in $\mathcal{B}(\mathcal{F}_p)$. This together with (2.29) and (2.11) imply that $\Phi(T) = \sigma_{FMV}(T)$.

Next, we compute $\phi(S_{(u,\psi)})$. Let us first consider $a \ne 0$ and $b = 0$. Then $\phi(S_{(u,\psi)})$ agrees with the usual spectrum and the claim follows easily from the bijectively condition for $\lambda f - S_{(u,\psi)}$. Suppose now that $a \ne 0$ and $b \ne 0$. Given that $\phi(T)$ is contained in $\Phi(T)$, it suffices to show whether $\lambda = a\alpha$ belongs to $\phi(T)$. To this end, for each f in $\mathcal{F}_p$,

$$\lambda f - S_{(u,\psi)} f = b\alpha,$$

and, hence, $\lambda I - S_{(u,\psi)}$ is not surjective. Thus, the operator is not strictly epi on any set Ω in $\mathcal{B}(\mathcal{F}_p)$.

It remains to check when $a = 0$ and, hence, $\lambda f - S_{(u,\psi)} f = \lambda f - ub$. But this follows as in the case of $a \ne 0$ and $b \ne 0$. □

2.5 The spectrum of $S_{(u,\psi)}$ at a point

Motivated by the fact that many concepts in nonlinear analysis are of local nature, Calamai, Furi, and Vignoli [6] in 2009 introduced the notion of spectrum of a nonlinear operator at a point. Let us recall the definition, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Banach spaces and $T : U \to \mathcal{H}_2$ be continuous, where U is an open subset of $\mathcal{H}_1$. Let $q \in U$ and U_q be the open neighborhood $\{x \in \mathcal{H}_1 : x + q \in U\}$ of $0 \in \mathcal{H}_1$. We define a local continuous operator $T_q : U_p \to \mathcal{H}_2$ by $T_q(x) = T(x + q) - T(q)$. Let $B(q, r)$ denote the open ball in $\mathcal{H}_1$ centered at q with radius $r > 0$. Suppose that $B(q, r) \subseteq U$ and consider the number

$$\gamma\big(T | B(q, r)\big) = \sup \left\{ \frac{\gamma(T(A))}{\gamma(A)}, \quad A \subseteq B(q, r), \quad \gamma(A) > 0 \right\},$$

which is nondecreasing as a function of r. Thus, we define

$$\gamma_q(T) = \lim_{r \to 0} \gamma(T | B(q, r)).$$

Similarly, we define another characteristic $d_q(T)$ as

$$d_q(T) = \liminf_{x \to 0} \frac{\|T_q(x)\|}{\|x\|}.$$

A map T is said to be regular at point q if $d_q(T)$ and $\gamma_q(T)$ are positive and T_q is 0-epi at 0. We may now define the spectrum of T at the point q as the set

$$\sigma(T, q) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not regular at q}\}.$$

This spectrum is closed and agrees with the classical spectrum in the linear case. It also finds applications in tackling bifurcation problems; see [2, 6] for more.

For the weighted superposition operator on Fock spaces, the spectrum at a point coincides with some of the spectra studied above as shown in the next result.

Theorem 2.5 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $1 \leq p \leq \infty$. Let $S_{(u,\psi)}$ be continuous on $\mathcal{F}_p$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then, for any g in $\mathcal{F}_p$,*

$$\sigma(S_{(u,\psi)}, g) = \{a\alpha\}. \tag{2.31}$$

As a consequence, we notice that the result is independent of the point g. Moreover, the spectrum at a point is nonempty for all points in the space.

Proof We claim that $S_{(u,\psi)}$ is Fréchet differentiable at each point g in $\mathcal{H}$ with derivative given by

$$S'_{(u,\psi)}(g) = M_{a\alpha} g.$$

For $a \neq 0$, we compute

$$\lim_{\|h\|_p \to 0} \frac{\|S_{(u,\psi)}(g + h) - S_{(u,\psi)}(g) - M_{a\alpha}(h)\|_p}{\|h\|_p}$$

$$= \lim_{\|h\|_p \to 0} \frac{\|a\alpha h - M_{a\alpha}(h)\|_p}{\|h\|_p} = 0$$

as asserted. A similar argument holds when $a = 0$.

Now by [6, Corollary 4.25],

$$\sigma(S_{(u,\psi)}, g) = \sigma\left(S'_{(u,\psi)}(g)\right).$$

Thus, it suffices to find the spectrum of the operator $S'_{(u,\psi)}(g)$ using linear spectral approach. A simple computation shows that $a\alpha$ when $a \neq 0$ and 0 when $a = 0$ are the only eigenvalues of $S'_{(u,\psi)}(g)$. Except when λ equals to these values, we also notice that $\lambda I - S'_{(u,\psi)}(g)$ is a bijective map on $\mathcal{F}_p$. $\square$

2.6 The classical and asymptotic eigenvalues of $S_{(u,\psi)}$

Now, we turn to the notion of eigenvalues for $S_{(u,\psi)}$. Defining the notion as in the case of linear operators(classical), we first prove the following.

Proposition 2.6 *Let* $\psi, u \in \mathcal{H}(\mathbb{C})$ *and* $1 \leq p \leq \infty$. *Let* $S_{(u,\psi)}$ *be continuous on* $\mathcal{F}_p$ *and hence either* $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$ *and* $u = \alpha$ *is a constant or* $\psi = b$ *is a constant and* u *belongs to* $\mathcal{F}_p$. *Then,*

$$\sigma_{class}(S_{(u,\psi)}) = \begin{cases} \mathbb{C} \setminus \{a\alpha\}, & a \neq 0 \text{ and } b \neq 0 \\ \{a\alpha\}, & b = 0 \\ \mathbb{C} \setminus \{0\}, & a = 0. \end{cases} \tag{2.32}$$

Another noteworthy difference has been identified here in contrast to linear operators. Unlike the linear case, Theorem 1.1 and Proposition 2.6 show that most of the different nonlinear spectra considered so far fail to contain the point spectrum. Thus, $S_{(u,\psi)}$ is a good example of operators on spaces of analytic functions which illustrates several variations between linear and nonlinear theories. We remind that the Feng-spectrum was constructed in such a way that it contains all the point spectrum. In this case, if $S_{(u,\psi)}$ fixes the origin, then as noted before $b = 0$. Thus, the spectrum in (2.32) is clearly contained in (2.23).

Proof Let us first assume $a = 0$. Then, $S_{(u,\psi)}f - \lambda f = ub - \lambda f = 0$ implies $f = (ub)/\lambda$ is an eigenvector for every non-zero λ in $\mathbb{C}$ since ub is nonzero. On the other hand, if $\lambda = 0$, then either u or b must be zero which contradicts the hypothesis in the proposition.

Next, suppose $a \neq 0$ and

$$S_{(u,\psi)}f - \lambda f = a\alpha f + b\alpha - \lambda f = (a\alpha - \lambda)f + b\alpha = 0.$$

It follows that $f_\lambda = b\alpha(\lambda - a\alpha)^{-1}$ is an eigenvector for each λ in $\mathbb{C}$ such that $\lambda \neq a\alpha$ and $b \neq 0$. If $a\alpha = \lambda$, then b must be zero as u is nonzero. In this case, f_λ can be any nonzero vector. Therefore, the claim in (2.32) follows.

There has been also efforts to define the notion of eigenvalue differently than the classical linear way. A related notion in this regard has been asymptotic eigenvalue $\sigma_{app}(T)$. Let T be a continuous operator on a Banach space $\mathcal{H}$. A number λ in $\mathbb{C}$ belongs to $\sigma_{app}(T)$ if there exists an unbounded sequence x_n in $\mathcal{H}$ such that

$$\lim_{\|x_n\| \to \infty} \frac{\|\lambda x_n - T x_n\|}{\|x_n\|} = 0.$$

The next corollary shows the asymptotic point spectrum of $S_{(u,\psi)}$ coincides with some of the spectra in Theorem 2.1. $\square$

Corollary 2.7 *Let* $\psi, u \in \mathcal{H}(\mathbb{C})$ *be nonzero and* $1 \leq p \leq \infty$. *Let* $S_{(u,\psi)}$ *be continuous on* $\mathcal{F}_p$ *and hence either* $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$ *and* $u = \alpha$ *is a constant or*

$\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then,

$$\sigma_{app}(S_{(u,\psi)}) = \{a\alpha\}.$$

Proof If $a = 0$, then for λ in $\mathbb{C}$ and any unbounded sequence f_n in $\mathcal{F}_p$

$$\lim_{\|f_n\|_p \to \infty} \frac{\|\lambda f_n - S_{(u,\psi)} f_n\|_p}{\|f_n\|_p} = \lim_{\|f_n\|_p \to \infty} \frac{\|\lambda f_n - ub\|_p}{\|f_n\|_p}$$

$$\geq \lim_{\|f_n\|_p \to \infty} \left| |\lambda| - \frac{|b|\|u\|_p}{\|f_n\|_p} \right| = |\lambda|.$$

Here we can chose $f_n = n$. Therefore, λ belongs to the asymptotic point spectrum in this case only when $\lambda = 0$.

Similarly, if $a \neq 0$, then

$$\lim_{\|f_n\|_p \to \infty} \frac{\|\lambda f_n - S_{(u,\psi)} f_n\|_p}{\|f_n\|_p} = \lim_{\|f_n\|_p \to \infty} \frac{\|(\lambda - a\alpha) f_n - \alpha b\|_p}{\|f_n\|_p}$$

$$\geq \lim_{\|f_n\|_p \to \infty} \left| |\lambda - a\alpha| - \frac{|b\alpha|}{\|f_n\|_p} \right| = |\lambda - a\alpha|,$$

and hence the assertion holds taking $f_n = n$. $\qquad\square$

2.7 The point phantom and connected eigenvalues of $S_{(u,\psi)}$

The notation of asymptotic eigenvalues discussed above takes into account the structure of the FMV-spectrum while the classical eigenvalues are associated to the Feng-spectrum. In [3, p.193], another notion which takes into account the structure of the Väth Phantom has been introduced. A scalar λ in $\mathbb{C}$ is called a connected eigenvalue of an operator T on $\mathcal{H}$ if the nullset

$$N(\lambda I - T) = \{x \in \mathcal{H} : Tx = \lambda x\}$$

of $\lambda I - T$ contains an unbounded connected set containing the zero element. Then, the point phantom of T refers to the set

$$\phi_{conn}(T) = \{\lambda \in \mathbb{C} : \lambda \text{ connected eigenvalue for } T\}$$

In case of a bounded linear operator T, this gives again the familiar definition of eigenvalue.

Corollary 2.8 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $1 \leq p \leq \infty$. Let $S_{(u,\psi)}$ be continuous on $\mathcal{F}_p$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or*

$\psi = b$ is a constant and u belongs to $\mathcal{F}_p$. Then,

$$\phi_{conn}(S_{(u,\psi)}) = \begin{cases} \{a\alpha\}, & a \neq 0 \text{ and } b = 0 \\ \{\}, \text{ otherwise.} \end{cases}$$

Proof Clearly, if $a = 0$, then $\lambda f - S_{(u,\psi)}f = \lambda f - ub = 0$ holds only when $\lambda \neq 0$, and hence $f = ub/\lambda \neq 0$. In this case, the corresponding nullset does not contain even the zero element. Consequently, there exists no connected eigenvalue.

Assume $a \neq 0$ and $\lambda f - S_{(u,\psi)}f = (\lambda - a\alpha)f - \alpha b = 0$. Now if $\lambda \neq a\alpha$, then the equation gives that $f = b\alpha/(\lambda - a\alpha)$. Thus, the corresponding nullset contains only one element and no connected eigenvalue in this case either. On the other hand, if $\lambda = a\alpha$ and $b = 0$, then the corresponding null set contains the whole space $\mathcal{F}_p$. The case for $\lambda = a\alpha$ and $b \neq 0$ gives not connected eigenvalue. This proves the claim. □

2.8 Asymptotic linearity and some numerical ranges of $S_{(u,\psi)}$

We now turn our attention to the asymptotic linearity and the numerical range of weighted superposition operator on Fock spaces. A bounded operator T on a Banach space $\mathcal{H}$ is called asymptotically linear if there exists a bounded linear operator L on $\mathcal{H}$ such that

$$\lim_{\|x\| \to \infty} \frac{\|Tx - Lx\|}{\|x\|} = 0. \tag{2.33}$$

Such an operator is called the asymptotic derivative of T. For background materials on this topic, we again refer to [3]. Now, another immediate consequence of Theorem 1.1 shows the weighted superposition operator is asymptotically linear with a respect to the multiplication operator.

Corollary 2.9 *Let* $\psi, u \in \mathcal{H}(\mathbb{C})$ *be nonzero and* $1 \leq p \leq \infty$. *Let* $S_{(u,\psi)}$ *be continuous on* $\mathcal{F}_p$ *and hence either* $\psi(z) = az + b$ *for some* $a, b \in \mathbb{C}$ *and* $u = \alpha$ *is a constant or* $\psi = b$ *is a constant and* u *belongs to* $\mathcal{F}_p$. *Then,* $S_{(u,\psi)}$ *is asymptotically linear with the multiplication operator* $M_{a\alpha}f = \alpha af$ *for* $a \neq 0$ *and the zero operator otherwise.*

As in the case of spectrum, there exists so far no single definition of numerical range for nonlinear operators that encompasses all the basic properties from linear operator perspective. Many attempts have been made to define the concept and relate it to the various spectral sets. In this section, we use some of the definitions and compute numerical ranges for weighted superposition operator on the Fock space. Zarantonello [22] defined the notion of numerical range on Hilbert spaces as follows. Let T be continuous on a Hilbert space $\mathcal{H}$. Then, the numerical range of T is the set

$$N_Z(T) = \left\{ \frac{\langle Tx - Ty, x - y \rangle}{\|x - y\|^2}, x, y \in \mathcal{H} \right\}. \tag{2.34}$$

Later in 1978, Furi, Martelli, and Vignoli presented another definition [14] which finds its own application in solving differential equations. For a given continuous operator

T on $\mathcal{H}$, the authors defined another continuous operator

$$T^N x = \langle Tx, x \rangle \|x\|^{-2} x.$$

Then, the numerical range of T is

$$N_{FMV}(T) = \left\{ \lambda \in \mathbb{C} : \liminf_{\|x\| \to \infty} \frac{\|T^N x - \lambda x\|}{\|x\|} = 0 \right\}. \tag{2.35}$$

We note in passing that the definition in (2.34) coincides with the definition in the linear operator case where as the definition in (2.35) coincides with the closure of the numerical range. In contrast for linear operators, both $N_Z(T)$ and $N_{FMV}(T)$ are not necessarily bounded for nonlinear operators. In particular for the weighted superposition operator, a consequence of Theorem 1.1 shows the two definitions give the same numerical range.

Theorem 2.10 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $S_{(u,\psi)}$ be continuous on $\mathcal{F}_2$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_2$. Then*

$$N_Z(S_{(u,\psi)}) = N_{FMV}(S_{(u,\psi)}) = \{a\alpha\}. \tag{2.36}$$

Proof For $a \neq 0$, and $f, g \in \mathcal{F}_2$,

$$\frac{\langle S_{(u,\psi)}f - S_{(u,\psi)}g, f - g \rangle}{\|f - g\|_2^2} = \frac{a\alpha \langle f - g, f - g \rangle}{\|f - g\|_2^2} = a\alpha. \tag{2.37}$$

On the other hand, if $a = 0$, then

$$\frac{\langle S_{(u,\psi)}f - S_{(u,\psi)}g, f - g \rangle}{\|f - g\|_2^2} = 0. \tag{2.38}$$

From (2.37) and (2.38), the assertion for $N_Z(S_{(u,\psi)})$ in (2.36) holds.

Next, we proceed to compute $N_{FMV}(S_{(u,\psi)})$. Let $a \neq 0$ and consider the operator

$$S_{(u,\psi)}^N f = \frac{\langle S_{(u,\psi)}f, f \rangle}{\|f\|_2^2} f = \frac{\langle a\alpha f + b\alpha, f \rangle}{\|f\|_2^2} f = \left(a\alpha + \frac{\alpha b \overline{f(0)}}{\|f\|_2^2} \right) f.$$

This implies

$$\frac{\|S_{(u,\psi)}^N f - \lambda f\|_2}{\|f\|_2} = \left\| \frac{\left(a\alpha + \alpha b \overline{f(0)} \|f\|_2^{-2} \right) f}{\|f\|_2} - \frac{\lambda f}{\|f\|_2^2} \right\|_2$$

$$= \left| a\alpha - \lambda + \frac{\alpha b \overline{f(0)}}{\|f\|_2^2} \right| \geq \left| |a\alpha - \lambda| - \left| \frac{\alpha b \overline{f(0)}}{\|f\|_2^2} \right| \right|.$$

It follows that

$$\liminf_{\|f\|_2 \to \infty} \frac{\|S_{(u,\psi)}^N f - \lambda f\|_2}{\|f\|_2} \geq \liminf_{\|f\|_2 \to \infty} \left| |a\alpha - \lambda| - \left| \frac{\alpha b \overline{f(0)}}{\|f\|_2^2} \right| \right|$$

$$\geq |a\alpha - \lambda| \qquad (2.39)$$

from which we must have $a\alpha = \lambda$.

Similarly, for $a = 0$,

$$S_{(u,\psi)}^N f = \frac{b\langle u, f \rangle}{\|f\|_2^2} f$$

and

$$\liminf_{\|f\|_2 \to \infty} \frac{\|S_{(u,\psi)}^N f - \lambda f\|_2}{\|f\|_2} = \liminf_{\|f\|_2 \to \infty} \left| \frac{b\langle u, f \rangle}{\|f\|_2^2} - \lambda \right| \geq |\lambda|.$$

From this and (2.39), the second equality in (2.36) follows. □

2.9 Odd and asymptotically odd $S_{(u,\psi)}$

Another common property of all continuous linear operators is that they all are odd. Recall that a continuous operator T on a Banach space $\mathcal{H}$ is odd if $T(-x) = -T(x)$ for all x in $\mathcal{H}$. It is called asymptotically odd if there exists a continuous odd operator $\overline{T}$ on the space such that $[T - \overline{T}]_Q = 0$. The operator $\overline{T}$ is known as the asymptotic derivative of T. A natural question is whether there exists an odd nonlinear weighted superposition $S_{(u,\psi)}$ on Fock spaces. The following corollary answers the question negatively.

Corollary 2.11 *Let $\psi, u \in \mathcal{H}(\mathbb{C})$ be nonzero and $S_{(u,\psi)}$ be continuous on $\mathcal{F}_2$ and hence either $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $u = \alpha$ is a constant or $\psi = b$ is a constant and u belongs to $\mathcal{F}_2$. Then,*

1. *$S_{(u,\psi)}$ is odd if and only if it is linear.*
2. *$S_{(u,\psi)}$ is asymptotically odd with the multiplication operator $M_{a\alpha} f = \alpha a f$ for $a \neq 0$ and the zero operator otherwise.*

Proof The first part follows easily since $S_{(u,\psi)}(-f) = -S_{(u,\psi)}(f)$ implies $b = 0$. To verify (ii), note that for $a \neq 0$,

$$[S_{(u,\psi)} - M_{a\alpha}]_Q = \limsup_{\|f\|_p \to \infty} \frac{\|\alpha a f + \alpha b - a\alpha f\|_p}{\|f\|_p} = 0.$$

Similarly, for $a = 0$, we have

$$[S_{(u,\psi)} - M_0]_Q = \limsup_{\|f\|_p \to \infty} \frac{\|S_{(u,\psi)} f\|_q}{\|f\|_p} = \limsup_{\|f\|_p \to \infty} \frac{\|bu\|_q}{\|f\|_p} = 0.$$

Table 1 Various spectra and numerical ranges of $S_{(u,\psi)}$ on Fock spaces

Spectra & N-range	$a = 0$	$a \neq 0$	$a \neq 0, b = 0$	$a \neq 0, b \neq 0$	$b = 0$	$b \neq 0$
$\sigma_R(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\sigma_K(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\sigma_N(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\sigma_{FMV}(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\sigma_D(S_{(u,\psi)})$	$\mathbb{C}$		$\{a\alpha\}$	$\mathbb{C}$		
$\sigma_F(S_{(u,\psi)})$					$\{a\alpha\}$	$\mathbb{C}$
$\phi(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\Phi(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\phi_{conn}(S_{(u,\psi)})$	$\{\}$		$\{a\alpha\}$	$\{\}$		
$\sigma(S_{(u,\psi)}, g)$	$\{0\}$	$\{a\alpha\}$				
$N_Z(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$N_{FMV}(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				
$\sigma_{class}(S_{(u,\psi)})$	$\mathbb{C}\backslash\{0\}$		$\{a\alpha\}$	$\mathbb{C}\backslash\{a\alpha\}$		
$\sigma_{app}(S_{(u,\psi)})$	$\{0\}$	$\{a\alpha\}$				

$\square$

2.10 Concluding remarks

For the sake of easy comparison, we summarize the main results obtained about the various spectra and numerical ranges of the operator $S_{(u,\psi)}$ in the following table. The numbers a, b, and α are as in Theorem 1.1 and g is any point in $\mathcal{F}_p$.

The table clearly shows that several forms of the spectra and the numerical ranges for $S_{(u,\psi)}$ not only coincide but are singletons. Furthermore, the operator is nonlinear in four of the cases where the spectrum or classical eigenvalue set contain more than singletons. But the sets in these cases are either the whole complex plane or an element less. We also notice that the operator admits connected eigenvalues only when it is linear, that is when $b = 0$.

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