Completely order bounded maps on non-commutative L_p -spaces

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Abstract

We define norms on $L_p(\mathcal{M}) \otimes M_n$ where $\mathcal{M}$ is a von Neumann algebra and M_n is the space of complex $n \times n$ matrices. We show that a linear map $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ is decomposable if $\mathcal{N}$ is an injective von Neumann algebra, the maps $T \otimes Id_{M_n}$ have a common upper bound with respect to our defined norms, and $p = \infty$ or $q = 1$. For $2p < q < \infty$ we give an example of a map T with uniformly bounded maps $T \otimes Id_{M_n}$ which is not decomposable.

Keywords Non-commutative L_p -space · Matrix norm · Completely positive map · Decomposable map

Mathematics Subject Classification 46L07 · 47L25

1 Introduction

Completely positive maps on von Neumann algebras have been studied extensively and there are some nice results on such maps (see e.g. [12], Ch. IV.3 and [10], Ch. 11). When combining the order structure with the vector space structure, it is natural to investigate the linear span of completely positive maps. These are called decomposable maps in [4], §1. The decomposable maps from a von Neumann algebra $\mathcal{M}$ into an injective von Neumann algebra $\mathcal{N}$ are the completely bounded maps. This has been shown at about the same time by Haagerup [4], Paulsen [8],

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and Wittstock [14]. Completely bounded maps use the operator norm on matrices of elements of a von Neumann algebra.

For a von Neumann algebra $\mathcal{M}$, the non-commutative L_p -space $L_p(\mathcal{M})$, $1 \leq p < \infty$, can be realized as (in general) unbounded operators on a Hilbert space. Therefore the L_p -space itself as well as matrices with entries of a L_p -space have a natural order given by positive operators. This allows us to define completely positive maps and decomposable maps from one L_p -space into another one. Then there should be a description of decomposable maps using some norm conditions on matrices of L_p -spaces.

This question has been partially answered by Pisier [9] for linear maps from the Schatten classes S_p to S_p , and Arhancet and Kriegler [1] for linear maps from $L_p(\mathcal{M})$ to $L_p(\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are semifinite, approximately finite dimensional von Neumann algebras. Junge and Ruan [6] give a description of the decomposable norm for finite rank maps from $L_p(\mathcal{M})$ to $L_p(\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are arbitrary von Neumann algebras. All cited articles require the same p -index for domain and range space.

Our idea is to derive a norm on matrices of L_p -spaces from the order structure. This norm is quite similar to the norm used in [1] and [9] (see [9] equation (1.5)). Using this norm, we can characterize decomposable maps from $\mathcal{M}$ to $L_q(\mathcal{N})$ where $1 \leq q \leq \infty$ and from $L_p(\mathcal{M})$ to $L_1(\mathcal{N})$ where $1 \leq p \leq \infty$. In both cases $\mathcal{N}$ must be injective. Then we give an example of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ and a linear map from $L_p(\mathcal{M})$ to $L_q(\mathcal{N})$ where $1 \leq p, q < \infty, q > 2p$ which is completely order bounded with respect to our matricial norms but not decomposable. Therefore this norm cannot characterize decomposable maps for all combinations of p and q .

The structure of the article is as follows: In Sect. 2, we give a short description of non-commutative L_p -spaces in the Haagerup-Terp construction. We also show some properties of matrices of operators. In Sect. 3, we define our norm on matrices and derive some properties of this norm. In Sect. 4, we define completely order bounded maps and show that they are decomposable for some combinations of p and q .

2 Non-commutative L_p -spaces

Short descriptions of non-commutative L_p -spaces in the Haagerup-Terp construction can be found in several publications, but the main source is still [13]. Here, we cite some basic facts from [13] we will use. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let φ be a normal faithful semifinite weight on $\mathcal{M}$ with modular automorphism group σ_t^φ . Then the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ acts on the Hilbert space $L_2(\mathbb{R}, \mathcal{H})$ and is the von Neumann algebra generated by the operators $\pi(x)$ and $\lambda(s)$ where

$$(\pi(x)\xi)(t) = \sigma_{-t}^\varphi(x)\xi(t), \quad x \in \mathcal{M}, t \in \mathbb{R}, \xi \in L_2(\mathbb{R}, \mathcal{H})$$

and

$$(\lambda(s)\xi)(t) = \xi(t-s), \quad s, t \in \mathbb{R}, \xi \in L_2(\mathbb{R}, \mathcal{H}).$$

For $s \in \mathbb{R}$ let $W(s)$ be the unitary operator on $L_2(\mathbb{R}, \mathcal{H})$ which is defined by

$$(W(s)\xi)(t) = e^{-ist}\xi(t), \quad s, t \in \mathbb{R}, \xi \in L_2(\mathbb{R}, \mathcal{H}).$$

The dual action θ is then defined by

$$\theta_s(x) = W(s)xW(s)^*, \quad x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, s \in \mathbb{R}.$$

The elements of $\mathcal{M}$ are the fixed points under θ when $\mathcal{M}$ is identified with $\pi(\mathcal{M})$:

$$\pi(\mathcal{M}) = \{x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \mid \theta_s(x) = x \text{ for all } s \in \mathbb{R}\}.$$

The crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ has a unique normal faithful semifinite trace τ which satisfies

$$\tau(\theta_s(x)) = e^{-s}\tau(x) \text{ for all } x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})_+, s \in \mathbb{R}.$$

The existence of the trace τ allows to consider the τ -measurable operators. These are all closed densely defined operators a affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ which satisfy: For every $\varepsilon > 0$ there exists a projection $p \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ such that $pL_2(\mathbb{R}, \mathcal{H}) \subseteq \mathcal{D}(a)$ and $\tau(1-p) \leq \varepsilon$. A subspace $\mathcal{D}$ of $L_2(\mathbb{R}, \mathcal{H})$ is called τ -dense if for every $\varepsilon > 0$ there is a projection $p \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ such that $pL_2(\mathbb{R}, \mathcal{H}) \subseteq \mathcal{D}$ and $\tau(1-p) \leq \varepsilon$. Thus the τ -measurable operators are those which are affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ and have a τ -dense domain. The closure of a τ -measurable operator restricted to a τ -dense subspace is unique. Therefore, for proving some property of a τ -measurable operator it suffices to prove the property on a τ -dense subspace which is contained in the domain of the operator. The τ -measurable operators form a topological $*$ -algebra. When two τ -measurable operators are added or multiplied, we have to create the closure of the sum or the product which always exist and are unique. More details for τ -measurable operators can be found in [13], Chapter I or [12], Chapter IX.2.

The action θ can be extended to all τ -measurable operators. The space $L_p(\mathcal{M})$, $1 \leq p \leq \infty$, consists of all τ -measurable operators a for which

$$\theta_s(a) = e^{-\frac{s}{p}}(a) \text{ for all } s \in \mathbb{R}.$$

There is a one-to-one correspondence between the elements of the predual $\mathcal{M}_*$ and the elements of $L_1(\mathcal{M})$. This correspondence defines the linear functional $tr : L_1(\mathcal{M}) \rightarrow \mathbb{C}$ by $tr(h_\varphi) = \varphi(1)$, where $h_\varphi \in L_1(\mathcal{M})$ is the operator assigned to $\varphi \in \mathcal{M}_*$. This linear functional has the following properties:

$$\begin{aligned} tr(a) &\geq 0 \text{ if } a \geq 0, \\ tr(a) &= 0 \text{ and } a \geq 0 \text{ implies } a = 0, \\ |tr(a)| &\leq tr(|a|) \text{ for all } a \in L_1(\mathcal{M}). \end{aligned} \tag{2.1}$$

If $a \in L_p(\mathcal{M})$ has the polar decomposition $a = u|a|$, then $u \in \mathcal{M}$ and $|a| \in L_p(\mathcal{M})$. For $1 \leq p < \infty$, the norm on $L_p(\mathcal{M})$ is given by

$$\|a\|_p = \operatorname{tr}(|a|^p)^{\frac{1}{p}}, \quad a \in L_p(\mathcal{M}).$$

For $p = \infty$ and $a \in \mathcal{M}$ the norm $\|a\|_\infty$ is the usual operator norm. If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $a \in L_p(\mathcal{M})$ and $b \in L_q(\mathcal{M})$ then $ab \in L_r(\mathcal{M})$ and

$$\|ab\|_r \leq \|a\|_p \|b\|_q.$$

Especially, when $\frac{1}{p} + \frac{1}{q} = 1$ we get for $a \in L_p(\mathcal{M})$ and $b \in L_q(\mathcal{M})$

$$\begin{aligned} \operatorname{tr}(ab) &= \operatorname{tr}(ba), \\ |\operatorname{tr}(ab)| &\leq \|ab\|_1 \leq \|a\|_p \|b\|_q, \\ \operatorname{tr}(ab) &\geq 0 \text{ for all } b \in L_q(\mathcal{M})_+ \text{ if and only if } a \geq 0. \end{aligned} \quad (2.2)$$

Moreover, the space $L_q(\mathcal{M})$ is isometrically isomorphic to the dual space of $L_p(\mathcal{M})$. We denote this duality by

$$\langle a, b \rangle = \operatorname{tr}(ab) \text{ for } a \in L_p(\mathcal{M}) \text{ and } b \in L_q(\mathcal{M}).$$

If $\mathcal{M}$ has a normal faithful semifinite trace τ , we get the L_p -spaces $L_p(\mathcal{M}, \tau) = \{a | a \text{ is } \tau\text{-measurable}, \tau(|a|^p) < \infty\}$. It is shown in [13], pp. 62 - 63 that $L_p(\mathcal{M})$ is isomorphic to $\{a \otimes f_p | a \in L_p(\mathcal{M}, \tau), f_p : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto \exp(s/p)\}$. Therefore we can switch to the easier to handle spaces $L_p(\mathcal{M}, \tau)$ for semifinite von Neumann algebras.

For $n \in \mathbb{N}$, we denote the complex $n \times n$ matrices by M_n with the usual trace Tr . If $a = [a_{ij}]$ is an $n \times n$ matrix of τ -measurable operators and each a_{ij} acts on the Hilbert space $\mathcal{K} = L_2(\mathbb{R}, \mathcal{H})$, then a acts on the Hilbert space $\mathcal{K}^n$. The operator a is densely defined, has a unique closure which we denote again by a , and is $\tau \otimes \operatorname{Tr}$ -measurable. Especially, the elements of $L_p(\mathcal{M}) \otimes M_n$ are $\tau \otimes \operatorname{Tr}$ -measurable operators. The elements of $L_p(\mathcal{M}) \otimes M_n$ have a natural ordering as being densely defined operators on the Hilbert space $\mathcal{K}^n$. The positive operators in $L_p(\mathcal{M})$ will be denoted by $L_p(\mathcal{M})_+$, and the positive operators in $L_p(\mathcal{M}) \otimes M_n$ will be denoted by $(L_p(\mathcal{M}) \otimes M_n)_+$. Following the description in [7], p. 70, $L_p(\mathcal{M}) \otimes M_n$ is linearly isomorphic to $L_p(\mathcal{M} \otimes M_n)$ and this isomorphism maps the positive cone $(L_p(\mathcal{M}) \otimes M_n)_+$ to $L_p(\mathcal{M} \otimes M_n)_+$. Furthermore for $\frac{1}{p} + \frac{1}{p'} = 1$, this isomorphism defines the duality of $L_p(\mathcal{M}) \otimes M_n$ and $L_{p'}(\mathcal{M}) \otimes M_n$ by

$$\langle a, b \rangle = \sum_{i,j=1}^n \langle a_{ij}, b_{ji} \rangle, \quad a = [a_{ij}] \in L_p(\mathcal{M}) \otimes M_n, \quad b = [b_{ij}] \in L_{p'}(\mathcal{M}) \otimes M_n.$$

By (2.2),

$$a \geq 0 \text{ if and only if } \langle a, b \rangle \geq 0 \text{ for all } b \in (L_{p'}(\mathcal{M}) \otimes M_n)_+. \quad (2.3)$$

Next, we describe some relationships between matrices and their elements.

Lemma 2.1 *Let $\mathcal{M}_1$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal faithful semifinite trace τ , and let a and b be self-adjoint τ -measurable operators. Then the following are equivalent:*

- (i) $-a \leq b \leq a$.
- (ii) The matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is positive.

Proof We show first that (ii) implies (i): Let $\mathcal{D}(a)$ and $\mathcal{D}(b)$ denote the domains of a and b . Let $\mathcal{D} = \mathcal{D}(a) \cap \mathcal{D}(b)$. Then $\mathcal{D}$ is a τ -dense subspace of $\mathcal{H}$. For $\xi \in \mathcal{D}$, we get

$$0 \leq \frac{1}{2} \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \xi \\ \xi \end{bmatrix} \middle| \begin{bmatrix} \xi \\ \xi \end{bmatrix} \right) = ((a+b)\xi | \xi). \quad (2.4)$$

Hence $-a \leq b$ on $\mathcal{D}$. By replacing the vector $\begin{bmatrix} \xi \\ \xi \end{bmatrix}$ in (2.4) with $\begin{bmatrix} \xi \\ -\xi \end{bmatrix}$, we get $b \leq a$.

For the implication i) $\Rightarrow$ ii), we assume first that a and b are bounded. Then it follows from [3], Proposition 1.3.5 that $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is positive. Since $\mathcal{D}(a) \cap \mathcal{D}(b)$ is τ -dense, there is a sequence of projections $(p_n)_{n=1}^\infty$ in $\mathcal{M}_1$ such that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$, $\tau(1 - p_n) \rightarrow 0$ as $n \rightarrow \infty$, and $p_n \mathcal{H} \subseteq \mathcal{D}(a) \cap \mathcal{D}(b)$. Hence $p_n a p_n$ and $p_n b p_n$ are bounded operators and for all $n \in \mathbb{N}$

$$-p_n a p_n \leq p_n b p_n \leq p_n a p_n.$$

Therefore, for $\xi, \eta \in p_n \mathcal{H}$, we have

$$\begin{aligned} \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \middle| \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right) &= \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} p_n \xi \\ p_n \eta \end{bmatrix} \middle| \begin{bmatrix} p_n \xi \\ p_n \eta \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} p_n a p_n & p_n b p_n \\ p_n b p_n & p_n a p_n \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \middle| \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right) \geq 0. \end{aligned}$$

Since the union $\bigcup_{n \in \mathbb{N}} p_n \mathcal{H}$ is τ -dense, $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is positive. □

For $n \in \mathbb{N}$, 1_n denotes the unit matrix in M_n . For a τ -measurable operator a let $\text{supp}(a)$ denote the smallest projection which fulfills $\text{supp}(a) \cdot a = a \cdot \text{supp}(a) = a$. If $a \in L_p(\mathcal{M})$ then $\text{supp}(a) \in \mathcal{M}$ although a is τ -measurable with respect to a larger algebra (see [13], Proposition II.4).

Lemma 2.2 *Let $\mathcal{M}_1$ be a von Neumann algebra with a normal faithful semifinite trace τ . Let a, b, c_1, c_2 be τ -measurable operators for which holds:*

- (i) The operators a and b are positive.
- (ii) $\text{supp}(a) \cdot c_1 \cdot \text{supp}(b) = c_1$ and $\text{supp}(a) \cdot c_2 \cdot \text{supp}(b) = c_2$.
- (iii) $ac_1 b = ac_2 b$.

Then $c_1 = c_2$.

Proof By putting $c = c_1 - c_2$, we may assume that $c_2 = 0$. First let $a = b$. Then ii) can be formulated as $\text{supp}(a) \cdot c \cdot \text{supp}(a) = c$ and iii) as $aca = 0$. This means that the left support and the right support of c are less than or equal to $\text{supp}(a)$. Thus c fulfills the conditions of [11], Lemma 2.2 (c), and therefore $c = 0$. For the general case, we put $a' = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $c' = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} \text{supp}(a') &= \begin{bmatrix} \text{supp}(a) & 0 \\ 0 & \text{supp}(b) \end{bmatrix}, \\ a'c'a' &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & acb \\ 0 & 0 \end{bmatrix} = 0, \end{aligned}$$

and

$$\text{supp}(a') \cdot c' \cdot \text{supp}(a') = \begin{bmatrix} 0 & \text{supp}(a) \cdot c \cdot \text{supp}(b) \\ 0 & 0 \end{bmatrix} = 0.$$

Hence, by the first part of the proof, $c' = 0$ which implies $c = 0$. $\square$

Theorem 2.3 Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful semifinite weight φ and acting on the Hilbert space $\mathcal{H}$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, $f, g \in L_p(\mathcal{M})_+$, and $x \in L_p(\mathcal{M}) \otimes M_n$ such that

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0.$$

Then there is an operator $y \in \mathcal{M} \otimes M_n$ such that

$$x = (f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n). \quad (2.5)$$

We have $\|y\|_\infty \leq 1$ and y is unique subject to the condition $(\text{supp}(f) \otimes 1_n) \cdot y \cdot (\text{supp}(g) \otimes 1_n) = y$ and to (2.5). If $f = g$ and x is self-adjoint, then y is self-adjoint. If $f = g$ and x is positive, then y is positive.

Proof Let $\mathcal{M}_1 = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, $\mathcal{K} = L_2(\mathbb{R}, \mathcal{H})$, and τ be the canonical trace on $\mathcal{M}_1$. Let $x = [x_{ij}]$ and

$$\mathcal{D} = \mathcal{D}(f) \cap \mathcal{D}(f^{\frac{1}{2}}) \cap \mathcal{D}(g) \cap \mathcal{D}(g^{\frac{1}{2}}) \cap \bigcap_{i,j=1}^n \mathcal{D}(x_{ij}).$$

Then $\mathcal{D}$ is τ -dense in $\mathcal{K}$ and consequently, $\mathcal{D}^n$ is $\tau \otimes \text{Tr}$ -dense in $\mathcal{K}^n$. For $\xi, \eta \in \mathcal{D}^n$, we get

$$\begin{aligned}
0 &\leq \left(\left[\begin{array}{cc} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{array} \right] \begin{bmatrix} \xi \\ \eta \end{bmatrix} \middle| \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right) \\
&= ((f \otimes 1_n)\xi|\xi) + (x\eta|\xi) + (x^*\xi|\eta) + ((g \otimes 1_n)\eta|\eta).
\end{aligned}$$

This implies

$$-2\operatorname{Re}((x\eta|\xi)) \leq ((f \otimes 1_n)\xi|\xi) + ((g \otimes 1_n)\eta|\eta). \quad (2.6)$$

We replace η in (2.6) by $e^{it}\eta$, and choose a suitable value for $t \in \mathbb{R}$ to get

$$2|(x\eta|\xi)| \leq ((f \otimes 1_n)\xi|\xi) + ((g \otimes 1_n)\eta|\eta). \quad (2.7)$$

Then we replace ξ by $\lambda\xi$, η by $\frac{1}{\lambda}\eta$ in (2.7), minimize over $\lambda \in \mathbb{R}_+$, and get

$$|(x\eta|\xi)|^2 \leq ((f \otimes 1_n)\xi|\xi)((g \otimes 1_n)\eta|\eta). \quad (2.8)$$

So, we can define the sesquilinear form

$$\begin{aligned}
B : (g^{\frac{1}{2}} \otimes 1_n)\mathcal{D}^n \times (f^{\frac{1}{2}} \otimes 1_n)\mathcal{D}^n &\rightarrow \mathbb{C} \\
((g^{\frac{1}{2}} \otimes 1_n)\eta, (f^{\frac{1}{2}} \otimes 1_n)\xi) &\mapsto (x\eta|\xi).
\end{aligned}$$

If $\xi, \xi', \eta, \eta' \in \mathcal{D}^n$ with $(f^{\frac{1}{2}} \otimes 1_n)\xi = (f^{\frac{1}{2}} \otimes 1_n)\xi'$ and $(g^{\frac{1}{2}} \otimes 1_n)\eta = (g^{\frac{1}{2}} \otimes 1_n)\eta'$, we get

$$\begin{aligned}
|(x\eta|\xi) - (x\eta'|\xi')| &\leq |(x\eta|\xi - \xi')| + |(x(\eta - \eta')|\xi')| \\
&\leq ((f \otimes 1_n)(\xi - \xi')|\xi - \xi')^{\frac{1}{2}} ((g \otimes 1_n)\eta|\eta)^{\frac{1}{2}} \\
&\quad + ((f \otimes 1_n)\xi'|\xi')^{\frac{1}{2}} ((g \otimes 1_n)(\eta - \eta')|\eta - \eta')^{\frac{1}{2}} \\
&= 0.
\end{aligned}$$

This shows that B is well defined, and by (2.8), we get

$$\begin{aligned}
|B((g^{\frac{1}{2}} \otimes 1_n)\eta, (f^{\frac{1}{2}} \otimes 1_n)\xi)|^2 &= |(x\eta|\xi)|^2 \\
&\leq ((g^{\frac{1}{2}} \otimes 1_n)\eta|(g^{\frac{1}{2}} \otimes 1_n)\eta)((f^{\frac{1}{2}} \otimes 1_n)\xi|(f^{\frac{1}{2}} \otimes 1_n)\xi).
\end{aligned}$$

Thus, B can be extended to a bounded sesquilinear form

$$B : (\operatorname{supp}(g) \otimes 1_n)\mathcal{K}^n \times (\operatorname{supp}(f) \otimes 1_n)\mathcal{K}^n \rightarrow \mathbb{C}$$

with norm $\|B\| \leq 1$. Next we extend B to a bounded sesquilinear form on $\mathcal{K}^n \times \mathcal{K}^n$ by $B(\eta, \xi) = B((\operatorname{supp}(g) \otimes 1_n)\eta, (\operatorname{supp}(f) \otimes 1_n)\xi)$ for $\xi, \eta \in \mathcal{K}^n$. Then B has still norm $\|B\| \leq 1$. Therefore, there is an operator $y \in \mathcal{B}(\mathcal{K}^n)$ with $(y\eta|\xi) = B(\eta, \xi)$ and $\|y\|_\infty = \|B\| \leq 1$. By construction, we have

$$(\operatorname{supp}(f) \otimes 1_n) \cdot y \cdot (\operatorname{supp}(g) \otimes 1_n) = y.$$

We still have to show that $y \in \mathcal{M} \otimes M_n$. Let $u \in \mathcal{M}'_1$ be a unitary element of the

commutant of $\mathcal{M}_1$, $\xi, \eta \in \mathcal{K}^n$. Then there are sequences $(\xi_i)_{i=1}^\infty$ and $(\eta_i)_{i=1}^\infty$ with $\xi_i, \eta_i \in \mathcal{D}^n$ for all $i \in \mathbb{N}$,

$$(\text{supp}(f) \otimes 1_n)\xi = \lim_{i \rightarrow \infty} (f^{\frac{1}{2}} \otimes 1_n)\xi_i,$$

and

$$(\text{supp}(g) \otimes 1_n)\eta = \lim_{i \rightarrow \infty} (g^{\frac{1}{2}} \otimes 1_n)\eta_i.$$

Since u commutes with f , g , $\text{supp}(f)$, and $\text{supp}(g)$, we get for all $i \in \mathbb{N}$

$$\begin{aligned} & ((u^* \otimes 1_n)y(u \otimes 1_n)(g^{\frac{1}{2}} \otimes 1_n)\eta_i | (f^{\frac{1}{2}} \otimes 1_n)\xi_i) \\ &= ((u^* \otimes 1_n)(f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n)(u \otimes 1_n)\eta_i | \xi_i) \\ &= ((u^* \otimes 1_n)x(u \otimes 1_n)\eta_i | \xi_i) \\ &= (x\eta_i | \xi_i) \\ &= ((f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n)\eta_i | \xi_i) \end{aligned}$$

and therefore

$$\begin{aligned} & ((u^* \otimes 1_n)y(u \otimes 1_n)\eta | \xi) \\ &= ((u^* \otimes 1_n)(\text{supp}(f) \otimes 1_n)y(\text{supp}(g) \otimes 1_n)(u \otimes 1_n)\eta | \xi) \\ &= ((u^* \otimes 1_n)y(u \otimes 1_n)(\text{supp}(g) \otimes 1_n)\eta | (\text{supp}(f) \otimes 1_n)\xi) \\ &= \lim_{i \rightarrow \infty} ((u^* \otimes 1_n)y(u \otimes 1_n)(g^{\frac{1}{2}} \otimes 1_n)\eta_i | (f^{\frac{1}{2}} \otimes 1_n)\xi_i) \\ &= \lim_{i \rightarrow \infty} ((f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n)\eta_i | \xi_i) \\ &= (y\eta | \xi). \end{aligned}$$

Every operator in $\mathcal{M}'_1$ can be written as a finite linear combination of unitaries and $(\mathcal{M}_1 \otimes M_n)' = \mathcal{M}'_1 \otimes \mathbb{C}$. Hence $y \in \mathcal{M}_1 \otimes M_n$. We still have to show that $y \in \mathcal{M} \otimes M_n$. Let $y = [y_{ij}]$ with $y_{ij} \in \mathcal{M}_1$ for $i, j \in \{1, \dots, n\}$. Let $s \in \mathbb{R}$. Then we have

$$\begin{aligned} f^{\frac{1}{2}}y_{ij}g^{\frac{1}{2}} &= x_{ij} = e^{\frac{i}{p}}\theta_s(x_{ij}) \\ &= e^{\frac{i}{p}}\theta_s(f^{\frac{1}{2}})\theta_s(y_{ij})\theta_s(g^{\frac{1}{2}}) = f^{\frac{1}{2}}\theta_s(y_{ij})g^{\frac{1}{2}}. \end{aligned}$$

Since

$$y_{ij} = \text{supp}(f) \cdot y_{ij} \cdot \text{supp}(g)$$

and

$$\theta_s(y_{ij}) = \text{supp}(f) \cdot \theta_s(y_{ij}) \cdot \text{supp}(g),$$

we can apply Lemma 2.2 and get $y_{ij} = \theta_s(y_{ij})$. Hence $y_{ij} \in \mathcal{M}$. For the uniqueness of the decomposition, suppose that there is another $\tilde{y} \in \mathcal{M} \otimes M_n$ such that

$$x = (f^{\frac{1}{2}} \otimes 1_n) \tilde{y} (g^{\frac{1}{2}} \otimes 1_n)$$

and

$$(\text{supp}(f) \otimes 1_n) \cdot \tilde{y} \cdot (\text{supp}(g) \otimes 1_n) = \tilde{y}.$$

Then we can apply Lemma 2.2 and get $y = \tilde{y}$. Now assume that x is self-adjoint and $f = g$. Since $x = x^*$, we get

$$\begin{aligned} (f^{\frac{1}{2}} \otimes 1_n) y (f^{\frac{1}{2}} \otimes 1_n) &= x = \frac{1}{2}(x + x^*) \\ &= (f^{\frac{1}{2}} \otimes 1_n) \frac{1}{2}(y + y^*) (f^{\frac{1}{2}} \otimes 1_n). \end{aligned}$$

Again, we apply Lemma 2.2, use the uniqueness of y , and get $y = y^*$. Finally, assume that x is positive. For $\xi \in \mathcal{K}^n$ there is a sequence $(\xi_i)_{i=1}^\infty$ with $\xi_i \in \mathcal{D}^n$ for all $i \in \mathbb{N}$ and

$$(\text{supp}(f) \otimes 1_n) \xi = \lim_{i \rightarrow \infty} (f^{\frac{1}{2}} \otimes 1_n) \xi_i.$$

Then

$$\begin{aligned} (y \xi | \xi) &= (y(\text{supp}(f) \otimes 1_n) \xi | (\text{supp}(f) \otimes 1_n) \xi) \\ &= \lim_{i \rightarrow \infty} (y(f^{\frac{1}{2}} \otimes 1_n) \xi_i | (f^{\frac{1}{2}} \otimes 1_n) \xi_i | \xi_i) \\ &= \lim_{i \rightarrow \infty} (x \xi_i | \xi_i) \geq 0. \end{aligned}$$

Since ξ was arbitrary, y is positive. □

Lemma 2.4 *Let $\mathcal{M}$ be a von Neumann algebra, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, and $x \in L_p(\mathcal{M}) \otimes M_n$. Then there exists $f \in L_p(\mathcal{M})_+$ such that*

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & f \otimes 1_n \end{bmatrix} \geq 0.$$

Proof Since every $x \in L_p(\mathcal{M}) \otimes M_n$ is a finite linear combination of elements of the form $y \otimes \alpha$, where $y \in L_p(\mathcal{M})_+$ and $\alpha \in M_n$, it suffices to prove the statement for $x = y \otimes \alpha$. Let $\|\alpha\|$ denote the usual maximum norm of an $n \times n$ matrix acting on $\mathbb{C}^n$. Then we get

$$\begin{aligned}
& \begin{bmatrix} \|\alpha\|y \otimes 1_n & y \otimes \alpha \\ y \otimes \alpha^* & \|\alpha\|y \otimes 1_n \end{bmatrix} \\
&= \begin{bmatrix} y^{\frac{1}{2}} \otimes 1_n & 0 \\ 0 & y^{\frac{1}{2}} \otimes 1_n \end{bmatrix} \begin{bmatrix} \|\alpha\|1_n & \alpha \\ \alpha^* & \|\alpha\|1_n \end{bmatrix} \begin{bmatrix} y^{\frac{1}{2}} \otimes 1_n & 0 \\ 0 & y^{\frac{1}{2}} \otimes 1_n \end{bmatrix} \geq 0.
\end{aligned}$$

□

Lemma 2.5 *Let $\mathcal{M}$ be a von Neumann algebra, $1 \leq p \leq \infty$, $n \in \mathbb{N}$, and $x \in (L_p(\mathcal{M}) \otimes M_n)_+$. Then x is a finite sum of matrices of the form $[x_i^* x_j]$ where $x_i \in L_{2p}(\mathcal{M})$ for $i \in \{1, \dots, n\}$.*

Proof By Lemmas 2.4 and 2.3, there exists $f \in L_p(\mathcal{M})_+$ and $y \in (\mathcal{M} \otimes M_n)_+$ such that $x = (f^{\frac{1}{2}} \otimes 1_n)(f^{\frac{1}{2}} \otimes 1_n)$. By applying [12], page 193, Lemma 3.1, to y we get the desired result. □

3 Matrix norms on non-commutative L_p -spaces

In this section $\mathcal{M}$ always denotes a von Neumann algebra without any further restrictions. For each $n \in \mathbb{N}$, we will define a norm on $L_p(\mathcal{M}) \otimes M_n$ and derive some properties of this norm.

Definition 3.1 *Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$, and $x \in L_p(\mathcal{M}) \otimes M_n$. Then we define*

$$\|x\|_{p,n} = \inf \left\{ \frac{1}{2} \left(\|f\|_p + \|g\|_p \right) \mid f, g \in L_p(\mathcal{M})_+, \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0 \right\}. \quad (3.1)$$

Remark 3.2 By Lemma 2.4, the set on the right side of (3.1) is not empty, and therefore, the infimum is well defined.

If $p = \infty$, the norm $\|x\|_{\infty,n}$ is identical to the usual operator norm of x considered as a bounded operator on a Hilbert space.

The combination of this norm definition with Theorem 2.3 shows that this norm is quite similar to the norm used in [9], Eq. (1.5).

If $x \in L_p(\mathcal{M}) \otimes M_n$, $x = \sum_{j=1}^k x_j \otimes \beta_j$ and $\alpha \in M_n$, then we have $\alpha x = \sum_{j=1}^k x_j \otimes \alpha \beta_j$ and $x \alpha = \sum_{j=1}^k x_j \otimes \beta_j \alpha$. For $n \in \mathbb{N}$, $i, j \in \{1, \dots, n\}$ let

$$\varepsilon_{ij} \text{ be the } n \times n \text{ matrix with } 1 \text{ at position } (i, j) \text{ and } 0 \text{ else.} \quad (3.2)$$

Theorem 3.3 *Let $1 \leq p \leq \infty$ and $n \in \mathbb{N}$. Then the following holds:*

- (i) If $x, y \in L_p(\mathcal{M}) \otimes M_n$, then $\|x + y\|_{p,n} \leq \|x\|_{p,n} + \|y\|_{p,n}$.
- (ii) If $x \in L_p(\mathcal{M}) \otimes M_n$, then $\|x^*\|_{p,n} = \|x\|_{p,n}$.
- (iii) If $x \in L_p(\mathcal{M}) \otimes M_n$ and $\alpha \in M_n$, then

$$\|\alpha x\|_{p,n} \leq \|\alpha\| \|x\|_{p,n} \text{ and } \|x\alpha\|_{p,n} \leq \|\alpha\| \|x\|_{p,n}.$$

- (iv) If $x \in L_p(\mathcal{M}) \otimes M_n$ and $\lambda \in \mathbb{C}$, then $\|\lambda x\|_{p,n} = |\lambda| \|x\|_{p,n}$.
 (v) If $x = [x_{ij}] \in L_p(\mathcal{M}) \otimes M_n$, then

$$\max \left\{ \|x_{ij}\|_p \mid 1 \leq i, j \leq n \right\} \leq \|x\|_{p,n} \leq \sum_{i,j=1}^n \|x_{ij}\|_p.$$

- (vi) If $x \in L_p(\mathcal{M}) \otimes M_n$ with $\|x\|_{p,n} = 0$, then $x = 0$.

Proof Let $x, y \in L_p(\mathcal{M}) \otimes M_n$, and $\varepsilon > 0$. Then there exist $f_1, f_2, g_1, g_2 \in L_p(\mathcal{M})_+$ such that

$$\begin{bmatrix} f_1 \otimes 1_n & x \\ x^* & g_1 \otimes 1_n \end{bmatrix} \geq 0, \quad \frac{1}{2} (\|f_1\|_p + \|g_1\|_p) \leq \|x\|_{p,n} + \varepsilon,$$

and

$$\begin{bmatrix} f_2 \otimes 1_n & y \\ y^* & g_2 \otimes 1_n \end{bmatrix} \geq 0, \quad \frac{1}{2} (\|f_2\|_p + \|g_2\|_p) \leq \|y\|_{p,n} + \varepsilon.$$

Then we get

$$\begin{bmatrix} (f_1 + f_2) \otimes 1_n & x + y \\ (x + y)^* & (g_1 + g_2) \otimes 1_n \end{bmatrix} \geq 0$$

and

$$\|x + y\|_{p,n} \leq \frac{1}{2} (\|f_1 + f_2\|_p + \|g_1 + g_2\|_p) \leq \|x\|_{p,n} + \|y\|_{p,n} + 2\varepsilon.$$

Since ε is arbitrary, (i) is proved.

To prove (ii), let $x \in L_p(\mathcal{M}) \otimes M_n$ and $\varepsilon > 0$. Then there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0, \quad \frac{1}{2} (\|f\|_p + \|g\|_p) \leq \|x\|_{p,n} + \varepsilon.$$

Then we get

$$0 \leq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} g \otimes 1_n & x^* \\ x & f \otimes 1_n \end{bmatrix}.$$

Hence, we conclude $\|x^*\|_{p,n} \leq \frac{1}{2} (\|f\|_p + \|g\|_p) \leq \|x\|_{p,n} + \varepsilon$. Since ε is arbitrary, we get $\|x^*\|_{p,n} \leq \|x\|_{p,n}$. Since $\|x\|_{p,n} = \|x^{**}\|_{p,n} \leq \|x^*\|_{p,n}$, (ii) is proved.

Next, we prove (iii). Let $\alpha \in M_n$. If $\alpha = 0$ then $\alpha x = 0$ and the inequality is true. So let $\alpha \neq 0$. For $\varepsilon > 0$, there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0 \text{ and } \frac{1}{2}(\|f\|_p + \|g\|_p) \leq \|x\|_{p,n} + \varepsilon.$$

Then we have for $\lambda > 0$

$$\begin{aligned} 0 &\leq \begin{bmatrix} \frac{1}{\lambda} \alpha & 0 \\ 0 & \lambda 1_n \end{bmatrix} \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda} \alpha^* & 0 \\ 0 & \lambda 1_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\lambda^2} \alpha \alpha^* (f \otimes 1_n) & \alpha x \\ (\alpha x)^* & \lambda^2 g \otimes 1_n \end{bmatrix} \\ &\leq \begin{bmatrix} \frac{1}{\lambda^2} \|\alpha\|^2 f \otimes 1_n & \alpha x \\ (\alpha x)^* & \lambda^2 g \otimes 1_n \end{bmatrix}. \end{aligned}$$

We put $\lambda^2 = \|\alpha\|$ and get

$$\|\alpha x\|_{p,n} \leq \frac{1}{2} \left(\|\alpha\| \|f\|_p + \|\alpha\| \|g\|_p \right) \leq \|\alpha\| \left(\|x\|_{p,n} + \varepsilon \right).$$

Since ε was arbitrary, we get the desired result. A similar argument proves the second inequality in iii).

To prove (iv), let $x \in L_p(\mathcal{M}) \otimes M_n$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, we have

$$\|\lambda x\|_{p,n} = 0 = |\lambda| \|x\|_{p,n}.$$

For $\lambda \neq 0$, we put $\alpha = \lambda 1_n$, apply iii), and get

$$\begin{aligned} \|\lambda x\|_{p,n} &= \|\lambda 1_n x\|_{p,n} \leq |\lambda| \|x\|_{p,n} \text{ and} \\ |\lambda| \|x\|_{p,n} &= |\lambda| \left\| \frac{1}{\lambda} \lambda x \right\|_{p,n} \leq |\lambda| \frac{1}{|\lambda|} \|x\|_{p,n} \leq \|\lambda x\|_{p,n}. \end{aligned}$$

For v), let $x = [x_{ij}] \in L_p(\mathcal{M}) \otimes M_n$. For $\varepsilon > 0$, there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0, \quad \frac{1}{2} \left(\|f\|_p + \|g\|_p \right) \leq \|x\|_{p,n} + \varepsilon.$$

By Theorem 2.3, there exists $y \in \mathcal{M} \otimes M_n$, such that $\|y\|_\infty \leq 1$ and $x = (f^{\frac{1}{2}} \otimes 1_n) y (g^{\frac{1}{2}} \otimes 1_n)$. Then $x_{ij} = f^{\frac{1}{2}} y_{ij} g^{\frac{1}{2}}$ and $\|y_{ij}\|_\infty \leq \|y\|_\infty \leq 1$ for all $i, j \in \{1, \dots, n\}$. Hence

$$\begin{aligned} \|x_{ij}\|_p &= \|f^{\frac{1}{2}} y_{ij} g^{\frac{1}{2}}\|_p \leq \|f^{\frac{1}{2}}\|_{2p} \|g^{\frac{1}{2}}\|_{2p} = \sqrt{\|f\|_p \|g\|_p} \\ &\leq \frac{1}{2} \left(\|f\|_p + \|g\|_p \right) \leq \|x\|_{p,n} + \varepsilon, \quad i, j \in \{1, \dots, n\}. \end{aligned}$$

For the second inequality of v), let $y \in L_p(\mathcal{M})$ with polar decomposition $y = v|y|$. Then $|y^*|v = v|y|$, $v^*|y^*|v = |y|$, and

$$0 \leq \begin{bmatrix} 1 & 0 \\ v^* & 0 \end{bmatrix} \begin{bmatrix} |y^*| & 0 \\ 0 & |y^*| \end{bmatrix} \begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} |y^*| & y \\ y^* & |y| \end{bmatrix}. \quad (3.3)$$

Now it follows that

$$\begin{bmatrix} |y^*| \otimes 1_n & y \otimes 1_n \\ y^* \otimes 1_n & |y| \otimes 1_n \end{bmatrix} \geq 0$$

and therefore $\|y \otimes 1_n\|_{p,n} \leq \|y\|_p$. For $i \in \{1, \dots, n\}$, let ε_{ij} be as in (3.2). Then we get for $x = [x_{ij}] \in L_p(\mathcal{M}) \otimes M_n$

$$\|x\|_{p,n} = \left\| \sum_{i,j=1}^n x_{ij} \otimes \varepsilon_{ij} \right\|_{p,n} \leq \sum_{i,j=1}^n \|x_{ij} \otimes 1_n\|_{p,n} \|\varepsilon_{ij}\| \leq \sum_{i,j=1}^n \|x_{ij}\|_p.$$

To prove (vi), let $\|x\|_{p,n} = 0$. From (v), it follows that $x_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$. Hence $x = 0$. $\square$

The next theorem shows that the infimum in Definition 3.1 is actually a minimum.

Theorem 3.4 *Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$, and $x \in L_p(\mathcal{M}) \otimes M_n$. Then there exist $f, g \in L_p(\mathcal{M})_+$ such that*

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0 \text{ and } \|f\|_p = \|g\|_p = \|x\|_{p,n}.$$

Proof If $x = 0$, we can take $f = g = 0$. So suppose that $x \neq 0$. Let $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $L_q(\mathcal{M})^*$ be the dual space of $L_q(\mathcal{M})$. Note that $L_q(\mathcal{M})^*$ is $L_p(\mathcal{M})$ when $q < \infty$ and $\mathcal{M}^*$ when $q = \infty$. For $\varepsilon > 0$ we define

$$K_\varepsilon = \left\{ (f, g) \in L_q(\mathcal{M})_+^* \times L_q(\mathcal{M})_+^* \left| \begin{array}{l} \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0, \\ \|f\| \leq \|x\|_{p,n} + \varepsilon, \\ \|g\| \leq \|x\|_{p,n} + \varepsilon \end{array} \right. \right\}.$$

The symbol $\|\cdot\|$ denotes the norm of $L_q(\mathcal{M})^*$. The sets K_ε , $\varepsilon > 0$, have the following properties:

Each $K_\varepsilon \neq \emptyset$: By definition of $\|x\|_{p,n}$, there exist $0 \neq f, g \in L_p(\mathcal{M})_+$ with

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0 \text{ and } \frac{1}{2}(\|f\|_p + \|g\|_p) < \|x\|_{p,n} + \varepsilon.$$

For $\lambda > 0$, we get

$$0 \leq \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} = \begin{bmatrix} \lambda^2 f \otimes 1_n & x \\ x^* & \frac{1}{\lambda^2} g \otimes 1_n \end{bmatrix}.$$

We put $\lambda^2 = \sqrt{\frac{\|g\|_p}{\|f\|_p}}$, $f' = \lambda^2 f$, and $g' = \frac{1}{\lambda^2} g$. Then

$$\|f'\|_p = \|g'\|_p = \sqrt{\|f\|_p \|g\|_p} \leq \frac{1}{2} (\|f\|_p + \|g\|_p) < \|x\|_{p,n} + \varepsilon$$

and

$$\begin{bmatrix} f' \otimes 1_n & x \\ x^* & g' \otimes 1_n \end{bmatrix} \geq 0.$$

This shows that $(f', g') \in K_\varepsilon$. Next we show that K_ε is weak*-closed for every $\varepsilon > 0$. We fix $\varepsilon > 0$ and (f, g) as an element of the weak* closure of K_ε . For $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in (L_q(\mathcal{M}) \otimes M_{2n})_+$ there exist sequences $(f_m)_{m=1}^\infty$ and $(g_m)_{m=1}^\infty$ in K_ε such that

$$\langle f \otimes 1_n, a_{11} \rangle = \lim_{m \rightarrow \infty} \langle f_m \otimes 1_n, a_{11} \rangle$$

and

$$\langle g \otimes 1_n, a_{22} \rangle = \lim_{m \rightarrow \infty} \langle g_m \otimes 1_n, a_{22} \rangle,$$

and $(f_m, g_m) \in K_\varepsilon$ for all $m \in \mathbb{N}$. Hence, we get

$$\left\langle \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} a \right\rangle = \lim_{m \rightarrow \infty} \left\langle \begin{bmatrix} f_m \otimes 1_n & x \\ x^* & g_m \otimes 1_n \end{bmatrix} a \right\rangle \geq 0.$$

Since this holds for every $a \in (L_q(\mathcal{M}) \otimes M_{2n})_+$, we can apply (2.3), and conclude that

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0.$$

If $q = \infty$, we have to do this in $\mathcal{M}^*$ which is isomorphic to $L_1(\mathcal{M}^{**})$. Especially, f and g are positive. Taking $a \in L_q(\mathcal{M})$ with $\|a\|_q \leq 1$, we can find a sequence $(f_m)_{m=1}^\infty$ in K_ε such that

$$\langle f, a \rangle = \lim_{m \rightarrow \infty} \langle f_m, a \rangle$$

Hence $|\langle f, a \rangle| \leq \|x\|_{p,n} + \varepsilon$ and

$$\|f\| = \sup\{|\langle f, a \rangle| : \|a\|_q \leq 1\} \leq \|x\|_{p,n} + \varepsilon.$$

Similarly, $\|g\| \leq \|x\|_{p,n} + \varepsilon$ and therefore $(f, g) \in K_\varepsilon$. By the Banach-Alaoglu

theorem, the unit ball of $L_q(\mathcal{M})$ is compact in the weak*-topology. Hence all sets K_ε are compact in the weak*-topology.

The sets K_ε , $\varepsilon > 0$, have the finite intersection property: Given $k \in \mathbb{N}$, $\varepsilon_1, \dots, \varepsilon_k > 0$, we put $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_k\}$ and get

$$K_\varepsilon \subseteq \bigcap_{i=1}^k K_{\varepsilon_i}.$$

Combining the finite intersection property and the weak*-compactness, we get

$$\bigcap_{\varepsilon > 0} K_\varepsilon \neq \emptyset. \quad (3.4)$$

Let (f, g) be in the set defined in Eq. (3.4). Then we have

$$\begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0 \quad (3.5)$$

If $q < \infty$, then $f, g \in L_p(\mathcal{M})$. If $q = \infty$, there is a central projection in $\mathcal{M}^{**}$ which works as projection from $\mathcal{M}^*$ to $L_1(\mathcal{M})$. Hence we may assume that $f, g \in L_1(\mathcal{M})$. By construction, we have $\|f\|_p, \|g\|_p \leq \|x\|_{p,n}$. From Eq. (3.5), we get $2\|x\|_{p,n} \leq \|f\|_p + \|g\|_p$. Combining both gives

$$\|f\|_p = \|x\|_{p,n} = \|g\|_p.$$

□

Theorem 3.5 *Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$, and $x = x^* \in L_p(\mathcal{M}) \otimes M_n$.*

- (i) *We have $\|x\|_{p,n} = \inf \left\{ \|f\|_p \mid f \in L_p(\mathcal{M})_+, f \otimes 1_n \pm x \geq 0 \right\}$.*
- (ii) *There exists $f \in L_p(\mathcal{M})_+$ such that*

$$f \otimes 1_n \pm x \geq 0 \text{ and } \|f\|_p = \|x\|_{p,n}.$$

Proof Let $A = \inf \left\{ \|f\|_p \mid f \in L_p(\mathcal{M})_+, f \otimes 1_n \pm x \geq 0 \right\}$. By Theorem 3.4, there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\|f\|_p = \|g\|_p = \|x\|_{p,n} \text{ and } \begin{bmatrix} f \otimes 1_n & x \\ x & g \otimes 1_n \end{bmatrix} \geq 0. \quad (3.6)$$

It follows then

$$0 \leq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f \otimes 1_n & x \\ x & g \otimes 1_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} g \otimes 1_n & x \\ x & f \otimes 1_n \end{bmatrix}$$

Hence we conclude

$$\begin{bmatrix} \frac{1}{2}(f+g) \otimes 1_n & x \\ x & \frac{1}{2}(f+g) \otimes 1_n \end{bmatrix} \geq 0.$$

Then Lemma 2.1 implies that $\frac{1}{2}(f+g) \otimes 1_n \pm x \geq 0$. This shows that

$$A \leq \left\| \frac{1}{2}(f+g) \right\|_p \leq \|x\|_{p,n}.$$

For the converse direction, let $\varepsilon > 0$. Then there exist $f \in L_p(\mathcal{M})_+$ such that $f \otimes 1_n \pm x \geq 0$ and $\|f\|_p \leq A + \varepsilon$. It follows from Lemma 2.1 that

$$\begin{bmatrix} f \otimes 1_n & x \\ x & f \otimes 1_n \end{bmatrix} \geq 0.$$

Hence, we get $\|x\|_{p,n} \leq \|f\|_p \leq A + \varepsilon$. Since ε is arbitrary, we get $\|x\|_{p,n} \leq A$. This proves (i). To prove (ii), we take f and g from (3.6). Then we have

$$\frac{1}{2}(f+g) \pm x \geq 0$$

and

$$\|x\|_{p,n} \leq \frac{1}{2}\|f+g\|_p \leq \frac{1}{2}(\|f\|_p + \|g\|_p) = \|x\|_{p,n}.$$

This shows that $\left\| \frac{1}{2}(f+g) \right\|_p = \|x\|_{p,n}$. □

Theorem 3.6 *Let $1 \leq p \leq \infty$ and $x \in L_p(\mathcal{M})$. Then*

$$\|x\|_{p,1} = \|x\|_p.$$

Proof Let $x = v|x|$ be the polar decomposition of x . By Eq. (3.3), we get $\begin{bmatrix} |x^*| & x \\ x^* & |x| \end{bmatrix} \geq 0$. Hence,

$$\|x\|_{p,1} \leq \frac{1}{2}(\|x\|_p + \|x^*\|_p) = \|x\|_p.$$

To prove the converse inequality, we apply Theorems 3.4 and 2.3 and get $f, g \in L_p(\mathcal{M})_+$, $\|f\|_p = \|g\|_p = \|x\|_{p,1}$, $y \in \mathcal{M}$, $\|y\|_\infty \leq 1$ and $x = f^{\frac{1}{2}}yg^{\frac{1}{2}}$. Hence

$$\|x\|_p = \|f^{\frac{1}{2}}yg^{\frac{1}{2}}\|_p \leq \|f\|_p^{\frac{1}{2}}\|y\|_\infty\|g\|_p^{\frac{1}{2}} = \|x\|_{p,1}.$$

□

4 Completely order bounded maps

In this section, we define completely order bounded maps from L_p to L_q and show the decomposition of such maps for $p = \infty, q$ arbitrary and for p arbitrary, $q = 1$. For $2p < q < \infty$ we give an example of a completely order bounded map which is not decomposable.

Throughout this Sects. $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras with no further restrictions unless stated explicitly. If $1 \leq p, q \leq \infty, n \in \mathbb{N}$, and $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ is a linear map, then

$$T_n : L_p(\mathcal{M}) \otimes M_n \rightarrow L_q(\mathcal{N}) \otimes M_n, [x_{ij}] \mapsto [T(x_{ij})].$$

We need the notion of decomposable maps which were introduced for C^* -algebras in [5] and extended to non-commutative L_p -spaces in [6, 9]. The above map T is called decomposable if there exist completely positive maps $S_1, S_2 : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ such that the induced map

$$\Phi : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_q(\mathcal{N}) \otimes M_2, \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & T(x_{12}) \\ T(x_{21})^* & S_2(x_{22}) \end{bmatrix} \quad (4.1)$$

is completely positive. The decomposable norm $\|T\|_{dec}$ is defined by

$$\|T\|_{dec} = \inf \{ \max \{ \|S_1\|, \|S_2\| \} \}$$

where the infimum is taken over all completely positive maps S_1 and S_2 in (4.1).

Definition 4.1 Let $1 \leq p, q \leq \infty$. A linear map $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ is called completely order bounded, if

$$\|T\|_{cob} = \sup \left\{ \|T_n(x)\|_{q,n} \mid x \in L_p(\mathcal{M}) \otimes M_n, \|x\|_{p,n} \leq 1, n \in \mathbb{N} \right\} < \infty.$$

The name completely order bounded will be justified by Theorem 4.5 where we show that a completely order bounded map maps order intervals to order intervals uniformly over all matrix levels.

Proposition 4.2 Let $1 \leq p, q \leq \infty$ and $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ be completely positive. Then T is completely order bounded and

$$\|T\|_{cob} = \|T\| = \sup \left\{ \|T(x)\|_q \mid x \in L_p(\mathcal{M})_+, \|x\|_p \leq 1 \right\}.$$

Here, $\|T\|$ means the usual operator norm of T as a bounded operator on a normed vector space.

Proof Let $\Lambda = \sup \left\{ \|T(x)\|_q \mid x \in L_p(\mathcal{M})_+, \|x\|_p \leq 1 \right\}$. By Theorem 3.6, we have

$$\sup \left\{ \|T(x)\|_q \mid x \in L_p(\mathcal{M})_+, \|x\|_p \leq 1 \right\} \leq \|T\| \leq \|T\|_{cob}.$$

For the opposite inequality, let $n \in \mathbb{N}$, $x \in L_p(\mathcal{M}) \otimes M_n$ with $\|x\|_{p,n} \leq 1$. By Theorem 3.4, there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\|f\|_p = \|g\|_p = \|x\|_{p,n} \text{ and } \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0.$$

Since T is completely positive, we have $T_n(x^*) = T_n(x)^*$ and

$$\begin{bmatrix} T(f) \otimes 1_n & T_n(x) \\ T_n(x)^* & T(g) \otimes 1_n \end{bmatrix} \geq 0$$

Hence,

$$\|T_n(x)\|_{q,n} \leq \frac{1}{2}(\|T(f)\|_q + \|T(g)\|_q) \leq \Lambda \|x\|_{p,n} \leq \Lambda.$$

□

Theorem 4.3 *Let $1 \leq p, q \leq \infty$, $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ be a decomposable map. Then T is completely order bounded and $\|T\|_{cob} \leq \|T\|_{dec}$.*

Proof For $\varepsilon > 0$ there exist completely positive maps $S_1, S_2 : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ such that

$$\Phi : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_q(\mathcal{N}) \otimes M_2, \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & T(x_{12}) \\ T(x_{21})^* & S_2(x_{22}) \end{bmatrix}$$

is completely positive and $\|S_1\|, \|S_2\| \leq \|T\|_{dec} + \varepsilon$. By Theorem 3.4, for $n \in \mathbb{N}$, $x = [x_{ij}] \in L_p(\mathcal{M}) \otimes M_n$, there exist $f, g \in L_p(\mathcal{M})_+$, $\|f\|_p = \|g\|_p = \|x\|_{p,n}$

such that $a = \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0$. We multiply a from left and right with a

suitable permutation matrix to get $b = [b_{ij}]$ with $b_{ij} = \begin{bmatrix} f & x_{ij} \\ x_{ji}^* & g \end{bmatrix}$ for $i, j = 1, \dots, n$.

Then b is positive too, so $\Phi_n(b)$ is positive and

$\Phi(b_{ij}) = \begin{bmatrix} S_1(f) & T(x_{ij}) \\ T(x_{ji})^* & S_2(g) \end{bmatrix}$ for $i, j = 1, \dots, n$. Again, we multiply $\Phi_n(b)$ from left

and right with a permutation matrix and get $\begin{bmatrix} S_1(f) \otimes 1_n & T_n(x) \\ T_n(x)^* & S_2(g) \otimes 1_n \end{bmatrix} \geq 0$. Thus

$$\|T_n(x)\|_{q,n} \leq \frac{1}{2}(\|S_1(f)\|_q + \|S_2(g)\|_q) \leq (\|T\|_{dec} + \varepsilon) \|x\|_{p,n}.$$

Since ε was arbitrary, the proof is finished. □

Next, we show that the composition of complete order bounded maps is completely order bounded.

Theorem 4.4 Let $1 \leq p_1, p_2, p_3 \leq \infty$, $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be von Neumann algebras, and $T_1 : L_{p_1}(\mathcal{M}_1) \rightarrow L_{p_2}(\mathcal{M}_2), T_2 : L_{p_2}(\mathcal{M}_2) \rightarrow L_{p_3}(\mathcal{M}_3)$ be completely order bounded maps. Then the composition $T_2 \circ T_1$ is completely order bounded and

$$\|T_2 \circ T_1\|_{cob} \leq \|T_2\|_{cob} \|T_1\|_{cob}.$$

Proof Let $n \in \mathbb{N}$ and $x \in L_{p_1}(\mathcal{M}_1) \otimes M_n$ such that $\|x\|_{p_1, n} \leq 1$. Then

$$\|T_{2,n}(T_{1,n}(x))\|_{p_3, n} \leq \|T_2\|_{cob} \|T_{1,n}(x)\|_{p_2, n} \leq \|T_2\|_{cob} \|T_1\|_{cob}.$$

□

The next theorem justifies the name completely order bounded: Completely order bounded maps map order intervals into order intervals uniformly over all matrix levels.

Theorem 4.5 Let $1 \leq p, q \leq \infty$, $T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ be completely order bounded, and $f_1, f_2 \in L_p(\mathcal{M})_+$. Then there exist $g_1, g_2 \in L_q(\mathcal{N})_+$ such that $\|g_1\|_q, \|g_2\|_q \leq \frac{1}{2} \|T\|_{cob} (\|f_1\|_p + \|f_2\|_p)$ and for all $n \in \mathbb{N}, x \in L_p(\mathcal{M}) \otimes M_n$

$$\begin{bmatrix} f_1 \otimes 1_n & x \\ x^* & f_2 \otimes 1_n \end{bmatrix} \geq 0 \text{ implies } \begin{bmatrix} g_1 \otimes 1_n & T_n(x) \\ T_n(x)^* & g_2 \otimes 1_n \end{bmatrix} \geq 0. \quad (4.2)$$

Proof For $q > 1, n \in \mathbb{N}, x \in L_p(\mathcal{M}) \otimes M_n$ with $\begin{bmatrix} f_1 \otimes 1_n & x \\ x^* & f_2 \otimes 1_n \end{bmatrix} \geq 0$, we put

$$\begin{aligned} K(x) = & \{(g_1, g_2) \in (L_q(\mathcal{N})_+, L_q(\mathcal{N})_+) \mid \\ & \begin{bmatrix} g_1 \otimes 1_n & T_n(x) \\ T_n(x)^* & g_2 \otimes 1_n \end{bmatrix} \geq 0, \\ & \|g_1\|_q, \|g_2\|_q \leq \frac{1}{2} \|T\|_{cob} (\|f_1\|_p + \|f_2\|_p)\}. \end{aligned}$$

For $q = 1$ we define a similar set, but take pairs $(g_1, g_2) \in \mathcal{N}_+^* \times \mathcal{N}_+^*$ instead of $L_1(\mathcal{N})_+ \times L_1(\mathcal{N})_+$. By Theorem 3.4, $K(x)$ is not empty. Further, $K(x)$ is weak*-closed. This is proved similarly as in the proof of Theorem 3.4. By the Banach-Alaoglu theorem, the unit ball of $L_q(\mathcal{N})$ is compact in the weak*-topology for $q > 1$. The same holds for $\mathcal{N}^*$. Hence all sets $K(x)$ are compact in the weak*-topology. The sets $K(x), x \in L_p(\mathcal{N}) \otimes M_n, n \in \mathbb{N}$, have the finite intersection property: For $k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{N}$, and $x_i \in L_p(\mathcal{M}) \otimes M_{n_i}$ with

$$\begin{bmatrix} f_1 \otimes 1_{n_i} & x_i \\ x_i^* & f_2 \otimes 1_{n_i} \end{bmatrix} \geq 0, i \in \{1, \dots, k\},$$

we put all x_i in the diagonal matrix $x = \text{diag}(x_1, \dots, x_k)$ and set $n = \sum_{i=1}^k n_i$. Then

$$\begin{bmatrix} f_1 \otimes 1_n & x \\ x^* & f_2 \otimes 1_n \end{bmatrix} \geq 0.$$

By Theorem 3.4, there exist $g_1, g_2 \in L_q(\mathcal{N})_+$ such that

$$\begin{bmatrix} g_1 \otimes 1_n & T_n(x) \\ T_n(x)^* & g_2 \otimes 1_n \end{bmatrix} \geq 0 \text{ and } \|g_1\|, \|g_2\| \leq \frac{1}{2} \|T\|_{cob} (\|f_1\|_p + \|f_2\|_p).$$

Hence, $(g_1, g_2) \in K(x_i)$ for all $i \in \{1, \dots, k\}$. We conclude that

$$\bigcap_{n \in \mathbb{N}, x \in L_p(\mathcal{M}) \otimes M_n} K(x) \neq \emptyset.$$

We take a pair (g_1, g_2) of this set. If $q > 1$, this pair fulfills (4.2). If $q = 1$ there is a central projection $z \in \mathcal{N}^{**}$ which maps $\mathcal{N}^*$ to $L_1(\mathcal{N})$. Then the pair (zg_1, zg_2) is in $L_1(\mathcal{N})_+ \times L_1(\mathcal{N})_+$ and fulfills (4.2). $\square$

Theorem 4.6 *Let $1 \leq q \leq \infty$ and let $T : \mathcal{M} \rightarrow L_q(\mathcal{N})$ be completely order bounded. Then there exist $[f, g \in L_q(\mathcal{N})_+$ and a completely order bounded map $S : \mathcal{M} \rightarrow \mathcal{N}$ such that*

$$T(x) = f^{\frac{1}{2}} S(x) g^{\frac{1}{2}} \text{ for all } x \in \mathcal{M}$$

and

$$\|f\|_q, \|g\|_q \leq \|T\|_{cob}, \|S\|_{cob} \leq 1.$$

Note that in case of a linear map from $\mathcal{M}$ to $\mathcal{N}$ completely order bounded is identical to completely bounded.

Proof Since the norm $\|\cdot\|_{\infty, n}$ is the usual operator norm on $\mathcal{M} \otimes M_n$, we get $\begin{bmatrix} \mathbf{1} \otimes 1_n & x \\ x^* & \mathbf{1} \otimes 1_n \end{bmatrix} \geq 0$ for all $n \in \mathbb{N}, x \in \mathcal{M} \otimes M_n, \|x\|_{\infty, n} \leq 1$ where $\mathbf{1}$ is the unit in $\mathcal{M}$. Hence we can apply Theorem 4.5 with $f_1 = f_2 = \mathbf{1}$ and get $f, g \in L_q(\mathcal{N})$, such that $\|f\|_q, \|g\|_q \leq \|T\|_{cob}$ and

$$\begin{bmatrix} f \otimes 1_n & T_n(x) \\ T_n(x)^* & g \otimes 1_n \end{bmatrix} \geq 0 \text{ for all } n \in \mathbb{N}, x \in \mathcal{M} \otimes M_n, \|x\|_{\infty, n} \leq 1.$$

Let $x \in \mathcal{M}$. We apply Theorem 2.3 with $n = 1$ to x , and get a unique $y \in \mathcal{N}, \|y\|_{\infty} \leq 1$ such that $T(x) = f^{\frac{1}{2}} y g^{\frac{1}{2}}$ and $\text{supp}(f) \cdot y \cdot \text{supp}(g) = y$. We put $S(x) = y$. Then S is a map from $\mathcal{M}$ to $\mathcal{N}$. If $x, x_1, x_2 \in \mathcal{M}, x = x_1 + x_2$, then there are according to Theorem 2.3 $y, y_1, y_2 \in \mathcal{N}$ such that $T(x) = f^{\frac{1}{2}} y g^{\frac{1}{2}}, \text{supp}(f) \cdot y \cdot \text{supp}(g) = y$ and $T(x_i) = f^{\frac{1}{2}} y_i g^{\frac{1}{2}}, \text{supp}(f) \cdot y_i \cdot \text{supp}(g) = y_i, i = 1, 2$. Then we get

$$f^{\frac{1}{2}}yg^{\frac{1}{2}} = T(x) = T(x_1) + T(x_2) = f^{\frac{1}{2}}(y_1 + y_2)(g^{\frac{1}{2}}),$$

$$\text{supp}(f) \cdot (y_1 + y_2) \cdot \text{supp}(g) = y_1 + y_2.$$

Hence we conclude from Lemma 2.2 that $y = y_1 + y_2$ which means that S is additive. Similarly, we show that $S(\lambda x) = \lambda S(x)$ for $\lambda \in \mathbb{C}, x \in \mathcal{M}$. Now for $n \in \mathbb{N}$, let $x \in \mathcal{M} \otimes M_n$ with $\|x\|_{\infty, n} \leq 1$. According to Theorem 2.3, there is $y \in \mathcal{N} \otimes M_n$ such that $(\text{supp}(f) \otimes 1_n) \cdot y \cdot (\text{supp}(g) \otimes 1_n) = y$ and $T_n(x) = (f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n)$. Then, we have

$$(f^{\frac{1}{2}} \otimes 1_n)S_n(x)(g^{\frac{1}{2}} \otimes 1_n) = T_n(x) = (f^{\frac{1}{2}} \otimes 1_n)y(g^{\frac{1}{2}} \otimes 1_n).$$

Hence, we conclude that $S_n(x) = y$ which shows that $\|S_n(x)\|_{\infty, n} \leq 1$. So $\|S\|_{cob} \leq 1$. $\square$

Theorem 4.7 *Let $\mathcal{N}$ be injective, $1 \leq q \leq \infty$ and let $T : \mathcal{M} \rightarrow L_q(\mathcal{N})$ be completely order bounded. Then there exist linear maps $T_i : \mathcal{M} \rightarrow L_q(\mathcal{N})$ such that the map*

$$\Phi : \mathcal{M} \otimes M_2 \rightarrow L_q(\mathcal{N}) \otimes M_2$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} T_1(x_{11}) & T(x_{12}) \\ T(x_{21})^* & T_2(x_{22}) \end{bmatrix}$$

is complete positive and $\|T_1\|, \|T_2\| \leq \|T\|_{cob}$. Thus T is decomposable and $\|T\|_{dec} = \|T\|_{cob}$.

Proof By Theorem 4.6, there exist $f, g \in L_q(\mathcal{N})_+, \|f\|_q, \|g\|_q \leq \|T\|_{cob}$ and $S : \mathcal{M} \rightarrow \mathcal{N}, \|S\|_{cob} \leq 1$ such that

$$T(x) = f^{\frac{1}{2}}S(x)g^{\frac{1}{2}}, \quad x \in \mathcal{M}.$$

Since for a linear map from $\mathcal{M}$ to $\mathcal{N}$ completely bounded is the same as completely order bounded, we can apply [14], Theorem 4.5, and get linear maps $S_1, S_2 : \mathcal{M} \rightarrow \mathcal{N}$ such that $\|S_i\| \leq 1, i = 1, 2$, and the map

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & S(x_{12}) \\ S(x_{21})^* & S_2(x_{22}) \end{bmatrix}$$

is completely positive. The linear maps $T_i : \mathcal{M} \rightarrow L_q(\mathcal{N})$ where $T_1(x) = f^{\frac{1}{2}}S_1(x)f^{\frac{1}{2}}, T_2(x) = g^{\frac{1}{2}}S_2(x)g^{\frac{1}{2}}, x \in \mathcal{M}$, fulfill $\|T_1\| \leq \|f\|_q \|S_1\| \leq \|T\|_{cob}, \|T_2\| \leq \|g\|_q \|S_2\| \leq \|T\|_{cob}$, and

$$\mathcal{M} \otimes M_2 \ni \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} T_1(x_{11}) & T(x_{12}) \\ T(x_{21})^* & T_2(x_{22}) \end{bmatrix} \in L_q(\mathcal{N}) \otimes M_2$$

is completely positive. Together with Theorem 4.3 we get $\|T\|_{dec} = \|T\|_{cob}$ $\square$

Remark 4.8 Theorem 4.5 in [14] states the decomposition for self-adjoint maps. The proof of Proposition 1.3 in [5] shows that this is also true for maps which are not self-adjoint.

The next goal is to prove that completely order bounded maps from $L_p(\mathcal{M})$ to $L_1(\mathcal{N})$ are decomposable. This proof is divided into several steps.

Lemma 4.9 *Let $k \in \mathbb{N}$ and $a \in L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k}$ be self-adjoint. Then a can be written in the form*

$$a = \sum_{j=1}^n a_j \otimes \alpha_j$$

where all $a_j \in L_p(\mathcal{M}) \otimes M_2$ and all $\alpha_j \in M_{2k}$ are self-adjoint.

Proof We mimic the proof of [12], Lemma IV.4.4. If $a = \sum_{j=1}^n b_j \otimes \beta_j$ with $b_j \in \mathcal{M} \otimes M_2$ and $\beta_j \in M_{2k}$ for $j = 1, \dots, n$ and a is self-adjoint then

$$a = \frac{1}{4} \sum_{j=1}^n \left((b_j + b_j^*) \otimes (\beta_j + \beta_j^*) - i(b_j - b_j^*) \otimes i(\beta_j - \beta_j^*) \right)$$

□

Let Tr be the usual trace on M_{2k} . We define the duality between $L_1(M_k) \otimes M_2$ and M_{2k} by $\langle a, b \rangle = Tr(ab)$, $a \in L_1(M_k) \otimes M_2$, $b \in M_{2k}$. If $n \in \mathbb{N}$, $a = [a_{ij}]$, $b = [b_{ij}]$, where $a_{ij} \in L_1(M_k) \otimes M_2$ and $b_{ij} \in M_k \otimes M_2$ then $\langle [a_{ij}], [b_{ij}] \rangle = \sum_{i,j=1}^{2k} \langle a_{ij}, b_{ji} \rangle$. We define the linear functional

$$\omega : L_1(M_k) \otimes M_2 \otimes M_{2k} \rightarrow \mathbb{C}$$

$$\sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n \langle a_i, b_i^t \rangle \quad (4.3)$$

where b_i^t denotes the transposed matrix of b_i .

Lemma 4.10 *Let $a \in (L_1(M_k) \otimes M_2 \otimes M_{2k})_+$. Then $\omega(a) \geq 0$.*

Proof Let $a^{\frac{1}{2}} = \sum_{i=1}^n a_i \otimes \alpha_i$. Then $a^{\frac{1}{2}}$ is self-adjoint, $a = \sum_{i,j=1}^n a_i^* a_j \otimes \alpha_i^* \alpha_j$, and

$$\begin{aligned} \omega(a) &= \sum_{i,j=1}^n \omega(a_i^* a_j \otimes \alpha_i^* \alpha_j) = \sum_{i,j=1}^n \left\langle a_i^* a_j, \alpha_j^t \alpha_i^{*t} \right\rangle \\ &= \left\langle [a_i^* a_j], [\alpha_i^t \alpha_j^{*t}] \right\rangle \end{aligned}$$

Since $[a_i^* a_j]$ and $[\alpha_i^t \alpha_j^{*t}]$ are positive matrices, the last expression is positive by (2.3). □

For a linear map $T : L_p(\mathcal{M}) \rightarrow L_1(M_k)$ we define

$$\begin{aligned} \tilde{T}: L_p(\mathcal{M}) \otimes M_2 &\rightarrow L_1(M_k) \otimes M_2 \\ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} &\mapsto \begin{bmatrix} 0 & T(x_{12}) \\ T(x_{21}^*)^* & 0 \end{bmatrix}. \end{aligned} \quad (4.4)$$

We define the linear functional

$$\begin{aligned} \varphi_T: L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k} &\rightarrow \mathbb{C} \\ x &\mapsto \omega \circ \tilde{T}_{2k}(x). \end{aligned} \quad (4.5)$$

Lemma 4.11 *Let $T: L_p(\mathcal{M}) \rightarrow L_1(M_k)$ be completely order bounded and φ_T as in (4.5). Let $a, b \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_+$, $f, g \in L_p(\mathcal{M})_+$ such that*

$$0 \leq a + b \leq f \otimes \varepsilon_{11} \otimes 1_{2k} + g \otimes \varepsilon_{22} \otimes 1_{2k}. \quad (4.6)$$

Then

$$\|T\|_{cob}(\|f\|_p + \|g\|_p) + \varphi_T(a - b) \geq 0.$$

Here ε_{ij} , $i, j = 1, 2$ are the 2×2 matrices defined in (3.2).

Proof Since $a - b$ is self-adjoint, by Lemma 4.9, we can write

$$a - b = \sum_{i=1}^n a_i \otimes \alpha_i$$

where all $a_i \in L_p(\mathcal{M}) \otimes M_2$ and all $\alpha_i \in M_{2k}$ are self-adjoint. For $i \in \{1, \dots, n\}$ let

$$a_i = \begin{bmatrix} a_{i,11} & a_{i,12} \\ a_{i,21} & a_{i,22} \end{bmatrix}, \alpha_i = \begin{bmatrix} \alpha_{i,11} & \alpha_{i,12} \\ \alpha_{i,21} & \alpha_{i,22} \end{bmatrix},$$

where $a_{i,st} \in L_p(\mathcal{M})$ and $\alpha_{i,st} \in M_k$ for $s, t = 1, 2$. Since a_i and α_i are self-adjoint for all i , we have $a_{i,21}^* = a_{i,12}$, $\alpha_{i,21}^* = \alpha_{i,12}$, and $\langle T(a_{i,21}^*)^*, \alpha_{i,21}^t \rangle = \langle T(a_{i,12})^*, \alpha_{i,12}^t \rangle = \overline{\langle T(a_{i,12}), \alpha_{i,12}^t \rangle}$ for all $i \in \{1, \dots, n\}$. Therefore

$$\begin{aligned} \varphi_T(a - b) &= \sum_{i=1}^n \varphi_T(a_i \otimes \alpha_i) \\ &= \sum_{i=1}^n \left\langle \begin{bmatrix} 0 & T(a_{i,12}) \\ T(a_{i,21}^*)^* & 0 \end{bmatrix}, \begin{bmatrix} \alpha_{i,11}^t & \alpha_{i,21}^t \\ \alpha_{i,12}^t & \alpha_{i,22}^t \end{bmatrix} \right\rangle \\ &= \sum_{i=1}^n \left(\langle T(a_{i,12}), \alpha_{i,12}^t \rangle + \langle T(a_{i,12})^*, \alpha_{i,12}^{*t} \rangle \right). \end{aligned}$$

By Proposition 2.1, we have

$$\begin{bmatrix} (f \otimes \varepsilon_{11} + g \otimes \varepsilon_{22}) \otimes 1_{2k} & a - b \\ a - b & (f \otimes \varepsilon_{11} + g \otimes \varepsilon_{22}) \otimes 1_{2k} \end{bmatrix} \geq 0. \quad (4.7)$$

We multiply the matrix in (4.7) from left with the matrix γ and from right with the transposed matrix γ^t , where γ is the $8k \times 4k$ matrix $\begin{bmatrix} \varepsilon_{11} \otimes 1_{2k} \\ \varepsilon_{22} \otimes 1_{2k} \end{bmatrix}$ and get

$$\begin{aligned} 0 &\leq \varepsilon_{11}(f \otimes \varepsilon_{11} + g \otimes \varepsilon_{22})\varepsilon_{11} \otimes 1_{2k} + (\varepsilon_{11} \otimes 1_{2k})(a - b)(\varepsilon_{22} \otimes 1_{2k}) \\ &\quad + (\varepsilon_{22} \otimes 1_{2k})(a - b)(\varepsilon_{11} \otimes 1_{2k}) + \varepsilon_{22}(f \otimes \varepsilon_{11} + g \otimes \varepsilon_{22})\varepsilon_{22} \otimes 1_{2k} \\ &= f \otimes \varepsilon_{11} \otimes 1_{2k} + \sum_{i=1}^n a_{i,12} \otimes \varepsilon_{12} \otimes \alpha_i \\ &\quad + \sum_{i=1}^n a_{i,21} \otimes \varepsilon_{21} \otimes \alpha_i + g \otimes \varepsilon_{22} \otimes 1_{2k}. \end{aligned} \quad (4.8)$$

This inequality can be written as $\|\sum_{i=1}^n a_{i,12} \otimes \alpha_i\|_{p,2k} \leq \frac{1}{2}(\|f\|_p + \|g\|_p)$. Hence $\|T(\sum_{i=1}^n a_{i,12} \otimes \alpha_i)\|_{1,2k} \leq \frac{1}{2}\|T\|_{cob}(\|f\|_p + \|g\|_p)$. By Theorem 3.4, there exist $f_1, g_1 \in L_1(M_k)_+$ such that $\|f_1\|_1 = \|g_1\|_1 \leq \frac{1}{2}\|T\|_{cob}(\|f\|_p + \|g\|_p)$ and

$$\begin{bmatrix} f_1 \otimes 1_{2k} & \sum_{i=1}^n T(a_{i,12}) \otimes \alpha_i \\ \sum_{i=1}^n T(a_{i,12})^* \otimes \alpha_i & g_1 \otimes 1_{2k} \end{bmatrix} \geq 0. \quad (4.9)$$

We apply ω to (4.9) and get

$$\begin{aligned} 0 &\leq \langle f_1 \otimes \varepsilon_{11}, 1_{2k} \rangle + \langle g_1 \otimes \varepsilon_{22}, 1_{2k} \rangle \\ &\quad + \sum_{i=1}^n (\langle T(a_{i,12}) \otimes \varepsilon_{12}, \alpha_i^t \rangle + \langle T(a_{i,12})^* \otimes \varepsilon_{21}, \alpha_i^t \rangle) \\ &= \|f_1\|_1 + \|g_1\|_1 + \sum_{i=1}^n (\langle T(a_{i,12}), \alpha_i^t \rangle + \langle T(a_{i,12})^*, \alpha_{i,12}^{*t} \rangle) \\ &= \|f_1\|_1 + \|g_1\|_1 + \varphi_T(a - b). \end{aligned}$$

Since $\|f_1\|_1 + \|g_1\|_1 \leq \|T\|_{cob}(\|f\|_p + \|g\|_p)$, the proof is finished. $\square$

Lemma 4.12 Let $x \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_n)_+$ and $y \in (M_k \otimes M_2 \otimes M_n)_+$ where $x = [x_{ij}]$ with $x_{ij} \in L_p(\mathcal{M} \otimes M_2)$ and $y = [y_{ij}]$ with $y_{ij} \in M_k \otimes M_2$ for $i, j \in \{1, \dots, n\}$. Then

$$\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \geq 0.$$

Proof By Lemma 2.3, there is $f \in L_p(\mathcal{M})_+$ and $a \in (\mathcal{M} \otimes M_2 \otimes M_n)_+$ such that $x = (f^{\frac{1}{2}} \otimes 1_{2n})a(f^{\frac{1}{2}} \otimes 1_{2n})$. By [12], Lemma IV.3.1, a can be written as a finite sum

of matrices of the form $[a_i^* a_j]$ where $a_1, \dots, a_n \in \mathcal{M} \otimes M_2$. Similarly, y can be written as a finite sum of matrices of the form $[y_i^* y_j]$ where $y_1, \dots, y_n \in M_k \otimes M_2$. Thus $\sum_{i,j=1}^n x_{ij} \otimes y_{ij}^t$ is a finite sum of elements of the form $\sum_{i,j=1}^n (f^{\frac{1}{2}} \otimes 1_2) a_i^* a_j (f^{\frac{1}{2}} \otimes 1_2) \otimes (y_i^* y_j)^t$. Then we have

$$\begin{aligned} & \sum_{i,j=1}^n (f^{\frac{1}{2}} \otimes 1_2) a_i^* a_j (f^{\frac{1}{2}} \otimes 1_2) \otimes (y_i^* y_j)^t \\ &= \sum_{i,j=1}^n (f^{\frac{1}{2}} \otimes 1_2) a_i^* a_j (f^{\frac{1}{2}} \otimes 1_2) \otimes y_i^t y_j^{*t} \\ &= \left(\sum_{i=1}^n a_i (f^{\frac{1}{2}} \otimes 1_2) \otimes y_i^{*t} \right)^* \left(\sum_{j=1}^n a_j (f^{\frac{1}{2}} \otimes 1_2) \otimes y_j^{*t} \right) \geq 0 \end{aligned}$$

□

Proposition 4.13 *Let $T : L_p(\mathcal{M}) \rightarrow L_1(M_k)$ be completely order bounded and let $\tilde{T}$ be as in (4.4). Then there is a linear map $S : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(M_k) \otimes M_2$ such that $S \pm \tilde{T}$ are completely positive and*

$$\begin{aligned} 0 &\leq \left\langle S \left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \right), \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right\rangle \leq \|T\|_{cob} \|c\|_p \|y\|_\infty \\ 0 &\leq \left\langle S \left(\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right), \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right\rangle \leq \|T\|_{cob} \|c\|_p \|y\|_\infty \end{aligned} \quad (4.10)$$

for all $c \in L_p(\mathcal{M})_+$ and all $y \in M_{k+}$.

Proof Let $(L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h$ denote the self-adjoint part of $L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k}$ and let φ_T be the linear functional defined by (4.5). For $x \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h$ we define

$$\begin{aligned} \theta(x) &= \inf \{ \|T\|_{cob} (\|f\|_p + \|g\|_p) + \varphi_T(a - b) \mid \\ &\quad a, b \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_+, f, g \in L_p(\mathcal{M})_+, \\ &\quad x + a + b \leq f \otimes \varepsilon_{11} \otimes 1_{2k} + g \otimes \varepsilon_{22} \otimes 1_{2k} \}. \end{aligned} \quad (4.11)$$

By Lemma 2.4, the set on the right side of (4.11) is not empty, so θ is well defined. We will show that θ is sublinear. To do this, let $x_1, x_2 \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h$ and $\varepsilon > 0$. Then there exist $a_1, b_1, a_2, b_2 \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_+$ and $f_1, g_1, f_2, g_2 \in L_p(\mathcal{M})_+$ such that

$$\begin{aligned}
x_1 + a_1 + b_1 &\leq f_1 \otimes \varepsilon_{11} \otimes 1_{2k} + g_1 \otimes \varepsilon_{22} \otimes 1_{2k}, \\
x_2 + a_2 + b_2 &\leq f_2 \otimes \varepsilon_{11} \otimes 1_{2k} + g_2 \otimes \varepsilon_{22} \otimes 1_{2k}, \\
\theta(x_1) &\geq \|T\|_{cob}(\|f_1\|_p + \|g_1\|_p) + \varphi_T(a_1 - b_1) - \varepsilon, \\
\theta(x_2) &\geq \|T\|_{cob}(\|f_2\|_p + \|g_2\|_p) + \varphi_T(a_2 - b_2) - \varepsilon.
\end{aligned}$$

Then we get

$$x_1 + x_2 + a_1 + a_2 + b_1 + b_2 \leq (f_1 + f_2) \otimes \varepsilon_{11} \otimes 1_{2k} + (g_1 + g_2) \otimes \varepsilon_{22} \otimes 1_{2k}.$$

This implies

$$\theta(x_1 + x_2) \leq \theta(x_1) + \theta(x_2) - 2\varepsilon.$$

Since ε was arbitrary, θ is sub-additive. Similarly, we show for $0 < \lambda \in \mathbb{R}$ and $x \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h$ that

$$\theta(\lambda x) \leq \lambda \theta(x)$$

and hence

$$\lambda \theta(x) = \lambda \theta\left(\frac{1}{\lambda} \lambda x\right) \leq \lambda \frac{1}{\lambda} \theta(\lambda x) = \theta(\lambda x).$$

It remains to show that $\theta(0) = 0$. If we put $x = a = b = 0, f = g = 0$ in (4.11), we get $\theta(0) \leq 0$. Lemma 4.11 states that every element in the set on the right side of (4.11) is not negative for $x = 0$. Hence $\theta(0) = 0$. By the Hahn-Banach theorem there is a real-linear functional

$$\psi : (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h \rightarrow \mathbb{R} \quad (4.12)$$

such that $\psi(x) \leq \theta(x)$ for all $x \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k})_h$. Now we can extend ψ to a complex linear functional on $L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k}$ by putting $\psi(x) = \frac{1}{2}\psi(x + x^*) + \frac{1}{2}i\psi(ix^* - ix)$ for $x \in L_p(\mathcal{M}) \otimes M_2 \otimes M_{2k}$. For $c \in (L_p(\mathcal{M}) \otimes M_2)_+$, $\|c\|_{p,2} \leq 1$, by Theorem 3.4, there exists $f \in L_p(\mathcal{M})_+$ such that

$$f \otimes 1_2 - c \geq 0 \text{ and } \|f\|_p \leq 1.$$

For $y \in (M_{2k})_+, 0 \leq y \leq 1_{2k}$, we put $x = c \otimes y^t, a = b = 0$, apply (4.11), and get

$$\psi(c \otimes y^t) \leq \theta(c \otimes y^t) \leq 2\|T\|_{cob}\|f\|_p \leq 2\|T\|_{cob}. \quad (4.13)$$

Then we put $x = -c \otimes y^t, a = c \otimes y^t, b = 0, f = g = 0$, apply (4.11), and get

$$\psi(-c \otimes y^t) \leq \theta(-c \otimes y^t) \leq \varphi_T(c \otimes y^t) \quad (4.14)$$

For $x = -c \otimes y^t, a = 0, b = c \otimes y^t, f = g = 0$, we apply (4.11) and (4.12) and get

$$\psi(-c \otimes y^t) \leq \theta(-c \otimes y^t) \leq -\varphi_T(c \otimes y^t) \quad (4.15)$$

We combine (4.13), (4.14), (4.15), and get

$$0 \leq \psi(c \otimes y^t) \leq 2\|T\|_{cob}.$$

By Theorem 2.3 and [2], Proposition II.3.1.2, we can write $c \in L_p(\mathcal{M}) \otimes M_2$, $\|c\|_{p,2} \leq 1$ as $c = c_1 - c_2 + i(c_3 - c_4)$ where $c_i \geq 0$ and $\|c_i\|_{p,2} \leq 1$ for $i = 1, \dots, 4$. A similar decomposition holds for $y \in M_{2k}$. Thus the bilinear map

$$B : L_p(\mathcal{M}) \otimes M_2 \times M_{2k} \rightarrow \mathbb{C}, B(c, y) = \psi(c \otimes y^t)$$

is bounded. Hence there is a linear map $S : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(M_k) \otimes M_2$ such that $\langle S(c), y \rangle = \psi(c \otimes y^t)$. Next we show that $S \pm \tilde{T}$ are completely positive. By (2.3), it suffices to show that for $n \in \mathbb{N}$, $x \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_n)_+$, $y \in (M_k \otimes M_2 \otimes M_n)_+$, where $x = [x_{ij}]$ with $x_{ij} \in L_p(\mathcal{M}) \otimes M_2$ and $y = [y_{ij}] \in M_k \otimes M_2$ the expression $\langle S_n(x) \pm \tilde{T}_n(x), y \rangle$ is positive. Now we have

$$\begin{aligned} \langle S_n(x) \pm \tilde{T}_n(x), y \rangle &= \sum_{i,j=1}^n \langle S(x_{ij}) \pm \tilde{T}(x_{ij}), y_{ji} \rangle \\ &= \psi \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right) \pm \varphi_T \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right). \end{aligned} \quad (4.16)$$

By Lemma 4.12, $\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t$ is positive. We put $x = -\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t$, $a = -x$, $b = 0$, $f = g = 0$ in (4.11) and get

$$-\psi \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right) \leq \varphi_T \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right). \quad (4.17)$$

Similarly, we put $x = \sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t$, $a = 0$, $b = -x$, $f = g = 0$ and get

$$-\psi \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right) \leq -\varphi_T \left(\sum_{i,j=1}^n x_{ij} \otimes y_{ji}^t \right). \quad (4.18)$$

The combination of (4.16), (4.17), and (4.18) gives

$$\langle S_n(x) \pm \tilde{T}_n(x), y \rangle \geq 0.$$

Now let $c \in L_p(\mathcal{M})_+$ and $y \in M_{k+}$. We put $x = c \otimes \varepsilon_{11} \otimes y^t \otimes \varepsilon_{11}$, $a = b = 0$, $f = \|y\|_\infty \cdot c$, $g = 0$. Then (4.11) gives

$$0 \leq \left\langle S \left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \right), \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = \psi(x) \leq \theta(x) \leq \|T\|_{cob} \|c\|_p \|y\|_\infty.$$

The second part of (4.10) is shown similarly. $\square$

Proposition 4.14 *Let $k \in \mathbb{N}$ and $T : L_p(\mathcal{M}) \rightarrow L_1(M_k)$ be completely order bounded. Then there are linear maps $S_1, S_2 : L_p(\mathcal{M}) \rightarrow L_1(M_k)$ such that the map*

$$\Phi : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(M_k) \otimes M_2$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & T(x_{12}) \\ T(x_{21})^* & S_2(x_{22}) \end{bmatrix}$$

is completely positive and $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$.

Proof Let $\tilde{T}$ be as in (4.4). By Proposition 4.13, there is a linear map $S : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(M_k) \otimes M_2$ such that $S \pm \tilde{T}$ are completely positive and for $c \in L_p(\mathcal{M})_+, y \in M_{k+}$

$$0 \leq \left\langle S \left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \right), \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right\rangle \leq \|T\|_{cob} \|c\|_p \|y\|_\infty$$

$$0 \leq \left\langle S \left(\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \right), \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right\rangle \leq \|T\|_{cob} \|c\|_p \|y\|_\infty.$$

The next steps are quite similar to the proof of [1], Proposition 3.18. Let α be the scalar 1×2 matrix $[10]$ and β be the scalar 1×2 matrix $[01]$, and let α^*, β^* be the adjointed matrices. We put

$$S_1 : L_p(\mathcal{M}) \rightarrow L_1(M_k), a \mapsto \alpha S \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \alpha^*$$

and

$$S_2 : L_p(\mathcal{M}) \rightarrow L_1(M_k), a \mapsto \beta S \left(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right) \beta^*$$

Since symmetric multiplication with a matrix and its adjoint is completely positive, the maps S_1 and S_2 are completely positive. For $a \in L_p(\mathcal{M})_+$, we have

$$\begin{aligned} \|S_1(a)\|_1 &= \langle S_1(a), 1_k \rangle = \left\langle \alpha S \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) \alpha^*, 1_k \right\rangle \\ &= \left\langle S \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right), \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle \leq \|T\|_{cob} \|a\|_p. \end{aligned}$$

Hence $\|S_1\| \leq \|T\|_{cob}$. Similarly, $\|S_2\| \leq \|T\|_{cob}$. Let γ_1 be the scalar 2×4 matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. For $a = [a_{ij}] \in L_p(\mathcal{M}) \otimes M_2$ we have

$$\gamma_1^* \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \gamma_1 = \begin{bmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & a_{22} \end{bmatrix}$$

and the map

$$\Phi_1 : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_p(\mathcal{M}) \otimes M_4, a \mapsto \gamma_1^* a \gamma_1$$

is completely positive. Next, we show that the map

$$\Phi_2 : L_p(\mathcal{M}) \otimes M_4 \rightarrow L_1(M_k) \otimes M_4$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S(x_{11}) & \tilde{T}(x_{12}) \\ \tilde{T}(x_{21}) & S(x_{22}) \end{bmatrix} \text{ where } x_{ij} \in L_p(\mathcal{M}) \otimes M_2$$

is completely positive. So let $n \in \mathbb{N}$ and $x = [x_{ij}] \in (L_p(\mathcal{M}) \otimes M_4 \otimes M_n)_+$ where $x_{ij} \in L_p(\mathcal{M}) \otimes M_4$. Then $S_{2n}(x) \pm \tilde{T}_{2n}(x) \geq 0$, and, by Lemma 2.1

$$\begin{bmatrix} S_{2n}(x) & \tilde{T}_{2n}(x) \\ \tilde{T}_{2n}(x) & S_{2n}(x) \end{bmatrix} \geq 0.$$

We write x as

$$x = \begin{bmatrix} x_{ij11} & x_{ij12} \\ x_{ij21} & x_{ij22} \end{bmatrix}$$

where $x_{ijlm} \in L_p(\mathcal{M}) \otimes M_2, i, j \in \{1, \dots, n\}, l, m = 1, 2$. Then

$$\begin{bmatrix} S_{2n}(x) & \tilde{T}_{2n}(x) \\ \tilde{T}_{2n}(x) & S_{2n}(x) \end{bmatrix} = \begin{bmatrix} S_n(x_{ij11}) & S_n(x_{ij12}) & \tilde{T}_n(x_{ij11}) & \tilde{T}_n(x_{ij12}) \\ S_n(x_{ij21}) & S_n(x_{ij22}) & \tilde{T}_n(x_{ij21}) & \tilde{T}_n(x_{ij22}) \\ \tilde{T}_n(x_{ij11}) & \tilde{T}_n(x_{ij12}) & S_n(x_{ij11}) & S_n(x_{ij12}) \\ \tilde{T}_n(x_{ij21}) & \tilde{T}_n(x_{ij22}) & S_n(x_{ij21}) & S_n(x_{ij22}) \end{bmatrix}.$$

We multiply this matrix from left by the scalar matrix $\gamma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and from right the by its adjointed matrix γ_2^* , and get

$$0 \leq \begin{bmatrix} S_n([x_{ij11}]) & \tilde{T}_n([x_{ij12}]) \\ \tilde{T}_n([x_{ij21}]) & S_n([x_{ij22}]) \end{bmatrix}. \quad (4.19)$$

We multiply the matrix in (4.19) from right by the scalar $2n \times 2n$ matrix γ_3 and from left by γ_3^* , where γ_3 has 1 at position (l, m) when $(l, m) = (2i - 1, i)$ or $(l, m) = (2i, n + i)$ for $i \in \{1, \dots, n\}$ and 0 else. Then the resulting matrix is $\Phi_{2,n}(x)$. This shows that Φ_2 is completely positive. Next we define the linear map

$$\Phi_3 : L_1(M_k) \otimes M_4 \rightarrow L_1(M_k) \otimes M_2, b \mapsto \gamma_1 b \gamma_1^*$$

where γ_1 is the 2×4 matrix used in the beginning of the proof. Then Φ_3 is completely positive. Since $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, Φ is completely positive. This finishes the proof. $\square$

For linear maps $S_1, S_2, T : L_p(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ we define

$$\begin{aligned} \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix} : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(\mathcal{N}) \otimes M_2 \\ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & T(x_{12}) \\ T(x_{21}^*)^* & S_2(x_{22}) \end{bmatrix} \end{aligned} \quad (4.20)$$

Lemma 4.15 *Let $\mathcal{N}$ be injective, $T : L_p(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be completely order bounded, $m, n_1, \dots, n_m \in \mathbb{N}$, $x_l \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{n_l})_+$, $y_l \in (\mathcal{N} \otimes M_2 \otimes M_{n_l})_+$, for $l = 1, \dots, m$ and $\varepsilon > 0$. Then there exist completely positive maps $S_1, S_2 : L_p(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ such that $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$ and for the map $\begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}$ defined in (4.20) holds*

$$\left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle \geq -\varepsilon, \quad l = 1, \dots, m.$$

Proof Let $x_l = [x_{l,ij}]$, where $x_{l,ij} \in L_p(\mathcal{M}) \otimes M_2$, and $y_l = [y_{l,ij}]$, where $y_{l,ij} \in \mathcal{N} \otimes M_2$, and for $i, j = 1, \dots, n_l$, $l = 1, \dots, m$

$$x_{l,ij} = \begin{bmatrix} x_{l,ij11} & x_{l,ij12} \\ x_{l,ij21} & x_{l,ij22} \end{bmatrix}, y_{l,ij} = \begin{bmatrix} y_{l,ij11} & y_{l,ij12} \\ y_{l,ij21} & y_{l,ij22} \end{bmatrix}.$$

Since $\mathcal{N}$ is injective, there exist by [2], Theorem IV.2.4.4, $k \in \mathbb{N}$ and completely positive contractions $\sigma_1 : \mathcal{N} \rightarrow M_k$ and $\sigma_2 : M_k \rightarrow \mathcal{N}$ such that σ_1 is continuous in the σ -weak topology and

$$\begin{aligned} & \left| \sum_{i,j=1}^{n_l} (\langle T(x_{l,ij21}^*)^*, y_{l,ji12} - \sigma_2 \circ \sigma_1(y_{l,ji12}) \rangle \right. \\ & \quad \left. + \langle T(x_{l,ij12}), y_{l,ji21} - \sigma_2 \circ \sigma_1(y_{l,ji21}) \rangle) \right| < \varepsilon, \quad l = 1, \dots, m. \end{aligned} \quad (4.21)$$

Since x_l and y_l are positive, we have $x_{l,ij21}^* = x_{l,ji12}$ and $y_{l,ij12}^* = y_{l,ji21}$ for all $l = 1, \dots, m$. Thus, $\langle T(x_{l,ij21}^*)^*, y_{l,ji12} \rangle$ and $\langle T(x_{l,ji12}), y_{l,ij21} \rangle$ are conjugate complex numbers for all $i, j = 1, \dots, n_l$, $l = 1, \dots, m$, and the sum on the left side of this inequality is a real number. Hence the sum is greater than $-\varepsilon$.

Since σ_1 is continuous in the σ -weak topology, its adjoint map σ'_1 maps $L_1(M_k)$ to $L_1(\mathcal{N})$ and is a completely positive contraction. Similarly, the adjoint map σ'_2 maps $L_1(\mathcal{N})$ to $L_1(M_k)$ and is a completely positive contraction, and $\sigma'_2 \circ T$ is completely order bounded with $\|\sigma'_2 \circ T\|_{cob} \leq \|T\|_{cob}$. Now, we apply Proposition 4.14 and get linear maps $S'_1, S'_2 : L_p(\mathcal{M}) \rightarrow L_1(M_k)$ such that $\|S'_1\|, \|S'_2\| \leq \|T\|_{cob}$ and $\begin{bmatrix} S'_1 & \sigma'_2 \circ T \\ \sigma'_2 \circ T^* & S'_2 \end{bmatrix}$ is completely positive. We put $S_1 = \sigma'_1 \circ S'_1$ and $S_2 = \sigma'_1 \circ S'_2$. Then $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$, S_1 and S_2 are completely positive, and for all $l = 1, \dots, m$

$$\begin{aligned} \left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle &= \sum_{i,j=1}^{n_l} (\langle S_1(x_{l,ij11}), y_{l,ji11} \rangle \\ &\quad + \langle T(x_{l,ij12}), y_{l,ji21} \rangle \\ &\quad + \langle T(x_{l,ij21}^*)^*, y_{l,ji12} \rangle \\ &\quad + \langle S_2(x_{l,ij22}), y_{l,ji22} \rangle) \\ &= \sum_{i,j=1}^{n_l} (\langle \sigma'_1 \circ S'_1(x_{l,ij11}), y_{l,ji11} \rangle \\ &\quad + \langle T(x_{l,ij12}), \sigma_2 \circ \sigma_1(y_{l,ji21}) \rangle \\ &\quad + \langle T(x_{l,ij21}^*)^*, \sigma_2 \circ \sigma_1(y_{l,ji12}) \rangle \\ &\quad + \langle \sigma'_1 \circ S'_2(x_{l,ij22}), y_{l,ji22} \rangle) \\ &\quad + \sum_{i,j=1}^{n_l} (\langle T(x_{l,ij21}^*)^*, y_{l,ji12} - \sigma_2 \circ \sigma_1(y_{l,ji12}) \rangle \\ &\quad + \langle T(x_{l,ij12}), y_{l,ji21} - \sigma_2 \circ \sigma_1(y_{l,ji21}) \rangle). \end{aligned} \quad (4.22)$$

Now the first sum of the last expression in (4.22) is equal to

$$\left\langle (\sigma'_1)_{2n_l} \circ \begin{bmatrix} S'_1 & \sigma'_2 \circ T \\ (\sigma'_2 \circ T)^* & S'_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle,$$

which is greater than or equal to 0, because it is a composition of two completely positive maps applied to a positive element. The second sum is greater than $-\varepsilon$ by (4.21). This finishes the proof. $\square$

Theorem 4.16 *Let $1 \leq p \leq \infty$, $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, $\mathcal{N}$ injective, and $T : L_p(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be a completely order bounded map. Then there exist linear maps $S_1, S_2 : L_p(\mathcal{M}) \rightarrow L_1(\mathcal{N})$, $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$ such that the map*

$$\Phi : L_p(\mathcal{M}) \otimes M_2 \rightarrow L_1(\mathcal{N}) \otimes M_2$$

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \mapsto \begin{bmatrix} S_1(x_{11}) & T(x_{12}) \\ T(x_{21}^*)^* & S_2(x_{22}) \end{bmatrix}$$

is completely positive.

Proof Let $\mathcal{N}^*$ be the dual space of $\mathcal{N}$ and $\mathcal{B}(L_p(\mathcal{M}), \mathcal{N}^*)$ be the vector space of all bounded linear maps from $L_p(\mathcal{M})$ to $\mathcal{N}^*$. By [12], Theorem IV.2.3, $\mathcal{B}(L_p(\mathcal{M}), \mathcal{N}^*)$ is isomorphic to the dual space of $L_p(\mathcal{M}) \otimes_\gamma \mathcal{N}$, where $\|\cdot\|_\gamma$ denotes the projective tensor norm. The weak*-topology on $\mathcal{B}(L_p(\mathcal{M}), \mathcal{N}^*)$ is given by the seminorms $|\langle S(x), y \rangle|$, where $x \in L_p(\mathcal{M})$, $y \in \mathcal{N}$, and $S \in \mathcal{B}(L_p(\mathcal{M}), \mathcal{N}^*)$. Let $\mathcal{U} = \{S \in \mathcal{B}(L_p(\mathcal{M}), \mathcal{N}^*) \mid \|S\| \leq \|T\|_{cob}\}$. By the Banach-Alaoglu theorem $\mathcal{U}$ is compact in the weak*-topology. Hence $\mathcal{U} \times \mathcal{U}$ is compact in the product weak*-topology. For $\varepsilon > 0$, $m \in \mathbb{N}$, $n_1, \dots, n_m \in \mathbb{N}$, $l = 1, \dots, m$, $X = \{(x_l, y_l) \mid x_l \in (L_p(\mathcal{M}) \otimes M_2 \otimes M_{n_l})_+, y_l \in (\mathcal{N} \otimes M_2 \otimes M_{n_l})_+\}$, let

$$K(X, \varepsilon) = \left\{ (S_1, S_2) \in \mathcal{U} \times \mathcal{U} \mid S_1, S_2 \text{ completely positive,} \right.$$

$$\left. \left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle \geq -\varepsilon \text{ for all } (x_l, y_l) \in X \right\}.$$

We will show that $K(X, \varepsilon)$ is closed in the weak*-topology and hence compact. So let (S_1, S_2) be in the closure of $K(X, \varepsilon)$. Let $\delta > 0$ and $(x_l, y_l) \in X$, where

$$x_l = [x_{l,ij}], \quad x_{l,ij} = \begin{bmatrix} x_{l,ij11} & x_{l,ij12} \\ x_{l,ij21} & x_{l,ij22} \end{bmatrix} \in L_p(\mathcal{M}) \otimes M_2,$$

$$y_l = [y_{l,ij}], \quad y_{l,ij} = \begin{bmatrix} y_{l,ij11} & y_{l,ij12} \\ y_{l,ij21} & y_{l,ij22} \end{bmatrix} \in \mathcal{N} \otimes M_2.$$

Then we can find $(S_{1\delta}, S_{2\delta}) \in K(X, \varepsilon)$ such that

$$\sum_{i,j=1}^{n_l} |\langle S_1(x_{l,ij11}) - S_{1\delta}(x_{l,ij11}), y_{l,ij11} \rangle| < \delta$$

$$\sum_{i,j=1}^{n_l} |\langle S_2(x_{l,ij22}) - S_{2\delta}(x_{l,ij22}), y_{l,ij22} \rangle| < \delta,$$

and therefore

$$\begin{aligned}
& \left| \left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle - \left\langle \begin{bmatrix} S_{1\delta} & T \\ T^* & S_{2\delta} \end{bmatrix}_{n_l} (x_l), y_l \right\rangle \right| \\
&= \left| \sum_{i,j=1}^{n_l} \langle S_1(x_{l,ij11}) - S_{1\delta}(x_{l,ij11}), y_{l,ji11} \rangle \right. \\
&\quad \left. + \sum_{i,j=1}^{n_l} \langle S_2(x_{l,ij22}) - S_{2\delta}(x_{l,ij22}), y_{l,ji22} \rangle \right| < 2\delta
\end{aligned}$$

Now $\left\langle \begin{bmatrix} S_{1\delta} & T \\ T^* & S_{2\delta} \end{bmatrix}_{n_l} (x_l), y_l \right\rangle$ is a positive number. Hence the imaginary part

$$\left| \operatorname{Im} \left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle \right| < 2\delta$$

and the real part

$$\operatorname{Re} \left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle > \left\langle \begin{bmatrix} S_{1\delta} & T \\ T^* & S_{2\delta} \end{bmatrix}_{n_l} (x_l), y_l \right\rangle - 2\delta \geq \varepsilon - 2\delta.$$

Since δ is arbitrary, $\left\langle \begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}_{n_l} (x_l), y_l \right\rangle$ is a real number which is greater than or equal to $-\varepsilon$. Similarly, we show that $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$ and S_1, S_2 are completely positive. So $(S_1, S_2) \in K(X, \varepsilon)$.

The sets $K(X, \varepsilon), m \in \mathbb{N}, X = \{(x_1, y_1), \dots, (x_m, y_m)\}, \varepsilon > 0$ have the finite intersection property. If we have sets $K(X_l, \varepsilon_l), l = 1, \dots, m$, we put $X = \bigcup_{l=1}^m X_l$ and $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_m\}$. Then $K(X, \varepsilon) \subseteq \bigcap_{l=1}^m K(X_l, \varepsilon_l)$. Combining the finite intersection property and the compactness in the weak*-topology, we get

$$\bigcap_{X, \varepsilon} K(X, \varepsilon) \neq \emptyset.$$

Any pair (S_1, S_2) in this intersection has the property that $\begin{bmatrix} S_1 & T \\ T^* & S_2 \end{bmatrix}$ is completely positive and $\|S_1\|, \|S_2\| \leq \|T\|_{cob}$. Since there is a projection $z \in \mathcal{N}^{**}$ which works as a projection from $\mathcal{N}^*$ to $L_1(\mathcal{N})$, the maps $S'_1 = zS_1$ and $S'_2 = zS_2$ have the desired properties. $\square$

We close this section with an example of a completely order bounded map which is not decomposable when $q > 2p$. The boundaries for p and q are not sharp. The goal is to show that there are many combinations of p and q for which completely order bounded does not imply decomposable. Since we are working with semifinite von Neumann algebras in the example, we use the L_p -spaces associated with the trace, because they are easier to handle than the Haagerup-Terp construction. For

$m \in \mathbb{N}$, let l_q^m be the vector space $\mathbb{C}^m$ equipped with norm $\|c\|_q = (\sum_{i=1}^m |c_i|^q)^{1/q}$ where $c = (c_1, \dots, c_m)$. Let

$$e_1, \dots, e_m \text{ be the standard base of } l_q^m. \quad (4.23)$$

Before we show our example, we need some formulas to estimate the completely order bounded norm and the decomposable norm for linear maps from $L_p(\mathcal{M})$ to l_q^m .

Lemma 4.17 *Let $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $m \in \mathbb{N}$, $g_1, \dots, g_m \in L_{p'}(\mathcal{M})$ and*

$$T : L_p(\mathcal{M}) \rightarrow l_q^m, \quad f \mapsto \sum_{k=1}^m \langle f, g_k \rangle e_k.$$

Then

$$\|T\|_{cob} \leq \sup \left\{ \left(\sum_{k=1}^m \|ag_k b\|_1^q \right)^{1/q} \mid a, b \in L_{2p}(\mathcal{M}), \|a\|_{2p}, \|b\|_{2p} \leq 1 \right\}. \quad (4.24)$$

Proof Let $n \in \mathbb{N}$ and $x \in L_p(\mathcal{M}) \otimes M_n$, $\|x\|_{p,n} \leq 1$. By Theorem 3.4 and Theorem 2.3, there exist $a, b \in L_{2p}(\mathcal{M})$, $\|a\|_{2p}, \|b\|_{2p} \leq 1$, $y \in \mathcal{M} \otimes M_n$, $\|y\|_\infty \leq 1$ such that $x = (b \otimes 1_n)y(a \otimes 1_n)$. Let ε_{ij} be as in (3.2) and $y = \sum_{i,j=1}^n y_{ij} \otimes \varepsilon_{ij}$. Then we have

$$T_n(x) = \sum_{k=1}^m \sum_{i,j=1}^n \langle ag_k b, y_{ij} \rangle e_k \otimes \varepsilon_{ij}.$$

For $k = 1, \dots, m$ let $\varphi_k : \mathcal{M} \rightarrow \mathbb{C}, f \mapsto \langle ag_k b, f \rangle$. Then φ_k is completely order bounded and $\|\varphi_k\|_{cob} = \|\varphi_k\| = \|ag_k b\|_1$. Since $\|y\|_\infty \leq 1$, we have

$$0 \leq \begin{bmatrix} \|\varphi_k\| 1_n & (\varphi_k)_n(y) \\ (\varphi_k)_n(y)^* & \|\varphi_k\| 1_n \end{bmatrix}.$$

Then we get

$$\begin{aligned} 0 &\leq \sum_{k=1}^m e_k \otimes \begin{bmatrix} \|\varphi_k\| 1_n & (\varphi_k)_n(y) \\ (\varphi_k)_n(y)^* & \|\varphi_k\| 1_n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^m \|\varphi_k\| e_k \otimes 1_n & T_n(x) \\ T_n(x)^* & \sum_{k=1}^m \|\varphi_k\| e_k \otimes 1_n \end{bmatrix} \end{aligned}$$

Hence

$$\|T_n(x)\|_{q,n} \leq \left\| \sum_{k=1}^m \|ag_k b\|_1 e_k \right\|_q = \left(\sum_{k=1}^m \|ag_k b\|_1^q \right)^{\frac{1}{q}}$$

which proves (4.24). $\square$

Lemma 4.18 Let $2 \leq q < \infty$, $\alpha \in M_n$, and ε_{ii} be as in (3.2) for $i = 1, \dots, n$. Then

$$\sum_{i=1}^n \|\alpha \varepsilon_{ii}\|_q^q \leq \|\alpha\|_q^q.$$

Proof Let $\alpha^* \alpha = \beta = [\beta_{ij}]_{i,j=1}^n$. Let $|\alpha \varepsilon_{ii}|$ be the positive matrix of the polar decomposition of $\alpha \varepsilon_{ii}$. Then $|\alpha \varepsilon_{ii}|^2 = \varepsilon_{ii} \beta \varepsilon_{ii} = \beta_{ii} \varepsilon_{ii}$ which implies $|\alpha \varepsilon_{ii}|^q = \beta_{ii}^{q/2} \varepsilon_{ii}$ and

$$\sum_{i=1}^n \|\alpha \varepsilon_{ii}\|_q^q = \sum_{i=1}^n \text{Tr}(\beta_{ii}^{q/2} \varepsilon_{ii}) = \sum_{i=1}^n \beta_{ii}^{q/2}.$$

Since β is positive, it has eigenvalues $\lambda_1, \dots, \lambda_n \geq 0$, and there is a unitary matrix $u = [u_{ij}] \in M_n$ such that $\beta = u^* \text{diag}(\lambda_1, \dots, \lambda_n) u$. Especially, we have $\beta_{ii} = \sum_{l=1}^n |u_{li}|^2 \lambda_l$. From u being unitary, it follows that $\sum_{i=1}^n |u_{li}|^2 = 1$ for $l = 1, \dots, n$. Since $q \geq 2$, the function $\phi : [0, \infty) \rightarrow \mathbb{R}, t \mapsto t^{q/2}$ is convex. Hence we get

$$\begin{aligned} \sum_{i=1}^n \beta_{ii}^{q/2} &= \sum_{i=1}^n \phi \left(\sum_{l=1}^n \lambda_l |u_{li}|^2 \right) \leq \sum_{i=1}^n \sum_{l=1}^n |u_{li}|^2 \phi(\lambda_l) \\ &= \sum_{l=1}^n \lambda_l^{q/2} \sum_{i=1}^n |u_{li}|^2 = \sum_{l=1}^n \lambda_l^{q/2} = \text{Tr}(\beta^{q/2}) = \|\alpha\|_q^q. \end{aligned}$$

$\square$

For $n \in \mathbb{N}$ let ε_{ij} be as in (3.2) and e_i as in (4.23). Then we define the linear map

$$T : L_p(M_n) \rightarrow l_n^q, \quad T(x) = \sum_{i=1}^n \text{Tr}((\varepsilon_{1i} + \varepsilon_{i1})x) e_i. \quad (4.25)$$

Proposition 4.19 Let $1 \leq p < \infty$, $2p < q < \infty$, and T as in (4.25). Then $\|T\|_{cob} \leq 2$.

Proof Let $\varepsilon > 0$. By Lemma 4.17, there exist $a, b \in L_{2p}(M_n)$, $\|a\|_{2p}, \|b\|_{2p} \leq 1$ such that

$$\|T\|_{cob} \leq \left(\sum_{i=1}^n \|a(\varepsilon_{i1} + \varepsilon_{1i})b\|_1^q \right)^{1/q} + \varepsilon.$$

Since for all $i = 1, \dots, n$ we have $\|a(\varepsilon_{i1} + \varepsilon_{1i})b\|_1 \leq \|a\varepsilon_{i1}b\|_1 + \|a\varepsilon_{1i}b\|_1$ we get

$$\left(\sum_{i=1}^n \|a(\varepsilon_{i1} + \varepsilon_{1i})b\|_1^q \right)^{1/q} \leq \left(\sum_{i=1}^n \|a\varepsilon_{i1}b\|_1^q \right)^{1/q} + \left(\sum_{i=1}^n \|a\varepsilon_{1i}b\|_1^q \right)^{1/q}$$

Let $\frac{1}{q} + \frac{1}{q'} = 1$. Then $\|a\varepsilon_{i1}b\|_1 = \|a\varepsilon_{ii}\varepsilon_{i1}b\|_1 \leq \|a\varepsilon_{ii}\|_q \cdot \|\varepsilon_{i1}b\|_{q'}$. Since $q > 2p \geq 2$ we have $q' < 2 \leq 2p$. Hence there exists a real number $s > 1$ such that $\frac{1}{q'} = \frac{1}{2p} + \frac{1}{s}$. By the generalized Hölder's inequality, we have $\|\varepsilon_{i1}b\|_{q'} \leq \|\varepsilon_{i1}\|_s \|b\|_{2p} \leq 1$. We apply Lemma 4.18 and get

$$\sum_{i=1}^n \|a\varepsilon_{i1}b\|_1^q \leq \sum_{i=1}^n \|a\varepsilon_{ii}\|_1^q \leq \|a\|_q^q.$$

Since $q > 2p \geq 2$, we have $\|a\|_q \leq \|a\|_{2p} \leq 1$. Hence

$$\left(\sum_{i=1}^n \|a\varepsilon_{i1}b\|_1^q \right)^{1/q} \leq 1.$$

Similarly, we get

$$\left(\sum_{i=1}^n \|a\varepsilon_{1i}b\|_1^q \right)^{1/q} \leq 1.$$

Since ε was arbitrary, this finishes the proof. $\square$

Proposition 4.20 *Let $1 \leq p < \infty$, $1 \leq q < \infty$, and T as in (4.25). Then $\|T\|_{dec} \geq n^{1/2q}$.*

Proof Since T is self-adjoint, we can apply [1], Lemma 2.18 and 2.19, and get

$$\|T\|_{dec} = \inf\{\|S\| \mid S : L_p(M_n) \rightarrow l_q^n, S \pm T \text{ is completely positive}\}.$$

In [1], the case $p = q$ is considered only. But the proof also works for $p \neq q$. Let $S : L_p(M_n) \rightarrow l_q^n$ be a linear map such that $S \pm T$ is completely positive. There exist $b_1, \dots, b_n \in M_{n+}$ such that $S(f) = \sum_{k=1}^n \langle b_k, f \rangle e_k$ for all $f \in L_p(M_n)$. Since $S \pm T$ are positive, we have

$$\sum_{k=1}^n \langle b_k \pm (\varepsilon_{1k} + \varepsilon_{k1}), f \rangle e_k \geq 0 \text{ for all } f \in L_p(M_n)_+.$$

This means that $b_k \pm (\varepsilon_{1k} + \varepsilon_{k1}) \geq 0$ for $k = 1, \dots, n$. Let $b_k = [b_{k,ij}]$. For $k = 1$ we have $b_1 - 2\varepsilon_{11} \geq 0$ which implies $b_{1,11} \geq 2$. For $k > 1$ we have

$$\begin{aligned}
0 &\leq \begin{bmatrix} e_k' & 0 \\ 0 & e_1' \end{bmatrix} \begin{bmatrix} b_k & \varepsilon_{1k} + \varepsilon_{k1} \\ \varepsilon_{1k} + \varepsilon_{k1} & b_k \end{bmatrix} \begin{bmatrix} e_k & 0 \\ 0 & e_1 \end{bmatrix} \\
&= \begin{bmatrix} e_k' b_k e_k & e_k' \varepsilon_{k1} e_1 \\ e_1' \varepsilon_{1k} e_k & e_1' b_k e_1 \end{bmatrix} = \begin{bmatrix} b_{k,kk} & 1 \\ 1 & b_{k,11} \end{bmatrix}
\end{aligned} \tag{4.26}$$

We compute the determinant of the last matrix in (4.26) and get $b_{k,kk}b_{k,11} \geq 1$. Since the diagonal elements of b_k are positive, we get $b_{k,kk} \geq \frac{1}{b_{k,11}}$ for all $k \geq 2$. For $k = 1, \dots, n$ we have

$$\|S(\varepsilon_{kk})\|_q^q = \left\| \sum_{j=1}^n \langle b_j, \varepsilon_{kk} \rangle e_j \right\|_q^q = \left\| \sum_{j=1}^n b_{j,kk} e_j \right\|_q^q = \sum_{j=1}^n (b_{j,kk})^q.$$

For $k = 1$ we have $\|S(\varepsilon_{11})\|_q^q = \sum_{j=1}^n (b_{j,11})^q$. For $k > 1$ we have $\|S(\varepsilon_{kk})\|_q^q \geq (b_{k,kk})^q \geq \frac{1}{(b_{k,11})^q}$. Hence

$$\|S\|^q \geq \max \left\{ \sum_{j=1}^n (b_{j,11})^q, \frac{1}{(b_{k,11})^q}, k = 2, \dots, n \right\}.$$

If there is some $k, 2 \leq k \leq n$ such that $b_{k,11} \leq n^{-1/2q}$, then

$$\|S\|^q \geq \frac{1}{(b_{k,11})^q} \geq \sqrt{n}. \tag{4.27}$$

If $b_{k,11} \geq n^{-1/2q}$ for all $k \geq 2$, we have

$$\begin{aligned}
\|S\|^q &\geq \sum_{k=1}^n (b_{k,11})^q \geq 2^q + (n-1)n^{-1/2} \\
&= \frac{2^q \sqrt{n} + n - 1}{\sqrt{n}} \\
&\geq \sqrt{n}
\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28), we get

$$\|S\| \geq n^{\frac{1}{2q}}.$$

Since S was arbitrary with $S \pm T$ completely positive, this finishes the proof. $\square$

Now we can show our counterexample. Let $\mathcal{M} = \oplus_{k=1}^{\infty} M_k$. On $\mathcal{M}$ we have the semifinite, normal, faithful trace $\tau_1(x) = \sum_{k=1}^{\infty} \text{Tr}_k(x_k)$ where $x = \oplus_{k=1}^{\infty} x_k$ and Tr_k is the usual trace on M_k . For any projection $e \in \mathcal{M}$, we have $\tau_1(e) \geq 1$ and therefore every τ_1 -measurable operator affiliated with $\mathcal{M}$ is a bounded operator. Hence for all $1 \leq p < \infty$, we can write $L_p(\mathcal{M}) = \{x = \oplus_{k=1}^{\infty} x_k \mid \sum_{k=1}^{\infty} \|x_k\|_p^p < \infty\}$ with norm $\|x\|_p = (\sum_{k=1}^{\infty} \|x_k\|_p^p)^{1/p}$. Let $\mathcal{N} = \oplus_{k=1}^{\infty} l_{\infty}^k$. Then $L_q(\mathcal{N}) = \{f = \oplus_{k=1}^{\infty} f_k \mid$

$\sum_{k=1}^{\infty} \|f_k\|_q^q < \infty\}$ with norm $\|f\|_q = (\sum_{k=1}^{\infty} \|f_k\|_q^q)^{1/q}$. For $1 \leq p, q < \infty, q > 2p$ let

$$T : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N}), \quad T(\oplus_{k=1}^{\infty} x_k) = \oplus_{k=1}^{\infty} T_k(x_k)$$

where T_k is defined as in (4.25). We claim that T is completely order bounded and $\|T\|_{cob} \leq 2$. To show this, let $n \in \mathbb{N}$ and $x \in L_p(\mathcal{M} \otimes M_n)$, $\|x\|_{p,n} \leq 1$. According to Theorem 3.4, there exist $f, g \in L_p(\mathcal{M})_+$ such that

$$\|f\|_p = \|g\|_p = \|x\|_{p,n} \text{ and } \begin{bmatrix} f \otimes 1_n & x \\ x^* & g \otimes 1_n \end{bmatrix} \geq 0.$$

Then we have $x = \oplus_{k=1}^{\infty} x_k, x_k \in L_p(M_k) \otimes M_n, f = \oplus_{k=1}^{\infty} f_k, f_k \in L_p(M_k)_+$ and $g = \oplus_{k=1}^{\infty} g_k, g_k \in L_p(M_k)_+$ with

$$\begin{bmatrix} f_k \otimes 1_n & x_k \\ x_k^* & g_k \otimes 1_n \end{bmatrix} \geq 0 \text{ for all } k \in \mathbb{N}.$$

By Proposition 4.19, we have $\|T_k\|_{cob} \leq 2$ for all $k \in \mathbb{N}$. By definition of completely order boundedness, for all $k \in \mathbb{N}$, there exist $h_{1,k}, h_{2,k} \in (\ell_k^q)_+$, such that

$$\begin{bmatrix} h_{1,k} \otimes 1_n & T_{k,n}(x_k) \\ T_{k,n}(x_k)^* & h_{2,k} \otimes 1_n \end{bmatrix} \geq 0 \text{ and } \|h_{1,k}\|_q, \|h_{2,k}\|_q \leq \frac{1}{2} \|T_k\|_{cob} (\|f_k\|_p + \|g_k\|_p).$$

We put $h_1 = \oplus_{k=1}^{\infty} h_{1,k}$ and $h_2 = \oplus_{k=1}^{\infty} h_{2,k}$. Then $\|h_1\|_q, \|h_2\|_q \leq 2$ and

$$\begin{bmatrix} h_1 \otimes 1_n & T_n(x) \\ T_n(x)^* & h_2 \otimes 1_n \end{bmatrix} \geq 0$$

which shows that $\|T\|_{cob} \leq 2$.

Next, suppose that T is decomposable. Then there exists a completely positive map $S : L_p(\mathcal{M}) \rightarrow L_q(\mathcal{N})$, such that $S \pm T$ are completely positive. For every $j \in \mathbb{N}$, the embedding $I_j : L_p(M_j) \rightarrow L_p(\mathcal{M}), x \mapsto (\dots, 0, x, 0, \dots)$ and the projection $P_j : L_q(\mathcal{N}) \rightarrow \ell_q^j, \oplus_{k=1}^{\infty} y_k \mapsto y_j$ are completely positive and have norm less than or equal to 1. We have $T_j = P_j \circ T \circ I_j$ and put $S_j = P_j \circ S \circ I_j$. Then $S_j \pm T_j$ are completely positive. We apply Proposition 4.20 and get

$$\infty > \|S\| \geq \|S_j\| \geq \|T_j\|_{dec} \geq j^{1/2q} \text{ for all } j \in \mathbb{N}$$

which gives a contradiction. Thus T is not decomposable.

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References

1. Arhancet, C., Kriegler, C.: Projections, multipliers and decomposable maps on noncommutative L^p -spaces, [arXiv:1707.05591v14](https://arxiv.org/abs/1707.05591v14) (2018)
2. Blackadar, B.: Operator Algebras: Theory of C^* -Algebras and von Neumann Algebras. Springer, New York (2006)
3. Effros, E., Ruan, Z.-J.: Operator Spaces. Oxford University Press, Oxford (2000)
4. Haagerup, U.: Decomposition of completely bounded maps on operator algebras, Unpublished manuscript (1980)
5. Haagerup, U.: Injectivity and decomposition of completely bounded maps, pp. 170–222, In: Operator Algebras and their connection with Topology and Ergodic Theory. Lecture Notes in Mathematics 1132, Springer (1985)
6. Junge, M., Ruan, Z.-J.: Decomposable maps on non-commutative L_p -spaces, 355–381, In: Operator algebras, quantization, and noncommutative geometry. Contemporary Mathematics 365, Journal of the American Mathematical Society, Providence, RI (2004)
7. Junge, M., Ruan, Z.-J., Xu, Q.: Rigid $\mathcal{OL}_\infty$ structures of non-commutative L_p -spaces associated with hyperfinite von Neumann algebras. Math. Scand. **95**, 63–95 (2005)
8. Paulsen, V.: Completely bounded maps On C^* -algebras and invariant operator ranges. Proc. Am. Math. Soc. **86**, 91–96 (1982)
9. Pisier, G.: Regular operators between non-commutative L_p -spaces. Bull. Sci. Math. **119**, 95–118 (1995)
10. Pisier, G.: Introduction to operator space theory. Cambridge University Press, Cambridge (2003)
11. Schmitt, L.M.: The Radon–Nikodym theorem for L^p -spaces of W^* -algebras. Publ. RIMS Kyoto Univ. **22**, 1025–1034 (1986)
12. Takesaki, M.: Theory of operator algebras I and II. In: Encyclopaedia of Mathematical Sciences, vol. 124 and 125. Springer (2002 and 2003)
13. Terp, M.: L^p spaces associated with von Neumann algebras. Copenhagen University, Copenhagen (1981)
14. Wittstock, G.: Ein operatorwertiger Hahn–Banach Satz. J. Funct. Anal. **40**, 127–150 (1981)