Boundedness of dyadic maximal operators on variable Lebesgue spaces

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Abstract

We introduce three types of dyadic maximal operators and prove that under some conditions on the variable exponent $p(\cdot)$, they are bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$. Here we correct Theorem 4.2 of the paper, Szarvas and Weisz (Banach J Math Anal 13:675–696, 2019).

Keywords Variable exponent · Dyadic maximal operators · Walsh-Fourier series · Cesàro and Riesz means

Mathematics Subject Classification 42B25 · 42C10

1 Introduction

For a measurable function $p(\cdot)$, the variable Lebesgue space $L_{p(\cdot)}$ consists of all measurable functions f for which $\int_0^1 |f(x)|^{p(x)} dx < \infty$. If $p(\cdot)$ is a constant, we get back the usual L_p space. This topic needs essentially new ideas and is investigated very intensively in the literature nowadays (see e.g., Cruz-Uribe and Fiorenza [1], Diening et al. [2], Nakai and Sawano [9, 10], Jiao et al. [4–6], Liu et al. [7, 8]). Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications (see the references in Jiao et al. [4]).

Usually, we suppose that $p(\cdot)$ satisfies the Hölder continuity condition. In [4, 5, 12] as well as in this paper, we suppose a slightly more general condition. It is known (see [4, 5]) that the usual dyadic (or martingale) maximal operator is

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bounded on $L_{p(\cdot)}$ with $1 < p_- \leq p_+ < \infty$. For the boundedness of the classical Hardy-Littlewood maximal operator see e.g., Cruz-Uribe and Fiorenza [1] and Diening et al. [2].

In [12], we investigated two more dyadic maximal operators denoted by $U_{\beta,s}$ and V_β , where β and s are positive parameters. We stated there that $U_{\beta,s}$ is bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$ and $\beta, s > 0$. However, there is a mistake in the proof, as we applied Lemma 2 in a wrong way. In this paper, we correct the proof and prove that, for $1 < p_- \leq p_+ < \infty$, $U_{\beta,s}$ is bounded on $L_{p(\cdot)}$ under the additional condition $\frac{1}{p_-} - \frac{1}{p_+} < \beta + s$. We show also that without this last additional condition, the boundedness of $U_{\beta,s}$ does not hold. Next, we verify that V_β is bounded on $L_{p(\cdot)}$ if $1 < p_- \leq p_+ < \infty$ and $\beta > 0$, without any additional condition. This was stated in [12] without proof.

In [12], we used the boundedness of the above dyadic maximal operators to prove that the maximal Cesàro (or (C, α)) and Riesz operators of the Walsh-Fourier series are bounded from the variable Hardy space $H_{p(\cdot)}$ to $L_{p(\cdot)}$ if $1/(\alpha + 1) < p_- < \infty$. Here we correct the theorem and show that under this last condition, the boundedness holds if and only if $\frac{1}{p_-} - \frac{1}{p_+} < 1$.

2 Variable Lebesgue spaces

In this section, we recall some basic notations on variable Lebesgue spaces and give some elementary and necessary facts about these spaces. Our main references are Cruz-Uribe and Fiorenza [1] and Diening et al. [2].

For a constant p , the L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p dx \right)^{1/p} \quad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure λ .

We are going to generalize these spaces. A measurable function $p(\cdot) : [0, 1) \rightarrow (0, \infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$ and any measurable set $A \subset [0, 1)$, we will use the notation

$$p_-(A) := \operatorname{ess\,inf}_{x \in A} p(x) \quad \text{and} \quad p_+(A) := \operatorname{ess\,sup}_{x \in A} p(x).$$

If $A = [0, 1)$, then the numbers $p_-(A)$ and $p_+(A)$ are denoted simply by p_- and p_+ . Denote by $\mathcal{P}$ the collection of all variable exponents $p(\cdot)$ satisfying

$$0 < p_- \leq p_+ < \infty.$$

The variable Lebesgue space $L_{p(\cdot)}$ contains all measurable functions f , for which

$$\|f\|_{L_{p(\cdot)}} := \inf \left\{ \rho \in (0, \infty) : \int_0^1 \left(\frac{|f(x)|}{\rho} \right)^{p(x)} dx \leq 1 \right\} < \infty.$$

Instead of the log-Hölder continuity condition, in [4, 12], we introduced the slightly more general condition

$$\lambda(I)^{p_-(I)-p_+(I)} \leq C \quad (1)$$

for all dyadic intervals $I \subset [0, 1)$. By a dyadic interval, we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \leq k < 2^n$.

Remark 1 There exist a lot of functions $p(\cdot)$ satisfying (1). For concrete examples we mention the function $a + cx$ for parameters a and c such that the function is positive ($x \in [0, 1)$). All positive Lipschitz functions with order $0 < \beta \leq 1$ also satisfy (1).

The following lemma was proved in Cruze-Urbe and Fiorenza [1] and Hao and Jiao [3].

Lemma 1 Let $p(\cdot) \in \mathcal{P}$ satisfy (1). Then, for any dyadic interval $I \subset [0, 1)$,

$$\lambda(I)^{1/p_-(I)} \sim \lambda(I)^{1/p(x)} \sim \lambda(I)^{1/p_+(I)} \sim \|\chi_I\|_{p(\cdot)} \quad (\forall x \in I),$$

where $\sim$ denotes the equivalence of the numbers.

The following lemma can be found in Jiao et al. [4, 5].

Lemma 2 Let $p(\cdot) \in \mathcal{P}$, $1 \leq p_- \leq p_+ < \infty$, satisfy (1). Suppose that $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \leq 1/2$ and $f = f\chi_{\{|f| \geq 1\}}$. Then, for any dyadic interval $I \subset [0, 1)$ and $x \in I$,

$$\left(\frac{1}{\lambda(I)} \int_I |f(t)| dt \right)^{p(x)} \leq \left(\frac{C}{\lambda(I)} \int_I |f(t)|^{p(t)} dt \right).$$

In this paper the constants C are absolute constants and the constants $C_{p(\cdot)}$ are depending only on $p(\cdot)$ and may denote different constants in different contexts. For two positive numbers A and B , we use also the notation $A \lesssim B$, which means that there exists a constant C such that $A \leq CB$.

3 Dyadic maximal operators

In this section, in addition to the well known Doob's maximal operator, we introduce two new types of maximal operators. The Doob's maximal operator is given by

$$Mf(x) := \sup_{x \in I} \frac{1}{\lambda(I)} \left| \int_I f d\lambda \right|,$$

where the supremum is taken over all dyadic intervals. The following result was proved in Jiao et al. [5]. For the boundedness of the classical Hardy–Littlewood maximal operator see e.g. Cruz-Urbe and Fiorenza [1] and Diening et al. [2].

Theorem 1 If $p(\cdot) \in \mathcal{P}$ satisfies (1) and $1 < p_- \leq p_+ < \infty$, then

$$\|Mf\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

For an integrable function $f \in L_1$, we define the second maximal operator by

$$U_{\beta,s}f(x) := \sup_{x \in I} \sum_{m=1}^n \sum_{j=0}^{m-1} 2^{(j-n)\beta} \sum_{i=j}^{m-1} 2^{(j-i)s} \frac{1}{\lambda(I^{j,i})} \left| \int_{I^{j,i}} f d\lambda \right|,$$

where I is a dyadic interval with length 2^{-n} , β, s are positive constants and

$$I^{j,i} := I + [2^{-j-1}, 2^{-j-1} + 2^{-i}).$$

Here $+$ denotes the dyadic addition (see e.g., Schipp, Wade, Simon and Pál [11]). Let us define $I_{k,n} := [k2^{-n}, (k+1)2^{-n})$ with $0 \leq k < 2^n$, $n \in \mathbb{N}$. The preceding definition can be rewritten to

$$U_{\beta,s}f := \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=1}^n \sum_{j=0}^{m-1} 2^{(j-n)\beta} \sum_{i=j}^{m-1} 2^{(j-i)s} \frac{1}{\lambda(I_{k,n}^{j,i})} \left| \int_{I_{k,n}^{j,i}} f d\lambda \right|.$$

The following theorem was proved in [12].

Theorem 2 For all $1 < p < \infty$ and all $0 < \beta, s < \infty$, we have

$$\|U_{\beta,s}f\|_p \leq C_p \|f\|_p \quad (f \in L_p).$$

Now we generalize this theorem to variable Lebesgue spaces.

Theorem 3 Let $p(\cdot) \in \mathcal{P}$ satisfy (1), $1 < p_- \leq p_+ < \infty$ and $0 < \beta, s < \infty$. If

$$\frac{1}{p_-} - \frac{1}{p_+} < \beta + s, \quad (2)$$

then

$$\|U_{\beta,s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

Proof It is easy to see that we may suppose the conditions $\|f\|_{p(\cdot)} \leq 1/2$, $|f| \geq 1$ or $f = 0$ and

$$\frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt > 1.$$

We denote by $I_{k,n,j,i,1}$ (resp. $I_{k,n,j,i,2}$) those points $x \in I_{k,n}$ for which $p(x) \leq p_+(I_{k,n}^{j,i})$ (resp. $p(x) > p_+(I_{k,n}^{j,i})$). Then

$$\begin{aligned}
\int_0^1 |U_{\beta,s} f(x)|^{p(x)} dx &\lesssim \sum_{l=1}^2 \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \\
&\quad \left. \frac{\chi_{I_{k,n,j,i,l}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{p(x)} dx \\
&=: (A) + (B).
\end{aligned}$$

Let $q(x) := p(x)/p_0 > 1$ for some $1 < p_0 < p_-$. Using convexity and the fact that $q(x) \leq q_+(I_{k,n}^{j,i})$ on $I_{k,n,j,i,l,1}$, we get that

$$\begin{aligned}
(A) &\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \left(\sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \right. \\
&\quad \left. \left. \frac{\chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx \\
&\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \\
&\quad \left. \left(\frac{\chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx \\
&\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \\
&\quad \left. \left(\frac{\chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{q_+(I_{k,n}^{j,i})} \right)^{p_0} dx.
\end{aligned}$$

Lemma 2 and Theorem 2 imply

$$\begin{aligned}
(A) &\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \\
&\quad \left. \frac{\chi_{I_{k,n,j,i,1}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q(t)} dt \right)^{p_0} dx \\
&\lesssim \left\| U_{\beta,s} (|f|^{q(\cdot)}) \right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \leq C.
\end{aligned}$$

Choosing $0 < \beta_0 < \beta$ and $0 < r < s + \beta_0$, we obtain

$$\begin{aligned}
(B) &\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \left(\sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)\beta} 2^{(j-i)s} \right. \right. \\
&\quad \left. \left. \frac{\chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx \\
&\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} \right. \\
&\quad \left. \left(2^{(j-i)r} \frac{\chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
(B) &\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} \right. \\
&\quad \left. 2^{(j-i)r q(x)} \left(\frac{\chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q_-(I_{k,n}^{j,i})} dt \right)^{q(x)/q_-(I_{k,n}^{j,i})} \right)^{p_0} dx \\
&\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
&\quad \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} 2^{(j-i)r q(x)} 2^{i q(x)/q_-(I_{k,n}^{j,i})} \\
&\quad \left. \chi_{I_{k,n,j,i,2}}(x) \left(\int_{I_{k,n}^{j,i}} |f(t)|^{q_-(I_{k,n}^{j,i})} dt \right)^{q(x)/q_-(I_{k,n}^{j,i})} \right)^{p_0} dx.
\end{aligned}$$

Since $|f| \geq 1$ or $f = 0$, $q(x) > q_-(I_{k,n}^{j,i})$ on $I_{k,n,j,i,2}$, $q_-(I_{k,n}^{j,i}) \leq q(t) < p(t)$ for all $t \in I_{k,n}^{j,i}$ and

$$\int_{I_{k,n}^{j,i}} |f(t)|^{q_-(I_{k,n}^{j,i})} dt \leq \int_{I_{k,n}^{j,i}} |f(t)|^{p(t)} dt \leq \frac{1}{2},$$

we can see that

$$(B) \lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} 2^{(j-i)r q(x)} 2^{i q(x)/q_-(I_{k,n}^{j,i})-i} \frac{\chi_{I_{k,n,j,i,2}}(x)}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q_-(I_{k,n}^{j,i})} dt \right)^{p_0} dx.$$

For fixed k and n let J_j denote the dyadic interval with length 2^{-j} and $I_{k,n} \subset J_j$. Then $I_{k,n}^{j,i} \subset J_j + 2^{-j-1} = J_j$. Inequality (1) implies that $2^{-jp(x)} \sim 2^{-jp_-(I_{k,n}^{j,i})}$ for $x \in I_{k,n}$. It is easy to check that for $x \in I_{k,n,j,l,2}$,

$$\begin{aligned} 2^{jr q(x)} &= 2^{jr q(x)} 2^{jq(x)} 2^{-jq(x)} \lesssim 2^{jr q(x)} 2^{jq_-(I_{k,n}^{j,i})} 2^{-jq(x)} \\ &< 2^j \left(r q(x) - \frac{q(x) - q_-(I_{k,n}^{j,i})}{q_-(I_{k,n}^{j,i})} \right) = 2^j \left(r q(x) - \frac{q(x)}{q_-(I_{k,n}^{j,i})} + 1 \right). \end{aligned}$$

Furthermore,

$$\begin{aligned} r q(x) - \frac{q(x)}{q_-(I_{k,n}^j)} + 1 &\geq q(x) \left(r - \frac{1}{q_-} \right) + 1 \\ &\geq \begin{cases} 1, & \text{if } r - \frac{1}{q_-} \geq 0; \\ q_+ \left(r - \frac{1}{q_-} \right) + 1, & \text{if } r - \frac{1}{q_-} < 0. \end{cases} \end{aligned}$$

Let $r_0 := \min \left(1, q_+ \left(r - \frac{1}{q_-} \right) + 1 \right)$. Then $r_0 > 0$ if and only if

$$\frac{1}{q_-} - \frac{1}{q_+} < r. \quad (3)$$

Hence

$$\begin{aligned}
(B) &\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \right. \\
&\quad \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r)} 2^{(j-i) \left(r q(x) - \frac{q(x)}{q_-(I_{k,n}^{j,i})} + 1 \right)} \\
&\quad \left. \frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q(t)} dt \right)^{p_0} dx \\
&\lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=1}^n \sum_{j=0}^{m-1} \sum_{i=j}^{m-1} 2^{(j-n)(\beta-\beta_0)} 2^{(j-i)(\beta_0+s-r+r_0)} \right. \\
&\quad \left. \frac{1}{\lambda(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q(t)} dt \right)^{p_0} dx \\
&\lesssim \left\| U_{\beta-\beta_0, \beta_0+s-r+r_0}(|f|^{q(\cdot)}) \right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \leq C,
\end{aligned}$$

whenever (3) holds. Since r can be arbitrarily near to $s + \beta_0$ and β_0 to β , this completes the proof. $\square$

Remark 2 Inequality (2) and Theorem 3 hold if $p_- > \max(1/(\beta + s), 1)$.

In [12], we used the parameters $\beta = (1 + \alpha)t - r/(r - t) > 0$, $s = r/(r - t) - \alpha t > 0$ and stated that Theorem 3 holds without the condition (2). However, this is not the case.

Theorem 4 Let $p(\cdot) \in \mathcal{P}$ satisfy (1), $1 < p_- \leq p_+ < \infty$ and $0 < \beta, s < \infty$. If

$$\frac{1}{p_-(I_{0,n}^{1,n-1})} - \frac{1}{p_+(I_{0,n})} > \beta + s \quad (4)$$

for all $n \in \mathbb{N}$, then $U_{\beta,s}$ is not bounded on $L_{p(\cdot)}$.

Proof Choosing $m = n$, $j = 0$ and $i = n - 1$, we can see that

$$\begin{aligned}
&\int_0^1 |U_{\beta,s} f(x)|^{p(x)} dx \\
&\geq \int_0^1 \chi_{I_{0,n}}(x) \left(2^{-n(\beta+s)} \frac{1}{\lambda(I_{0,n}^{1,n-1})} \left| \int_{I_{0,n}^{1,n-1}} f(t) dt \right| \right)^{p(x)} dx.
\end{aligned}$$

Let

$$f(t) := \chi_{I_{0,n}^{1,n-1}}(t) 2^{n/p_-(I_{0,n}^{1,n-1})}.$$

Lemma 1 implies that

$$\|f\|_{p(\cdot)} = 2^{n/p_-(I_{0,n}^{1,n-1})} \|\chi_{I_{0,n}^{1,n-1}}\|_{p(\cdot)} \leq C.$$

Thus, by (1),

$$\begin{aligned} \int_0^1 |U_{\beta,s} f(x)|^{p(x)} dx &\geq \int_{I_{0,n}} 2^{-n(\beta+s)p(x)} 2^{np(x)/p_-(I_{0,n}^{1,n-1})} dx \\ &\geq C \int_{I_{0,n}} 2^{np_+(I_{0,n})(1/p_-(I_{0,n}^{1,n-1})-\beta-s)} dx \\ &= C 2^{np_+(I_{0,n})(1/p_-(I_{0,n}^{1,n-1})-\beta-s)} 2^{-n} \end{aligned}$$

which tends to infinity as $n \rightarrow \infty$ if (4) holds. $\square$

The third dyadic maximal operator is introduced by

$$V_{\beta} f(x) := \sup_{x \in I} \sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I^m)} \left| \int_{I^m} f \right|,$$

where $f \in L_1$, I is a dyadic interval with length 2^{-n} , β is a positive constant and

$$I^m := I + [0, 2^{-m}).$$

Then

$$V_{\beta} f = \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}} \sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I_{k,n}^m)} \left| \int_{I_{k,n}^m} f \right|.$$

The following theorem can be found in [12].

Theorem 5 For all $1 < p < \infty$ and all $0 < \beta < \infty$, we have

$$\|V_{\beta} f\|_p \leq C_p \|f\|_p \quad (f \in L_p).$$

The generalization of this result reads as follows.

Theorem 6 If $p(\cdot) \in \mathcal{P}$ satisfies (1), $1 < p_- \leq p_+ < \infty$ and $0 < \beta < \infty$, then

$$\|V_{\beta} f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

Proof Similarly to the proof of Theorem 3, we may suppose again that $\|f\|_{p(\cdot)} \leq 1/2$ and

$$\frac{1}{\lambda(I_{k,n}^m)} \int_{I_{k,n}^m} |f(t)| dt > 1.$$

Since $I_{k,n} \subset I_{k,n}^m$ ($m = 0, \dots, n-1$), we can apply Lemma 2 and Theorem 5 to obtain

$$\begin{aligned}
& \int_0^1 |V_\beta f(x)|^{p(x)} dx \\
& \lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \left(\sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I_{k,n}^m)} \int_{I_{k,n}^m} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx \\
& \lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^{n-1} 2^{(m-n)\beta} \left(\frac{1}{\lambda(I_{k,n}^m)} \int_{I_{k,n}^m} |f(t)| dt \right)^{q(x)} \right)^{p_0} dx \\
& \lesssim \int_0^1 \left(\sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{m=0}^{n-1} 2^{(m-n)\beta} \frac{1}{\lambda(I_{k,n}^m)} \int_{I_{k,n}^m} |f(t)|^{q(t)} dt \right)^{p_0} dx \\
& \lesssim \|V_\beta(|f|^{q(\cdot)})\|_{p_0}^{p_0} \lesssim \| |f|^{q(\cdot)} \|_{p_0}^{p_0} \leq C,
\end{aligned}$$

which proves the theorem. $\square$

4 The maximal Cesàro and Riesz operator on $H_{p(\cdot)}$

Using Theorems 3 and 6, we can prove the boundedness of the maximal Cesàro and Riesz operators of Walsh-Fourier series as in [12].

Theorem 7 *Let $p(\cdot) \in \mathcal{P}$ satisfy (1) and*

$$\frac{1}{p_-} - \frac{1}{p_+} < 1. \quad (5)$$

If $0 < \alpha \leq 1 \leq \gamma$ and $1/(\alpha + 1) < p_- < \infty$, then

$$\| \sigma_*^\alpha f \|_{p(\cdot)} + \| \sigma_*^{\alpha, \gamma} f \|_{p(\cdot)} \lesssim \| f \|_{H_{p(\cdot)}} \quad (f \in H_{p(\cdot)}).$$

The same holds for the space $H_{p(\cdot), q}$ ($0 < q \leq \infty$).

For the definitions, details and proof see [12]. We stated there that Theorem 7 holds without the condition (5). However, as we have seen in Jiao et al. [4], this is not true for $\alpha = \gamma = 1$.

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References

1. Cruz-Uribe, D.V., Fiorenza, A.: Variable Lebesgue spaces. Foundations and harmonic analysis. Birkhäuser/Springer, New York (2013)
2. Diening, L., Harjulehto, P., Hästö, P., Ružička, M.: Lebesgue and Sobolev spaces with variable exponents. Springer, Berlin (2011)
3. Hao, Z., Jiao, Y.: Fractional integral on martingale Hardy spaces with variable exponents. *Fract. Calc. Appl. Anal.* **18**(5), 1128–1145 (2015)
4. Jiao, Y., Weisz, F., Wu, L., Zhou, D.: Variable martingale Hardy spaces and their applications in Fourier analysis. *Dissertationes Math. (to appear)*
5. Jiao, Y., Zhou, D., Hao, Z., Chen, W.: Martingale Hardy spaces with variable exponents. *Banach J. Math* **10**, 750–770 (2016)
6. Jiao, Y., Zuo, Y., Zhou, D., Wu, L.: Variable Hardy–Lorentz spaces $H^{p(\cdot),q}(\mathbb{R}^n)$. *Math. Nachr.* **292**, 309–349 (2019)
7. Liu, J., Weisz, F., Yang, D., Yuan, W.: Variable anisotropic Hardy spaces and their applications. *Taiwanese J. Math.* **22**, 1173–1216 (2018)
8. Liu, J., Weisz, F., Yang, D., Yuan, W.: Littlewood–Paley and finite atomic characterizations of anisotropic variable Hardy–Lorentz spaces and their applications. *J. Fourier Anal. Appl.* **25**, 874–922 (2019)
9. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. *J. Funct. Anal.* **262**(9), 3665–3748 (2012)
10. Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators. *Integral Equ. Oper. Theory* **77**, 123–148 (2013)
11. Schipp, F., Wade, W.R., Simon, P., Pál, J.: Walsh Series: An Introduction to Dyadic Harmonic Analysis. Adam Hilger, Bristol, New York (1990)
12. Szarvas, K., Weisz, F.: The boundedness of the Cesaro- and Riesz means in variable dyadic Hardy spaces. *Banach J. Math. Anal.* **13**, 675–696 (2019)