Lebesgue points of ℓ_1 -Cesàro summability of d -dimensional Fourier series

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Abstract

We generalize the classical Lebesgue's theorem and prove that the ℓ_1 -Cesàro means of the Fourier series of the multi-dimensional function $f \in L_1(\mathbb{T}^d)$ converge to f at each strong ω -Lebesgue point.

Keywords Cesàro summability · Fejér summability · ℓ_1 -summability · Hardy–Littlewood maximal function · Lebesgue points

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1 Introduction

In 1905, Lebesgue [13] proved in his famous result that the so called Fejér means

$$\sigma_n f(x) := \sum_{k=-n}^n \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{ikx}$$

of the trigonometric Fourier series of a one-dimensional integrable function f converge to $f(x)$ at each Lebesgue point, thus almost everywhere. Here $\widehat{f}(k)$ denotes the k th Fourier coefficient. Some years later Riesz [17] generalized this theorem for the Cesàro means of one-dimensional integrable functions. The Cesàro summability was investigated in several papers (see e.g. Gát [5–7], Goginava [8–10], Simon [18, 19], Nagy et al. [15, 16], Weisz [22, 23] and Zygmund [27]).

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In this paper, we will generalize this result to the Cesàro summability of multi-dimensional functions. More exactly, we investigate the ℓ_1 -Fejér means

$$\sigma_n f(x) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{ik \cdot x}$$

and their generalization, the ℓ_1 -Cesàro means of d -dimensional functions, where $|k| = |k_1| + \dots + |k_d|$. These means were investigated e.g. in Berens et al. [1, 2, 14, 26], Szili and Vértési [20]. Recently, using Hardy spaces, we [21, 22] proved that $\sigma_n f \rightarrow f$ almost everywhere if $f \in L_1(\mathbb{T}^d)$.

Here we partly characterize the set of this convergence. We generalize the Lebesgue points and introduce a new type Lebesgue points, the so called strong ω -Lebesgue points. In [24], we verified that almost every point is a strong ω -Lebesgue point of $f \in L_1(\mathbb{T}^d)$. Our main result reads as follows. If the Hardy–Littlewood maximal function $\mathcal{M}^\omega f(x)$ is finite and x is a strong ω -Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then

$$\lim_{n \rightarrow \infty} \sigma_n^\omega f(x) = f(x),$$

where $\sigma_n^\omega f$ denotes the n th ℓ_1 -Cesàro mean of the Fourier series of f . A similar result was shown for the ℓ_1 - θ -means of Fourier transforms in [25]. However, the θ -summability does not contain the Cesàro summability and in this paper we consider Fourier series. So in the present proof, we have to use new ideas and the parts that are similar to [25] are omitted.

2 Hardy–Littlewood maximal function and strong ω -Lebesgue points

Let us fix $d \geq 3$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \dots \times \mathbb{Y}$ taken with itself d times. We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p d\lambda \right)^{1/p} \quad (1 \leq p < \infty),$$

with the usual modification for $p = \infty$, where λ is the Lebesgue measure. We identify the torus $\mathbb{T}$ with $[-\pi, \pi]$.

By a diagonal, we understand any diagonal of the two-dimensional faces of the cube $[0, \pi]^d$. Let us denote by $P_{2^{i_1}h, \dots, 2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose k th side length is $2^{i_k+1}h$ if the k th side is parallel to an axis and $\sqrt{2}2^{i_k+1}h$ if the k th side is parallel to a diagonal ($i \in \mathbb{N}^d, h > 0, k = 1, \dots, d$). In [24], we introduced the next Hardy–Littlewood maximal function. For some $\omega > 0$ and $f \in L_1(\mathbb{T}^d)$, let

$$\mathcal{M}^\omega f(x) := \sup_{P_{2^{i_1}h, \dots, 2^{i_d}h}, i \in \mathbb{N}^d, h > 0} 2^{-\omega|i|} \times \frac{1}{|P_{2^{i_1}h, \dots, 2^{i_d}h}|} \int_{P_{2^{i_1}h, \dots, 2^{i_d}h}} |f(x-t)| dt,$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h, \dots, 2^{i_d}h}$ ($i \in \mathbb{N}^d, h > 0$) just defined. Taking the supremum over all parallelepipeds whose sides are parallel to the axes and $\omega = 0$, we obtain the strong Hardy–Littlewood maximal function, and, if in addition $i_1 = \dots = i_d$, the usual Hardy–Littlewood maximal function (for more about these maximal functions see e.g. Feichtinger and Weisz [4] and the references therein). We have proved in [24] that

$$\sup_{\rho > 0} \rho \lambda(\mathcal{M}^\omega f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbb{T}^d)) \quad (2.1)$$

and, for $1 < p \leq \infty$,

$$\|\mathcal{M}^\omega f\|_p \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T}^d)).$$

In this paper the constants C and C_p may vary from line to line.

Based on the Hardy–Littlewood maximal function $\mathcal{M}^\omega$, we introduced the following type of Lebesgue points in [25]. Let

$$U_r^\omega f(x) := \sup_{\substack{P_{2^{i_1}h, \dots, 2^{i_d}h}, i \in \mathbb{N}^d, h > 0 \\ 2^{i_k}h < r, k = 1, \dots, d}} 2^{-\omega|i|} \times \frac{1}{|P_{2^{i_1}h, \dots, 2^{i_d}h}|} \int_{P_{2^{i_1}h, \dots, 2^{i_d}h}} |f(x-t) - f(x)| dt,$$

where the supremum is taken over all parallelepipeds mentioned above. For $\omega > 0$, a point $x \in \mathbb{T}^d$ is called a strong ω -Lebesgue point of $f \in L_1(\mathbb{T}^d)$ if

$$\lim_{r \rightarrow 0} U_r^\omega f(x) = 0.$$

Taking the supremum in $U_r^\omega f$ over all parallelepipeds whose sides are parallel to the axes and $\omega = 0$, we obtain the strong Lebesgue points. Moreover, if in addition $i_1 = \dots = i_d$, then we get the usual Lebesgue points, i.e.,

$$\lim_{h \rightarrow 0} \frac{1}{(2h)^d} \int_{-h}^h \dots \int_{-h}^h |f(x-t) - f(x)| dt = 0.$$

Note that every strong ω_2 -Lebesgue point is a strong ω_1 -Lebesgue point ($0 < \omega_2 < \omega_1 < \infty$), because of $U_r^{\omega_1} f \leq U_r^{\omega_2} f$. If f is continuous at x , then x is a strong ω -Lebesgue point of f . The next theorem was proved in [25].

Theorem 1 For $\omega > 0$, almost every point $x \in \mathbb{T}^d$ is a strong ω -Lebesgue point of $f \in L_1(\mathbb{T}^d)$.

3 The kernel functions

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ set $u \cdot x := \sum_{k=1}^d u_k x_k$ and $|x| := \sum_{k=1}^d |x_k|$. The k th Fourier coefficient of a d -dimensional integrable function $f \in L_1(\mathbb{T}^d)$ is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ik \cdot x} dx \quad (k \in \mathbb{Z}^d).$$

For $f \in L_1(\mathbb{T}^d)$ and $n \in \mathbb{N}$, the n th ℓ_1 -partial sum $s_n f$ of the Fourier series of f and the n th ℓ_1 -Dirichlet kernel D_n are given by

$$s_n f(x) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} \widehat{f}(k) e^{ik \cdot x}$$

and

$$D_n(t) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} e^{ik \cdot t},$$

respectively. It is known (see e.g. Grafakos [11] or Weisz [23]) that for $f \in L_p(\mathbb{T}^d)$, $1 < p < \infty$,

$$\lim_{n \rightarrow \infty} s_n f = f \quad \text{in the } L_p(\mathbb{T}^d)\text{-norm and a.e.}$$

Since this convergence does not hold for $p = 1$, we consider the Cesàro summation.

For $\alpha \neq -1, -2, \dots$ and $n \in \mathbb{N}$, let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2) \cdots (\alpha+n)}{n!}.$$

Then $A_0^\alpha = 1$, $A_n^0 = 1$ and $A_n^1 = n+1$ ($n \in \mathbb{N}$). Let $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}$ and $\alpha \geq 0$. The n th ℓ_1 -Cesàro means $\sigma_n^\alpha f$ of the Fourier series of f are introduced by

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d, |k| \leq n} A_{n-1-|k|}^\alpha \widehat{f}(k) e^{ik \cdot x}.$$

If $\alpha = 0$, we get $s_n f$, if $\alpha = 1$, then the ℓ_1 -Fejér means

$$\sigma_n^1 f(x) := \sum_{k \in \mathbb{Z}^d, |k| \leq n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{ik \cdot x}.$$

It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\alpha(t) dt,$$

where the ℓ_1 -Cesàro kernel is given by

$$K_n^\alpha(t) := \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d, |k| \leq n} A_{n-1-|k|}^\alpha e^{ik \cdot t} = \frac{1}{A_{n-1}^\alpha} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} D_j(t). \quad (3.1)$$

It is clear that

$$|D_n(t)| \leq Cn^d, \quad |K_n^\alpha(t)| \leq Cn^d \quad (t \in \mathbb{T}^d). \quad (3.2)$$

The next two lemmas can be found in Zygmund [27].

Lemma 1 For $0 < \alpha \leq 1$, $n \geq 1$, $m \in \mathbb{N}$ and $t \in \mathbb{T}$, $t \neq 0$,

$$\left| \sum_{k=0}^{n-1} k^m A_{n-1-k}^{\alpha-1} \sin((k+1/2)t) \right| \leq \frac{Cn^m}{|\sin(t/2)|^\alpha} + \frac{Cn^{\alpha+m-1}}{|\sin(t/2)|}$$

and

$$\left| \sum_{k=0}^{n-1} k^m A_{n-1-k}^{\alpha-1} \cos((k+1/2)t) \right| \leq \frac{Cn^m}{|\sin(t/2)|^\alpha} + \frac{Cn^{\alpha+m-1}}{|\sin(t/2)|}.$$

Lemma 2 For $\alpha > -1$ and $h > 0$, we have

$$\sigma_n^{\alpha+h} f = \frac{1}{A_{n-1}^{\alpha+h}} \sum_{k=1}^n A_{n-k}^{h-1} A_{k-1}^\alpha \sigma_k^\alpha f.$$

Lemma 3 was proved in Weisz [21, 22].

Lemma 3 If $0 < \alpha \leq 1$, then

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}^d} |K_n^\alpha| d\lambda \leq C.$$

We define the n th divided difference of a function f at the pairwise distinct knots $x_1, \dots, x_n \in \mathbb{R}$ by

$$[x_1]f := f(x_1), \quad [x_1, \dots, x_n]f := \frac{[x_1, \dots, x_{n-1}]f - [x_2, \dots, x_n]f}{x_1 - x_n}.$$

It is known (see e.g. DeVore and Lorentz [3, p. 120]) that if f is $(n-1)$ -times continuously differentiable on $[a, b]$ and $x_1, \dots, x_n \in [a, b]$, then there exists $\xi \in [a, b]$ such that

$$[x_1, \dots, x_n]f = \frac{f^{(n-1)}(\xi)}{(n-1)!}. \quad (3.3)$$

Herriot [12] and Berens and Xu [1, 26] proved that the ℓ_1 -Dirichlet kernels can be expressed as

$$D_n(x) = [\cos x_1, \dots, \cos x_d]G_n \quad (x \in \mathbb{T}^d), \quad (3.4)$$

where

$$G_n(\cos x) := (-1)^{[(d-1)/2]} 2 \cos(x/2) (\sin x)^{d-2} \text{soc}((n+1/2)x)$$

and

$$\text{soc } x := \begin{cases} \cos x, & \text{if } d \text{ is even;} \\ \sin x, & \text{if } d \text{ is odd.} \end{cases}$$

A sequence $(i_l, j_l)_l = (i_l, j_l)_{l=1}^{d-1}$ is called a proper index sequence, if

- $i_1 = 1, j_1 = d$,
- for any $l = 1, \dots, d-2$ we have that either $i_{l+1} = i_l$ and $j_{l+1} = j_l - 1$ or $i_{l+1} = i_l + 1$ and $j_{l+1} = j_l$.

Let us denote by $\mathcal{I}(1, \dots, d)$ the set of all proper index sequences. Then $(i_l)_l$ is non-decreasing and $(j_l)_l$ is non-increasing. Moreover, $i_l < j_l$ for all $l = 1, \dots, d-1$ and $j_l - i_l = d - l$ ($l = 1, \dots, d-1$). For $(i_l, j_l)_{l=1}^{d-1} \in \mathcal{I}(1, \dots, d)$, we say that the first k term of the sequence, i.e. $(i_l, j_l)_{l=1}^k$ is in $\mathcal{I}^{(k)}(1, \dots, d)$. Then $\mathcal{I}^{(d-1)}(1, \dots, d) = \mathcal{I}(1, \dots, d)$.

Using (3.4), we can easily prove the next lemma by induction (see also [21, 25]). For $k = 0$ the equation is the same as (3.4).

Lemma 4 For $k = 0, 1, \dots, d-2$, we have

$$\begin{aligned} D_n(x) = & \sum_{(i_l, j_l)_{l=1}^{k+1} \in \mathcal{I}^{(k+1)}(1, \dots, d)} (-1)^{i_{k+1}-1} \\ & \times \left(\prod_{l=1}^k (\cos x_{i_l} - \cos x_{j_l})^{-1} \right) [\cos x_{i_{k+1}}, \dots, \cos x_{j_{k+1}}] G_n. \end{aligned}$$

We estimate the Cesàro kernel K_n^α as follows.

Lemma 5 If $0 < \alpha \leq 1$, $\pi > x_1 > x_2 > \dots > x_d > 0$, $1 < n_1 < \dots < n_m < d$ and $m = 0, \dots, d-2$, then

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^m \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} x_{i_l}^{-1} \\
&\quad \times \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + Cn^m \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} (\pi - x_{j_l})^{-1} \\
&\quad \times \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-1}} > \pi/2\}}.
\end{aligned} \tag{3.5}$$

Proof In this proof we use the notation $D_n^d(x) := D_n(x)$, where d denotes the dimension. We apply Lemma 4 with $k = d - 2$ to get that

$$\begin{aligned}
|D_n^d(x)| &\leq \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \left| \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\
&\quad \times |G_n(\cos x_{i_{d-1}}) - G_n(\cos x_{j_{d-1}})| \\
&\leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \left| \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\
&\quad \times (\sin^{d-2} x_{i_{d-1}} + \sin^{d-2} x_{j_{d-1}}).
\end{aligned}$$

We use the trigonometric identity

$$\cos a - \cos b = -2 \sin((a - b)/2) \sin((a + b)/2) \tag{3.6}$$

and the estimations

$$\sin(a \pm b)/2 \sim a \pm b \quad \text{if} \quad 0 \leq b \leq \pi/2, b < a \leq \pi \tag{3.7}$$

and

$$\sin(a - b)/2 \sim a - b, \quad \sin(a + b)/2 \sim 2\pi - a - b \tag{3.8}$$

if $0 \leq b < a \leq \pi, \pi/2 < a$. Then

$$\begin{aligned}
|D_n^d(x)| &\leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=1}^{d-1} (x_{i_l} + x_{j_l})^{-1} \right) \\
&\quad \times (\sin^{d-2} x_{i_{d-1}} + \sin^{d-2} x_{j_{d-1}}) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=1}^{d-1} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times (\sin^{d-2} x_{i_{d-1}} + \sin^{d-2} x_{j_{d-1}}) 1_{\{x_{j_{d-1}} > \pi/2\}}.
\end{aligned}$$

Since $x_{i_1} + x_{j_1} \geq x_{i_1}$, $2\pi - x_{i_1} - x_{j_1} > \pi - x_{j_1}$, $x_{i_l} + x_{j_l} \geq x_{i_{d-1}}$ and $2\pi - x_{i_l} - x_{j_l} > \pi - x_{j_{d-1}}$ for $l = 2, \dots, d-1$, we have

$$\begin{aligned}
|D_n^d(x)| &\leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left(\prod_{l=2}^{d-1} (x_{i_l} + x_{j_l})^{-1} \right) x_{i_{d-1}}^{d-2} 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} (\pi - x_{j_1})^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left(\prod_{l=2}^{d-1} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) (\pi - x_{j_{d-1}})^{d-2} 1_{\{x_{j_{d-1}} > \pi/2\}} \\
&\leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} (\pi - x_{j_1})^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} > \pi/2\}}.
\end{aligned} \tag{3.9}$$

Then (3.5) with $m = 0$ follows from (3.1).

We may suppose that $n_1 = 2$. Observe that

$$\begin{aligned}
D_n^d(x) &= \sum_{k \in \mathbb{Z}^d, |k| \leq n} e^{ik \cdot x} \\
&= \sum_{k_2=-n}^n e^{ik_2 x_2} \sum_{k_1=-n+|k_2|}^{n-|k_2|} \\
&\quad \times \sum_{k_3=-n+|k_1|+|k_2|}^{n-|k_1|-|k_2|} \cdots \sum_{k_d=-n+|k_1|+\dots+|k_{d-1}|}^{n-|k_1|-\dots-|k_{d-1}|} e^{i(k_1 x_1 + k_3 x_3 + \dots + k_d x_d)} \\
&= \sum_{k_2=-n}^n e^{ik_2 x_2} D_{n-|k_2|}^{d-1}(x_1, x_3, \dots, x_d).
\end{aligned}$$

We apply (3.9) for the $(d-1)$ -dimensional Dirichlet kernel $D_{n-|k_2|}^{d-1}(x_1, x_3, \dots, x_d)$ and get that

$$\begin{aligned}
|D_n^d(x)| &\leq Cn \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,3,\dots,d)}} x_{i_1}^{-1} \left(\prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + Cn \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,3,\dots,d)}} (\pi - x_{j_1})^{-1} \left(\prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} > \pi/2\}}.
\end{aligned}$$

Hence (3.1) implies (3.5) with $m = 1$. The proof can be finished in the same way. $\square$

Lemma 6 Suppose that $0 < \alpha \leq 1$ and $\pi - 1/n > x_1 > x_2 > \dots > x_d > 1/n$. Then

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,\dots,d)}} x_{i_1}^{-1} x_{j_1}^{-\alpha} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,\dots,d)}} (\pi - x_{j_1})^{-1} (\pi - x_{i_1}^{-1})^{-\alpha} \\
&\quad \times \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-1}} > \pi/2\}}.
\end{aligned} \tag{3.10}$$

If $m = 1, \dots, d-2$, then

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^{m-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,\dots,d)}} x_{i_1}^{-1} x_{j_1}^{-\alpha} \\
&\quad \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-m}} \leq \pi/2\}} \\
&\quad + Cn^{m-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1,\dots,d)}} (\pi - x_{j_1})^{-1} (\pi - x_{i_1}^{-1})^{-\alpha} \\
&\quad \times \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} 1_{\{x_{j_{d-m}} > \pi/2\}}.
\end{aligned} \tag{3.11}$$

Moreover,

$$|K_n^\alpha(x)| \leq Cn^{d-1-\alpha}x_{j_1}^{-1-\alpha}1_{\{x_{j_1} \leq \pi/2\}} + Cn^{d-1-\alpha}(\pi - x_{i_1})^{-1-\alpha}1_{\{x_{i_1} > \pi/2\}}.$$

Proof We use Lemma 4 with $k = d - 2$ and (3.1) to obtain

$$\begin{aligned} |K_n^\alpha(x)| &= \frac{1}{A_{n-1}^\alpha} \left| \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} \sum_{(i,j_l) \in \mathcal{I}} (-1)^{i_{d-1}-1} \right. \\ &\quad \times \left(\prod_{l=1}^{d-2} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right) [\cos x_{i_{d-1}}, \cos x_{j_{d-1}}] G_j \left. \right| \\ &\leq \frac{1}{A_{n-1}^\alpha} \sum_{(i,j_l) \in \mathcal{I}} \left| \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\ &\quad \times \left| \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} (G_j(x_{i_{d-1}}) - G_j(\cos x_{j_{d-1}})) \right|. \end{aligned}$$

Taking into account Lemma 1 and the definition of G_n ,

$$\begin{aligned} |K_n^\alpha(x)| &\leq \frac{C}{A_{n-1}^\alpha} \sum_{(i,j_l) \in \mathcal{I}} \left| \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\ &\quad \times \left(\sin^{d-2}(x_{i_{d-1}}) \sin^{-\alpha}(x_{i_{d-1}}/2) \right. \\ &\quad + Cn^{\alpha-1} \sin^{d-2}(x_{i_{d-1}}) \sin^{-1}(x_{i_{d-1}}/2) \\ &\quad + \sin^{d-2}(x_{j_{d-1}}) \sin^{-\alpha}(x_{j_{d-1}}/2) \\ &\quad \left. + Cn^{\alpha-1} \sin^{d-2}(x_{j_{d-1}}) \sin^{-1}(x_{j_{d-1}}/2) \right). \end{aligned}$$

Now we use again (3.6), (3.7) and (3.8) and conclude

$$\begin{aligned} |K_n^\alpha(x)| &\leq Cn^{-\alpha} \sum_{(i,j_l) \in \mathcal{I}} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=1}^{d-1} (x_{i_l} + x_{j_l})^{-1} \right) \\ &\quad \times \left(x_{i_{d-1}}^{d-2-\alpha} + Cn^{\alpha-1} x_{i_{d-1}}^{d-3} + x_{j_{d-1}}^{d-2-\alpha} + Cn^{\alpha-1} x_{j_{d-1}}^{d-3} \right) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\ &\quad + Cn^{-\alpha} \sum_{(i,j_l) \in \mathcal{I}} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=1}^{d-1} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) \\ &\quad \times \left((\pi - x_{i_{d-1}})^{d-2-\alpha} + Cn^{\alpha-1} (\pi - x_{i_{d-1}})^{d-3} \right. \\ &\quad \left. + (\pi - x_{j_{d-1}})^{d-2-\alpha} + Cn^{\alpha-1} (\pi - x_{j_{d-1}})^{d-3} \right) 1_{\{x_{j_{d-1}} > \pi/2\}}. \end{aligned}$$

Since $\pi - 1/n > x_1 > x_2 > \dots > x_d > 1/n$, we have

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^{-\alpha} \sum_{(i_l, j_l) \in \mathcal{I}} x_{i_l}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=2}^{d-1} (x_{i_l} + x_{j_l})^{-1} \right) \\
&\quad \times \left(x_{i_{d-1}}^{d-2-\alpha} + x_{j_{d-1}}^{d-2-\alpha} \right) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + Cn^{-\alpha} \sum_{(i_l, j_l) \in \mathcal{I}} (\pi - x_{j_l})^{-1} \\
&\quad \times \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=2}^{d-1} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left((\pi - x_{i_{d-1}})^{d-2-\alpha} + (\pi - x_{j_{d-1}})^{d-2-\alpha} \right) 1_{\{x_{j_{d-1}} > \pi/2\}} \\
&\leq Cn^{-\alpha} \sum_{(i_l, j_l) \in \mathcal{I}} x_{i_l}^{-1} x_{j_l}^{-\alpha} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} \leq \pi/2\}} \\
&\quad + Cn^{-\alpha} \sum_{(i_l, j_l) \in \mathcal{I}} (\pi - x_{j_l})^{-1} (\pi - x_{i_l})^{-\alpha} \\
&\quad \times \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{j_{d-1}} > \pi/2\}},
\end{aligned}$$

which shows (3.10).

After some basic calculation, we can see that

$$\begin{aligned}
G_n^{(m)}(y) &= C \sum_{r_1=0}^m \sum_{k_1=1}^{r_1} \sum_{l_1=(r_1-k_1)/2}^{r_1-k_1} \cos^{(k_1)} \left(\frac{\arccos y}{2} \right) \\
&\quad \times (1-y^2)^{-k_1/2-l_1} y^{l_1-(r_1-k_1-l_1)} \\
&\quad \times \sum_{r_2=0}^{m-r_1} \sum_{k_2=1}^{\min(r_2, d-2)} \sum_{l_2=\max(r_2/2-k_2, 0)}^{r_2-k_2} \sin^{d-2-k_2}(\arccos y) \\
&\quad \times (1-y^2)^{-k_2/2-l_2} y^{k_2+l_2-(r_2-k_2-l_2)} \\
&\quad \times \sum_{k_3=1}^{r_3} \sum_{l_3=(r_3-k_3)/2}^{r_3-k_3} (n+1/2)^{k_3} \operatorname{soc}^{(k_3)}((n+1/2) \arccos y) \\
&\quad \times (1-y^2)^{-k_3/2-l_3} y^{l_3-(r_3-k_3-l_3)},
\end{aligned}$$

where $r_1 + r_2 + r_3 = m$ and

$$G_n(y) := (-1)^{[(d-1)/2]} 2 \cos \left(\frac{\arccos y}{2} \right) \sin^{d-2}(\arccos y) \operatorname{soc}((n+1/2) \arccos y).$$

This implies for $y = \cos \xi$ that

$$\begin{aligned}
& G_n^{(m)}(\cos \xi) \\
&= C \sum_{r_1=0}^m \sum_{k_1=1}^{r_1} \sum_{l_1=(r_1-k_1)/2}^{r_1-k_1} \sum_{r_2=0}^{m-r_1} \sum_{k_2=1}^{\min(r_2, d-2)} \sum_{l_2=\max(r_2/2-k_2, 0)}^{r_2-k_2} \sum_{k_3=1}^{r_3} \sum_{l_3=(r_3-k_3)/2}^{r_3-k_3} \\
&\quad \times \cos^{(k_1)}(\xi/2) \cos^\beta(\xi) \sin^{d-2-m}(\xi) \sin^{-\beta}(\xi) (n+1/2)^{k_3} \text{soc}^{(k_3)}((n+1/2)\xi),
\end{aligned}$$

where

$$\begin{aligned}
\beta &= k_1 + 2l_1 + 2k_2 + 2l_2 + k_3 + 2l_3 - r_1 - r_2 - r_3 \\
&= k_1 + 2l_1 + 2k_2 + 2l_2 + k_3 + 2l_3 - m
\end{aligned}$$

and

$$0 \leq \beta \leq m - k_3.$$

Using this formula as well as (3.1) and Lemma 4 for $k = d - 1 - m$ ($m = 1, \dots, d - 1$), we conclude

$$\begin{aligned}
|K_n^\alpha(x)| &= \frac{1}{A_{n-1}^\alpha} \left| \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} \sum_{(i_l j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} (-1)^{i_{d-m-1}} \right. \\
&\quad \times \left(\prod_{l=1}^{d-1-m} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right) [\cos x_{i_{d-m}}, \dots, \cos x_{j_{d-m}}] G_j \Big| \\
&= \frac{1}{A_{n-1}^\alpha} \left| \sum_{(i_l j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} (-1)^{i_{d-m-1}} \left(\prod_{l=1}^{d-1-m} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right) \right. \\
&\quad \times [\cos x_{i_{d-m}}, \dots, \cos x_{j_{d-m}}] \left(\sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} G_j \right) \Big|.
\end{aligned}$$

Taking into account (3.3), we deduce

$$\begin{aligned}
|K_n^\alpha(x)| &\leq \frac{1}{A_{n-1}^\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} \left| \prod_{l=1}^{d-1-m} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\
&\quad \times \left| \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} \frac{G_j^{(m)}(\cos \xi)}{m!} \right| \\
&= \frac{C}{A_{n-1}^\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} \left| \prod_{l=1}^{d-1-m} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\
&\quad \times \left| \sum \cos^{(k_1)}(\xi/2) \cos^\beta(\xi) \sin^{d-2-m}(\xi) \sin^{-\beta}(\xi) \right. \\
&\quad \times \left. \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} (j+1/2)^{k_3} \operatorname{soc}^{(k_3)}((j+1/2)\xi) \right|,
\end{aligned}$$

where $x_{j_{d-m}}^2 \leq \xi = \xi_{(i_l, j_l)_l} \leq x_{i_{d-m}}^2$. By Lemma 1, (3.6), (3.7) and (3.8),

$$\begin{aligned}
|K_n^\alpha(x)| &\leq \frac{C}{A_{n-1}^\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} \left| \prod_{l=1}^{d-1-m} (\cos x_{i_l} - \cos x_{j_l})^{-1} \right| \\
&\quad \times \sin^{d-2-m}(\xi) \sin^{-\beta}(\xi) \left(n^{k_3} \sin(\xi/2)^{-\alpha} + n^{\alpha+k_3-1} \sin(\xi/2)^{-1} \right) \\
&\leq C n^{-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left(\prod_{l=1}^{d-1-m} (x_{i_l} + x_{j_l})^{-1} \right) \xi^{d-2-m-\beta} \\
&\quad \times \left(n^{k_3} \xi^{-\alpha} + n^{\alpha+k_3-1} \xi^{-1} \right) 1_{\{\xi \leq \pi/2\}} \\
&+ C n^{-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} \\
&\quad \times \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \left(\prod_{l=1}^{d-1-m} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times (\pi - \xi)^{d-2-m-\beta} \left(n^{k_3} (\pi - \xi)^{-\alpha} + n^{\alpha+k_3-1} (\pi - \xi)^{-1} \right) 1_{\{\xi > \pi/2\}}.
\end{aligned}$$

Since $1/n < \xi < \pi - 1/n$,

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} x_{i_l}^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left(\prod_{l=1}^{d-2-m} (x_{i_l} + x_{j_l})^{-1} \right) n^{k_3} \zeta^{d-2-m-\beta-\alpha} 1_{\{x_{i_{d-m}} \leq \pi/2\}} \\
&\quad + Cn^{-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} (\pi - x_{j_l})^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \left(\prod_{l=1}^{d-2-m} (2\pi - x_{i_l} - x_{j_l})^{-1} \right) n^{k_3} (\pi - \zeta)^{d-2-m-\beta-\alpha} 1_{\{x_{i_{d-m}} > \pi/2\}}
\end{aligned}$$

and

$$\begin{aligned}
|K_n^\alpha(x)| &\leq C \sum n^{k_3-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} x_{i_l}^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times \zeta^{-\beta-\alpha} 1_{\{x_{i_{d-m}} \leq \pi/2\}} \\
&\quad + C \sum n^{k_3-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} (\pi - x_{j_l})^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \\
&\quad \times (\pi - \zeta)^{-\beta-\alpha} 1_{\{x_{i_{d-m}} > \pi/2\}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|K_n^\alpha(x)| &\leq Cn^{m-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} x_{i_l}^{-1} x_{j_l}^{-\alpha} \\
&\quad \times \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{i_{d-m}} \leq \pi/2\}} \\
&\quad + Cn^{m-\alpha} \sum_{(i_l, j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1, \dots, d)} (\pi - x_{j_l})^{-1} (\pi - x_{i_l})^{-\alpha} \\
&\quad \times \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) 1_{\{x_{i_{d-m}} > \pi/2\}}.
\end{aligned}$$

which proves (3.11). $\square$

4 Convergence at strong ω -Lebesgue points

Now we are ready to prove our main theorem.

Theorem 2 Suppose that $0 < \alpha < \infty$, $0 < \omega < \min(\alpha, 1)/d$ and $\mathcal{M}^\omega f(x)$ is finite. If $f \in L_1(\mathbb{T}^d)$ is periodic with respect to π and x is a strong ω -Lebesgue point of f , then

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f(x) = f(x).$$

Proof By Lemma 2, it is enough to prove the theorem for $0 < \alpha \leq 1$. Since

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} K_n^\alpha(t) dt = 1,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leq \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} |f(x-t) - f(x)| |K_n^\alpha(t)| dt.$$

It is enough to integrate on the set

$$S := \{t \in \mathbb{T}^d : \pi > t_1 > \dots > t_d > 0\}.$$

We introduce the sets

$$A_0 := \{t : 8/n > t_1 > \dots > t_d > 0\} \cup \{t : \pi > t_1 > \dots > t_d > \pi - 8/n\},$$

$$A_1 := \{t : \pi > t_1 > \dots > t_d > 0, t_k - t_{k+1} > 2/n, k = 1, \dots, d-1\} \setminus A_0,$$

$$A_i := \{t : \pi > t_1 > \dots > t_d > 0, t_k - t_{k+i} > 2/n, k = 1, \dots, d-i$$

$$\text{and there exists } 1 \leq j \leq d-i+1 \text{ such that } t_j - t_{j+i-1} \leq 2/n\} \setminus A_0$$

for $i = 2, \dots, d-1$,

$$A_d := \{t : \pi > t_1 > \dots > t_d > 0, t_1 - t_d \leq 2/n\} \setminus A_0$$

and

$$B_1 := \{t : 0 < t_d \leq 1/n\}, \quad B_2 := \{t : \pi > t_1 \geq \pi - 1/n\}.$$

Since x is a strong ω -Lebesgue point of f , we can fix a number $r < \pi/2$ such that $U_r^\omega f(x) < \epsilon$. Let us introduce the cubes

$$S_{r/2} := \left[-\frac{r}{2}, \frac{r}{2}\right]^d, \quad T_{r/2} := \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right]^d$$

and suppose that $8/n < r/2$. Instead of S , we will integrate on the sets A_0 and

$$\bigcup_{j=1}^d \bigcup_{k_1=0}^1 \bigcup_{k_2=0}^1 \bigcup_{k_3=0}^1 \bigcup_{k_4=0}^1 (A_j \cap B_1^{k_1} \cap B_2^{k_2} \cap S_{r/2}^{k_3} \cap T_{r/2}^{k_4}),$$

where $H^0 := H$ and $H^1 := H^c$ denotes the complement of the set H .

Since $A_0 \subset S_{r/2} \cup T_{r/2}$, (3.2) and the periodicity of f imply

$$\begin{aligned}
& \int_{A_0} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^d \int_0^{8/n} \cdots \int_0^{8/n} |f(x-t) - f(x)| dt \\
& \quad + Cn^d \int_{\pi-8/n}^\pi \cdots \int_{\pi-8/n}^\pi |f(x-t) - f(x)| dt \\
& \leq CU_r^\omega f(x) < C\epsilon.
\end{aligned}$$

Note that $A_d \cap B_1 = A_d \cap B_2 = \emptyset$, $S_{r/2} \cap T_{r/2} = \emptyset$, $B_1 \cap T_{r/2} = \emptyset$ and $B_2 \cap S_{r/2} = \emptyset$. Let us start with the integral on $A_j \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}$. If $t \in T_{r/2}$, then each $t_k > \pi/2$, hence we have to use only the second summand of (3.5). Henceforth, for $j = 1, \dots, d-1$,

$$\begin{aligned}
& \int_{A_j \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq \int_{A_j \cap B_2 \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^m \int_{A_j \cap B_2 \cap T_{r/2}} |f(x-t) - f(x)| \\
& \quad \times \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} (\pi - t_{j_1})^{-1} \prod_{l=1}^{d-1-m} (t_{i_l} - t_{j_l})^{-1} dt.
\end{aligned} \tag{4.1}$$

For $j = 1$, we use this estimation with $m = 0$. On $B_2 \setminus A_0$, we have $\pi - t_d > 2(\pi - t_1)$ and so $t_1 - t_d > (\pi - t_d)/2$. This implies

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_2 \cap T_{r/2}} |f(x-t) - f(x)| (\pi - t_{j_1})^{-1} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt \\
& \leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_2 \cap T_{r/2}} |f(x-t) - f(x)| (\pi - t_{j_1})^{-2} \prod_{l=2}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt.
\end{aligned} \tag{4.2}$$

For a given proper index sequence $(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)$, we introduce a permutation $i'_1, \dots, i'_d$ of $1, \dots, d$ and then we integrate with respect to $t'_{i'_d}, t'_{i'_{d-1}}, \dots, t'_{i'_1}$, in this order. Let $i'_1 = j_1 = d$ and compute the integral

$$\sum_{k_1=3}^{r_0} \int_{\pi-2^{k_1+1}/n}^{\pi-2^{k_1}/n} \cdots (\pi - t_{j_1})^{-2} dt'_{i'_1},$$

where r_0 denotes the natural number i , for which $r/2 \leq 2^{i+1}/n < r$. Next let $i'_2 = i_1 = 1$ and compute the integral

$$\int_{\pi-1/n}^{\pi} \cdots dt_{i_2'}.$$

Let us continue with the next two cases. If $i_2 = i_1$ and $j_2 = j_1 - 1$, then let $i_3' = j_2$ and consider the integral

$$\sum_{k_2=0}^{k_1} \int_{t_{i_2}-2^{k_2+1}/n}^{t_{i_2}-2^{k_2}/n} \cdots (t_{i_2} - t_{j_2})^{-1} dt_{i_3'}.$$

If $i_2 = i_1 + 1$ and $j_2 = j_1$, then let $i_3' = i_2$ and consider the integral

$$\sum_{k_2=0}^{k_1} \int_{t_{j_2}+2^{k_2}/n}^{t_{j_2}+2^{k_2+1}/n} \cdots (t_{i_2} - t_{j_2})^{-1} dt_{i_3'}.$$

Continuing this process and substituting $t_j = u_j + \pi$, we get the integrals

$$\sum_{k_1=3}^{r_0} \int_{-2^{k_1+1}/n}^{-2^{k_1}/n} \cdots t_{j_1}^{-2} dt_{i_1'}, \quad \int_{-1/n}^0 \cdots dt_{i_2'}$$

and

$$\sum_{k_2=0}^{k_1} \int_{t_{i_2}-2^{k_2+1}/n}^{t_{i_2}-2^{k_2}/n} \cdots (t_{i_2} - t_{j_2})^{-1} dt_{i_3'}$$

or

$$\sum_{k_2=0}^{k_1} \int_{t_{j_2}+2^{k_2}/n}^{t_{j_2}+2^{k_2+1}/n} \cdots (t_{i_2} - t_{j_2})^{-1} dt_{i_3'}.$$

Consequently, we integrate over a parallelepiped $P_{2^{k_1}/n, \dots, 2^{k_d}/n}$, with $k_d = 0$ and obtain

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| \, dt \\
& \leq C \sum_{(i_l j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-2d-1} \prod_{l=2}^{d-1} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
& \quad \times \int_{P_{2^{k_1/n}, \dots, 2^{k_d/n}}} |f(x-t) - f(x)| \, dt \\
& \leq C \sum_{(i_l j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} 2^{(\omega-1/(d-1))|k|} \\
& \quad \times 2^{-\omega|k|} \frac{1}{|P_{2^{k_1/n}, \dots, 2^{k_d/n}}|} \int_{P_{2^{k_1/n}, \dots, 2^{k_d/n}}} |f(x-t) - f(x)| \, dt \\
& \leq C \sum_{(i_l j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} 2^{(\omega-1/(d-1))|k|} U_r^\omega f(x) < C\epsilon.
\end{aligned}$$

It is easy to see that $t_1 > (\pi - t_d)/2$ on B_2 . By (3.5),

$$\begin{aligned}
& \int_{A_1 \cap B_1^{k_1} \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| \, dt \\
& \leq C n^m \int_{A_j \cap B_2 \cap T_{r/2}^c} |f(x-t) - f(x)| \\
& \quad \times \sum_{(i_l j_l)_{l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})}} (\pi - t_{j_1})^{-1} \prod_{l=1}^{d-1-m} (t_{i_l} - t_{j_l})^{-1} \, dt
\end{aligned}$$

for $k_1 = 0, 1$. Similarly as above,

$$\begin{aligned}
& \int_{A_1 \cap B_1^{k_1} \cap B_2 \cap S_{r/2}^c \cap T_{r/2}^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-2d-1} \prod_{l=2}^{d-1} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
& \quad \times \int_{P_{2^{k_1/n}, \dots, 2^{k_d/n}}} |f(x-t) - f(x)| dt \\
& \leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} 2^{(\omega-1/(d-1))|k|} \\
& \quad \times 2^{-\omega|k|} \frac{1}{|P_{2^{k_1/n}, \dots, 2^{k_d/n}}|} \int_{P_{2^{k_1/n}, \dots, 2^{k_d/n}}} |f(x-t) - f(x)| dt \\
& \leq C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} 2^{(\omega-1/(d-1))|k|} \mathcal{M}^\omega f(x) \\
& \quad + C \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \dots \sum_{k_{d-1}=0}^{k_1} 2^{-|k|/(d-1)} |f(x)| \\
& \leq C \sum_{k_1=r_0}^{\infty} 2^{(\omega-1/(d-1))k_1} \mathcal{M}^\omega f(x) + C \sum_{k_1=r_0}^{\infty} 2^{-k_1/(d-1)} |f(x)| \\
& \leq C 2^{(\omega-1/(d-1))r_0} \mathcal{M}^\omega f(x) + C 2^{-r_0/(d-1)} |f(x)| \\
& \leq C(nr)^{\omega-1/(d-1)} \mathcal{M}^\omega f(x) + C(nr)^{-1/(d-1)} |f(x)| \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$.

To integrate on A_2 , we introduce the sets $A_2^{\alpha_1, \dots, \alpha_m}$ ($1 \leq \alpha_1 < \dots < \alpha_m \leq d-1$, $1 \leq m \leq d-1$) containing all $t \in A_2$ for which $t_{\alpha_j} - t_{\alpha_j+1} \leq 2/n$ ($j = 1, \dots, m$) and $t_k - t_{k+2} > 2/n$ for all $k = 1, \dots, d-2$. Instead of A_2 , we will integrate on $A_2^{\alpha_1, \dots, \alpha_m}$. If $m = d-1$, then the integrals on $A_2^{\alpha_1, \dots, \alpha_m} \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}$ and $A_2^{\alpha_1, \dots, \alpha_m} \cap B_1^{k_1} \cap B_2 \cap S_{r/2}^c \cap T_{r/2}^c$ are similar to the integrals above. Suppose that $1 \leq m \leq d-2$ and $\alpha_m + 1 < d$. Then let $n_j = \alpha_j + 1$, $j = 1, \dots, m$ and by (4.1) and (4.2),

$$\begin{aligned}
& \int_{A_2^{\alpha_1, \dots, \alpha_m} \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq C n^m \sum_{(i_l, j_l)_{l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})}} \\
& \quad \times \int_{A_2^{\alpha_1, \dots, \alpha_m} \cap B_2 \cap T_{r/2}} |f(x-t) - f(x)| (\pi - t_{i_1})^{-2} \prod_{l=2}^{d-1-m} (t_{i_l} - t_{j_l})^{-1} dt.
\end{aligned}$$

Here, we integrate first as follows:

$$\int_{t_{n_{m-1}}-2/n}^{t_{n_m}-1} \cdots dt_{n_m}, \quad \int_{t_{n_{m-1}-1}-2/n}^{t_{n_{m-1}}-1} \cdots dt_{n_{m-1}}, \quad \dots, \quad \int_{t_{n_1-1}-2/n}^{t_{n_1}-1} \cdots dt_{n_1}.$$

Then we continue with the integrals as before in (4.2) with $\mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})$ instead of $\mathcal{I}(\{1, \dots, d\})$. This means that we integrate over a parallelepiped $P_{2^{k_1}/n, \dots, 2^{k_d}/n}$ with $k_{n_1} = \dots = k_{n_m} = 1$, $k_d = 0$ and get that

$$\begin{aligned} & \int_{A_2^{\alpha_1, \dots, \alpha_m} \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C n^m \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} \\ & \quad \times \sum_{k_1=3}^{r_0} \sum_{k_{\beta_2}=0}^{k_1} \cdots \sum_{k_{\beta_{d-1-m}}=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-2d-1-m} \prod_{l=2}^{k_{\beta_l}} \left(\frac{2^{k_{\beta_l}}}{T}\right)^{-1} \\ & \quad \times \int_{P_{2^{k_1}/n, \dots, 2^{k_d}/n}} |f(x-t) - f(x)| dt \\ & \leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} \sum_{k_1=3}^{r_0} \sum_{k_{\beta_2}=0}^{k_1} \cdots \sum_{k_{\beta_{d-1-m}}=0}^{k_1} 2^{(\omega-1/(d-1-m))|k|} \\ & \quad \times 2^{-\omega|k|} \frac{1}{|P_{2^{k_1}/n, \dots, 2^{k_d}/n}|} \int_{P_{2^{k_1}/n, \dots, 2^{k_d}/n}} |f(x-t) - f(x)| dt \\ & \leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(\{1, \dots, d\} \setminus \{n_1, \dots, n_m\})} \\ & \quad \times \sum_{k_1=3}^{r_0} \sum_{k_{\beta_2}=0}^{k_1} \cdots \sum_{k_{\beta_{d-1-m}}=0}^{k_1} 2^{(\omega-1/(d-1-m))|k|} U_r^\omega f(x) < C\epsilon, \end{aligned}$$

where the indices $1 < \beta_2 < \dots < \beta_{d-1-m} < d$ are all different from $n_1, \dots, n_m$. It is easy to see that if $\alpha_m + 1 = d$, then we can also choose $1 < n_1 < \dots < n_m < d$ such that $n_j = \alpha_j$ or $n_j = \alpha_j + 1$, $j = 1, \dots, m$ and the estimation of the integral is the same. The integrals on the sets $A_2^{\alpha_1, \dots, \alpha_m} \cap B_1^{k_1} \cap B_2 \cap S_{r/2}^c \cap T_{r/2}^c$, $A_j \cap B_1^c \cap B_2 \cap S_{r/2}^c \cap T_{r/2}$ and on $A_j \cap B_1^{k_1} \cap B_2 \cap S_{r/2}^c \cap T_{r/2}^c$ ($j = 3, \dots, d-1, k_0 = 0, 1$) can be handled similarly.

We estimate the integrals on $A_j \cap B_1 \cap B_2^c \cap S_{r/2} \cap T_{r/2}^c$ and $A_j \cap B_1 \cap B_2^{k_2} \cap S_{r/2}^c \cap T_{r/2}^c$ ($j = 1, \dots, d-1, k_0 = 0, 1$) with the integrals on $A_j \cap B_1 \cap S_{r/2}$ and $A_j \cap B_1 \cap S_{r/2}^c$ and these integrals can be found in Weisz [25]. Here we use the first summand of (3.5).

Next, we integrate on $A_j \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}$ and introduce the set

$$F := \{t : (\pi - t_d)/2 > t_1 - t_d\} = \{t : \pi - t_1 > (\pi - t_d)/2\}.$$

For $j = 1$, we use inequality (3.10) to obtain

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l=1} \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-1} (\pi - t_{i_1})^{-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l=1} \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_2^c \cap T_{r/2}} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-1-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt.
\end{aligned}$$

We can integrate as in (4.2) with the only difference that, with respect to $t_{i'_2}$, we consider the integral $\sum_{k_d=0}^{k_1} \int_{t_{j_1} + 2^{k_d}/n}^{t_{j_1} + 2^{k_d+1}/n} \dots (t_{i_1} - t_{j_1})^{-1} dt_{i'_2}$. Thus

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l=1} \in \mathcal{I}(1, \dots, d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_d=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-1-\alpha} \prod_{l=2}^d \left(\frac{2^{k_l}}{T}\right)^{-1} \\
& \quad \times \int_{P_{2^{k_1}/n, \dots, 2^{k_d}/n}} |f(x-t) - f(x)| dt \\
& \leq C \sum_{(i_l, j_l)_{l=1} \in \mathcal{I}(1, \dots, d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_d=0}^{k_1} 2^{(\omega-\alpha/d)|k|} \\
& \quad \times 2^{-\omega|k|} \frac{1}{|P_{2^{k_1}/n, \dots, 2^{k_d}/n}|} \int_{P_{2^{k_1}/n, \dots, 2^{k_d}/n}} |f(x-t) - f(x)| dt \\
& \leq C \sum_{(i_l, j_l)_{l=1} \in \mathcal{I}(1, \dots, d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \dots \sum_{k_d=0}^{k_1} 2^{(\omega-\alpha/d)|k|} U_r^\omega f(x) < C\epsilon.
\end{aligned} \tag{4.3}$$

Moreover, on the set F^c , we have

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F^c} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-2} (\pi - t_{i_1})^{-\alpha} \prod_{l=2}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \int_{A_1 \cap B_2^c \cap T_{r/2}} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-1-\alpha} (\pi - t_{i_1})^{-1} \prod_{l=2}^{d-1} (t_{i_l} - t_{j_l})^{-1} dt.
\end{aligned}$$

We can use the same order as in (4.2) and we compute the integral

$$\sum_{k_d=0}^{k_1} \int_{\pi-2^{k_d+1}/n}^{\pi-2^{k_d}/n} \cdots (\pi - t_{i_1})^{-1} dt_{i_2}' .$$

Similarly to (4.3),

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2} \cap F^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-1-\alpha} \prod_{l=2}^d \left(\frac{2^{k_l}}{T}\right)^{-1} \\
& \quad \times \int_{P_{2^{k_1}/n, \dots, 2^{k_d}/n}} |f(x-t) - f(x)| dt \\
& \leq C \sum_{(i_l, j_l)_l \in \mathcal{I}(1, \dots, d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} 2^{(\omega-\alpha/d)|k|} U_r^{\omega f}(x) < C\epsilon.
\end{aligned}$$

On the set $A_j \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}$, we use the inequality (3.11) with $m = j - 1$. We omit the details because they are similar to those above and to those in [25]. We estimate the integrals on $A_j \cap B_1^c \cap B_2^c \cap S_{r/2} \cap T_{r/2}^c$ with the integrals on $A_j \cap B_1^c \cap S_{r/2}$ ($j = 1, \dots, d$). The latter integrals can be found in Weisz [25].

Finally, by (3.10),

$$\begin{aligned}
& \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}^c} |f(x-t) - f(x)| |K_n^\alpha(t)| \, dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}^c} |f(x-t) - f(x)| \\
& \quad \times t_{i_1}^{-1} t_{j_1}^{-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} 1_{\{t_{j_{d-1}} \leq \pi/2\}} \, dt \\
& \quad + Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \int_{A_1 \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}^c} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-1} (\pi - t_{i_1})^{-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} 1_{\{x_{j_{d-1}} > \pi/2\}} \, dt \\
& \leq Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \int_{A_1 \cap B_1^c \cap S_{r/2}^c} |f(x-t) - f(x)| \\
& \quad \times t_{i_1}^{-1} t_{j_1}^{-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} 1_{\{t_{j_{d-1}} \leq \pi/2\}} \, dt \\
& \quad + Cn^{-\alpha} \sum_{(i_l, j_l)_{l \in \mathcal{I}(1, \dots, d)}} \int_{A_1 \cap B_2^c \cap T_{r/2}^c} |f(x-t) - f(x)| \\
& \quad \times (\pi - t_{j_1})^{-1} (\pi - t_{i_1})^{-\alpha} \prod_{l=1}^{d-1} (t_{i_l} - t_{j_l})^{-1} 1_{\{x_{j_{d-1}} > \pi/2\}} \, dt.
\end{aligned}$$

These integrals as well as the integral on $A_j \cap B_1^c \cap B_2^c \cap S_{r/2}^c \cap T_{r/2}^c$ ($j = 1, \dots, d$) can be estimated as above. The details are left to the reader. This completes the proof of the theorem. $\square$

Theorems 1, 2 and (2.1) imply the next corollary, which was shown in Weisz [22].

Corollary 1 *If $0 < \alpha < \infty$ and $f \in L_1(\mathbb{T}^d)$ is periodic with respect to π , then*

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f = f \quad \text{a.e.}$$

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References

- Berens, H., Li, Z., Xu, Y.: On l -1 Riesz summability of the inverse Fourier integral. *Indag. Math. (N.S.)* **12**, 41–53 (2001)
- Berens, H., Xu, Y.: l -1 summability of multiple Fourier integrals and positivity. *Math. Proc. Camb. Philos. Soc.* **122**, 149–172 (1997)
- DeVore, R.A., Lorentz, G.G.: *Constructive Approximation*. Springer, Berlin (1993)
- Feichtinger, H.G., Weisz, F.: Wiener amalgams and pointwise summability of Fourier transforms and Fourier series. *Math. Proc. Camb. Philos. Soc.* **140**, 509–536 (2006)
- Gát, G.: Pointwise convergence of cone-like restricted two-dimensional $(C, 1)$ means of trigonometric Fourier series. *J. Approx. Theory* **149**, 74–102 (2007)
- Gát, G.: Almost everywhere convergence of sequences of Cesàro and Riesz means of integrable functions with respect to the multidimensional Walsh system. *Acta Math. Sin. Engl. Ser.* **30**(2), 311–322 (2014)
- Gát, G., Goginava, U., Nagy, K.: On the Marcinkiewicz–Fejér means of double Fourier series with respect to Walsh–Kaczmarz system. *Stud. Sci. Math. Hung.* **46**, 399–421 (2009)
- Goginava, U.: Marcinkiewicz–Fejér means of d -dimensional Walsh–Fourier series. *J. Math. Anal. Appl.* **307**, 206–218 (2005)
- Goginava, U.: Almost everywhere convergence of (C, α) -means of cubical partial sums of d -dimensional Walsh–Fourier series. *J. Approx. Theory* **141**, 8–28 (2006)
- Goginava, U.: The maximal operator of the Marcinkiewicz–Fejér means of d -dimensional Walsh–Fourier series. *East J. Approx.* **12**, 295–302 (2006)
- Grafakos, L.: *Classical and Modern Fourier Analysis*. Pearson Education, New Jersey (2004)
- Herriot, J.G.: Nörlund summability of multiple Fourier series. *Duke Math. J.* **11**, 735–754 (1944)
- Lebesgue, H.: Recherches sur la convergence des séries de Fourier. *Math. Ann.* **61**, 251–280 (1905)
- Li, Z., Xu, Y.: Summability of product Jacobi expansions. *J. Approx. Theory* **104**, 287–301 (2000)
- Nagy, K., Tephnadze, G.: The Walsh–Kaczmarz–Marcinkiewicz means and Hardy spaces. *Acta Math. Hung.* **149**, 346–374 (2016)
- Persson, L.E., Tephnadze, G., Wall, P.: Maximal operators of Vilenkin–Nörlund means. *J. Fourier Anal. Appl.* **21**(1), 76–94 (2015)
- Riesz, M.: Sur la sommation des séries de Fourier. *Acta Sci. Math. (Szeged)* **1**, 104–113 (1923)
- Simon, P.: Cesàro summability with respect to two-parameter Walsh systems. *Monatshefte Math.* **131**, 321–334 (2000)
- Simon, P.: (C, α) summability of Walsh–Kaczmarz–Fourier series. *J. Approx. Theory* **127**, 39–60 (2004)
- Szili, L., Vértési, P.: On multivariate projection operators. *J. Approx. Theory* **159**, 154–164 (2009)
- Weisz, F.: ℓ_1 -summability of higher-dimensional Fourier series. *J. Approx. Theory* **163**, 99–116 (2011)
- Weisz, F.: Cesàro-summability of higher-dimensional Fourier series. *Ann. Univ. Sci. Bp. Sect. Comput.* **37**, 47–64 (2012)
- Weisz, F.: Summability of multi-dimensional trigonometric Fourier series. *Surv. Approx. Theory* **7**, 1–179 (2012)
- Weisz, F.: Lebesgue points of two-dimensional Fourier transforms and strong summability. *J. Fourier Anal. Appl.* **21**, 885–914 (2015)
- Weisz, F.: ℓ_1 -summability and Lebesgue points of d -dimensional Fourier transforms. *Adv. Oper. Theory* **4**, 284–304 (2019)
- Xu, Y.: Christoffel functions and Fourier series for multivariate orthogonal polynomials. *J. Approx. Theory* **82**, 205–239 (1995)
- Zygmund, A.: *Trigonometric Series*, 3rd edn. Cambridge Press, London (2002)