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ORIGINAL PAPER

Generalized Cesàro operator acting on Hilbert spaces of analytic functions

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Abstract

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. We define the generalized Cesàro operator as follows:

$$C_\omega(f)(z) = \int_0^1 f(tz)\left(\frac{1}{z}\int_0^z B_t^\omega(u)\,\mathrm{d}u\right)\omega(t)\mathrm{d}t,$$

where $\{B_\zeta^\omega\}_{\zeta\in\mathbb{D}}$ are the reproducing kernels of the Bergman space A^2_ω induced by a radial weight ω in the unit disc $\mathbb{D}$. We study the action of the operator C_ω on weighted Hardy spaces of analytic functions $\mathcal{H}_\gamma$, $\gamma > 0$ and on general weighted Bergman spaces A^2_μ.

Keywords Cesàro operator · Hilbert spaces · Weighted Bergman spaces · Bergman reproducing kernel · Radial weight

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1 Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $\gamma > 0$, let $\mathcal{H}_\gamma$ denote the Hilbert space of analytic functions in $\mathbb{D}$ such that its reproducing kernels are given by

$$K_\omega(z) = \frac{1}{(1 - z\bar{\omega})^\gamma} = \sum_{n=0}^{\infty} \gamma(n)(\bar{\omega}z)^n, \ z, \ \omega \in \mathbb{D}.$$

It is clear that the sequence $\gamma(n)$ is given by $\gamma(0) = 1$, $\gamma(1) = \gamma$ and $\gamma(n) = \frac{\Gamma(n+\gamma)}{\Gamma(\gamma)n!}$, $n \in \mathbb{N}$. Actually, this family of spaces are well known: for $\gamma = 1$ the space $\mathcal{H}_\gamma$ is the Hardy space $\mathcal{H}_1 = H^2$ and $\gamma(n) = 1$ for all $n \in \mathbb{N}$. For $\gamma > 1$, $\mathcal{H}_\gamma$ consists of the standard weighted Bergman space $A^2_{\gamma-2}$ and for $\gamma < 1$, it is the weighted Dirichlet space $\mathcal{H}_\gamma = D^2_\gamma$.

Observe that for $\gamma = 0$, the corresponding space would be the classical Dirichlet space D^2, so it is not included in the definition of the spaces $\mathcal{H}_\gamma$.

In other words, the Hilbert space $\mathcal{H}_\gamma$ consists of all the analytic functions such that

$$\|f\|^2_{\mathcal{H}_\gamma} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1 - |z|)^\gamma \, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized area measure on $\mathbb{D}$. Moreover, a simple observation yields an equivalent norm in terms of the coefficients of an analytic function f. If $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k)z^k$,

$$\|f\|^2_{\mathcal{H}_\gamma} \asymp \sum_{n=0}^{\infty} |\widehat{f}(n)|^2(n+1)^{1-\gamma}.$$

Further, we can consider more general weighted Bergman spaces than the ones defined by $\mathcal{H}_\gamma$ with $\gamma > 1$. For a nonnegative function $\omega \in L^1_{[0,1)}$, the extension to $\mathbb{D}$, defined by $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, is called a radial weight. Let A^2_ω denote the weighted Bergman space of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|^2_{A^2_\omega} = \int_{\mathbb{D}} |f(z)|^2\omega(z)\, dA(z) < \infty$. Throughout this paper, we assume $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise $A^2_\omega = \mathcal{H}(\mathbb{D})$.

For any radial weight, the convergence in A^2_ω implies the uniform convergence in compact subsets, so the point evaluations L_z are bounded linear functionals in A^2_ω and by the Riesz Representation Theorem, there exist Bergman reproducing kernels $B^\omega_z \in A^2_\omega$ such that

$$L_z(f) = f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)dA(\zeta), \quad f \in A^2_\omega.$$

For a complex sequence $\{a_k\}_{k=0}^{\infty}$, the classic Cesàro operator is defined as follows:

$$\mathcal{C}(\{a_k\}) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\}_{n=0}^{\infty}.$$

It is well known that the Cesàro operator is bounded on l^p, $1 < p < \infty$. This result was mostly showed by Hardy, whose main aim was to provide a simpler proof of the Hilbert inequality in [11, 12] and Landau [14], whose contribution was obtaining the sharp constant in the inequality, that is, the norm of the operator, among other authors.

Further, it can be considered as an operator between analytic functions by identifying each analytic function with its Taylor coefficients as follows: for $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k$,

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \widehat{f}(k) \right) z^n, \quad z \in \mathbb{D}.$$

Observe that it defines an analytic function, and a simple calculation gives the following integral representation:

$$\mathcal{C}(f)(z) = \int_0^1 f(tz) \frac{1}{1-tz} \mathrm{d}t, \quad z \in \mathbb{D}. \tag{1.1}$$

This operator is bounded on H^p, $0 < p < \infty$. This result has been showed by several authors and on different ways such as Hardy [13], Siskakis [23, 25], Miao [16], Stempak [27] and Andersen [3], among others.

The boundedness of the Cesàro operator on Bergman spaces was studied in [3] and [24] where it is shown that the Cesàro operator is bounded from A^p_α into itself if $p > 0$ and $\alpha > -1$.

Regarding Dirichlet spaces, Galanopoulos [7] proved that it is bounded on the weighted Dirichlet spaces D^2_α if $0 < \alpha < 1$.

Due to the historical magnitude of this classical operator and the authors that have been working on it, different generalizations have been raised during the last decades [5, 8, 9, 27]. Bearing in mind the formula (1.1), we are interested in replacing the kernel $\frac{1}{1-tz}$ of the integral representation with a more general kernel. In that sense, we are going to focus on the following generalization of the kernel induced by radial weights, which was previously introduced in works regarding the Hilbert operator [15, 22].

For a radial weight ω, we consider the generalized Cesàro operator

$$C_\omega(f)(z) = \int_0^1 f(tz) \left(\frac{1}{z} \int_0^z B_t^\omega(\zeta) d\zeta \right) \omega(t) \mathrm{d}t, \tag{1.2}$$

where $\{B_z^\omega\}_{z \in \mathbb{D}} \subset A^2_\omega$ are the Bergman reproducing kernels of A^2_ω. Notice that this operator is well defined for any analytic function and the choice $\omega = 1$ gives (1.1).

One of the first and main obstacles that we find when dealing with the operator (1.2) is that Bergman reproducing kernels have not an explicit formula in general (this is not the case for standard weights $\nu_\alpha(z) = (1-|z|)^\alpha$, $\alpha > -1$, since Bergman reproducing kernels induced by ν_α have nice properties and they can be written as $B_z^{\nu_\alpha}(\zeta) = (1-\overline{z}\zeta)^{-(2+\alpha)}$). Consequently, we are forced to use that for any radial weight ω they can be written as $B_z^\omega(\zeta) = \sum \overline{e_n(z)} e_n(\zeta)$ for each orthonormal basis $\{e_n\}$ of A^2_ω, and therefore, using the normalized monomials as basis, we can obtain the following representation in terms of the odd moments of the weight, denoted by ω_{2n+1}:

$$B_z^\omega(\zeta) = \sum_{n=0}^\infty \frac{(\overline{z}\zeta)^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D}. \tag{1.3}$$

In general, from now on, we will write $\omega_x = \int_0^1 r^x \omega(r)\,dr$ for all $x \geq 0$. In addition, we can write the norm of A^2_ω in terms of the Taylor coefficients of an analytic function as follows:

$$\|f\|^2_{A^2_\omega} = \sum_{n=0}^\infty 2\omega_{2n+1} |\widehat{f}(n)|^2.$$

The primary purpose of this paper is to describe the radial weights ω so that C_ω is bounded on $\mathcal{H}_\gamma$, for $\gamma > 0$ and on general weighted Bergman spaces. It is worth mentioning that just as Galanopoulos [7] pointed out that $\mathcal{C}$ is not bounded in the Dirichlet space D^2, this fact is true not only for $\omega = 1$ but also for any radial weight. Indeed, using the formula (1.3), for any radial weight ω,

$$C_\omega(1)(z) = \sum_{n=0}^\infty \frac{\omega_n}{2(n+1)\omega_{2n+1}} z^n,$$

so, since the moments of a radial weight form a decreasing sequence, we have $\|C_\omega(1)\|^2_{D^2} \asymp \sum_{n=0}^\infty \frac{\omega_n^2}{4(n+1)\omega_{2n+1}^2} \geq \sum_{n=0}^\infty \frac{1}{4(n+1)}$ which implies that $C_\omega(1)$ does not belong to D^2.

Before stating the main result of the paper, we need to introduce some notation and definitions. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. This condition implies a restriction on the decay of the weight, for example, if $\omega \in \widehat{\mathcal{D}}$, ω cannot decrease exponentially. However, every increasing weight belongs to $\widehat{\mathcal{D}}$, and weights of $\widehat{\mathcal{D}}$ admit an oscillatory behavior. The study of the intrinsic nature of this class of weights entails a considerable difficulty, which has led to a deep research for years, collected in works such as [18, 20, 21].

A radial weight $\omega \in \check{\mathcal{D}}$ if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ for all $0 \leq r < 1$. We write the class $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$. Observe that standard weights $\upsilon_\alpha = (1-|z|)^\alpha$, $\alpha > -1$ belong to the class $\mathcal{D}$, which means that $H_\gamma = A^2_{\gamma-2}$, $\gamma > 1$ are particular cases of weighted Bergman spaces A^2_μ, $\mu \in \mathcal{D}$.

Moreover, a radial weight $\omega \in \mathcal{M}$ if there exist constants $C = C(\omega) > 1$ and $K = K(\omega) > 1$ such that $\omega_x \geq C\omega_{Kx}$ for all $x \geq 1$. Peláez and Rättyä showed that the classes $\check{\mathcal{D}}$ and $\mathcal{M}$ are closely related. They recently proved that $\check{\mathcal{D}} \subset \mathcal{M}$ [20, Proof of Theorem 3] but $\check{\mathcal{D}} \subsetneq \mathcal{M}$ [20, Proposition 14]. However, [20, Theorem 3] shows that $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}} = \widehat{\mathcal{D}} \cap \mathcal{M}$. The theory of these classes of weights has been basically developed by these authors in the work [20], and they have shown that these classes of weights arise on a natural way in significant questions of the operator theory and the weighted Bergman spaces. For instance, $\mathcal{D}$ describes the radial weights such that the following Littlewood–Paley formula holds

$$\|f\|_{A^p_\omega}^p \asymp \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|)^{np} \omega(z)\, dA(z) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \quad f \in \mathcal{H}(\mathbb{D})$$

for any $0 < p < \infty$, $n \in \mathbb{N}$; or the radial weights such that $P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_z^\omega(\zeta)}\, \omega(\zeta) dA(\zeta)$ is bounded and onto from L^∞ to the Bloch space, among other important results.

Theorem 1 *Let ω be a radial weight, $\gamma > 0$. Then $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is bounded if and only if $\omega \in \mathcal{D}$.*

The underlying nature of the spaces $\mathcal{H}_\gamma$ that we are considering and as far as we know, the almost unique formula for the Bergman reproducing kernels (1.3) lead us to address the problem by working on coefficients, so an appropriate expression for C_ω in terms of coefficients plays a key role in this work. Let $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{k=0}^\infty \widehat{f}(k) z^k$, by (1.3) and a change of variable,

$$\begin{aligned} C_\omega(f)(z) &= \sum_{n=0}^\infty \frac{1}{2(n+1)\omega_{2n+1}} \left(\sum_{k=0}^\infty \widehat{f}(k) \omega_{n+k} z^{n+k} \right) \\ &= \sum_{n=0}^\infty \left(\sum_{k=0}^n \frac{\widehat{f}(k)}{2(n-k+1)\omega_{2(n-k)+1}} \right) \omega_n z^n. \end{aligned} \tag{1.4}$$

The proof of the Theorem 1 for $\gamma = 1$ draws strongly on accurate estimates of the moments $\omega_{2(n-k)+1}$ and ω_n and on the Carleson measures theory.

For $0 < \gamma < 1$, the Carleson measures description was solved in [26], but the innocent looking condition that characterize such measures is not easy to work with, so we are forced to appeal to Littlewood–Paley formulas for non radial weights, specifically whose ν on $\mathbb{D}$ which belongs to one of the Bekollé classes $B_p(\alpha)$ for some $p > 1$ and $\alpha > -1$.

The proof of the case $\gamma > 1$ is slightly simpler since it is not necessary to use the Carleson measures tool. Going further, we are able to characterize the boundedness of the Cesàro-type operator C_ω in more general weighted Bergman spaces A^2_μ, $\mu \in \mathcal{D}$.

Theorem 2 *Let μ and ω be radial weights, $\mu \in \mathcal{D}$. Then $C_\omega : A^2_\mu \to A^2_\mu$ is bounded if and only if $\omega \in \mathcal{D}$.*

Finally, we are able to show in Theorem 5 that there does not exist radial weight ω such that $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$, $\gamma > 0$, is compact neither radial weight such that $C_\omega : A^2_\mu \to A^2_\mu$, $\mu \in \mathcal{D}$, is compact.

The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \le Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that a and b are comparable.

2 Previous results

2.1 Previous results of radial weights

Before tackling with the proof of Theorems 1 and 2, we gather the following two lemmas with some descriptions of the classes of weights $\widehat{\mathcal{D}}$ and $\mathcal{M}$ in terms of the moments or integral tails of the weights, which are useful for our purposes. The next one concerning the doubling properties of the weights in the class $\widehat{\mathcal{D}}$ can be found in [18, Lemma 2.1].

Lemma A *Let ω be a radial weight on $\mathbb{D}$. Then, the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) \ge 1$ and $\alpha_0 = \alpha_0(\omega) > 0$ such that*

$$\widehat{\omega}(r) \le C \left(\frac{1-r}{1-t}\right)^\alpha \widehat{\omega}(t), \quad 0 \le r \le t < 1;$$

for all $\alpha \ge \alpha_0$.

(iii)

$$\int_0^1 s^x \omega(s) ds \asymp \widehat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty);$$

(iv) *There exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that*

$$\omega_x \le C \left(\frac{y}{x}\right)^\alpha \omega_y, \quad 0 < x \le y < \infty;$$

(v) $\sup_{n \in \mathbb{N}} \frac{\omega_n}{\omega_{2n}} < \infty$.

The following lemma gives useful descriptions of the class $\mathcal{M}$. The results and their proofs can be found in [20, (2.16) and (2.17)]. To set notation, we will denote $\omega_{[\beta]}(z) = (1 - |z|)^\beta \omega(z)$.

Lemma B *Let ω be a radial weight. The following statements are equivalent:*

(i) $\omega \in \mathcal{M}$;

(ii) *There exist* $C = C(\omega) > 0$ *and* $\beta_0 = \beta_0(\omega) > 0$ *such that*

$$\omega_x \geq C\left(\frac{y}{x}\right)^{\beta} \omega_y, \quad 1 \leq x \leq y < \infty$$

for all $0 < \beta \leq \beta_0$*;*

(iii) *For some (equivalently for each)* $\beta > 0$*, there exists* $C = C(\omega, \beta) > 0$ *such that*

$$\omega_x \leq C x^{\beta} (\omega_{[\beta]})_x, \quad 1 \leq x < \infty.$$

2.2 Littlewood–Paley formula for general weights

Now, we are interested in the general weights ν that satisfy the following equivalence, called Littlewood–Paley formula:

$$\int_{\mathbb{D}} |f(z)|^p \nu(z) dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \nu(z) dA(z). \tag{2.1}$$

These kind of estimations are useful not only to obtain equivalent norms in terms of derivatives but also due to their relation with bounded Bergman projections, and this is one of the reasons why it is a prominent topic in the operator theory on spaces of analytic functions [1, 2, 4, 19, 20]. We are interested in the one proved in [1] en route to a description of the spectra of integration operators on weighted Bergman spaces, where Aleman and Constantin showed that (2.1) holds for every weight ν on $\mathbb{D}$ which belongs to one of the Bekollé classes $B_p(\alpha)$ for some $p > 1$ and $\alpha > -1$. In fact, they proved not only the belonging to one of the Bekollé classes is sufficient condition in order that (2.1) holds, but also it is necessary for sufficiently regular weights ν on $\mathbb{D}$.

Before stating the results, and to be self-contained, we will recall the definitions of the Bekollé class $B_p(\alpha)$ and also a closed related class $B_1^*(\eta)$. On the one hand, a weight ν on $\mathbb{D}$ belong to the Bekollé class $B_p(\alpha)$, $p > 1$ and $\alpha > -1$ if

$$\left(\int_{S(\theta,h)} \nu \, dA_\alpha\right)\left(\int_{S(\theta,h)} \nu^{-\frac{p'}{p}} dA_\alpha\right)^{\frac{p}{p'}} \lesssim (A_\alpha(S(\theta,h)))^p$$

for any Carleson square $S(\theta, h) = \{z = re^{i\alpha} : 1-h < r < 1, |\theta - \alpha| < h/2\}$, $\theta \in [0, 2\pi]$, $h \in (0, 1)$, where A_α denote the measure given by $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA$ and $1/p + 1/p' = 1$. On the other hand, a weight ν on $\mathbb{D}$ belongs to the class $B_1^\star(\eta)$, $\eta > -1$ if

$$\int_{\mathbb{D}} \frac{\nu(z)}{|1-\overline{a}z|^{\eta+2}} (1-|z|^2)^\eta \, dA(z) \lesssim \nu(a)$$

for almost every $a \in \mathbb{D}$.

Theorem C [1, Theorem 3.2] *Let ν be a strictly positive weight $\nu \in C^1(\mathbb{D})$ which satisfies that $(1-|z|^2)|\nabla \nu(z)| \le k_\nu \nu(z)$ for some constant $k_\nu > 0$ and all $z \in \mathbb{D}$. Then the following are equivalent:*

(i) *The estimate (2.1) holds for all $p > 0$;*
(ii) *The estimate (2.1) holds for some $p > 0$;*
(iii) *$\frac{\nu}{(1-|z|)^\alpha}$ belongs to $B_p(\alpha)$ for some $p > 1$ and $\alpha > -1$;*
(iv) *$\frac{\nu}{(1-|z|)^\eta}$ belongs to $B_1^\star(\eta)$ for some $\eta > -1$.*

3 Proof of the main results

Proof of Theorem 1 Since the Bergman case will be deal with in a more general way in Theorem 2, it is enough to prove the result for $0 < \gamma \le 1$.

Let us consider the following suitable formula for the generalized Cesàro operator (1.4), which is mainly followed by (1.3):

$$C_\omega(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\widehat{f}(k)}{2(n-k+1)\omega_{2(n-k)+1}} \right) \omega_n z^n, \; z \in \mathbb{D}.$$

Before we get into the proof, note that if we find a constant $C > 0$ satisfying $\|C_\omega(f)\|^2_{\mathcal{H}_\gamma} \le C \|f\|^2_{\mathcal{H}_\gamma}$ for any function $f \in \mathcal{H}(\mathbb{D})$ with $\widehat{f}(n) \ge 0$, $n \in \mathbb{N} \cup \{0\}$, we are done.

Assume $\omega \in \mathcal{D}$. By Lemma A(iv),

$$\begin{aligned} \|C_\omega(f)\|^2_{\mathcal{H}_\gamma} &\asymp \sum_{n=0}^{\infty} \omega_n^2 (n+1)^{1-\gamma} \left(\sum_{k=0}^{n} \frac{\widehat{f}(k)}{2(n-k+1)\omega_{2(n-k)+1}} \right)^2 \\ &\lesssim \sum_{n=0}^{\infty} \omega_n^2 (n+1)^{1-\gamma} \left(\sum_{k=0}^{n} \frac{\widehat{f}(k)}{2(n-k+1)\omega_{n-k+1}} \right)^2 \end{aligned}$$

and by Lemma B (ii), there exists $0 < \beta < 1$ such that $(n-k+1)^\beta \omega_{n-k+1} \gtrsim (n+1)^\beta \omega_{n+1}$ for all $k \le n$, so

$$\|C_\omega(f)\|^2_{\mathcal{H}_\gamma} \lesssim \sum_{n=0}^{\infty} \frac{1}{(n+1)^{2\beta+\gamma-1}} \left(\sum_{k=0}^{n} \frac{\widehat{f}(k)}{(n-k+1)^{1-\beta}} \right)^2. \tag{3.1}$$

Now, it is well known that $g(z) = \frac{1}{(1-z)^\beta} = \sum_{n=0}^{\infty} \alpha_n z^n \in \mathcal{H}(\mathbb{D})$, whose Taylor coefficients are given by $\alpha_n = \frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)}$, and folklore estimations for ratios of gamma functions yields $\alpha_n \asymp \frac{1}{(n+1)^{1-\beta}}$. In addition, a simple observation yields

$$\frac{f(z)}{(1-z)^\beta} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \widehat{f}(k) \alpha_{n-k} \right) z^n, \; z \in \mathbb{D}.$$

From now on, we will deal with the following two cases separately:

Case $\gamma = 1$: Bearing in mind that $\|z^n\|^2_{A^2_{2\beta-1}} \asymp \frac{1}{(n+1)^{2\beta}}$, $n \in \mathbb{N}$ and (3.1), we deduce

$$\|C_\omega(f)\|^2_{H^2} \lesssim \sum_{n=0}^{\infty} \|z^n\|^2_{A^2_{2\beta-1}} \left(\sum_{k=0}^{n} \widehat{f}(k)\alpha_{n-k}\right)^2 \lesssim \left\| \frac{f(z)}{(1-z)^\beta} \right\|^2_{A^2_{2\beta-1}} \lesssim \|f\|^2_{H^2},$$

where the last inequality holds if and only if $d\nu(z) = \frac{(1-|z|^2)^{2\beta-1}}{|1-z|^{2\beta}} dA(z)$ is a Carleson measure (see [6, Theorem 9.3]).

A direct computation using the Cauchy–Schwarz inequality shows that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2} d\nu(z) \\
&\quad \lesssim \sup_{a\in\mathbb{D}} (1-|a|^2) \int_0^1 (1-r^2)^{2\beta-1} \left(\int_0^{2\pi} \frac{1}{|1-\bar{a}re^{i\theta}|^2 |1-re^{i\theta}|^{2\beta}} d\theta \right) dr \\
&\quad \lesssim \sup_{a\in\mathbb{D}} \int_0^1 \frac{(1-|a|^2)}{(1-|a|r)^{\frac{3}{2}}(1-r)^{\frac{1}{2}}} < \infty,
\end{aligned}$$

so by [10, Lemma 6.1], ν is a Carleson measure and this finishes the proof of this case.

Case $0 < \gamma < 1$: For this range of gamma values, the space $\mathcal{H}_\gamma$ consists of the Dirichlet space $\mathcal{H}_\gamma = D^2_\gamma$. Observe that $\|z^n\|^2_{D^2_{2\beta+\gamma}} \asymp \frac{1}{(n+1)^{2\beta+\gamma-1}}$, so from (3.1) follows:

$$\|C_\omega(f)\|^2_{\mathcal{H}_\gamma} \lesssim \sum_{n=0}^{\infty} \|z^n\|^2_{D^2_{2\beta+\gamma}} \left(\sum_{k=0}^{n} \widehat{f}(k)\alpha_{n-k}\right)^2 \lesssim \left\| \frac{f(z)}{(1-z)^\beta} \right\|^2_{D^2_{2\beta+\gamma}} \lesssim I + II$$

where

$$I = \int_{\mathbb{D}} \frac{|f'(z)|^2}{|1-z|^{2\beta}} (1-|z|^2)^{\gamma+2\beta} dA(z)$$

and

$$II = \int_{\mathbb{D}} \frac{|f(z)|^2}{|1-z|^{2\beta+2}} (1-|z|^2)^{\gamma+2\beta} dA(z).$$

It is clear that $I \lesssim \|f\|^2_{D^2_\gamma}$. Therefore, the proof of the sufficiency for $0 < \gamma < 1$ boils down to prove the inequality

$$\int_{\mathbb{D}} \frac{|f(z)|^2}{|1-z|^{2\beta+2}} (1-|z|^2)^{\gamma+2\beta} dA(z) \lesssim \|f\|^2_{D^2_\gamma},$$

which is followed from Littlewood–Paley formula (2.1).

In order to simplify notation, let us denote by $\nu(z) = \frac{(1-|z|^2)^{\gamma+2\beta}}{|1-z|^{2\beta+2}}$. It is not difficult to show that $\nu \in C^1(\mathbb{D})$ and it satisfies the regularity condition $(1-|z|^2)|\nabla \nu(z)| \leq k_\nu \nu(z)$. In addition, the weight $\frac{\nu}{(1-|z|)^{2\gamma+2\beta}}$ belongs to the class $B_1^\star(2\gamma+2\beta)$.

Indeed, let $b_n = 1 - \frac{1}{n}$, by Fatou's Lemma and [17, Lemma 2.5],

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|z|^2)^{\gamma+2\beta}}{|1-\overline{a}z|^{2\gamma+2\beta+2}|1-z|^{2\beta+2}}\,dA(z)\\
&\quad \leq \liminf_{n\to\infty} \int_{\mathbb{D}} \frac{(1-|z|^2)^{\gamma+2\beta}}{|1-\overline{a}z|^{2\gamma+2\beta+2}|1-b_n z|^{2\beta+2}}\,dA(z)\\
&\quad \lesssim \liminf_{n\to\infty} \frac{1}{(1-|a|^2)^{\gamma}|1-b_n\overline{a}|^{2+2\beta}}\\
&\quad = \frac{1}{(1-|a|^2)^{\gamma}|1-a|^{2+2\beta}}\\
&= \frac{\nu(a)}{(1-|a|^2)^{2\gamma+2\beta}}.
\end{aligned}$$

This is $\frac{\nu}{(1-|z|)^{2\gamma+2\beta}} \in B_1^\star(2\gamma+2\beta)$, so by Theorem C the proof of the sufficiency is finished.

Conversely, assume $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is bounded. First, we proceed to show $\omega \in \widehat{\mathcal{D}}$. We consider the following family of test functions $f_N(z) = \sum_{n=0}^{N} (n+1)^{\frac{\gamma-1}{2}} z^n$, $N \in \mathbb{N}$. Then, $\|f_N\|^2_{\mathcal{H}_\gamma} \asymp (N+1)$ and

$$\begin{aligned}
\|C_\omega(f_N)\|^2_{\mathcal{H}_\gamma} &\asymp \sum_{n=0}^{\infty}(n+1)^{1-\gamma}\omega_n^2 \left(\sum_{k=0}^{n} \frac{\widehat{f_N}(k)}{2(n-k+1)\omega_{2(n-k)+1}}\right)^2\\
&\geq \sum_{n=7N}^{8N} (n+1)^{1-\gamma}\omega_n^2 \left(\sum_{k=0}^{N} \frac{(k+1)^{\frac{\gamma-1}{2}}}{2(n-k+1)\omega_{2(n-k)+1}}\right)^2\\
&\gtrsim \frac{\omega_{8N}^2}{\omega_{12N}^2} \frac{1}{(N+1)^2} \sum_{n=7N}^{8N} (n+1)^{1-\gamma} \left(\sum_{k=0}^{N}(k+1)^{\frac{\gamma-1}{2}}\right)^2,
\end{aligned}$$

for all $N \in \mathbb{N}$, hence,

$$\|C_\omega(f_N)\|^2_{\mathcal{H}_\gamma} \gtrsim \frac{\omega_{8N}^2}{\omega_{12N}^2} \frac{(N+1)^{\gamma+1}}{(N+1)^2} \sum_{n=7N}^{8N} (n+1)^{1-\gamma} \geq \frac{\omega_{8N}^2}{\omega_{12N}^2}(N+1), \;\; N \in \mathbb{N}.$$

Since $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is bounded, $\omega_{8N} \lesssim \omega_{12N}$, $N \in \mathbb{N}$ and this implies $\omega \in \widehat{\mathcal{D}}$ by Lemma A(v).

Now, to finish the proof, we will prove $\omega \in \mathcal{M}$, which together with $\omega \in \widehat{\mathcal{D}}$ gives $\omega \in \mathcal{D}$ by [20, Theorem 3]. We want to point out that from now on, the letter

$C = C(\gamma, \omega) > 0$ will denote a constant whose value depends on $\gamma > 0$ and ω, but does not depend on M or N, and may change from one occurrence to another.

Consider the family of test functions $f_{N,M}(z) = \sum_{n=0}^{MN} z^n,\ N,\ M \in \mathbb{N}$. On the one hand, observe that $\|f_{N,M}\|_{\mathcal{H}_\gamma}^2 \le C\sum_{n=0}^{MN}(n+1)^{1-\gamma}$, and on the other hand,

$$\begin{aligned}\|C_\omega(f_{M,N})\|_{\mathcal{H}_\gamma}^2 &\ge C\sum_{n=0}^{MN}(n+1)^{1-\gamma}\omega_n^2\left(\sum_{k=0}^{n}\frac{1}{(n-k+1)\omega_{2(n-k)+1}}\right)^2\\ &\ge C\omega_{MN}^2\sum_{n=0}^{MN}(n+1)^{1-\gamma}\left(\sum_{k=0}^{n}\frac{1}{(k+1)\omega_k}\right)^2,\ \ M, N \in \mathbb{N},\end{aligned}$$

hence, by hypothesis,

$$\omega_{MN}^2\left(\frac{1}{\sum_{n=0}^{MN}(n+1)^{1-\gamma}}\sum_{n=0}^{MN}(n+1)^{1-\gamma}\left(\sum_{k=0}^{n}\frac{1}{(k+1)\omega_k}\right)^2\right) \le C,\ \ M, N \in \mathbb{N}.$$

Therefore, using Jensen inequality,

$$\omega_{MN}\left(\frac{1}{\sum_{n=0}^{MN}(n+1)^{1-\gamma}}\sum_{n=0}^{MN}(n+1)^{1-\gamma}\left(\sum_{k=0}^{n}\frac{1}{(k+1)\omega_k}\right)\right) \le C,\ \ M, N \in \mathbb{N},$$

so,

$$\sum_{k=N}^{MN}\frac{1}{(k+1)}\sum_{n=k}^{MN}(n+1)^{1-\gamma} \le C\frac{\omega_N}{\omega_{MN}}\sum_{n=0}^{MN}(n+1)^{1-\gamma},\ \ M, N \in \mathbb{N}.$$

It remains to prove that there exists a sufficiently large $M \in \mathbb{N}$ and $C' > 1$ such that

$$\frac{1}{C}\frac{1}{\left(\sum_{n=0}^{MN}(n+1)^{1-\gamma}\right)}\left(\sum_{k=N}^{MN}\frac{1}{(k+1)}\sum_{n=k}^{MN}(n+1)^{1-\gamma}\right) > C' \text{ for all } N \in \mathbb{N}.$$

Indeed, for $M, N \in \mathbb{N}$

$$\frac{1}{\sum_{n=0}^{MN}(n+1)^{1-\gamma}}\left(\sum_{k=N}^{MN}\frac{1}{(k+1)}\sum_{n=k}^{MN}(n+1)^{1-\gamma}\right)$$
$$\gtrsim\left(\sum_{k=N}^{MN}\frac{1}{(k+1)}-\frac{1}{(MN)^{2-\gamma}}\sum_{k=N}^{MN}(k+1)^{1-\gamma}\right)$$

and due to $\frac{1}{(MN)^{2-\gamma}}\sum_{k=N}^{MN}(k+1)^{1-\gamma}$ is uniformly bounded for all $M, N \in \mathbb{N}$, there exist $C_1 = C_1(\gamma) > 0$ and $C_2 = C_2(\gamma) > 0$ such that

$$\frac{1}{\sum_{n=0}^{MN}(n+1)^{1-\gamma}}\left(\sum_{k=N}^{MN}\frac{1}{(k+1)}\sum_{n=k}^{MN}(n+1)^{1-\gamma}\right) > C_1\log M - C_2,\ \ M, N \in \mathbb{N}.$$

Then, take a sufficiently large $M \in \mathbb{N}$ satisfying $\log M > \frac{2C+C_2}{C_1}$ so that there exists $C' = C'(\omega,\gamma) > 1$ and $M = M(\omega,\gamma) > 1$ such that $\omega_N \geq C'\omega_{MN}$ for all $N \in \mathbb{N}$. This is $\omega \in \mathcal{M}$. □

Proof of Theorem 2 Assume $\omega \in \mathcal{D}$ and note that it is enough proving that there exists a constant $C > 0$ such that $\|C_\omega(f)\|^2_{A^2_\mu} \leq C\|f\|^2_{A^2_\mu}$ for any function $f \in \mathcal{H}(\mathbb{D})$ such that $\widehat{f}(n) \geq 0$, $n \in \mathbb{N}\cup\{0\}$.

By following the proof of (3.1), we obtain there exists $0 < \beta < 1$ such that

$$\|C_\omega(f)\|^2_{A^2_\mu} \lesssim \sum_{n=0}^{\infty}\frac{\mu_{2n+1}}{(n+1)^{2\beta}}\left(\sum_{k=0}^{n}\frac{\widehat{f}(k)}{(n-k+1)^{1-\beta}}\right)^2,$$

and by Lemma B(iii),

$$\|C_\omega(f)\|^2_{A^2_\mu} \lesssim \sum_{n=0}^{\infty}(\mu_{[2\beta]})_{2n+1}\left(\sum_{k=0}^{n}\widehat{f}(k)\alpha_{n-k}\right)^2 \lesssim \left\|\frac{f(z)}{(1-z)^\beta}\right\|^2_{A^2_{\mu_{[2\beta]}}} \lesssim \|f\|^2_{A^2_\mu},$$

where we recall that α_n, $n \in \mathbb{N}$, denote the Taylor coefficients of the function $g(z) = \frac{1}{(1-z)^\beta}$.

Reciprocally, let $\mu \in \widehat{\mathcal{D}}$ and assume $C_\omega : A^2_\mu \to A^2_\mu$ is bounded. First, we will show $\omega \in \widehat{\mathcal{D}}$. Now, we consider the following family of functions $f_N(z) =$

$\sum_{n=0}^{N}(\mu_{2n+1})^{-\frac{1}{2}}z^n$, $N \in \mathbb{N}$. Then, $\|f_N\|^2_{A^2_\mu} \asymp (N+1)$ and on the other hand

$$\begin{aligned}\|C_\omega(f_N)\|^2_{A^2_\mu} &\geq \sum_{n=4N}^{5N} \mu_{2n+1}\omega_n^2 \left(\sum_{k=0}^{N} \frac{\mu_{2k+1}^{-\frac{1}{2}}}{2(n-k+1)\omega_{2(n-k)+1}}\right)^2 \\ &\gtrsim \frac{\omega_{5N}^2}{\omega_{6N}^2}\frac{1}{(N+1)^2}\sum_{n=4N}^{5N}\mu_{2n+1}\left(\sum_{k=0}^{N}\mu_{2k+1}^{-\frac{1}{2}}\right)^2,\end{aligned}$$

for all $N \in \mathbb{N}$, so Lemma A(iv) yields there exists $\alpha = \alpha(\mu) > 0$ such that

$$\|C_\omega(f_N)\|^2_{A^2_\mu} \gtrsim \frac{\omega_{5N}^2}{\omega_{6N}^2}\frac{1}{(N+1)}\frac{\mu_{10N+1}}{\mu_{2N+1}}\left(\sum_{k=0}^{N}\left(\frac{2k+1}{2N+1}\right)^{\frac{\alpha}{2}}\right)^2 \gtrsim (N+1)\frac{\omega_{5N}^2}{\omega_{6N}^2}.$$

The boundedness of C_ω yields $\omega_{5N} \lesssim \omega_{6N}$, for all $N \in \mathbb{N}$ and this implies $\omega \in \widehat{\mathcal{D}}$ by Lemma A (v).

We proceed to prove $\omega \in \mathcal{M}$. Consider the family of functions $f_{N,M}(z) = \sum_{n=0}^{MN}(\mu_{2n+1})^{-\frac{1}{2}}z^n$, $N,\ M \in \mathbb{N}$. As before we obtain $\|f_{N,M}\|^2_{A^2_\mu} \asymp (MN+1)$ and by Lemma A(iv), there exists $\alpha > 2$ such that

$$\begin{aligned}\|C_\omega(f_{N,M})\|^2_{A^2_\mu} &\gtrsim \omega_{MN}^2 \sum_{n=0}^{MN}\mu_{2n+1}\left(\sum_{k=0}^{n}\frac{(\mu_{n-k+1})^{-\frac{1}{2}}}{(k+1)\omega_{k+1}}\right)^2 \\ &\gtrsim \omega_{MN}^2\sum_{n=0}^{MN}\frac{1}{(2n+1)^\alpha}\left(\sum_{k=0}^{n}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(k+1)\omega_{k+1}}\right)^2,\end{aligned}$$

for all $M, N \in \mathbb{N}$. Now, by Jensen inequality and the boundedness of the operator C_ω, it follows

$$\left(\frac{1}{MN+1}\sum_{n=0}^{MN}\frac{1}{(2n+1)^{\frac{\alpha}{2}}}\sum_{k=0}^{n}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(k+1)\omega_{k+1}}\right)^2 \lesssim \frac{1}{\omega_{MN}^2},\quad M, N \in \mathbb{N}.$$

As a consequence,

$$\frac{\omega_{MN}}{\omega_N}\sum_{k=N}^{MN}\frac{1}{k+1}\sum_{n=k}^{MN}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{\alpha}{2}}} \leq C(MN+1).$$

To complete the proof, we will show that there exists $M \in \mathbb{N}$ large enough such that

$$\frac{1}{C}\frac{1}{MN+1}\sum_{k=N}^{MN}\frac{1}{k+1}\sum_{n=k}^{MN}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{\alpha}{2}}} > 2.$$

Since $\alpha \geq 2$, $(n-k+1)^{\frac{\alpha}{2}} \geq (n+1)^{\frac{\alpha}{2}}(1-\frac{k+1}{n+1})^{\frac{\alpha}{2}} \geq (n+1)^{\frac{\alpha}{2}} - \frac{\alpha}{2}(k+1)(n+1)^{\frac{\alpha}{2}-1}$ for all $n \geq k$, so

$$\frac{1}{MN+1}\sum_{k=N}^{MN}\frac{1}{k+1}\sum_{n=k}^{MN}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{\alpha}{2}}}$$
$$\geq \frac{1}{MN+1}\left(\sum_{k=N}^{MN}\frac{MN-k+1}{k+1} - \frac{\alpha}{2}\sum_{k=N}^{MN}\sum_{n=k}^{MN}\frac{1}{n+1}\right) \asymp \sum_{k=N}^{MN}\frac{1}{k+1} - 1$$

and there exist $C_1, C_2 > 0$ such that

$$\frac{1}{MN+1}\sum_{k=N}^{MN}\frac{1}{k+1}\sum_{n=k}^{MN}\frac{(n-k+1)^{\frac{\alpha}{2}}}{(n+1)^{\frac{\alpha}{2}}} \geq C_1 \log M - C_2$$

Therefore, for a fixed $M \in \mathbb{N}$ such that $\log M > \frac{2C+C_2}{C_1}$, there exists $C' > 1$ such that $\omega_N \geq C'\omega_{MN}$ for all $N \in \mathbb{N}$. Then, $\omega \in \mathcal{M}$ and the proof is finished. □

4 Compactness

Once we have described the radial weights such that $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ and $C_\omega : A^2_\mu \to A^2_\mu$, $\mu \in \mathcal{D}$ is bounded, it is natural to think about the compactness of this operator.

Lemma 3 *Let ω be a radial weight and $\{f_k\}_{k=0}^{\infty} \subset \mathcal{H}(\mathbb{D})$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then, $C_\omega(f_k) \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

Proof Let be $M \subset \mathbb{D}$ a compact subset and $K_t^\omega(z) = \frac{1}{z}\int_0^z B_t^\omega(u)\,\mathrm{d}u$. If $z \in M$,

$$|C_\omega(f_k)(z)| \leq \int_0^1 |f_k(tz)||K_t^\omega(z)|\omega(t)\,\mathrm{d}t.$$

By following the proof of [15, Lemma 20], we obtain that there exists a $\rho_0 \in (0, 1)$ such that $M \subset \overline{D(0, \rho_0)}$ and

$$\sup_{\substack{z\in M\\ t\in[0,1)}} |K_t^\omega(z)| \leq C(\omega, \rho_0) < \infty.$$

Let $\varepsilon > 0$. By hypothesis, there exists a $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $tz \in \overline{D(0, \rho_0)}$, $|f_k(tz)| < \varepsilon$. Putting all together, we have that $|C_\omega(f_k)(z)| \leq \varepsilon \cdot C(\omega, \rho_0) \cdot \omega_0$, so $C_\omega(f_k) \to 0$ uniformly on M. □

Bearing in mind the previous lemma and by following a classic argument (see for example [15, Theorem 21]), we claim the following characterization of the compactness holds.

Theorem 4 *Let ω and μ be radial weights, $\gamma > 0$, $\mu \in \mathcal{D}$, and $X \in \{\mathcal{H}_\gamma, A^2_\mu\}$. Then, the following assertions are equivalent:*

(i) *$C_\omega : X \to X$ is compact;*
(ii) *For every sequence $\{f_k\}_{k=0}^{\infty} \subset X$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_X < \infty$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{k\to\infty} \|C_\omega(f_k)\|_X = 0$.*

Once we have the previous result, we are able to show that there does not exist radial weight ω such that $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma, \gamma > 0$ is compact neither $C_\omega : A^2_\mu \to A^2_\mu$, $\mu \in \mathcal{D}$, is compact.

Theorem 5 *Let ω and μ be radial weights, $\gamma > 0$, $\mu \in \mathcal{D}$, and $X \in \{\mathcal{H}_\gamma, A^2_\mu\}$. Then, $C_\omega : X \to X$ is not compact.*

Proof **Case** $X = \mathcal{H}_\gamma$. For each $a \in (0, 1)$, we set

$$f_a(z) = \sum_{n=0}^{\infty} (1-a^2)^{\frac{1}{2}} \frac{a^n}{(n+1)^{\frac{1-\gamma}{2}}} z^n.$$

Consequently, it is obvious that

$$\|f_a\|^2_{\mathcal{H}_\gamma} \asymp \sum_{n=0}^{\infty} |\widehat{f_a}(n)|^2 (n+1)^{1-\gamma} = \sum_{n=0}^{\infty} (1-a^2) a^{2n} = 1, \quad a \in (0,1).$$

Furthermore, it is clear that $f_a \to 0$ as $a \to 1^-$ uniformly on compact subsets of $\mathbb{D}$. In addition, we have

$$\begin{aligned}
\|C_\omega(f_a)\|^2_{\mathcal{H}_\gamma} &\asymp \sum_{n=0}^{\infty} \omega_n^2 \left(\sum_{k=0}^{n} \frac{\widehat{f_a}(k)}{2(n-k+1)\omega_{2(n-k)+1}} \right)^2 (n+1)^{1-\gamma} \\
&\asymp (1-a^2) \sum_{n=0}^{\infty} \omega_n^2 \left(\sum_{k=0}^{n} \frac{a^k (k+1)^{\frac{\gamma-1}{2}}}{(n-k+1)\omega_{2(n-k)+1}} \right)^2 (n+1)^{1-\gamma} \\
&\geq (1-a^2) \sum_{n=0}^{\infty} \frac{\omega_n^2 a^{2n}}{(n+1)^{1+\gamma}} \left(\sum_{k=0}^{n} \frac{(k+1)^{\frac{\gamma-1}{2}}}{\omega_{2(n-k)+1}} \right)^2 \\
&\gtrsim (1-a^2) \sum_{n=0}^{\infty} \frac{\omega_{2n}^2 a^{4n}}{(n+1)^{1+\gamma}} \left(\sum_{k=0}^{n} \frac{(k+1)^{\frac{\gamma-1}{2}}}{\omega_{2(2n-k)+1}} \right)^2 \\
&\geq (1-a^2) \sum_{n=0}^{\infty} \frac{a^{4n}}{(n+1)^{1+\gamma}} \left(\sum_{k=0}^{n} (k+1)^{\frac{\gamma-1}{2}} \right)^2 \\
&\asymp 1,
\end{aligned}$$

so using Theorem 4, we deduce that $C_\omega : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is not a compact operator.

Case $X = A^2_\mu$. For each $a \in (0, 1)$, we consider

$$f_a(z) = \sum_{n=0}^{\infty} (1-a^2)^{\frac{1}{2}} \mu_{2n+1}^{-\frac{1}{2}} a^n z^n.$$

As a result $\|f_a\|^2_{A^2_\mu} \asymp 1$, $a \in (0, 1)$ and it is clear that $f_a \to 0$ as $a \to 1^-$ uniformly on compact subsets of $\mathbb{D}$. By following the argument of the previous case, it is not difficult to show

$$\|C_\omega(f_a)\|^2_{A^2_\mu} \gtrsim (1-a^2) \sum_{n=0}^{\infty} \frac{a^{4n}}{(n+1)^2} \left(\sum_{k=0}^{n} \mu_{2k+1}^{-\frac{1}{2}} \right)^2 \mu_{2n+1}, \quad a \in (0, 1),$$

so Lemma A(iv) yields that there exists $\alpha = \alpha(\mu) > 0$ such that

$$\|C_\omega(f_a)\|^2_{A^2_\mu} \gtrsim (1-a^2) \sum_{n=0}^{\infty} \frac{a^{4n}}{(n+1)^2} \left(\sum_{k=0}^{n} \left(\frac{2k+1}{2n+1} \right)^{\frac{\alpha}{2}} \right)^2 \asymp 1.$$

Therefore, using Theorem 4 again, we deduce that $C_\omega : A^2_\mu \to A^2_\mu$ is not a compact operator. □

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References

1. Aleman, A., Constantin, O.: Spectra of integration operators on weighted Bergman spaces. J. Anal. Math. **109**, 199–231 (2009)
2. Aleman, A., Pott, S., Reguera, M.C.: Characterizations of a limiting class B_∞ of Bekollé–Bonami weights. Rev. Mat. Iberoam. **35**(6), 1677–1692 (2019)
3. Andersen, K.F.: Cesàro averaging operators on Hardy spaces. Proc. R. Soc. Edinb. Sect. A **126**(3), 617–624 (1996)
4. Bao, G., Wulan, H., Zhu, K.: A Hardy–Littlewood theorem for Bergman spaces. Ann. Acad. Sci. Fenn. Math. **43**(2), 807–821 (2018)

5. Blasco, O.: Cesàro-type operators on Hardy spaces. J. Math. Anal. Appl. **529**(2), 26 (2024) (**Paper No. 127017**)
6. Duren, P.: Theory of H^p spaces. In: Pure and Applied Mathematics, vol. 38, pp. 1970 xii+258. Academic Press, New York
7. Galanopoulos, P.: The Cesàro operator on Dirichlet spaces. Acta Sci. Math. (Szeged) **67**(1–2), 411–420 (2001)
8. Galanopoulos, P., Girela, D., Merchán, N.: Cesàro-like operators acting on spaces of analytic functions. Anal. Math. Phys. **12**(2), 29 (2022) (**Paper No. 51**)
9. Galanopoulos, P., Girela, D., Mas, A., Merchán, N.: Operators induced by radial measures acting on the Dirichlet space. Results Math. **78**(3), 106–24 (2023)
10. Girela, D.: Analytic functions of bounded mean oscillation. In: Complex Function Spaces (Mekrijärvi, 1999), pp. 61–170, Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, Joensuu (2001)
11. Hardy, G.H.: Note on a theorem of Hilbert. Math. Z. **6**(3–4), 314–317 (1920)
12. Hardy, G.H.: Notes on some points in the integral calculus LX: an inequality between integrals. Mess. Math. **60**(54), 150–156 (1925)
13. Hardy, G.H.: Notes on some points in the integral calculus LXVI: the arithmetic mean of a Fourier constant. Mess. Math. **58**, 50–52 (1929)
14. Landau, E.: A note on a theorem concerning series of positive terms: extract from a letter from Prof. E. Landau to Prof. I. Schur (communicated by G. H. Hardy). J. Lond. Math. Soc. **1**, 38–39 (1926)
15. Merchán, N., Peláez, J.A., de la Rosa, E.: Hilbert-type operator induced by radial weight on Hardy spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **1**, 2–36 (2024)
16. Miao, J.: The Cesàro operator is bounded on H^p for $0<p<1$. Proc. Am. Math. Soc. **116**, 1017–1019 (1992)
17. Ortega, J.M., Fàbrega, J.: Pointwise multipliers and corona type decomposition in $BMOA$. Ann. Inst. Fourier (Grenoble) **46**(1), 111–137 (1996)
18. Peláez, J.A.: Small weighted Bergman spaces. In: Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, pp. 29–98. Publ. Univ. East. Finl. Rep. Stud. For. Nat. Sci., 22, Univ. East. Finl., Fac. Sci. For., Joensuu (2016)
19. Pavlović, M., Peláez, J.A.: An equivalence for weighted integrals of an analytic function and its derivative. Math. Nachr. **281**(11), 1612–1623 (2008)
20. Peláez, J.A., Rättyä, J.: Bergman projection induced by a radial weight. Adv. Math. **391**, 107950 (2021) (70 p)
21. Peláez, J.A., Rättyä, J.: Weighted Bergman spaces induced by rapidly increasing weights. Mem. Am. Math. Soc. **227**, 1066 (2014)
22. Peláez, J.A., de la Rosa, E.: Hilbert-type operator induced by radial weight. J. Math. Anal. Appl. **495**(1), 124689 (2021)
23. Siskakis, A.G.: Composition semigroups and the Cesàro operator on H^p. J. Lond. Math. Soc. 36 **2**(1), 153–164 (1987)
24. Siskakis, A.G.: On the Bergman space norm of the Cesàro operator. Arch. Math. (Basel) **67**(4), 312–318 (1996)
25. Siskakis, A.G.: The Cesàro operator is bounded on H1. Proc. Am. Math. Soc. **110**(2), 461–462 (1990)
26. Stegenga, D.A.: Multipliers of the Dirichlet space. Ill. J. Math. **24**(1), 113–139 (1980)
27. Stempak, K.: Cesàro averaging operators. Proc. R. Soc. Edinb. Sect. A **124**(1), 121–126 (1994)